# Quantum algorithms

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"Quantum algorithm" means an algorithm that a quantum computer can run.

i.e. a sequence of instructions, where each instruction is in a quantum computer's supported instruction set.

**How do we know which instructions a quantum computer will support?**

Quantum computer type 1 (QC1):
stores many “qubits”;
can efficiently perform
“Hadamard gate”, “$T$ gate”,
“controlled NOT gate”.

**Making these instructions work is the main goal of quantum-computer engineering.**

Combine these instructions
to compute “Toffoli gate”;
. . . “Simon’s algorithm”;
. . . “Shor’s algorithm”;
. . . “Grover’s algorithm”; etc.

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General belief: any QC1 is a QC2. Partial proof: see, e.g., 2011 Jordan–Lee–Preskill "Quantum algorithms for quantum field theories".

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General belief: any QC3 is a QC1. Argument for belief: look, we're building a QC1.

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a nonzero element of $\mathbf{C}^{2^n}$.
Retrieving this vector is tough!

If $n$ qubits have state
$(a_0, a_1, \ldots, a_{2^n-1})$ then
**measuring** the qubits produces
an element of $\{0,1,\ldots,2^n-1\}$
and destroys the state.
Measurement produces element $q$
with probability $|a_q|^2/\sum_r |a_r|^2$.

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$(0, 0, 4, 0, 0, 0, 8, 0) = 4|2\rangle + 8|6\rangle$:
Measurement produces
2 with probability 20%,
6 with probability 80%.

# Fast quantum operations, part 1

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$
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$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$
is measured as $(q_0, q_1, q_2)$,
representing $q = q_0 + 2q_1 + 4q_2$,
with probability $|a_q|^2 / \sum_r |a_r|^2$.

$(a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6)$
is measured as $(q_0 \oplus 1, q_1, q_2)$,
representing $q \oplus 1$,
with probability $|a_q|^2 / \sum_r |a_r|^2$.

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$
$(a_4, a_5, a_6, a_7, a_0, a_1, a_2, a_3)$
is "complementing qubit 2":
$(q_0, q_1, q_2) \mapsto (q_0, q_1, q_2 \oplus 1)$.

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= swapping qubits 0 and 2
$\circ$ complementing qubit 0
$\circ$ swapping qubits 0 and 2.

Similarly: swapping qubits $i, j$.

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"controlled NOT gate":
$(q_0, q_1, q_2) \mapsto (q_0 \oplus q_1, q_1, q_2)$.

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$$a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15},$$
$$a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23},$$
$$a_{24}, a_{25}, a_{26}, a_{27}, a_{28}, a_{29}, a_{30}, a_{31})$$
$$\mapsto (a_0, a_1, a_3, a_2, a_4, a_5, a_7, a_6,$$
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## Reversible computation

Say $p$ is a permutation of $\{0, 1, \ldots, 2^n - 1\}$.

General strategy to compose these fast quantum operations to obtain index permutation $(a_0, a_1, \ldots, a_{2^n-1}) \mapsto (a_{p^{-1}(0)}, a_{p^{-1}(1)}, \ldots, a_{p^{-1}(2^n-1)})$:

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1. Build a traditional circuit to compute $j \mapsto p(j)$ using NOT/XOR/AND gates.

2. Convert into reversible gates: e.g., convert AND into Toffoli.

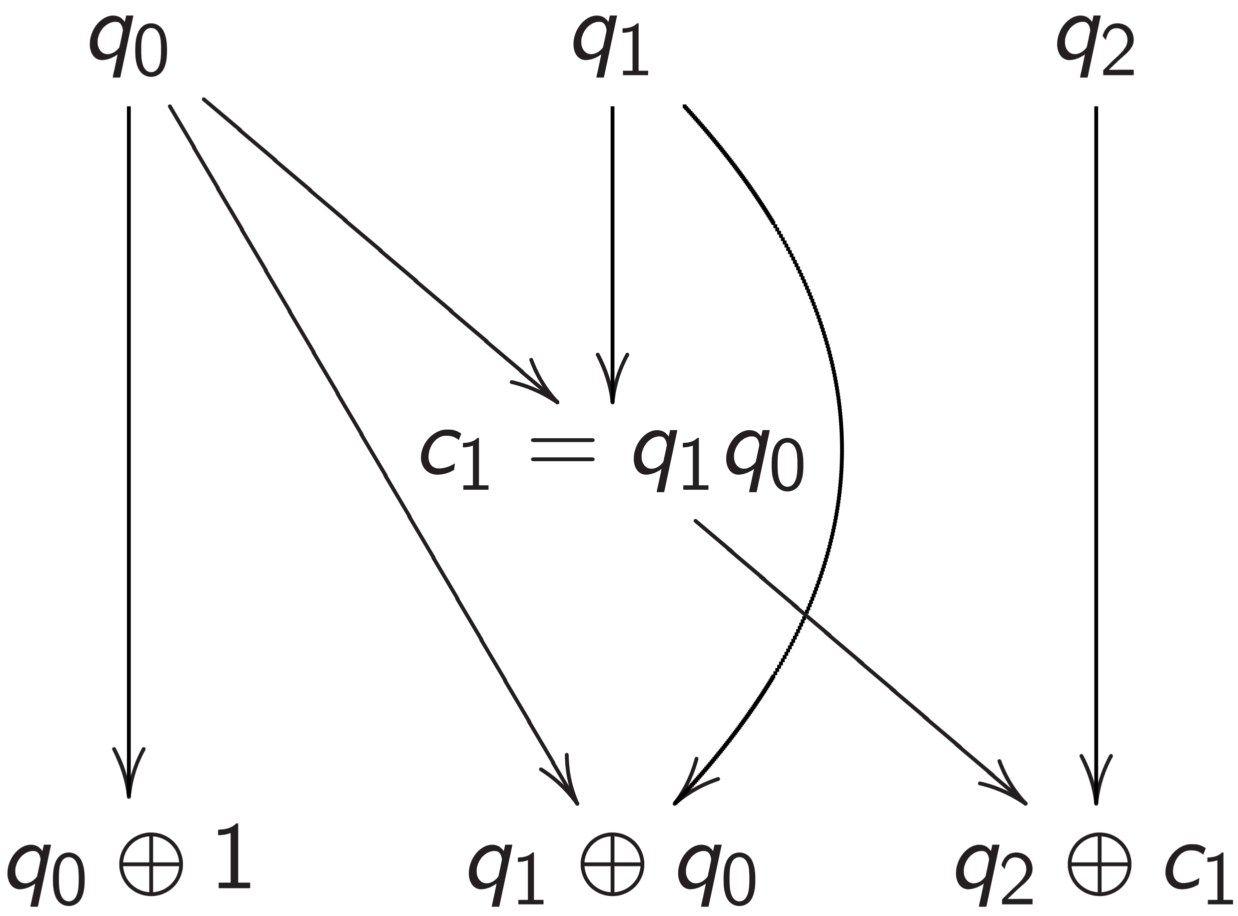
Example: Let's compute

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$

$(a_7, a_0, a_1, a_2, a_3, a_4, a_5, a_6)$;

permutation $q \mapsto q + 1 \bmod 8$.

1. Build a traditional circuit to compute $q \mapsto q + 1 \bmod 8$.



2. Convert into reversible gates.

Toffoli for $q_2 \leftarrow q_2 \oplus q_1 q_0$:
$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$
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NOT for $q_0 \leftarrow q_0 \oplus 1$:
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Also, it didn't need extra storage: circuit operated "in place" after computation $c_1 \leftarrow q_1 q_0$ was merged into $q_2 \leftarrow q_2 \oplus c_1$.

Typical circuits aren't in-place.

Start from any circuit:

inputs $b_1, b_2, \ldots, b_i$;

$b_{i+1} = 1 \oplus b_{f(i+1)} b_{g(i+1)}$;

$b_{i+2} = 1 \oplus b_{f(i+2)} b_{g(i+2)}$;

$\ldots$

$b_T = 1 \oplus b_{f(T)} b_{g(T)}$;

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Reversible but dirty:

inputs $b_1, b_2, \ldots, b_T$;

$b_{i+1} \leftarrow 1 \oplus b_{i+1} \oplus b_{f(i+1)} b_{g(i+1)}$;

$b_{i+2} \leftarrow 1 \oplus b_{i+2} \oplus b_{f(i+2)} b_{g(i+2)}$;

...

$b_T \leftarrow 1 \oplus b_T \oplus b_{f(T)} b_{g(T)}$.

Same outputs if all of

$b_{i+1}, \ldots, b_T$ started as 0.

Reversible and clean:
after finishing dirty computation,
set non-outputs back to 0,
by repeating same operations
on non-outputs in reverse order.

Original computation:
(inputs) $\mapsto$
(inputs, dirt, outputs).

Dirty reversible computation:
(inputs, zeros, zeros) $\mapsto$
(inputs, dirt, outputs).

Clean reversible computation:
(inputs, zeros, zeros) $\mapsto$
(inputs, zeros, outputs).

Given fast circuit for $p$
and fast circuit for $p^{-1}$,
build fast reversible circuit for
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Replace reversible bit operations
with Toffoli gates etc.
permuting $\mathbf{C}^{2^{n+z}} \to \mathbf{C}^{2^{n+z}}$.

Permutation on first $2^n$ entries is
$(a_0, a_1, \ldots, a_{2^n-1}) \mapsto$
$(a_{p^{-1}(0)}, a_{p^{-1}(1)}, \ldots, a_{p^{-1}(2^n-1)})$.

Typically prepare vectors
supported on first $2^n$ entries
so don't care how permutation
acts on last $2^{n+z} - 2^n$ entries.

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$\approx$ number of **bit operations**
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Crude "poly-time" analyses
don't care about this,
but serious cryptanalysis
is much more precise.

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Qubit 0 and then qubit 1:

$(a_0, a_1, a_2, a_3) \mapsto$
$(a_0 + a_1, a_0 - a_1, a_2 + a_3, a_2 - a_3) \mapsto$
$(a_0 + a_1 + a_2 + a_3, a_0 - a_1 + a_2 - a_3,$
$a_0 + a_1 - a_2 - a_3, a_0 - a_1 - a_2 + a_3)$.

Repeat $n$ times: e.g.,
$(1, 0, 0, \ldots, 0) \mapsto (1, 1, 1, \ldots, 1)$.

Measuring $(1, 0, 0, \ldots, 0)$
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Aside from "normalization"
(irrelevant to measurement),
have Hadamard = Hadamard$^{-1}$,
so easily work backwards
from "uniform superposition"
$(1, 1, 1, \ldots, 1)$ to "pure state"
$(1, 0, 0, \ldots, 0)$.

## Simon's algorithm

Assume: nonzero $s \in \{0,1\}^n$
satisfies $f(x) = f(x \oplus s)$
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Traditional solution:
Compute $f$ for many inputs,
sort, analyze collisions.
Success probability is very low
until #inputs approaches $2^{n/2}$.

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vector $(1, 0, 0, \ldots)$.

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Use $n$-fold Hadamard
to move first $n$ qubits
into uniform superposition:
$(1, 1, 1, \ldots, 1, 0, 0, \ldots)$
with $2^n$ entries 1, others 0.

Apply fast vector permutation
for reversible $f$ computation:
1 in position $(q, 0, 0)$
moves to position $(q, f(q), 0)$.

Note symmetry between
1 at $(q, f(q), 0)$ and
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Repeat $n + 10$ times.
Use Gaussian elimination
to (probably) find $s$.

# Example, 3 bits to 3 bits:

$f(0) = 4.$

$f(1) = 7.$

$f(2) = 2.$

$f(3) = 3.$

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$f(5) = 4.$

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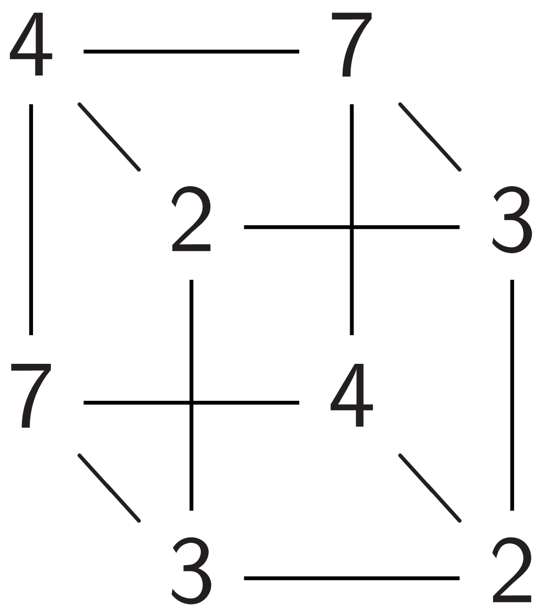

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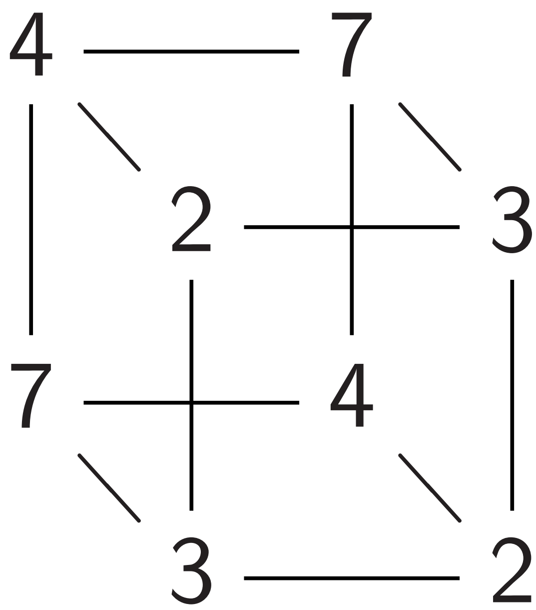



Complete table shows that $f(x) = f(x \oplus 5)$ for all $x$.

Let's watch Simon's algorithm for $f$, using 6 qubits.

Step 1. Set up pure zero state:

1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Step 2. Hadamard on qubit 0:

1, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Step 3. Hadamard on qubit 1:

1, 1, 1, 1, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Step 4. Hadamard on qubit 2:

1, 1, 1, 1, 1, 1, 1, 1,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Step 5. $(q, 0) \mapsto (q, f(q))$:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 1,

0, 0, 0, 1, 0, 0, 1, 0,

1, 0, 0, 0, 0, 1, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 1, 0, 0, 1, 0, 0, 0.

# Step 6. Hadamard on qubit 0:

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 1, 1, 0, 0, 1, \overline{1},$

$0, 0, 1, \overline{1}, 0, 0, 1, 1,$

$1, 1, 0, 0, 1, \overline{1}, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, \overline{1}, 0, 0, 1, 1, 0, 0.$

Notation: $\overline{1} = -1$.

Step 7. Hadamard on qubit 1:

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, 1, \overline{1}, \overline{1}, 1, \overline{1}, \overline{1}, 1,$

$1, \overline{1}, \overline{1}, 1, 1, 1, \overline{1}, \overline{1},$

$1, 1, 1, 1, 1, \overline{1}, 1, \overline{1},$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, \overline{1}, 1, \overline{1}, 1, 1, 1, 1.$

Step 8. Hadamard on qubit 2:

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$2, 0, \overline{2}, 0, 0, \overline{2}, 0, \overline{2},$

$2, 0, \overline{2}, 0, 0, \overline{2}, 0, 2,$

$2, 0, 2, 0, 0, 2, 0, 2,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$2, 0, 2, 0, 0, \overline{2}, 0, \overline{2}.$

Step 8. Hadamard on qubit 2:

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$2, 0, \overline{2}, 0, 0, \overline{2}, 0, \overline{2},$

$2, 0, \overline{2}, 0, 0, \overline{2}, 0, 2,$

$2, 0, 2, 0, 0, 2, 0, 2,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$2, 0, 2, 0, 0, \overline{2}, 0, \overline{2}.$

Step 9. Measure.

First 3 qubits are uniform random vector orthogonal to 101: i.e., 000, 010, 101, or 111.

# Grover's algorithm

Assume: unique $s \in \{0,1\}^n$
has $f(s) = 0$.

Traditional algorithm to find $s$:
compute $f$ for many inputs,
hope to find output 0.
Success probability is very low
until #inputs approaches $2^n$.

# Grover's algorithm

Assume: unique $s \in \{0,1\}^n$ has $f(s) = 0$.

Traditional algorithm to find $s$: compute $f$ for many inputs, hope to find output 0. Success probability is very low until #inputs approaches $2^n$.

Grover's algorithm takes only $2^{n/2}$ reversible computations of $f$. Typically: reversibility overhead is small enough that this easily beats traditional algorithm.

Start from uniform superposition over all $n$-bit strings $q$.

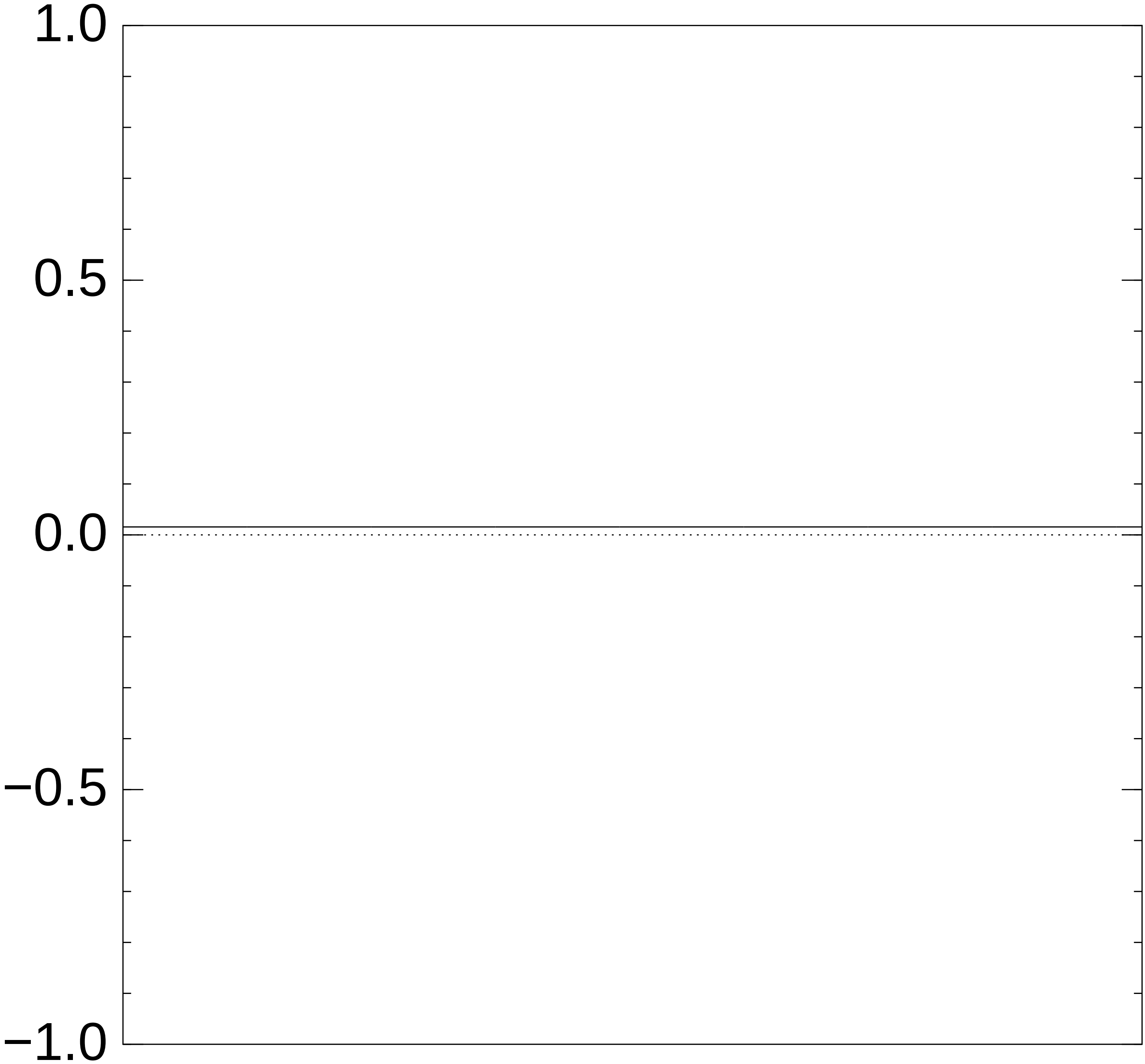
Step 1: Set $a \leftarrow b$ where
$b_q = -a_q$ if $f(q) = 0$,
$b_q = a_q$ otherwise.
This is fast.

Step 2: "Grover diffusion".
Negate $a$ around its average.
This is also fast.

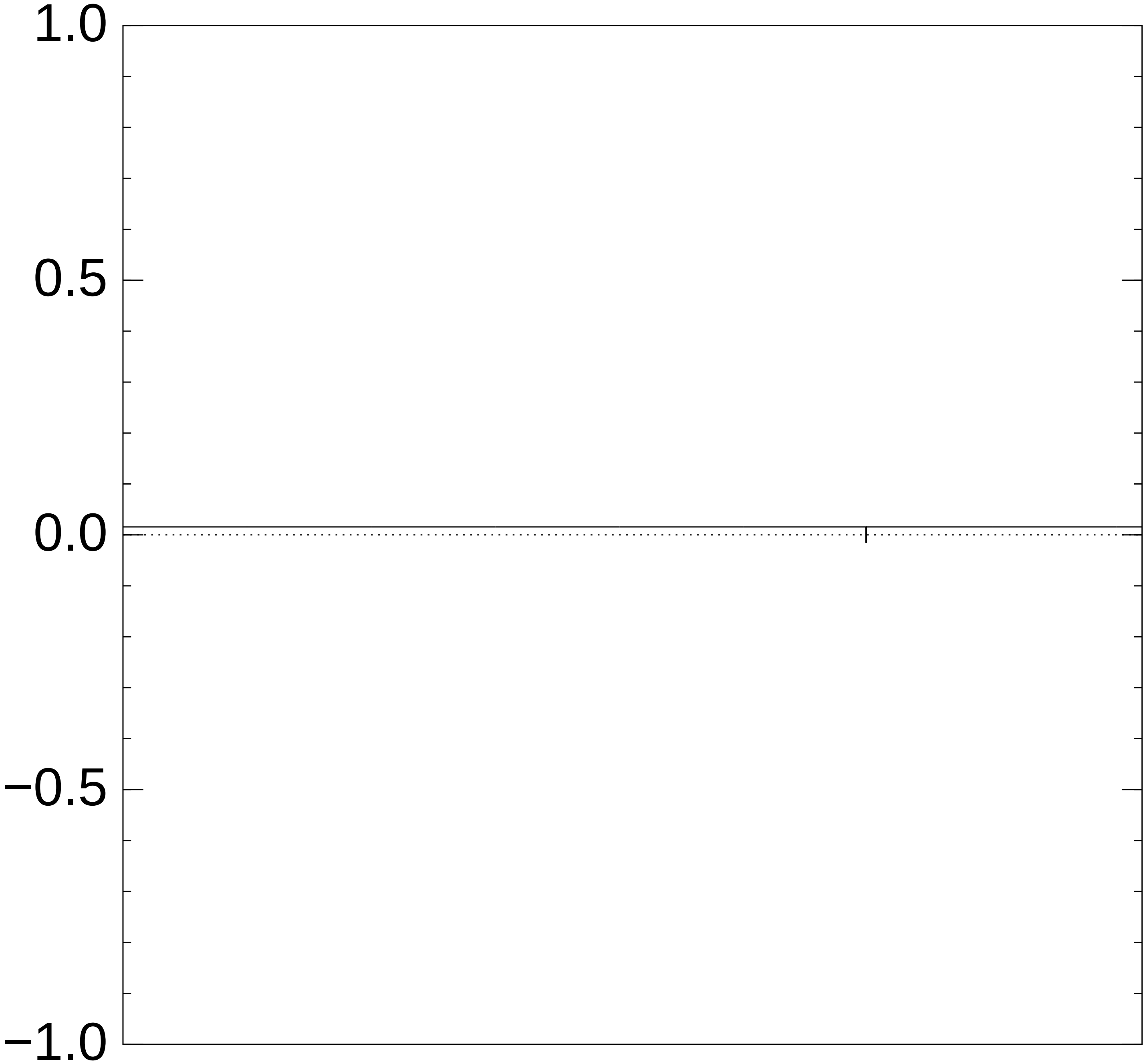
Repeat Step 1 + Step 2 about $0.58 \cdot 2^{0.5n}$ times.

Measure the $n$ qubits.
With high probability this finds $s$.

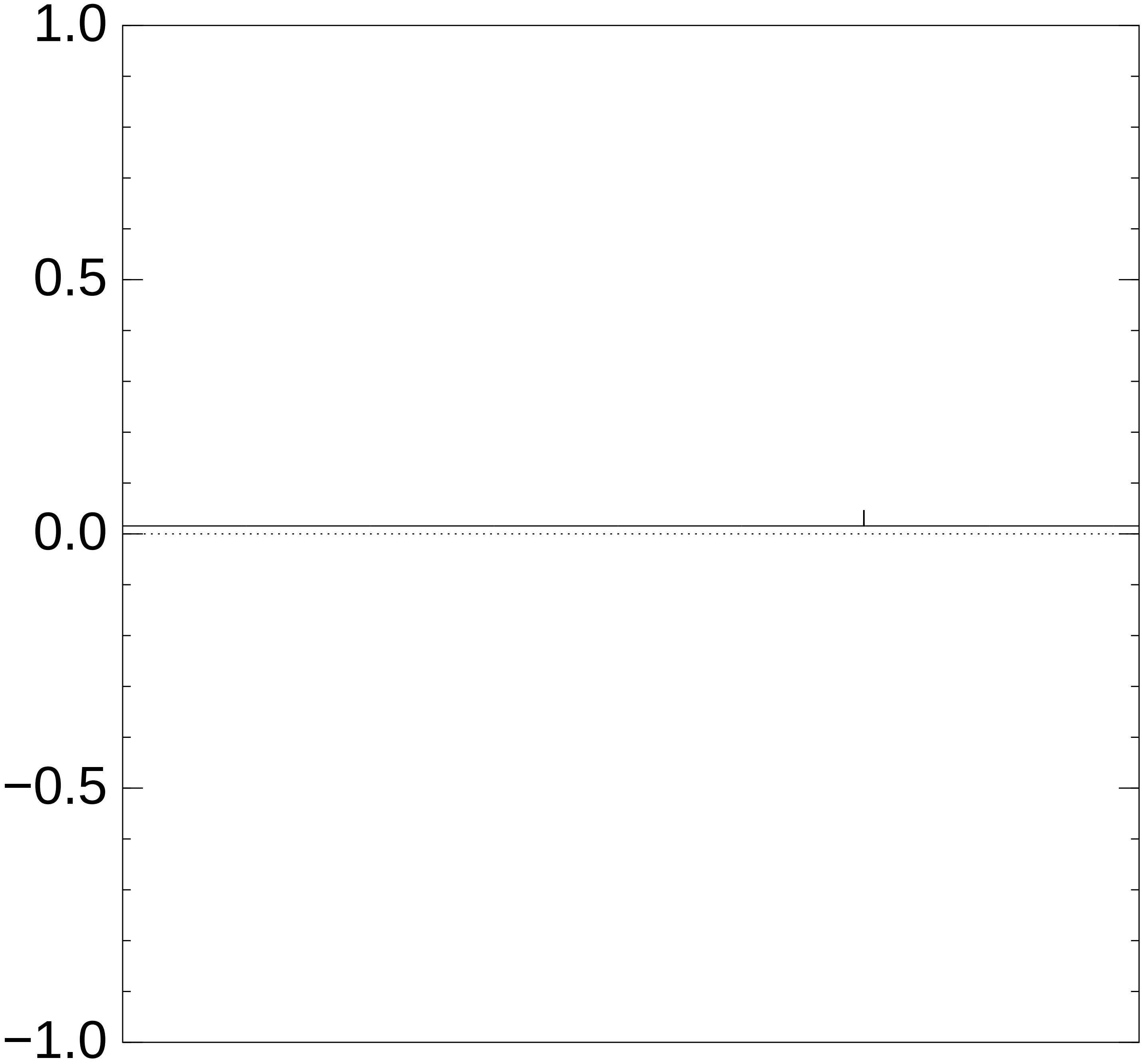
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after 0 steps:



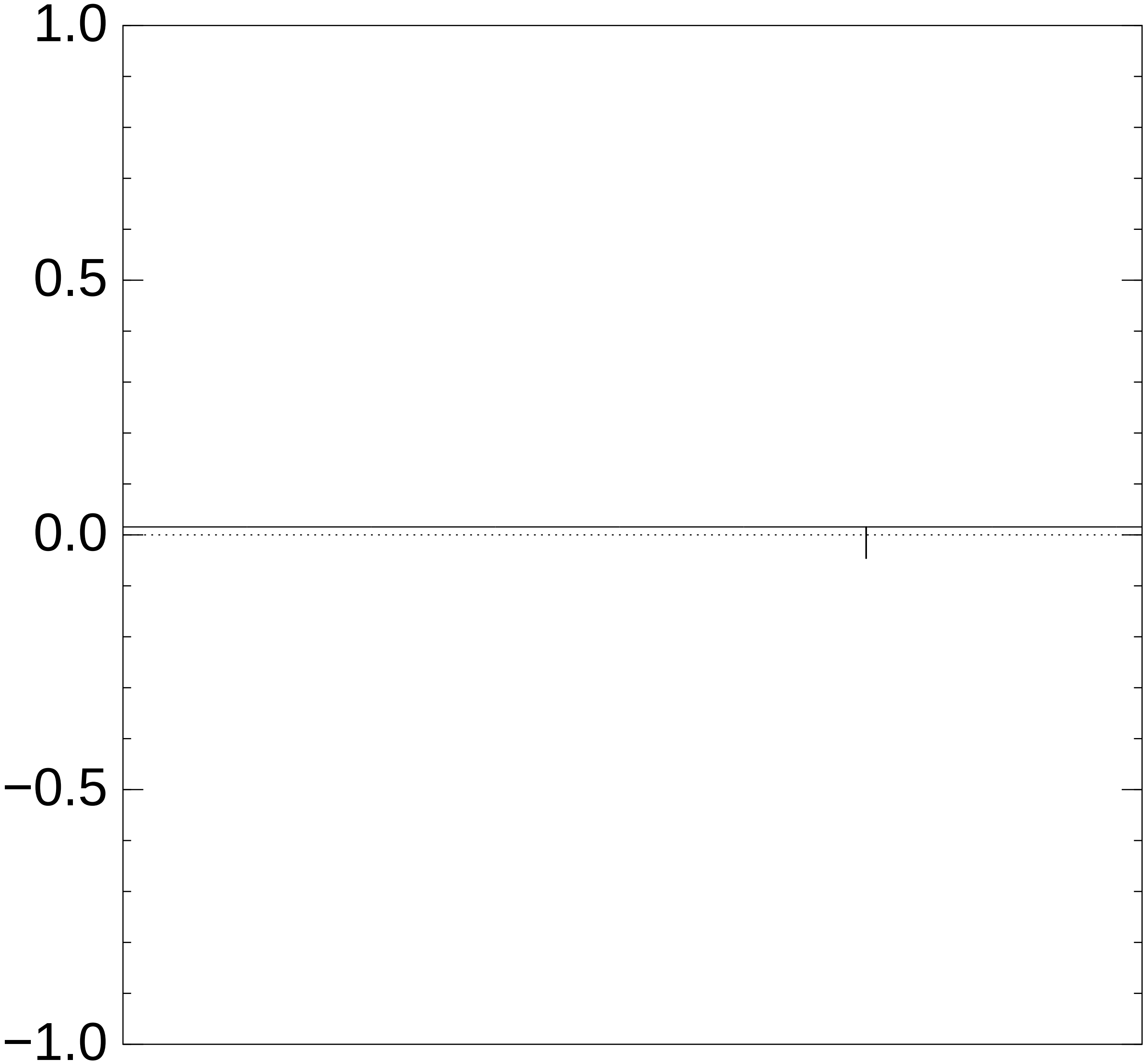
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after Step 1:



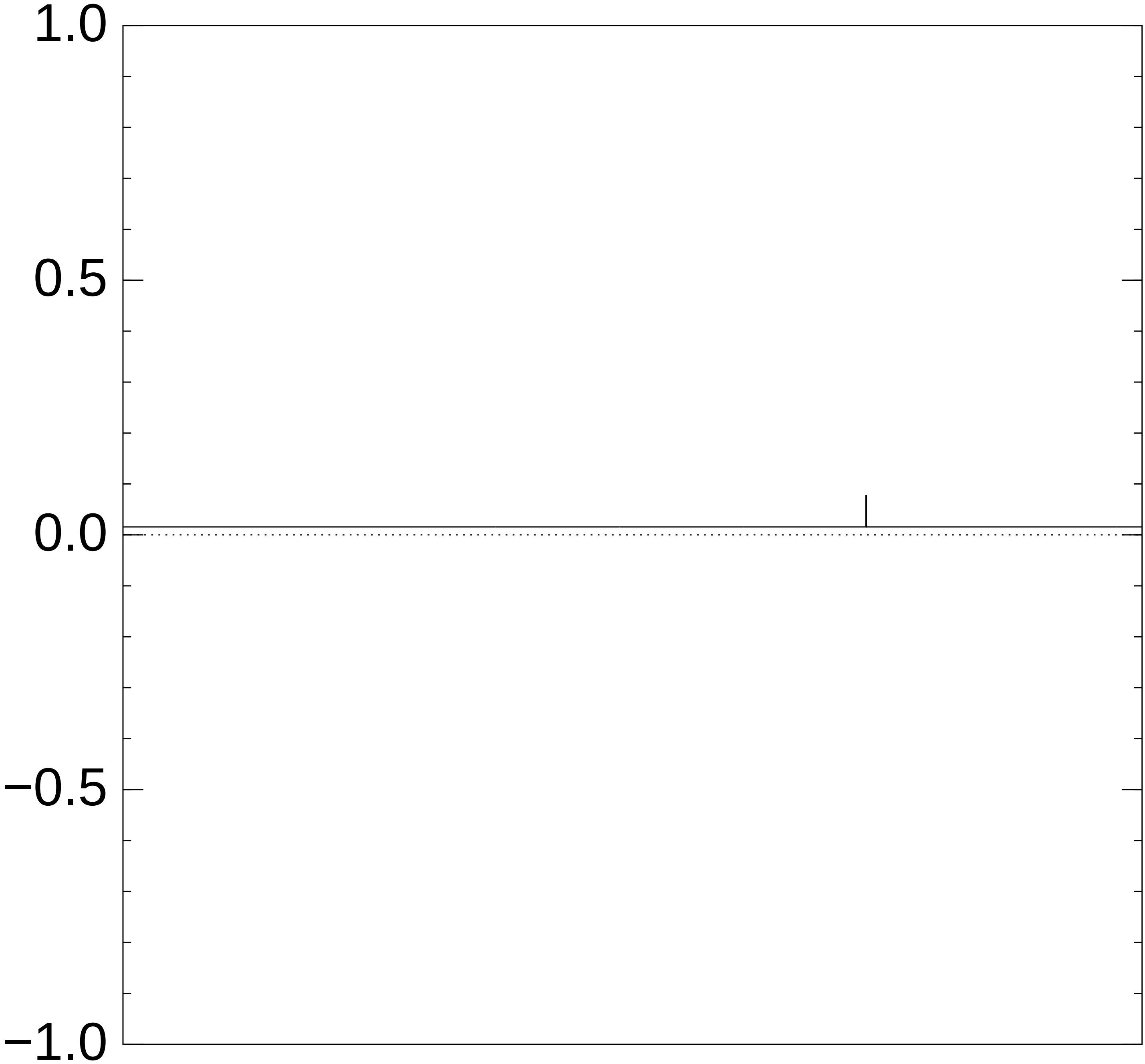
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after Step 1 + Step 2:



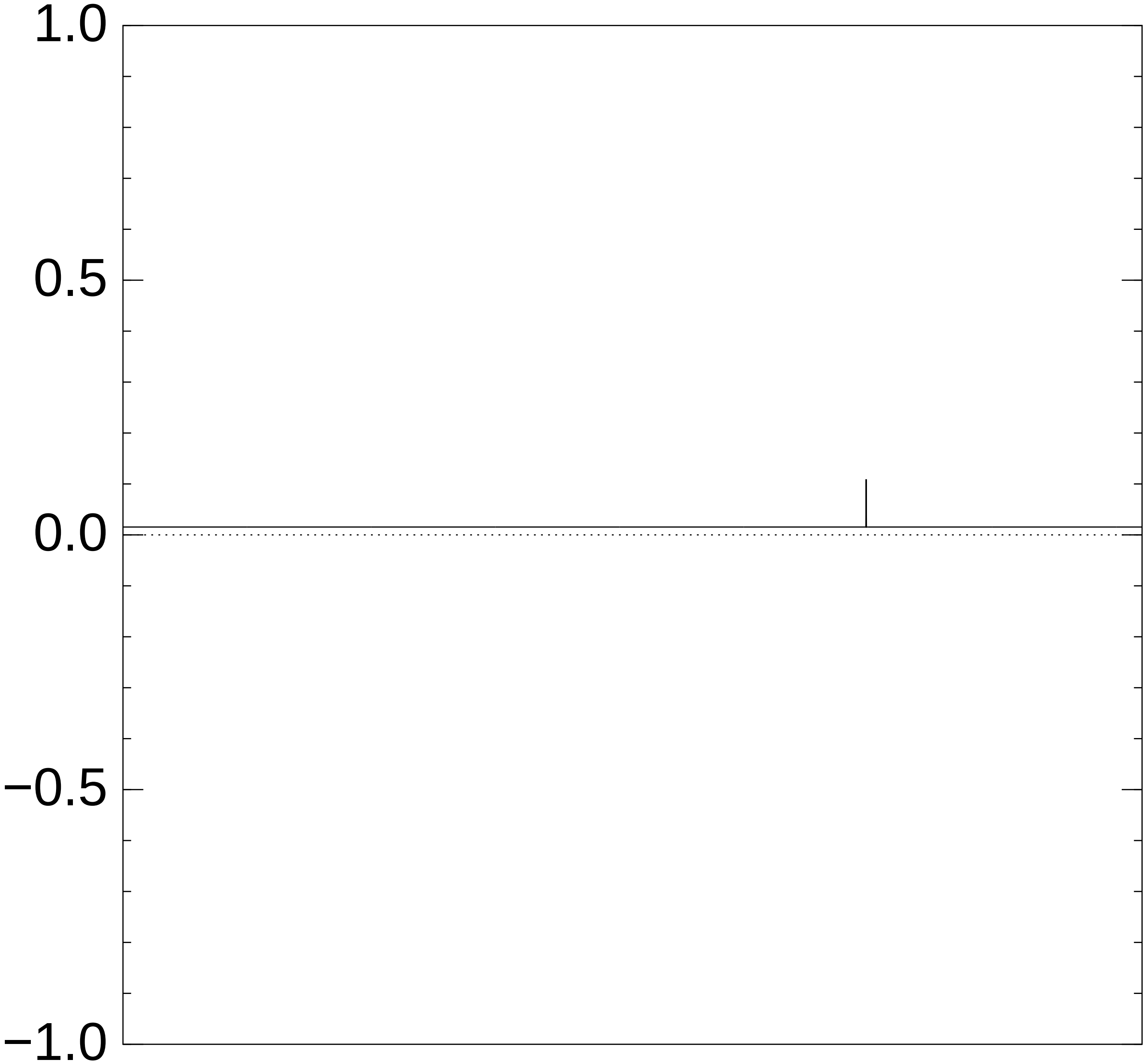
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after Step 1 + Step 2 + Step 1:



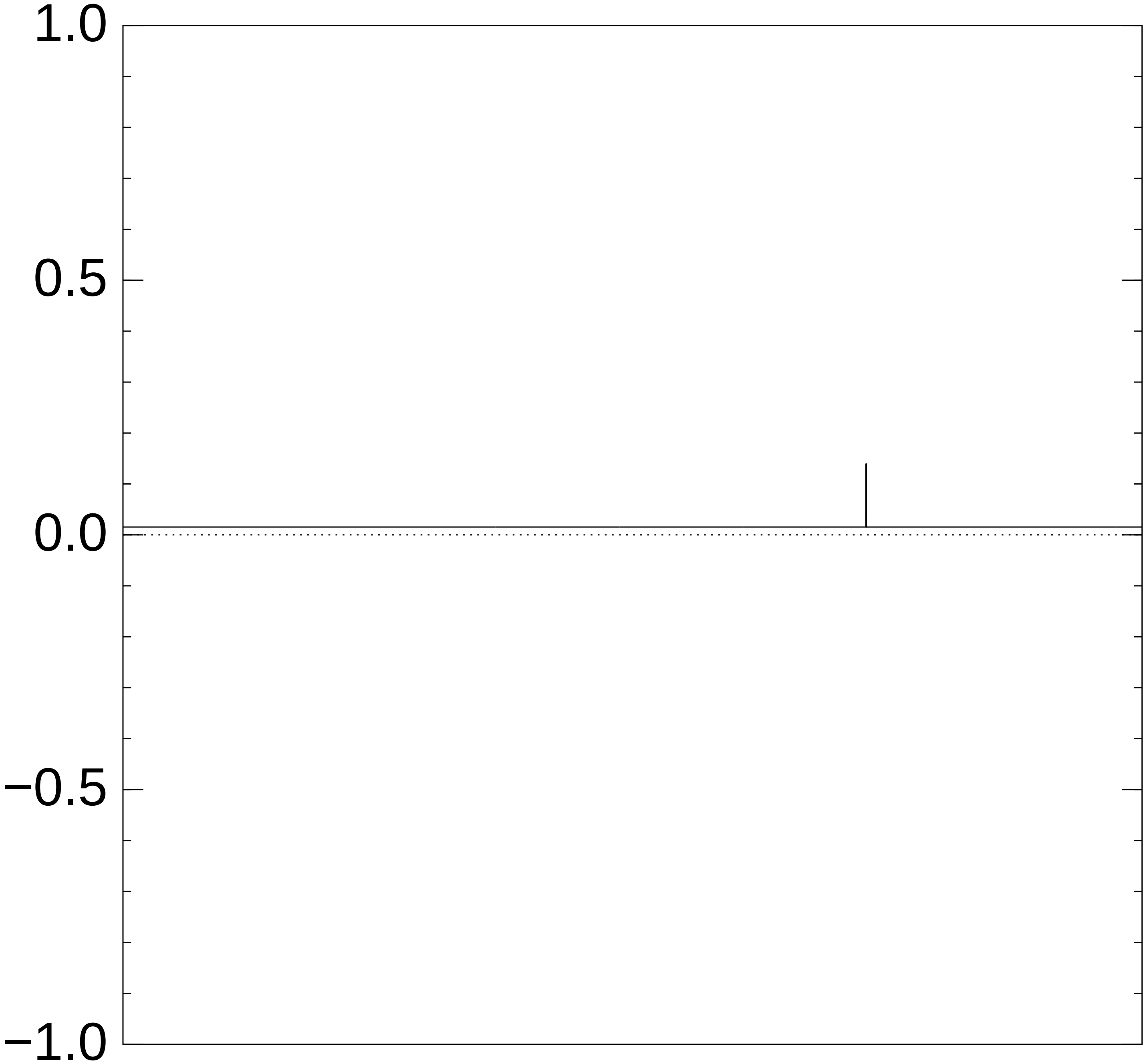
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $2 \times$ (Step 1 + Step 2):



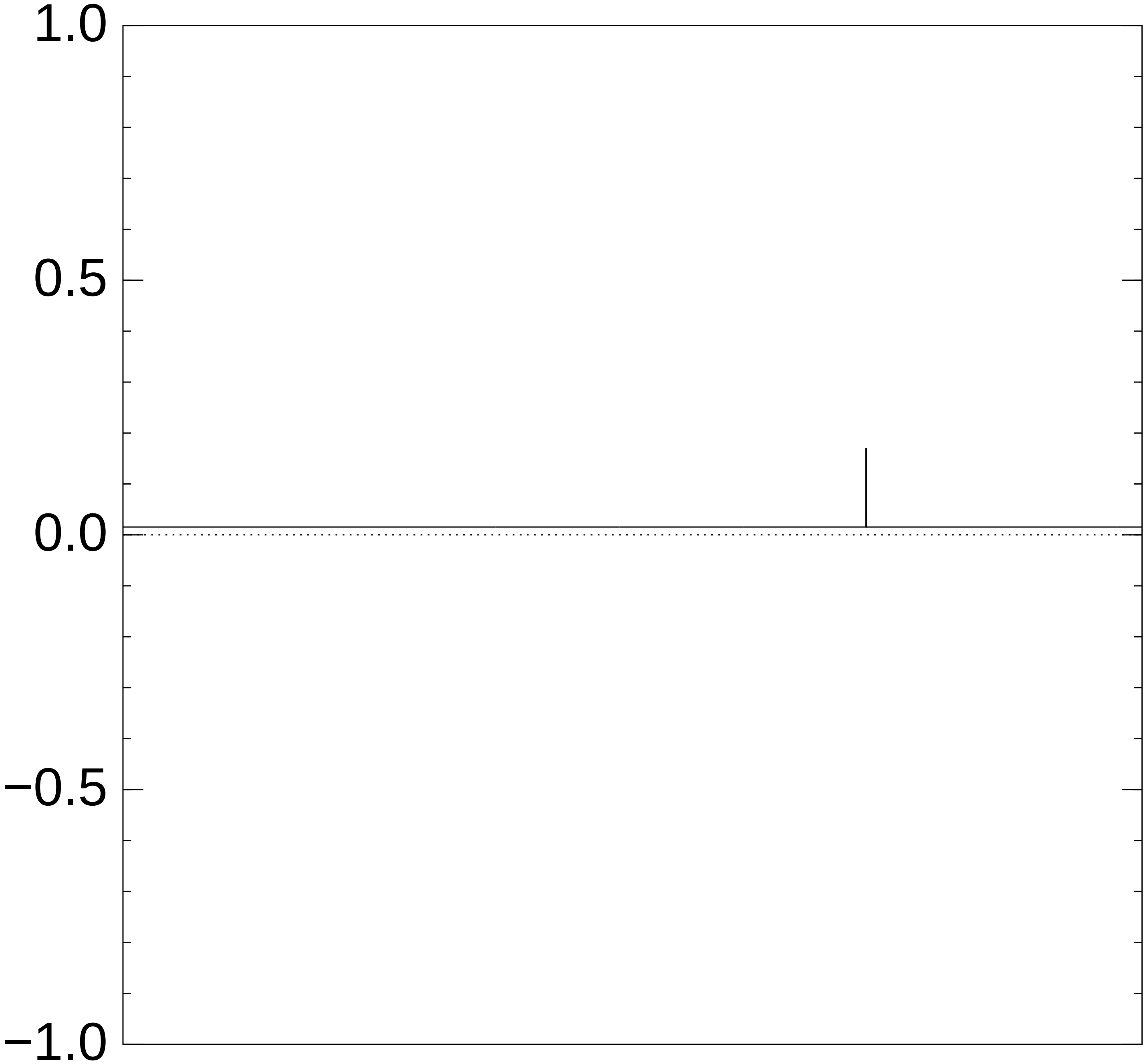
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $3 \times (\text{Step } 1 + \text{Step } 2)$:



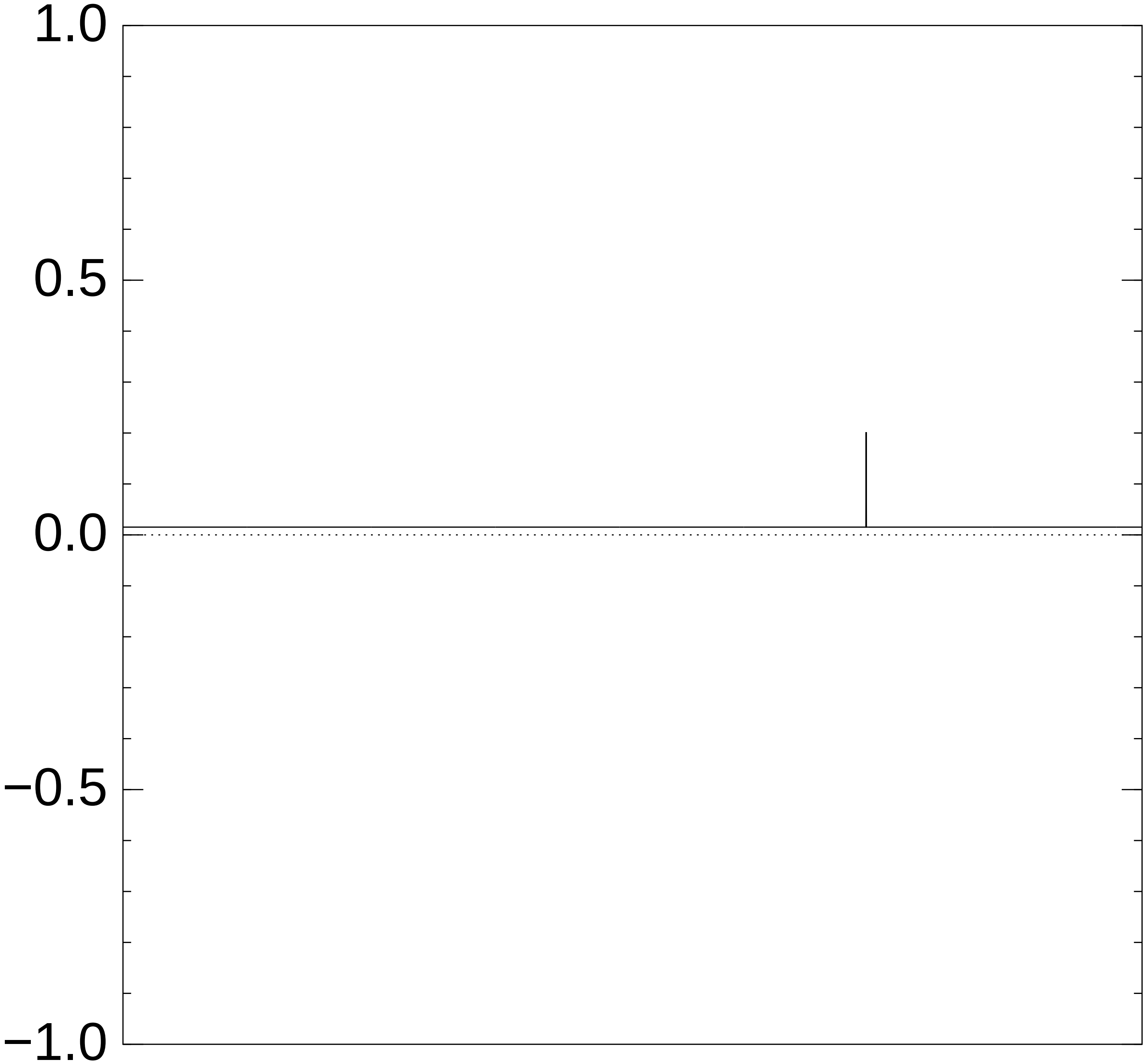
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $4 \times (\text{Step } 1 + \text{Step } 2)$:



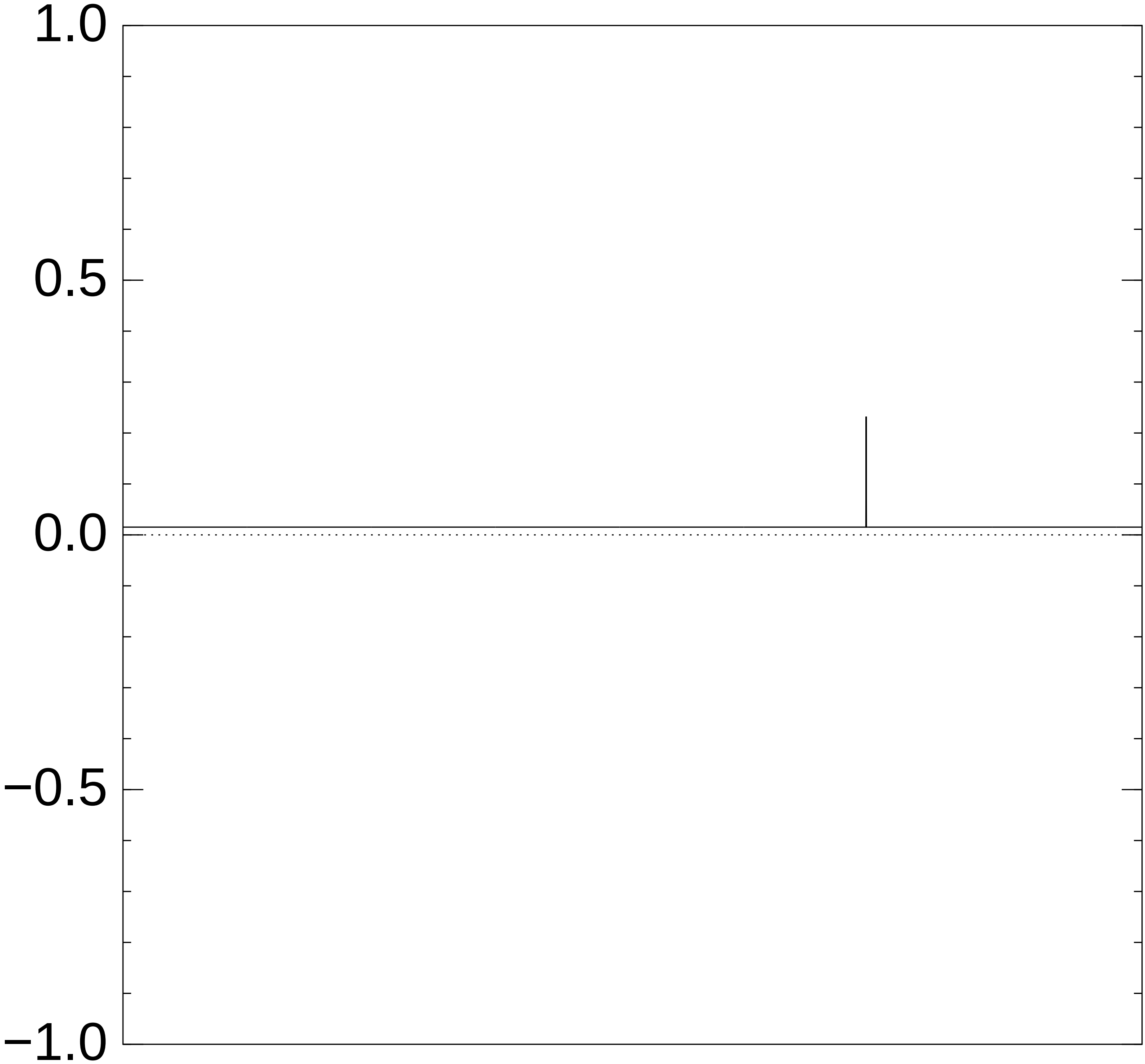
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $5 \times (\text{Step } 1 + \text{Step } 2)$:



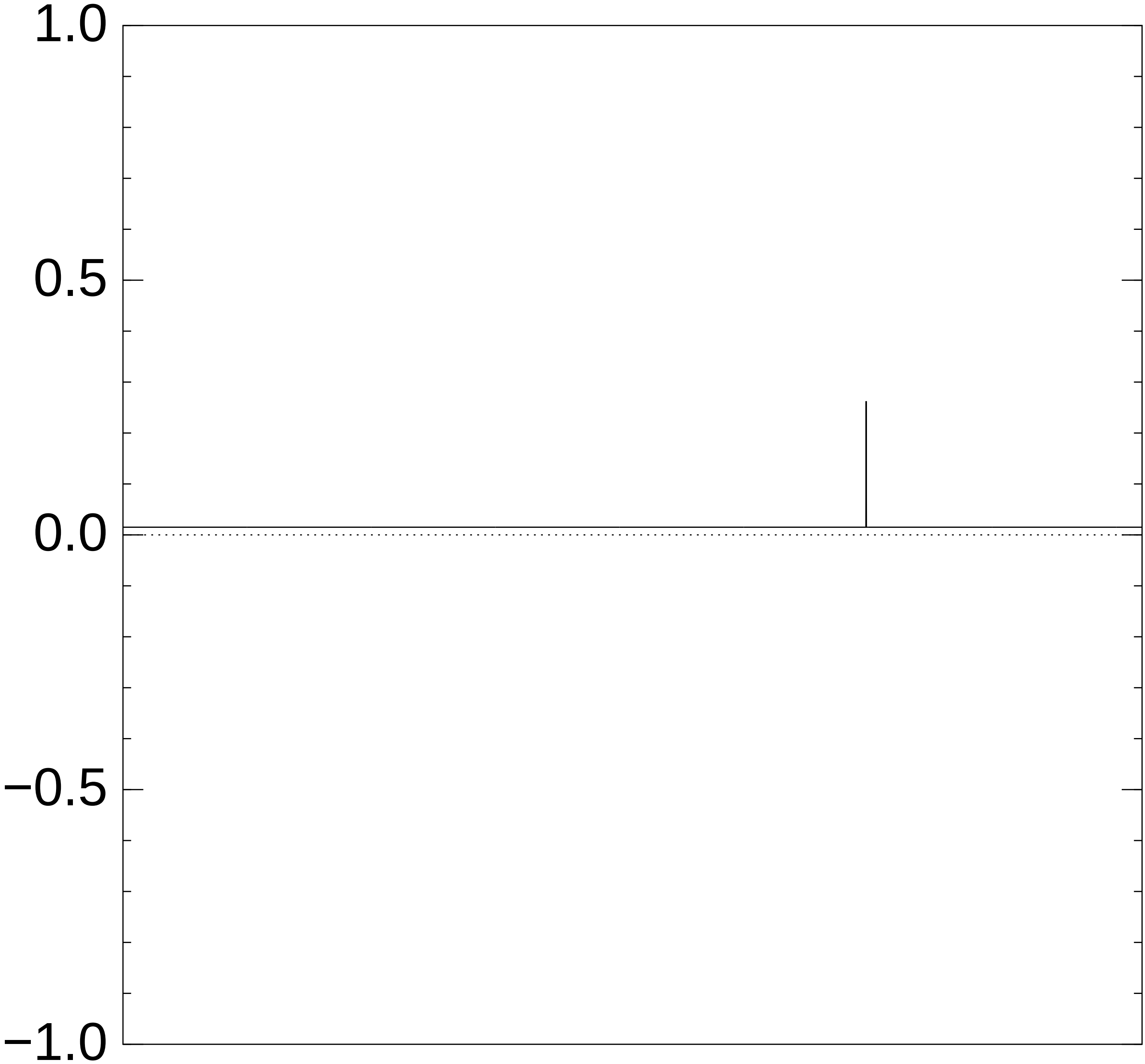
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $6 \times (\text{Step 1} + \text{Step 2})$:



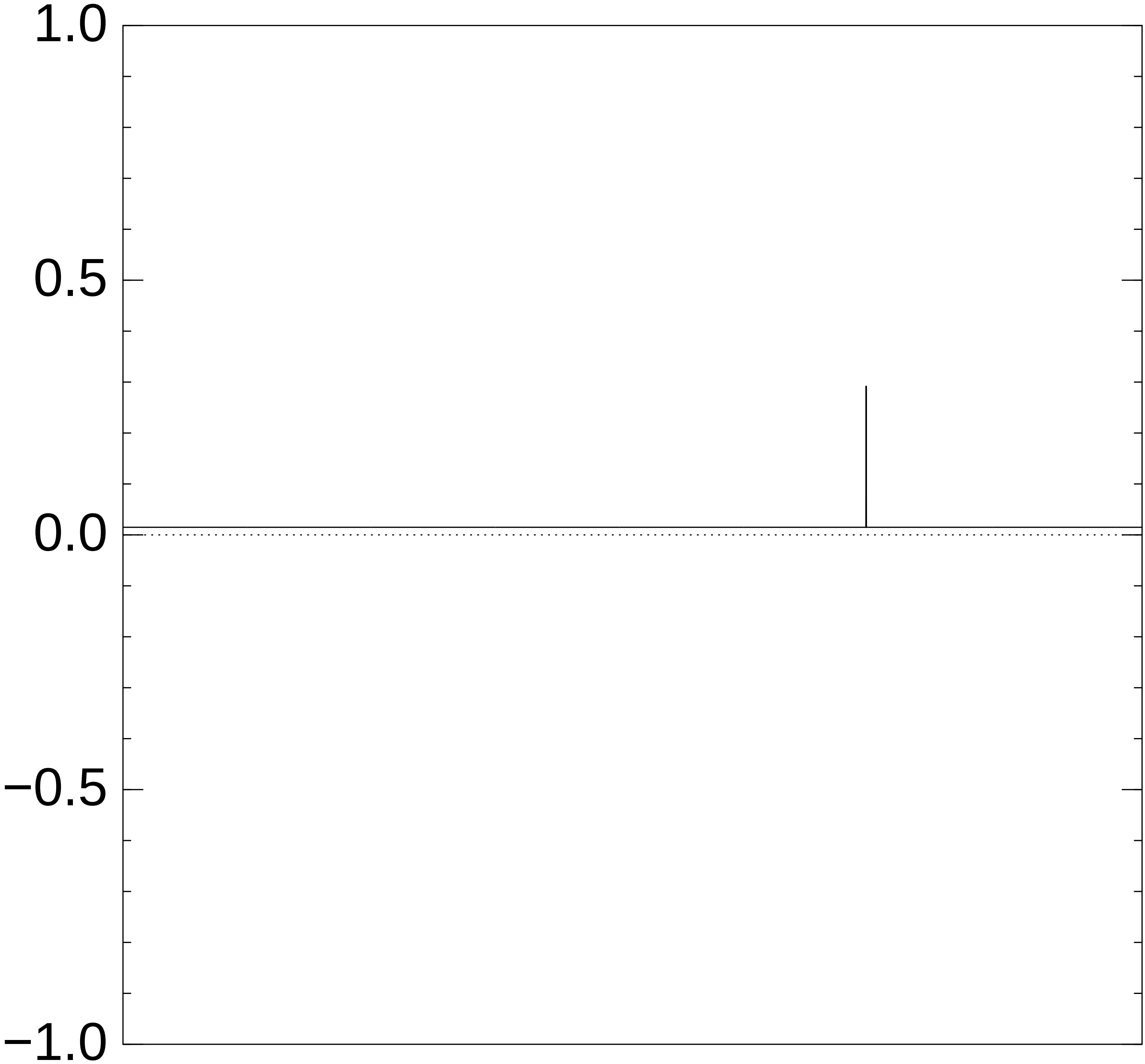
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $7 \times$ (Step $1 +$ Step 2):



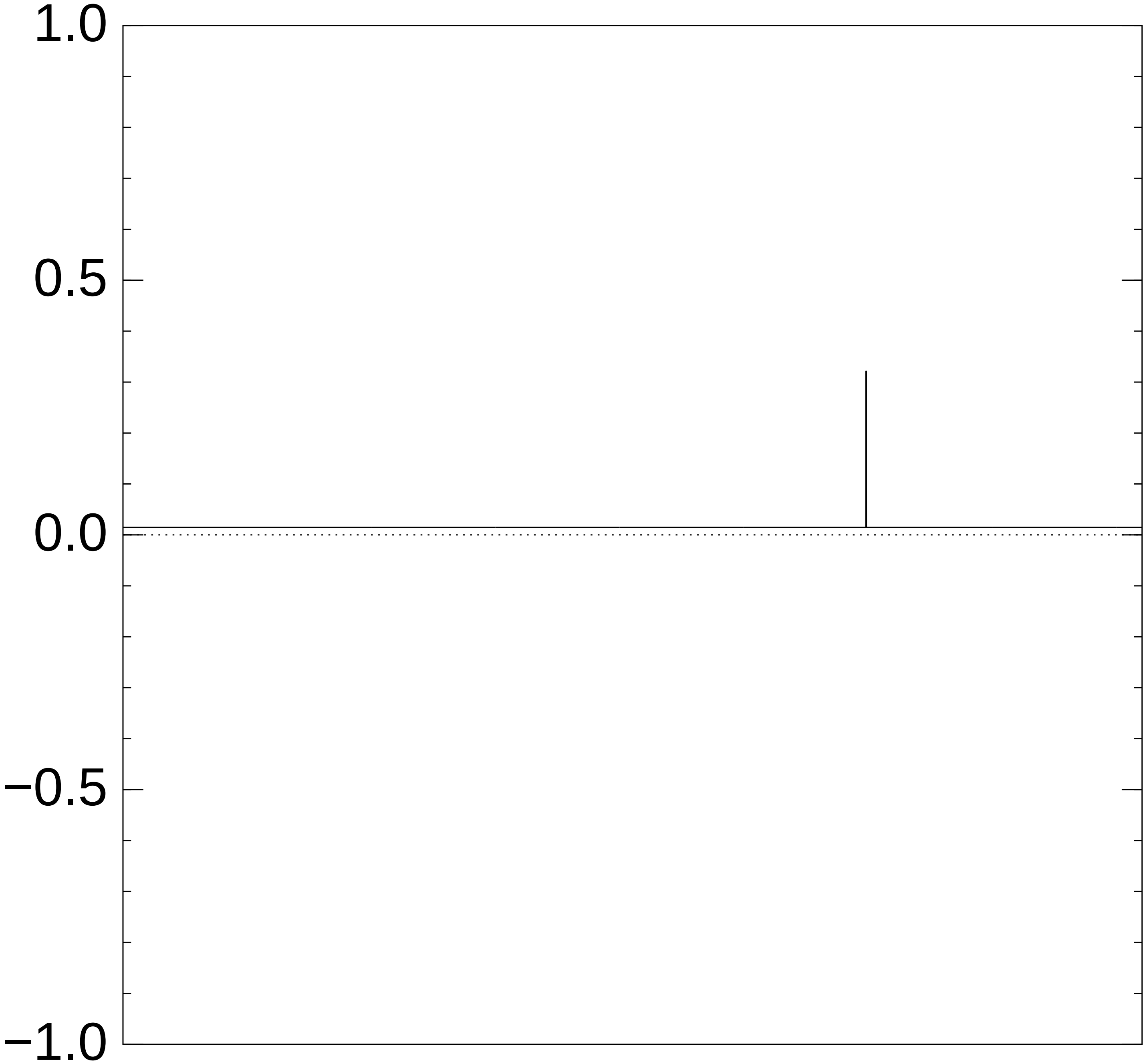
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $8 \times (\text{Step } 1 + \text{Step } 2)$:



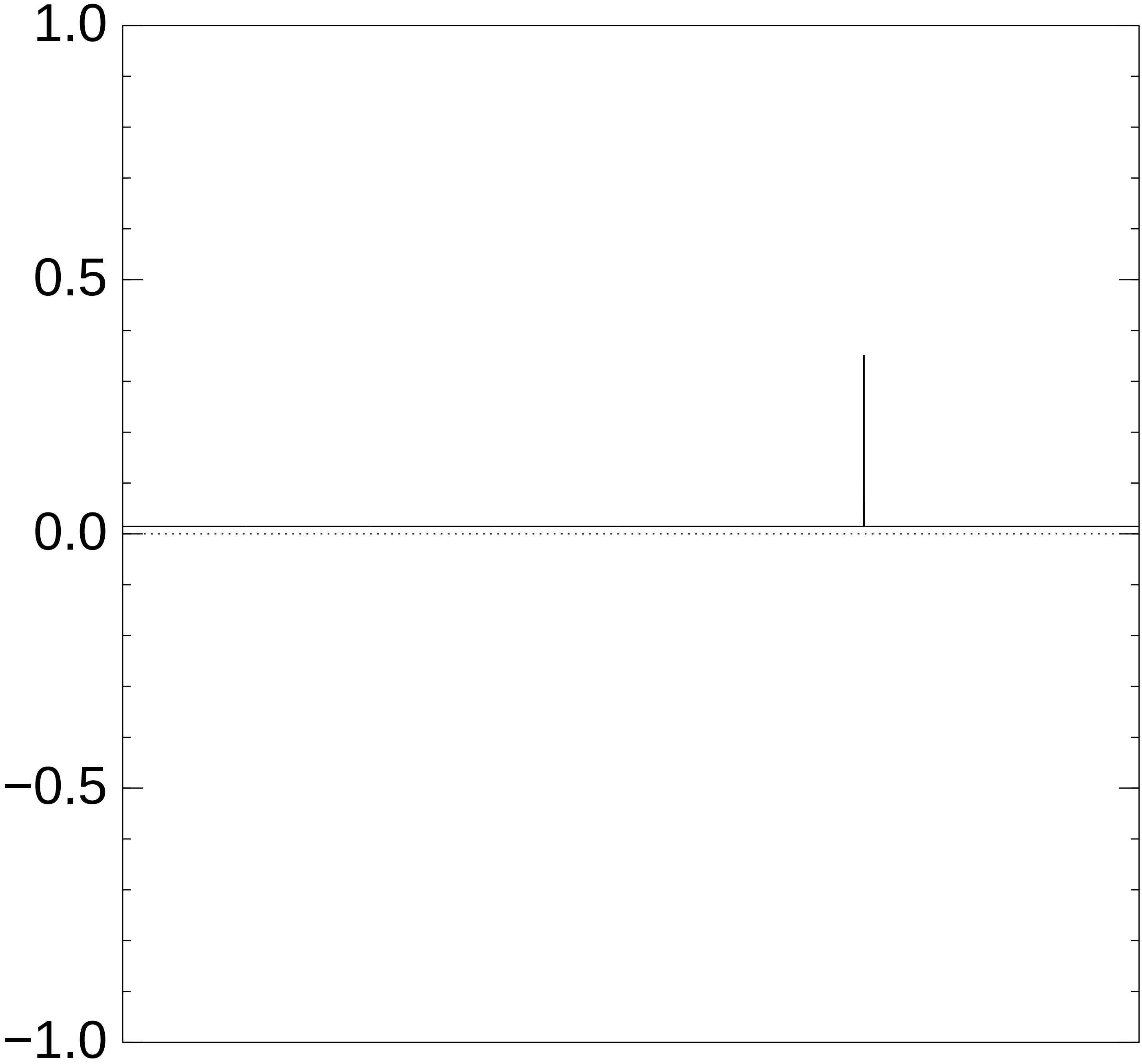
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $9 \times (\text{Step } 1 + \text{Step } 2)$:



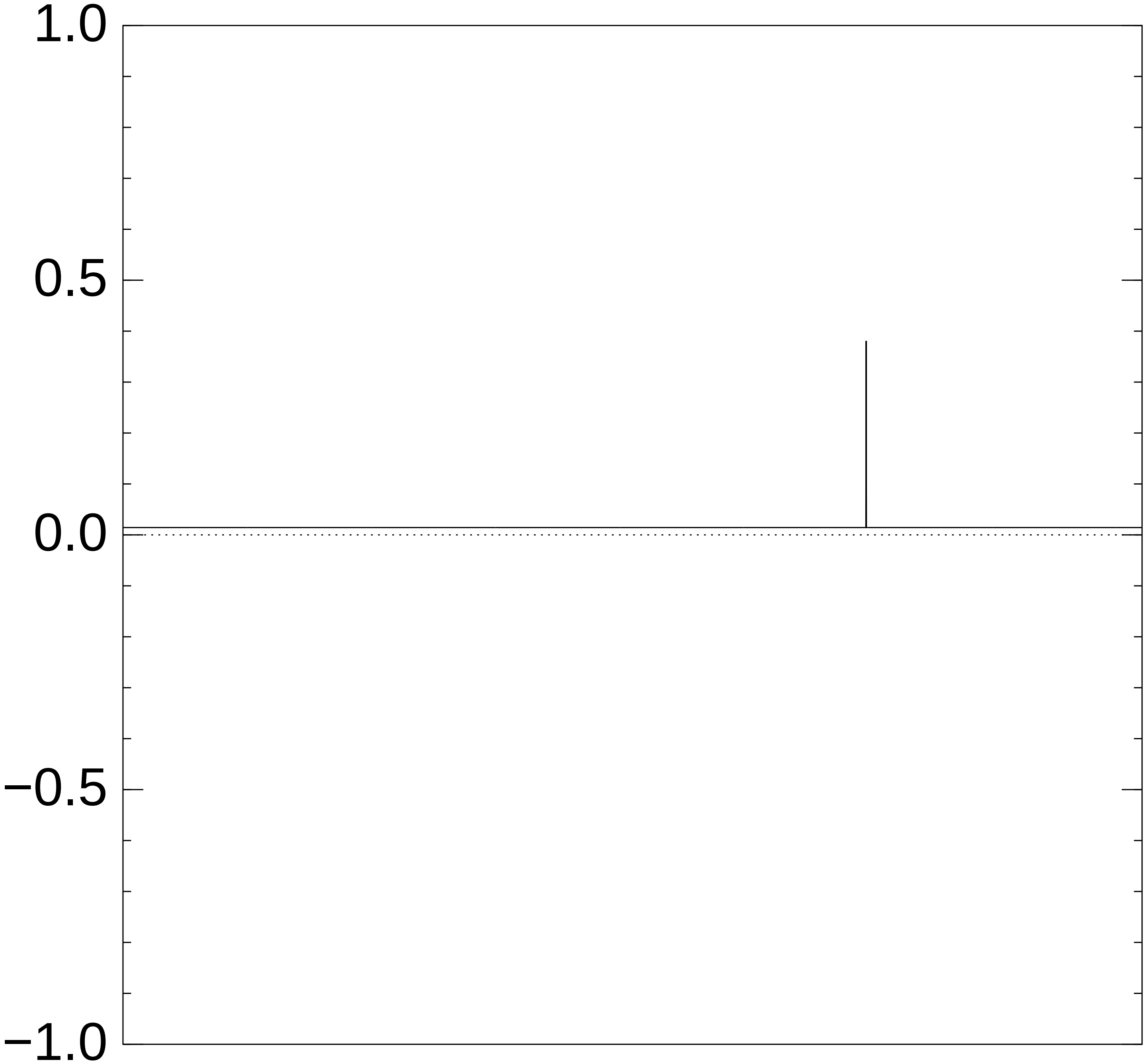
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $10 \times (\text{Step 1} + \text{Step 2})$:



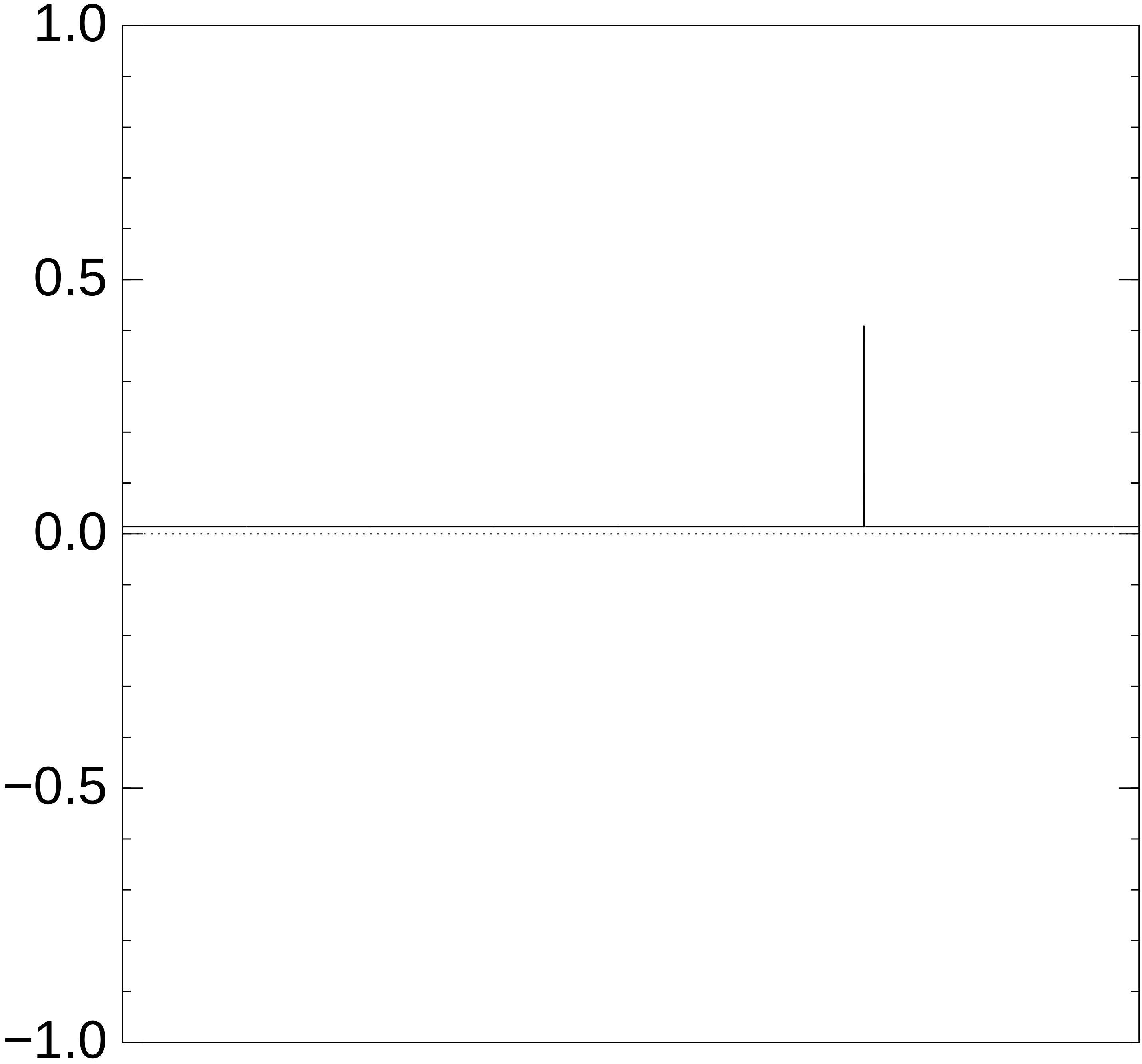
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $11 \times (\text{Step } 1 + \text{Step } 2)$:



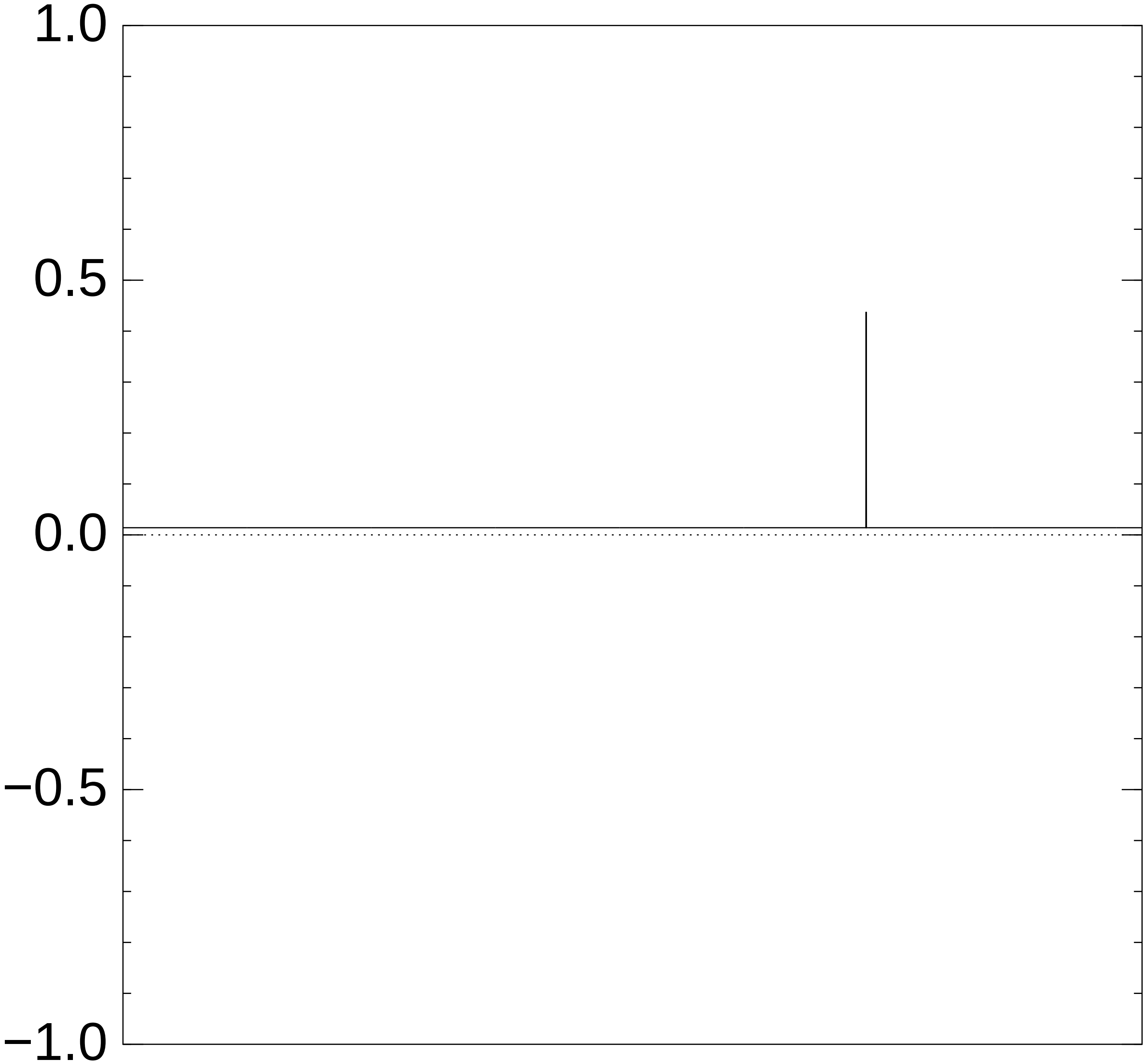
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $12 \times$ (Step 1 + Step 2):



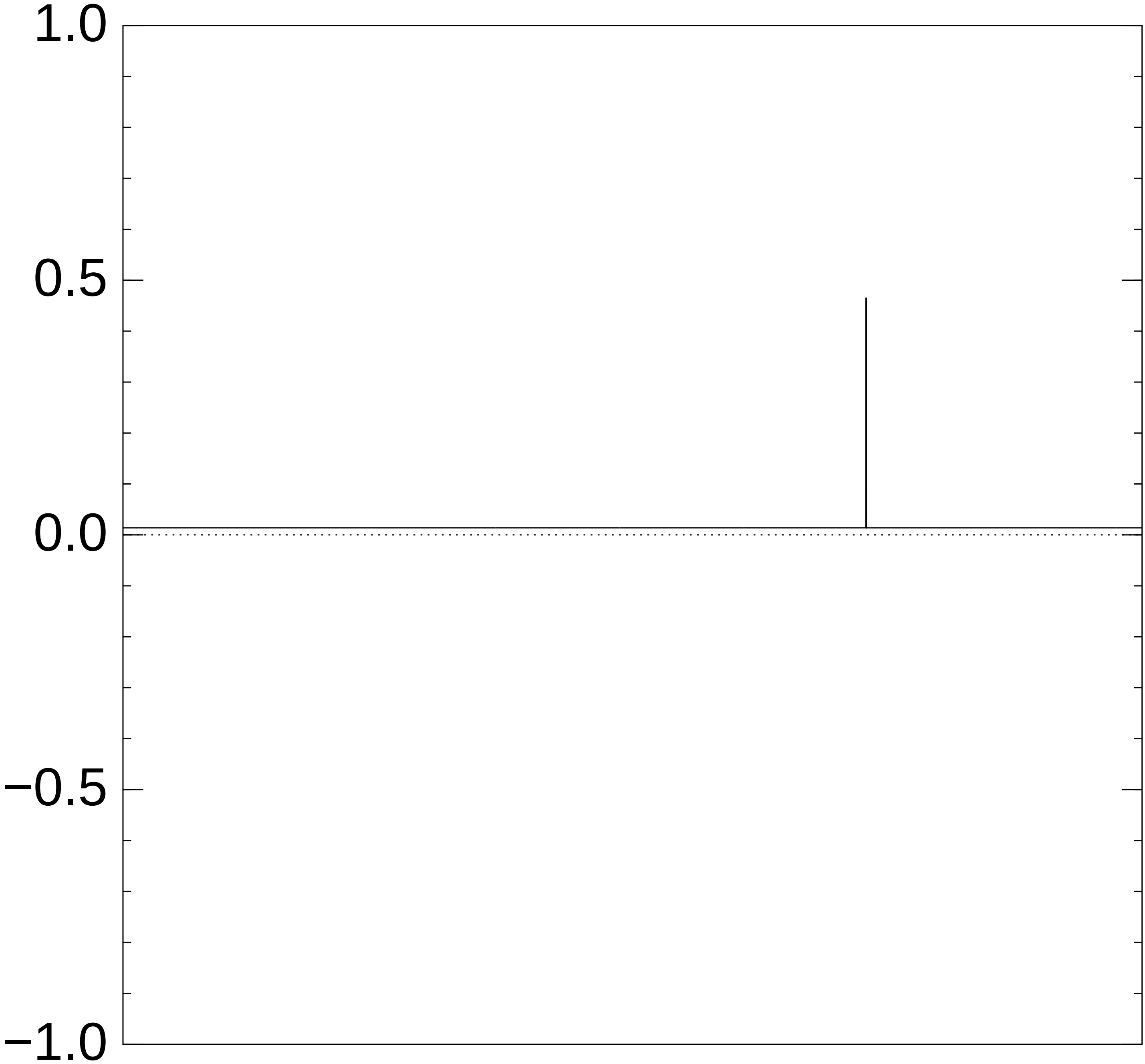
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $13 \times$ (Step $1 +$ Step 2):



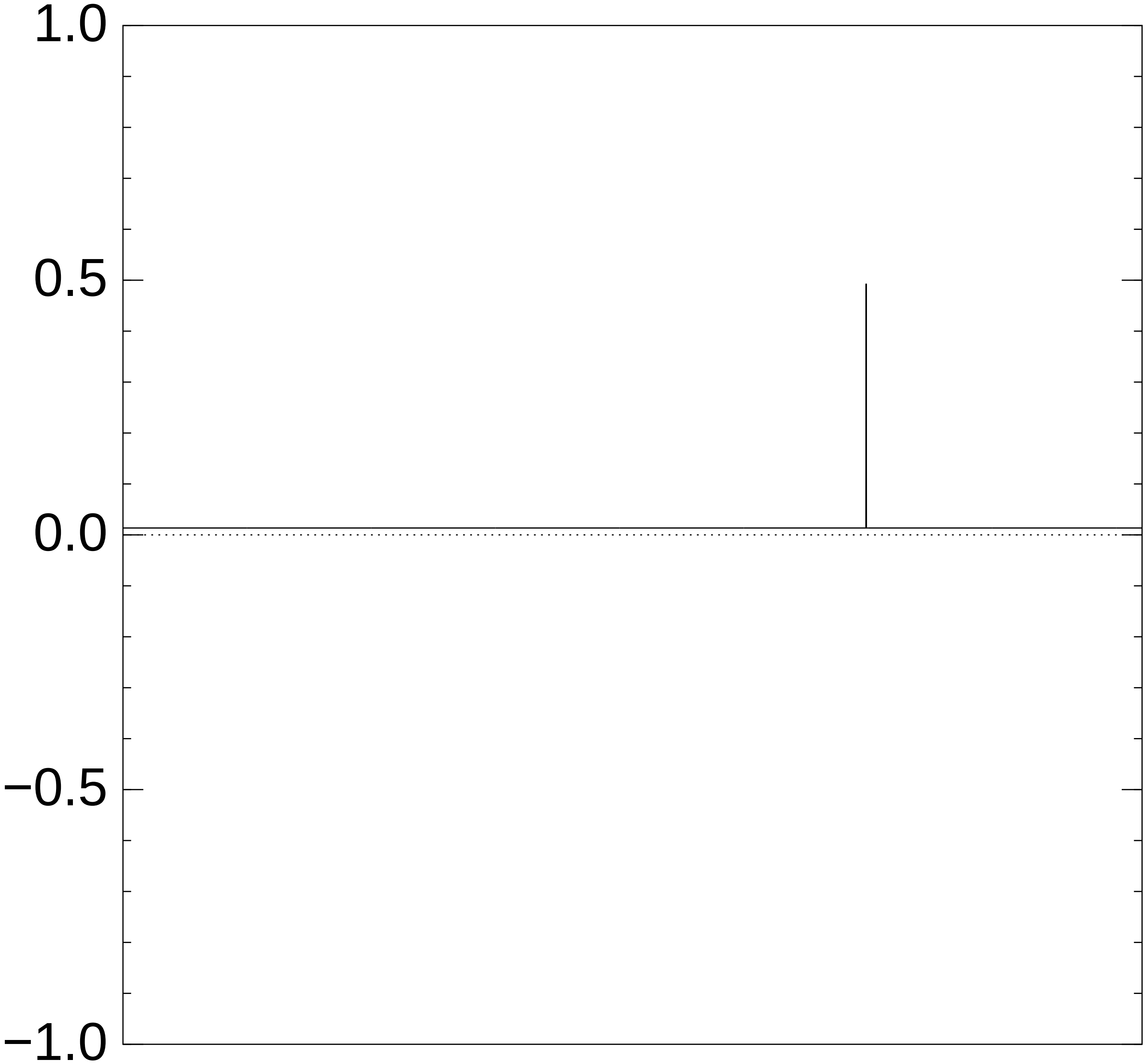
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $14 \times (\text{Step } 1 + \text{Step } 2)$:



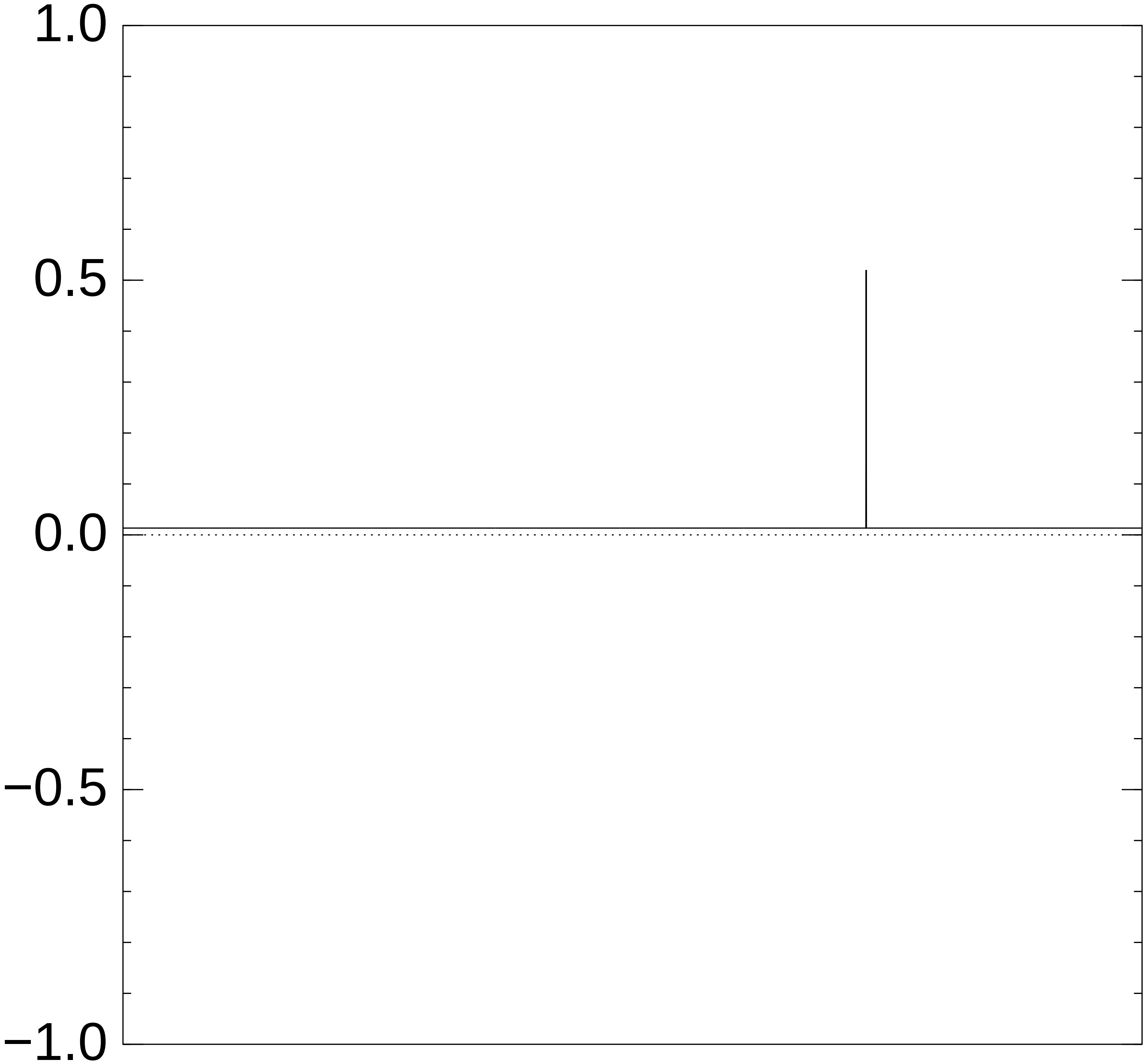
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $15 \times$ (Step $1 +$ Step 2):



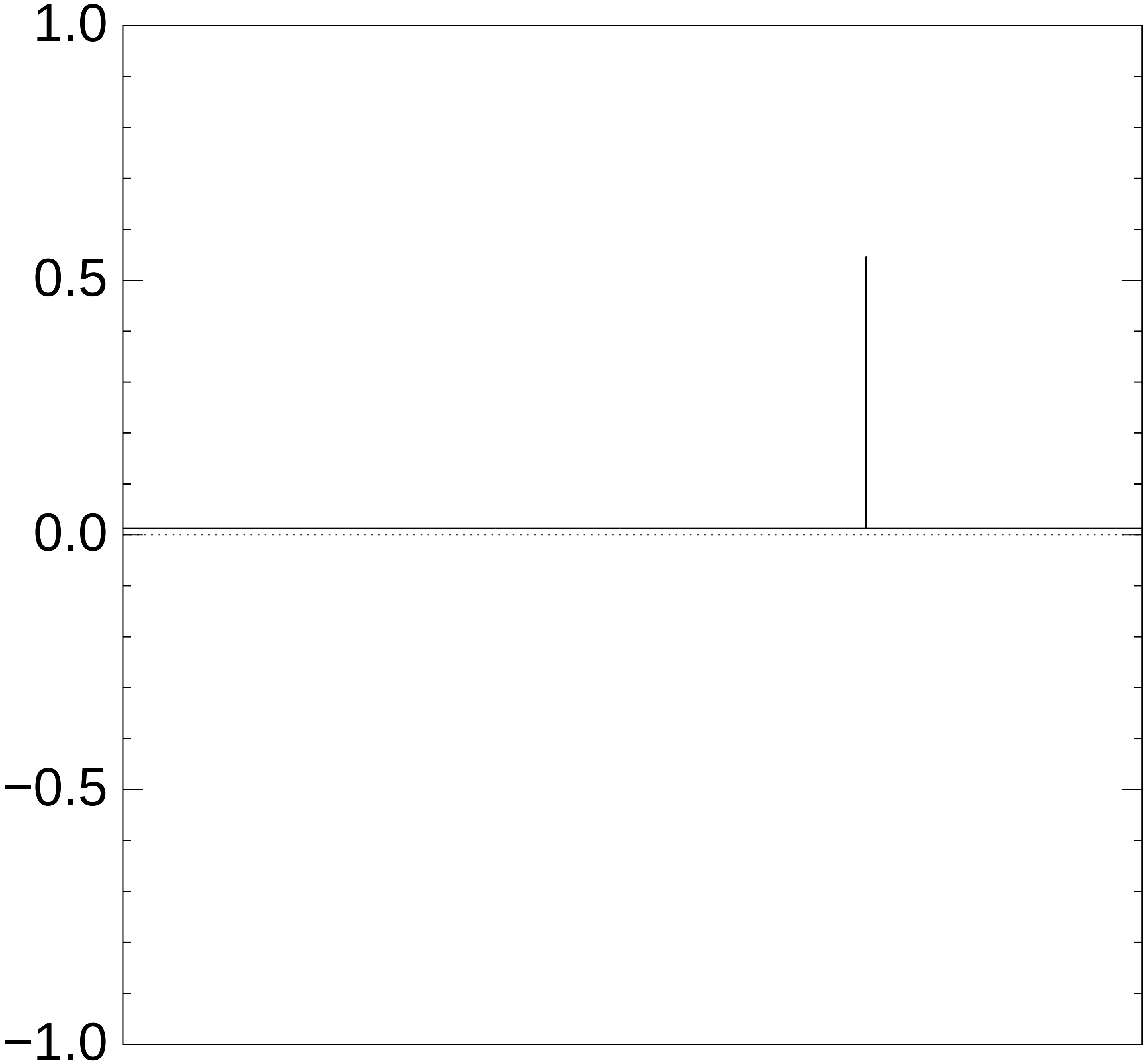
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $16 \times$ (Step 1 + Step 2):



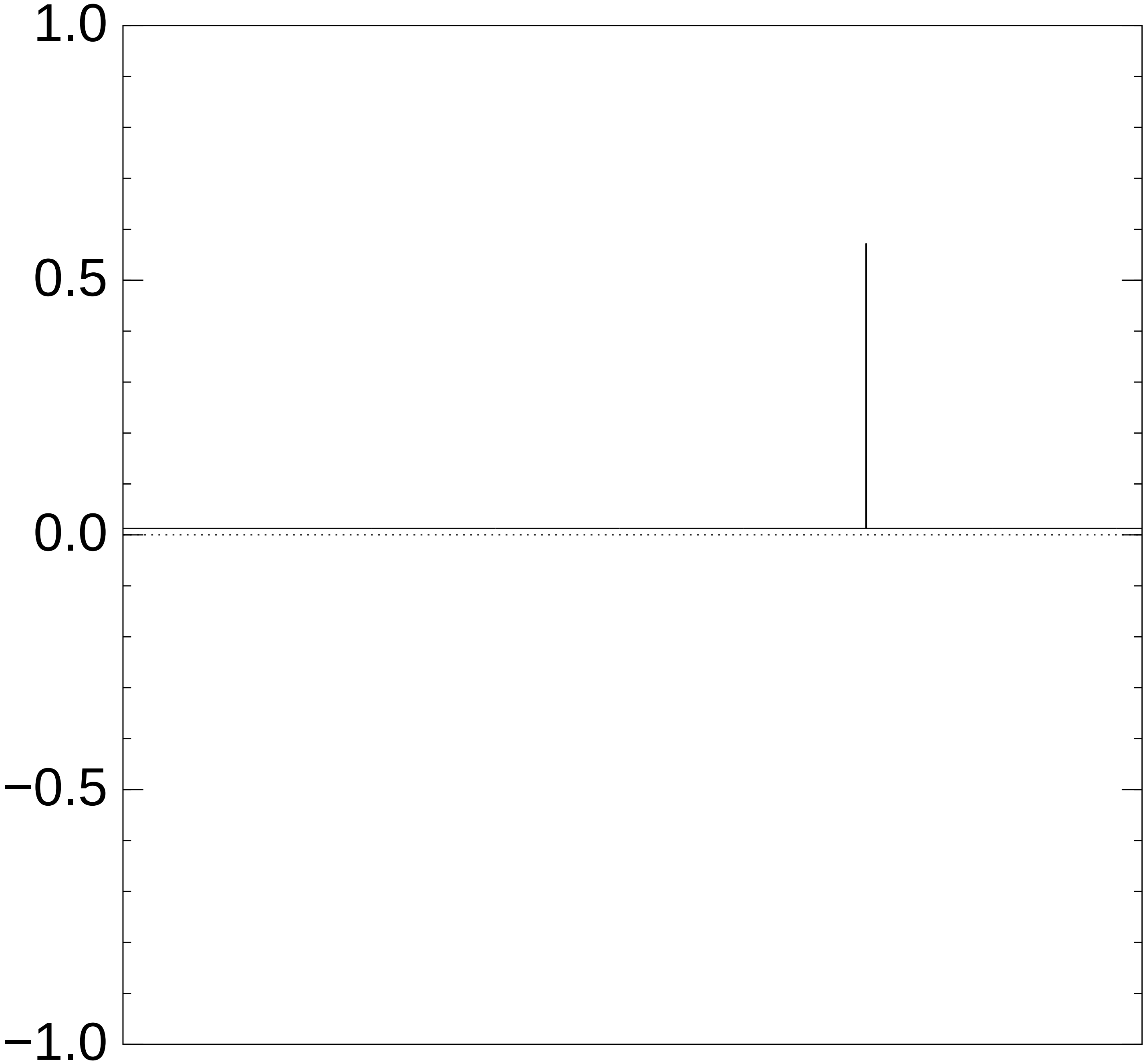
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $17 \times (\text{Step 1} + \text{Step 2})$:



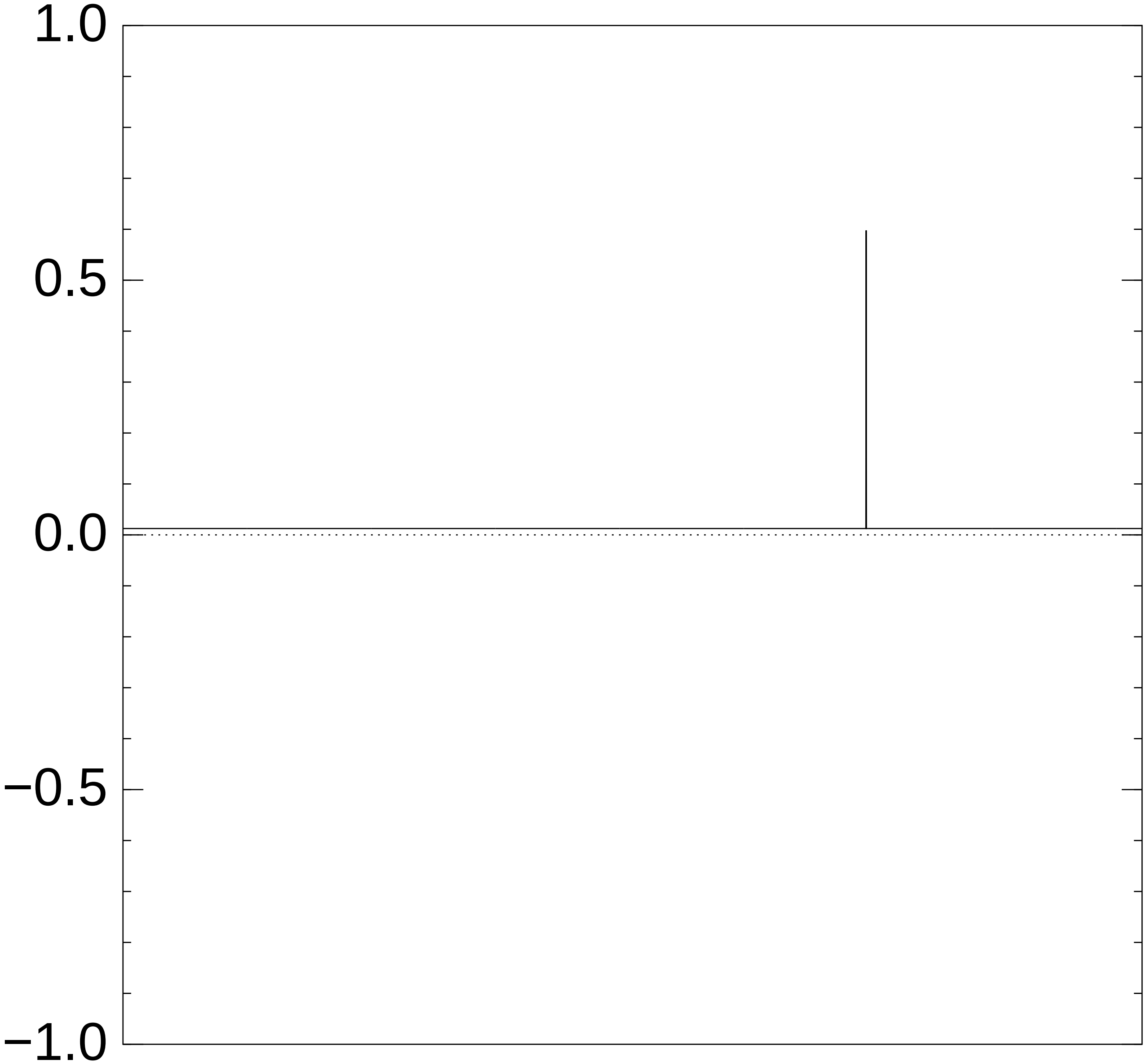
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $18 \times (\text{Step } 1 + \text{Step } 2)$:



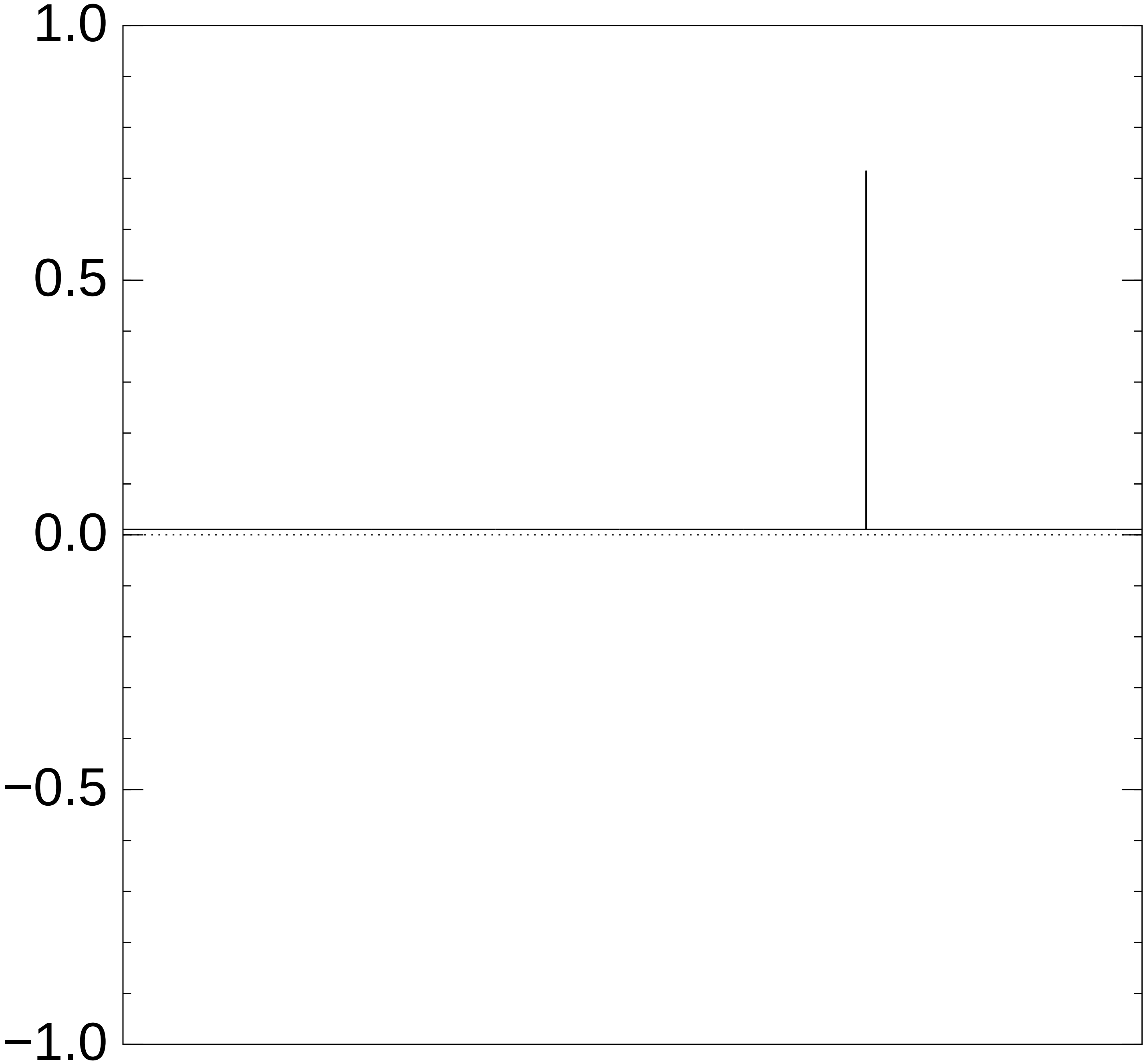
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $19 \times (\text{Step } 1 + \text{Step } 2)$:



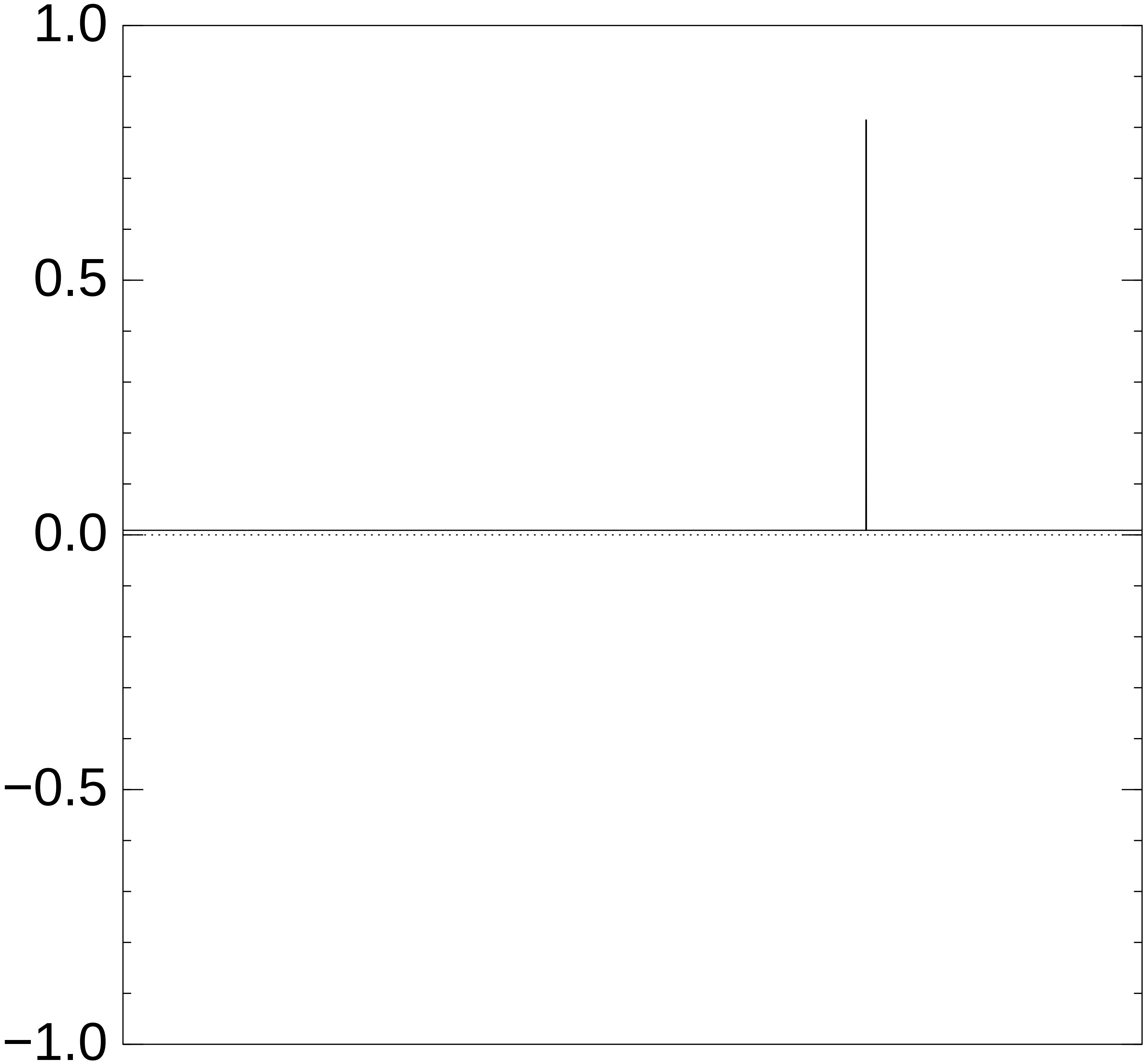
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $20 \times (\text{Step } 1 + \text{Step } 2)$:



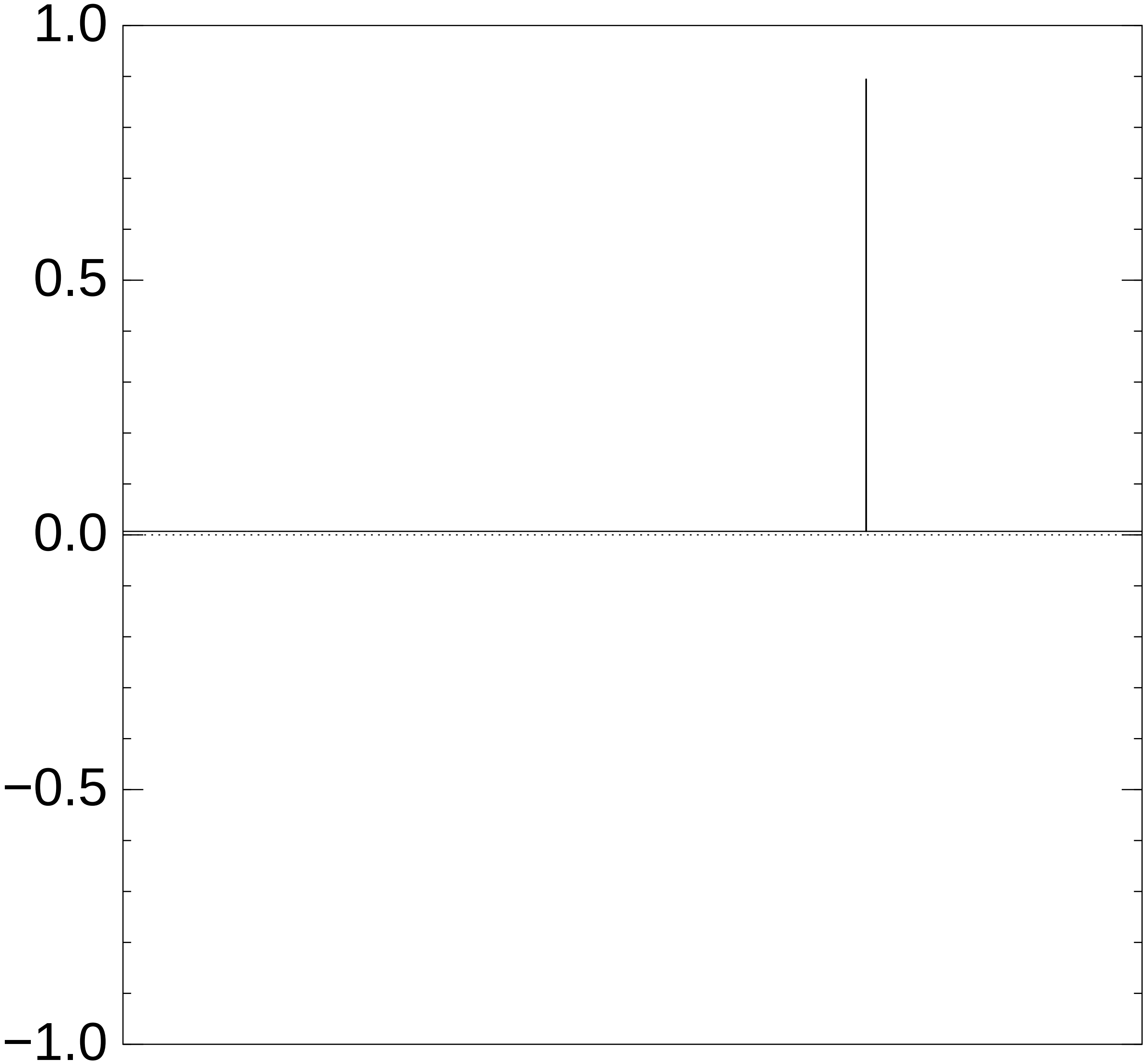
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $25 \times (\text{Step } 1 + \text{Step } 2)$:



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $30 \times (\text{Step } 1 + \text{Step } 2)$:

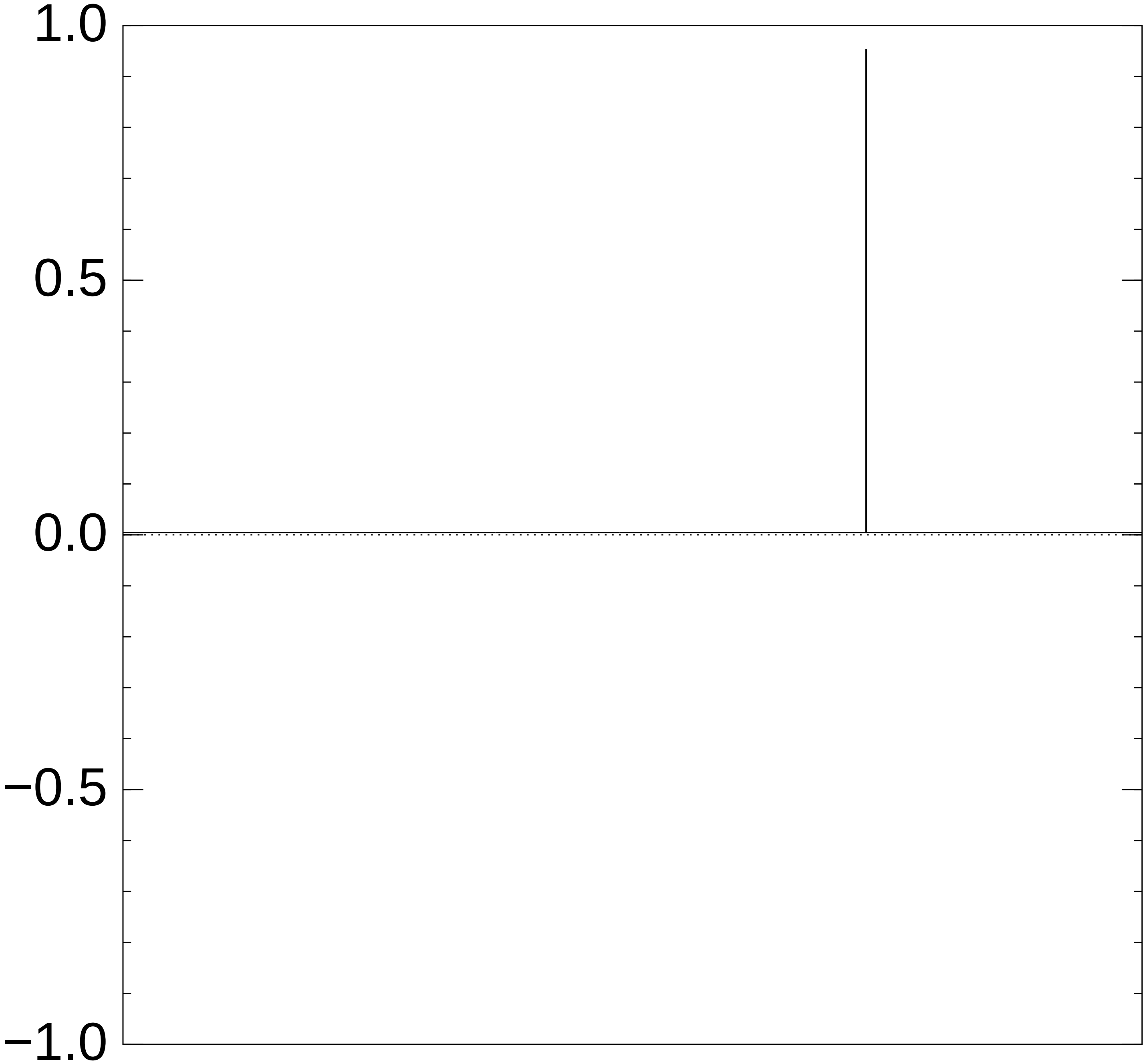


Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $35 \times$ (Step $1 +$ Step 2):

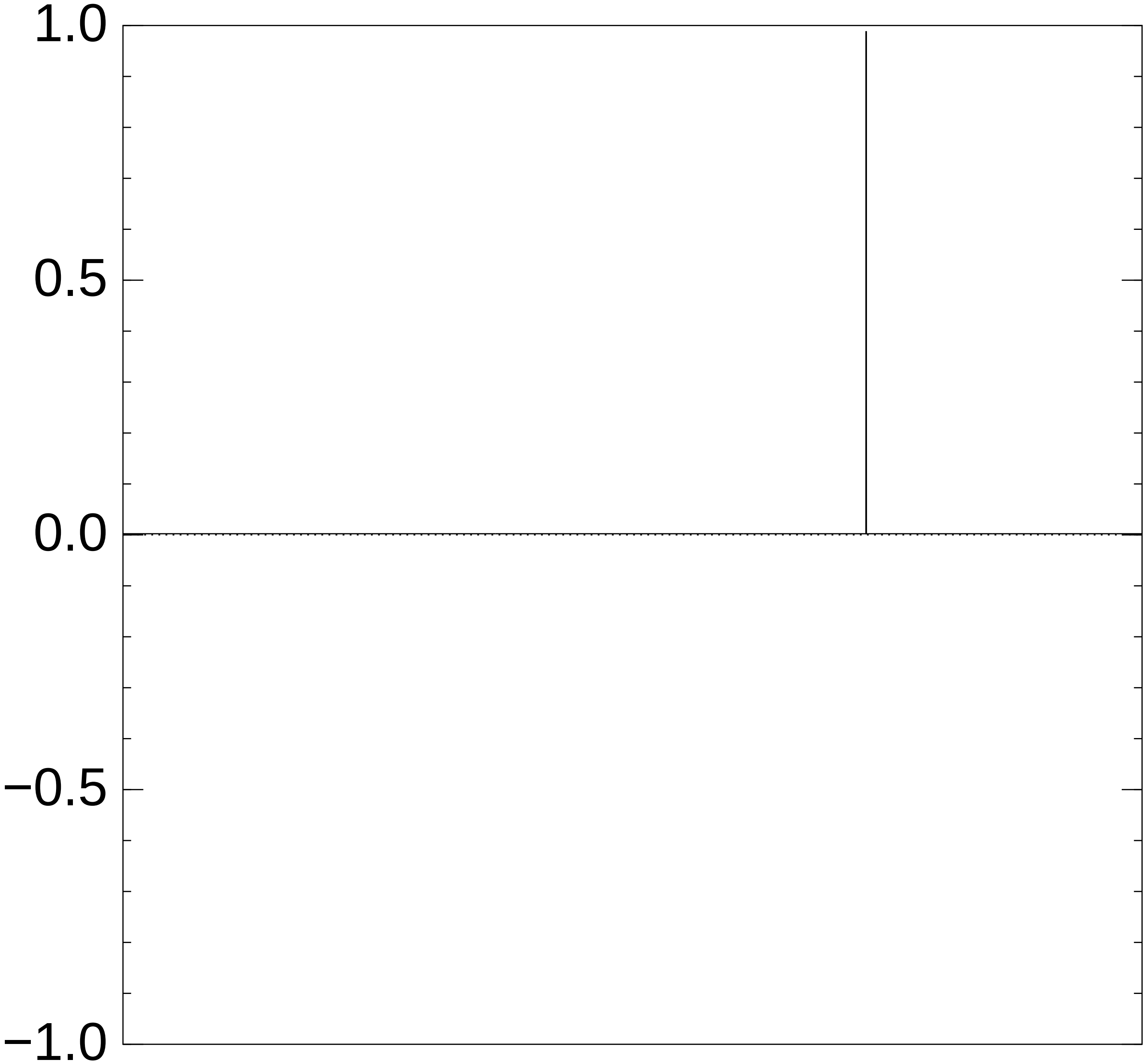



Good moment to stop, measure.

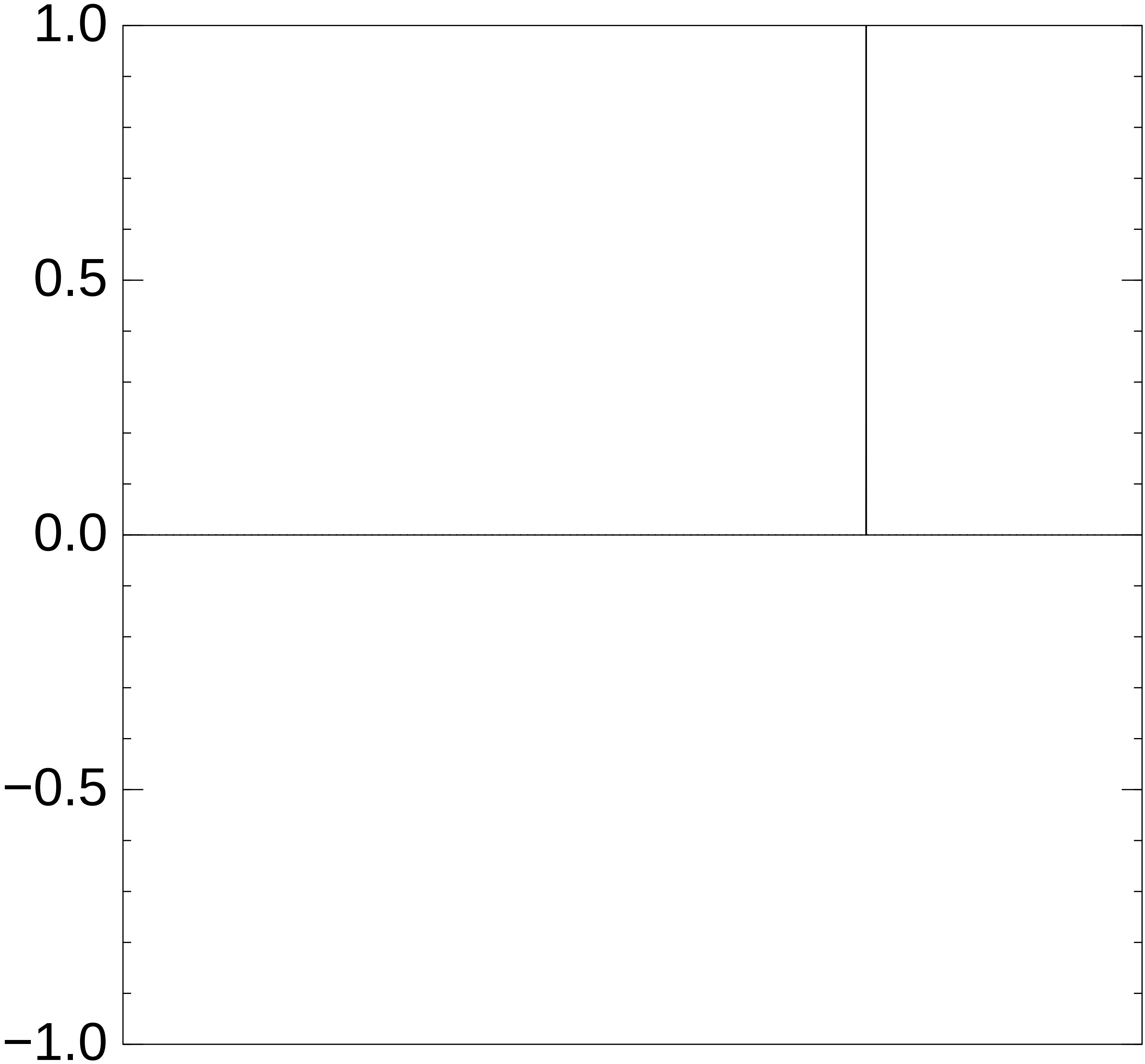
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $40 \times (\text{Step } 1 + \text{Step } 2)$:



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $45 \times$ (Step $1 +$ Step 2):

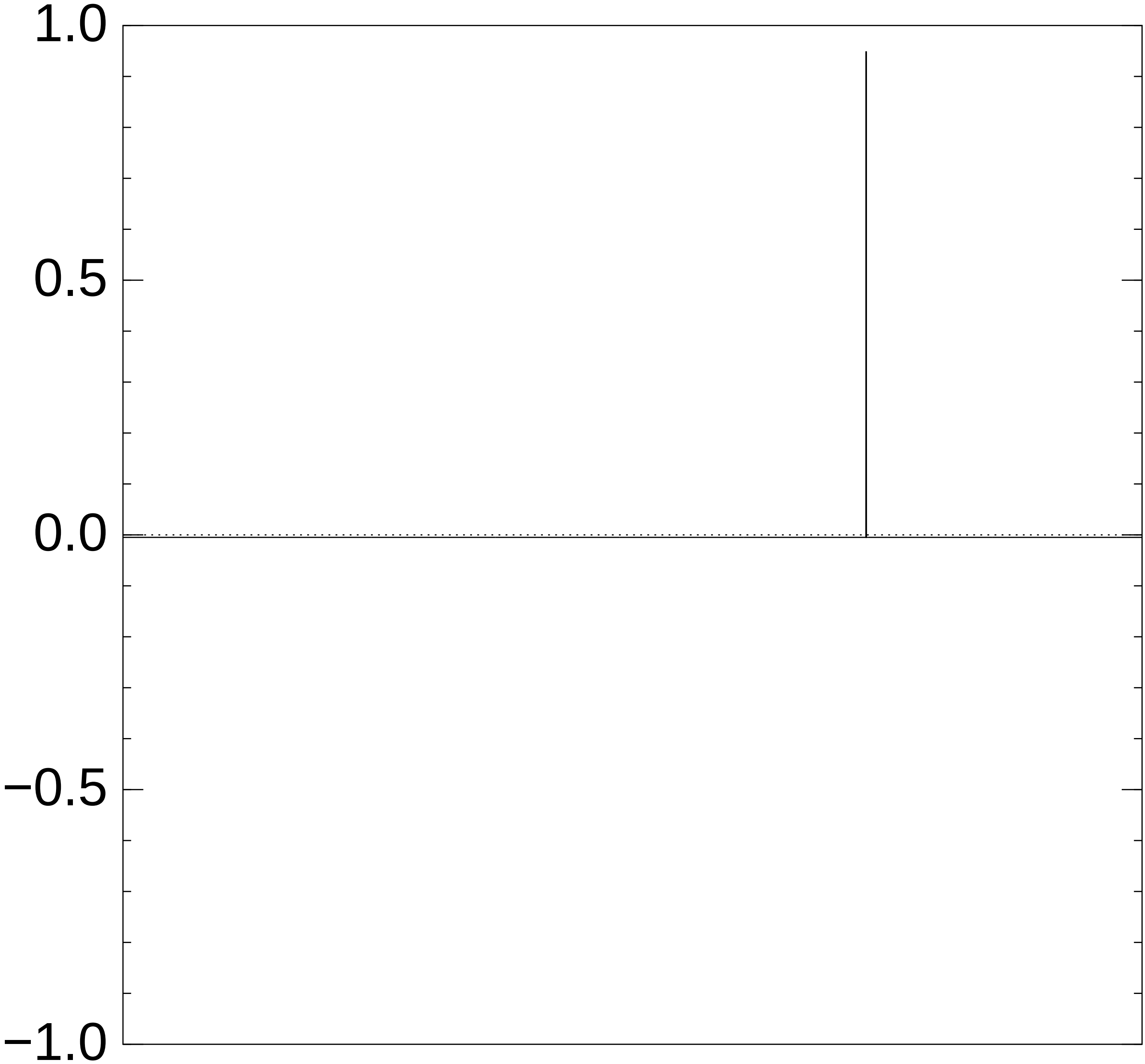


Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $50 \times$ (Step $1 +$ Step 2):

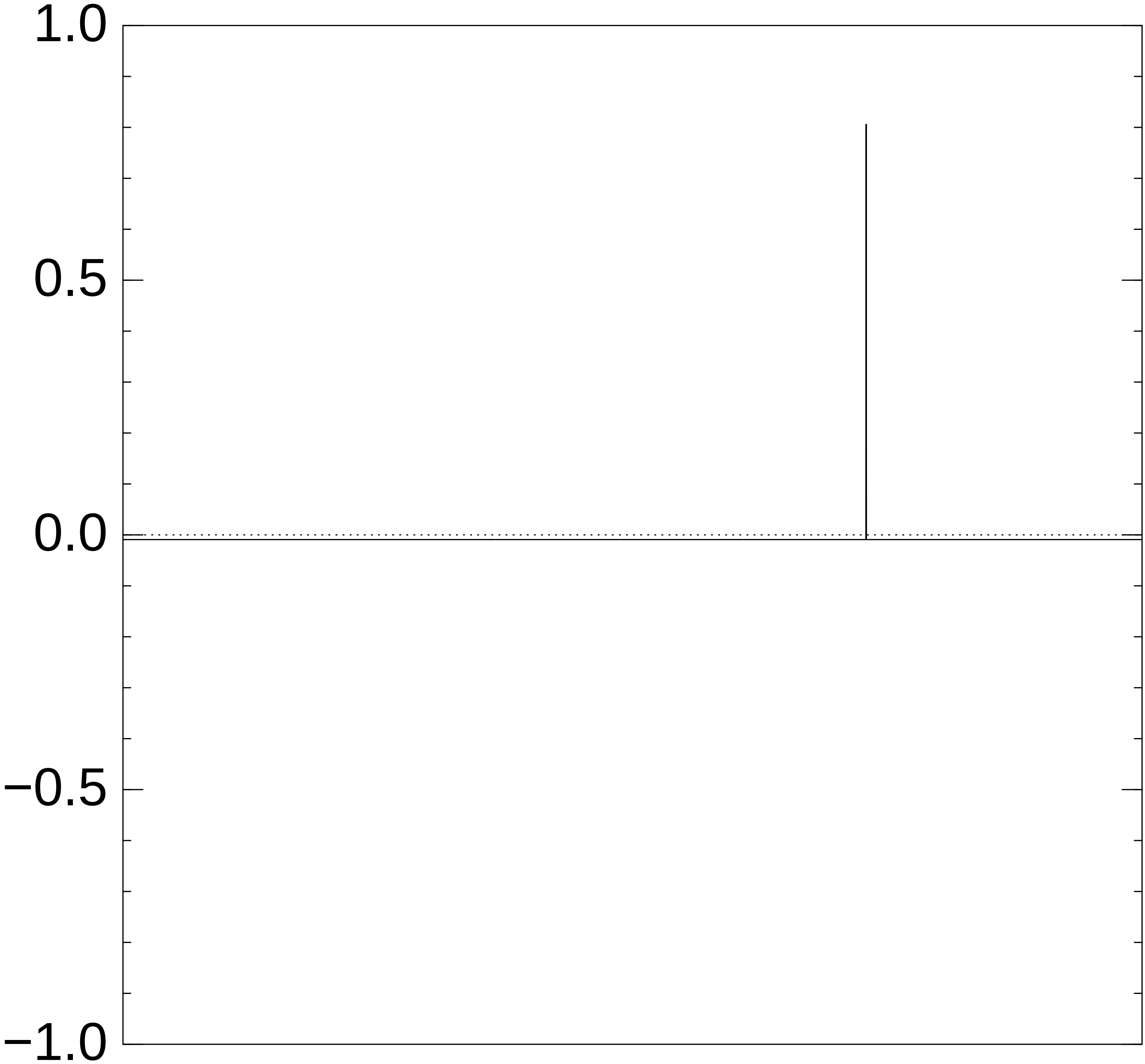



Traditional stopping point.

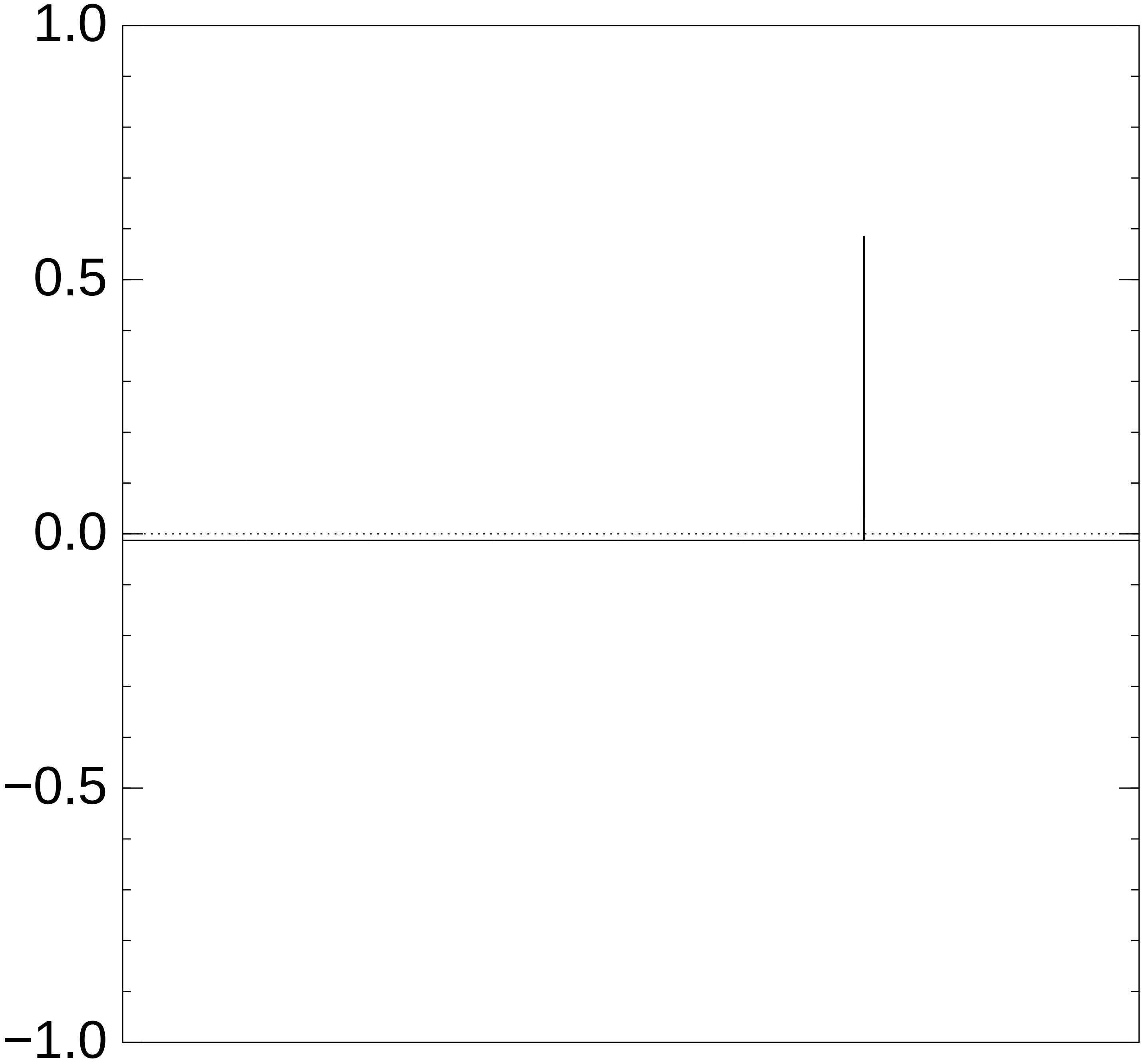
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $60 \times (\text{Step 1} + \text{Step 2})$:



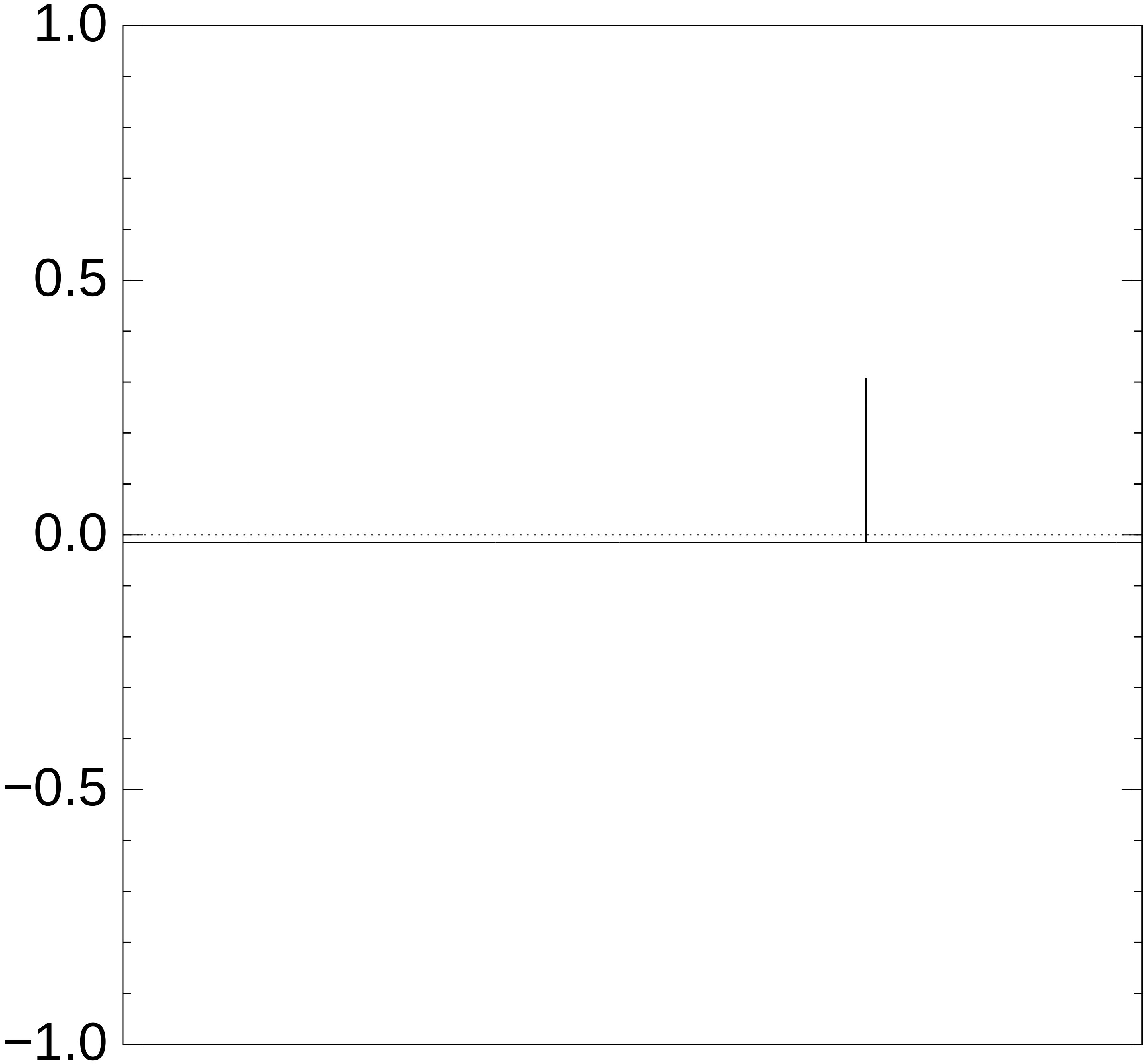
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $70 \times (\text{Step 1} + \text{Step 2})$:



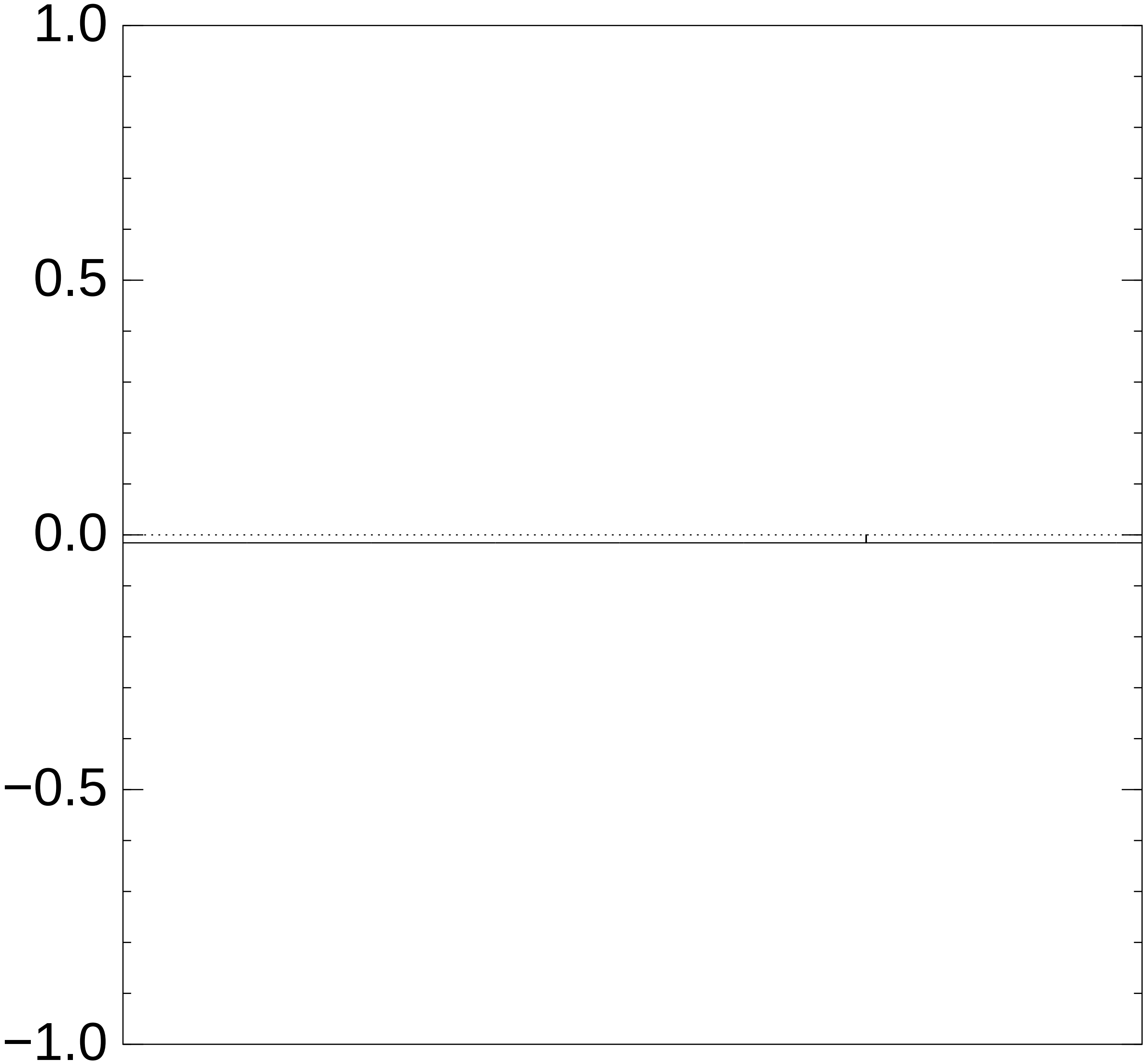
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $80 \times (\text{Step } 1 + \text{Step } 2)$:



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $90 \times$ (Step $1 +$ Step 2):



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $100 \times (\text{Step } 1 + \text{Step } 2)$:




Very bad stopping point.

$q \mapsto a_q$ is completely described
by a vector of two numbers
(with fixed multiplicities):
(1) $a_q$ for roots $q$;
(2) $a_q$ for non-roots $q$.

Step 1 + Step 2
act linearly on this vector.

Easily compute eigenvalues
and powers of this linear map
to understand evolution
of state of Grover's algorithm.
$\Rightarrow$ Probability is $\approx 1$
after $\approx (\pi/4)2^{0.5n}$ iterations.

# Textbook algorithm analysis

"WHAT is your algorithm?"

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“WHAT is your algorithm?”

“Heapsort. Here’s the code.”

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Here’s a proof.”

“WHAT is its run time?”

“$O(n \lg n)$ comparisons;
and $\Theta(n \lg n)$ comparisons
for most inputs. Here’s a proof.”

# Textbook algorithm analysis

"WHAT is your algorithm?"

"Heapsort. Here's the code."

"WHAT does it accomplish?"

"It sorts the input array in place.
Here's a proof."

"WHAT is its run time?"

"$O(n \lg n)$ comparisons;
and $\Theta(n \lg n)$ comparisons
for most inputs. Here's a proof."

"You may pass."

# Algorithms to attack crypto

Critical question for ECC security:
How hard is ECDLP?

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How hard is ECDLP?

Standard estimate for "strong"
ECC groups of prime order $\ell$:
Latest "negating" variants of
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break an average ECDLP instance
using $\approx 0.886\sqrt{\ell}$ additions.

## Algorithms to attack crypto

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Is this proven? No!

Is this provable? Maybe not!

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Is this proven? No!

Is this provable? Maybe not!

So why do we think it's true?

2000 Gallant–Lambert–Vanstone: inadequately specified statement of a negating rho algorithm.

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2010 Bos–Kleinjung–Lenstra: a plausible interpretation of that algorithm is *non-functional*.

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See 2011 Bernstein–Lange–Schwabe for more history and better algorithms.

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Why do we believe that the latest algorithms work at the claimed speeds?
**Experiments!**

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Similar story for RSA security: we don't have proofs for the best factoring algorithms.

Code-based cryptography: we don't have proofs for the best decoding algorithms.

Lattice-based cryptography: we don't have proofs for the best lattice algorithms.

MQ-based cryptography: we don't have proofs for the best system-solving algorithms.

Similar story for RSA security: we don't have proofs for the best factoring algorithms.

Code-based cryptography: we don't have proofs for the best decoding algorithms.

Lattice-based cryptography: we don't have proofs for the best lattice algorithms.

MQ-based cryptography: we don't have proofs for the best system-solving algorithms.

Confidence relies on experiments.

## Where's my quantum computer?

Quantum-algorithm design is moving beyond textbook stage into algorithms without proofs.

Example: subset-sum exponent $\approx 0.241$ from 2013 Bernstein–Jeffery–Lange–Meurer.

Don't expect proofs or provability for the best quantum algorithms to attack post-quantum crypto.

How do we obtain confidence in analysis of these algorithms? Quantum experiments are hard.

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Vastly larger extrapolation for the quantum situation. Imagine attacker performing $2^{80}$ operations on $2^{40}$ qubits; compare to today's challenges of $2^1$, $2^2$, $2^3$, $2^4$, $2^5$, $2^6$ qubits.

## Simulations

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Simulation shows error in
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Ambainis: Yes, thanks, will fix.

2014.04 Chou $\rightarrow$ Childs: Simulation shows that 2003 Childs–Eisenberg distinctness algorithm is non-functional; need to take half angle.

Childs: Yes. Typo, already fixed in 2005 journal version.

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Maybe, maybe not.

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Real-world security systems cannot avoid these questions.