How to multiply big integers

Standard idea: Use polynomial with coefficients in $\{0, 1, \ldots, 9\}$ to represent integer in radix 10.

Example of representation: $839 = 8 \cdot 10^2 + 3 \cdot 10^1 + 9 \cdot 10^0 =$ value (at $t = 10$) of polynomial $8t^2 + 3t^1 + 9t^0$.

Convenient to express polynomial inside computer as array $9, 3, 8$ (or $9, 3, 8, 0$ or $9, 3, 8, 0, 0$ or $\ldots$): "`p[0] = 9; p[1] = 3; p[2] = 8`"

Multiply two integers
by multiplying polynomials
that represent the integers.

Polynomial multiplication
involves *small* integer coefficients.
Have split one big multiplication
into many small operations.

Example, squaring 839:

$(8t^2 + 3t^1 + 9t^0)^2 =$
$8t^2(8t^2 + 3t^1 + 9t^0) +$
$3t^1(8t^2 + 3t^1 + 9t^0) +$
$9t^0(8t^2 + 3t^1 + 9t^0) =$
$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0.$

Oops, product polynomial usually has coefficients > 9.

So "carry" extra digits:

$ct^j \to \lfloor c/10 \rfloor t^{j+1} + (c \bmod 10)t^j$.

Example, squaring 839:

$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0$;

$64t^4 + 48t^3 + 153t^2 + 62t^1 + 1t^0$;

$64t^4 + 48t^3 + 159t^2 + 2t^1 + 1t^0$;

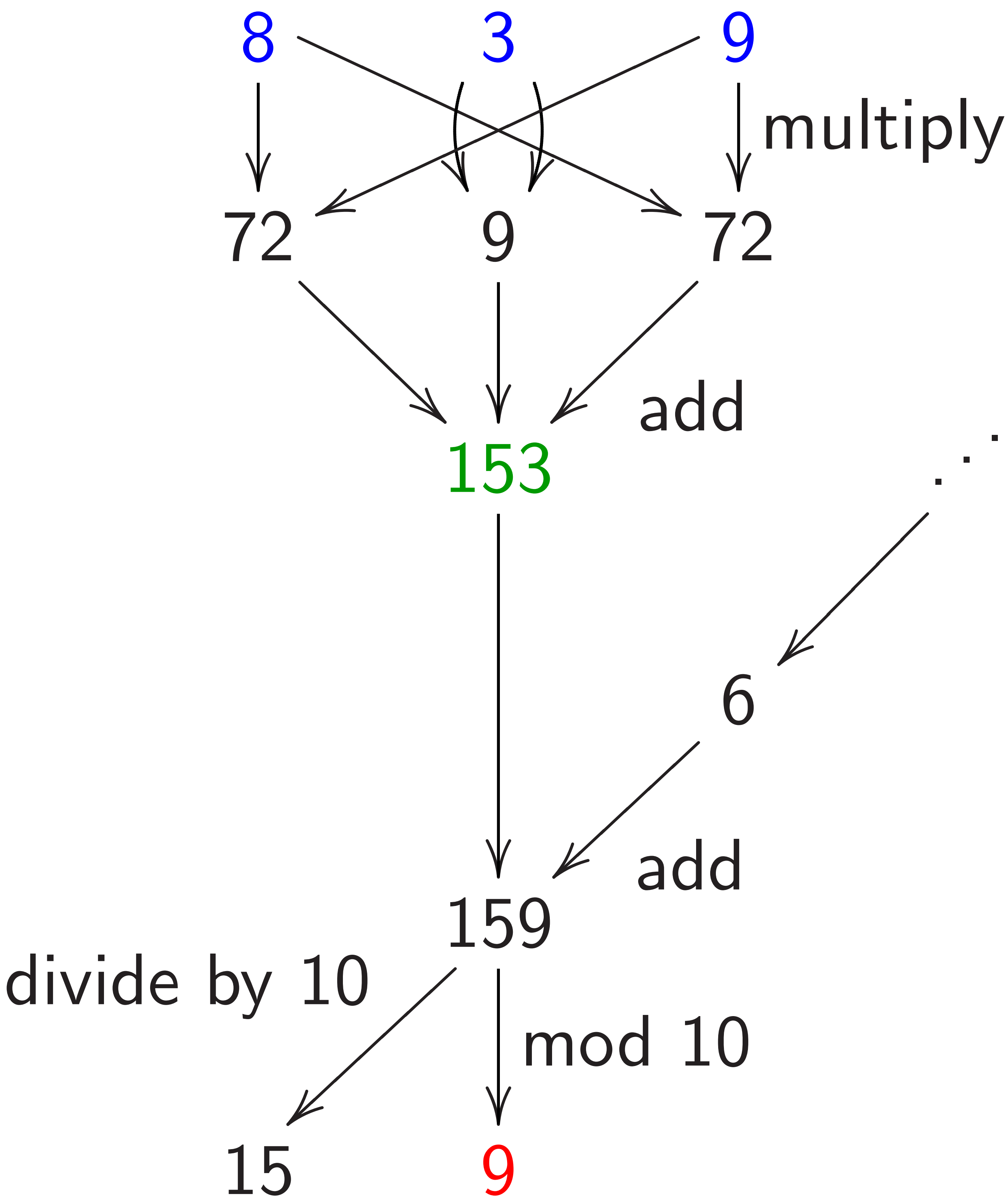
$64t^4 + 63t^3 + 9t^2 + 2t^1 + 1t^0$;

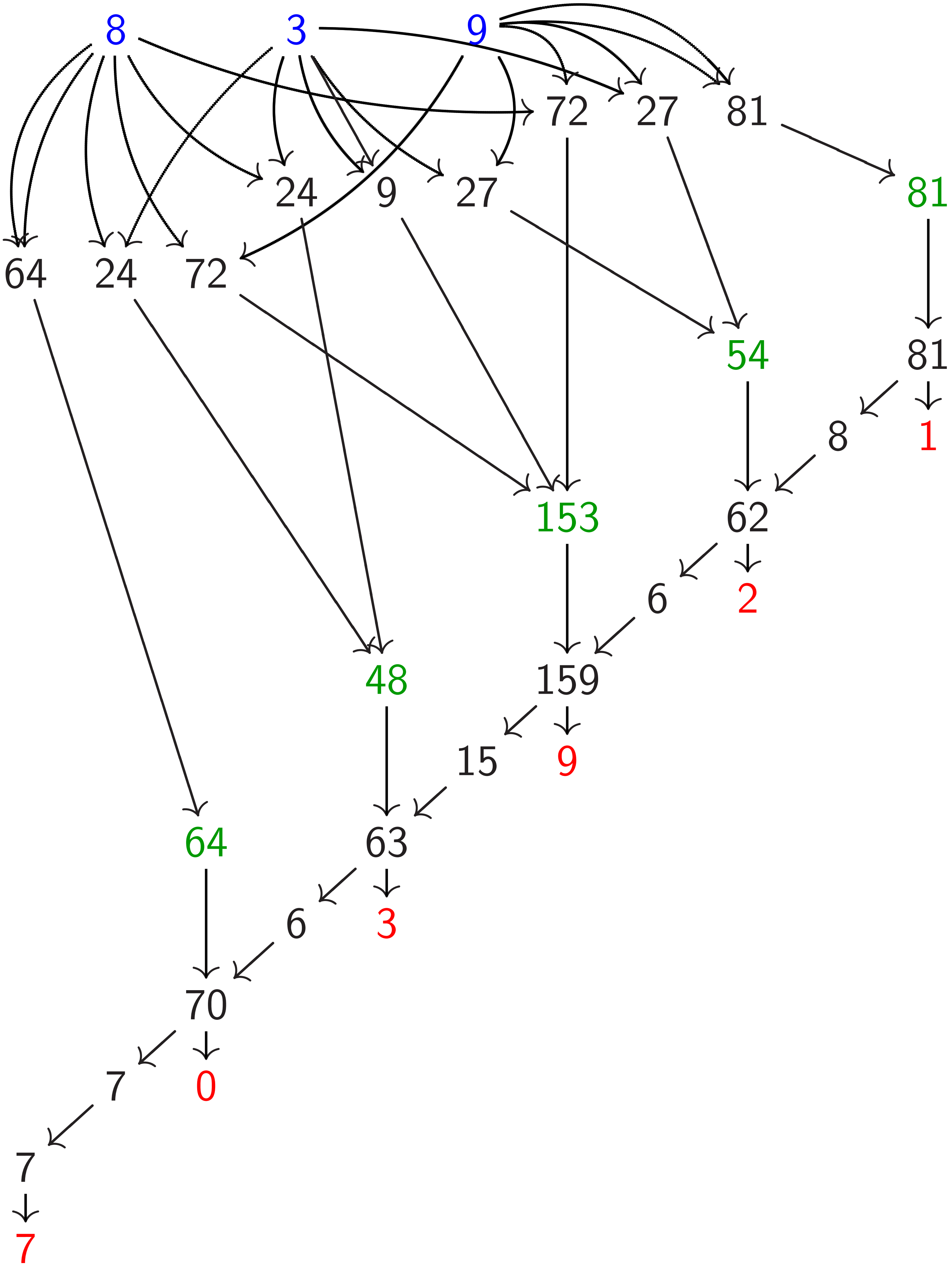
$70t^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$;

$7t^5 + 0t^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$.

In other words, $839^2 = 703921$.

What operations were used here?




8
3
9
72
27
81
24
9
27
81
64
24
72
54
81
8
1
153
62
6
2
48
159
15
9
64
63
6
3
70
7
0
7
7

The scaled variation

$839 = 800 + 30 + 9 =$

value (at $t = 1$) of polynomial

$800t^2 + 30t^1 + 9t^0$.

Squaring: $(800t^2 + 30t^1 + 9t^0)^2 =$

$640000t^4 + 48000t^3 + 15300t^2 +$

$540t^1 + 81t^0$.

Carrying:

$640000t^4 + 48000t^3 + 15300t^2 +$

$540t^1 + 81t^0$;

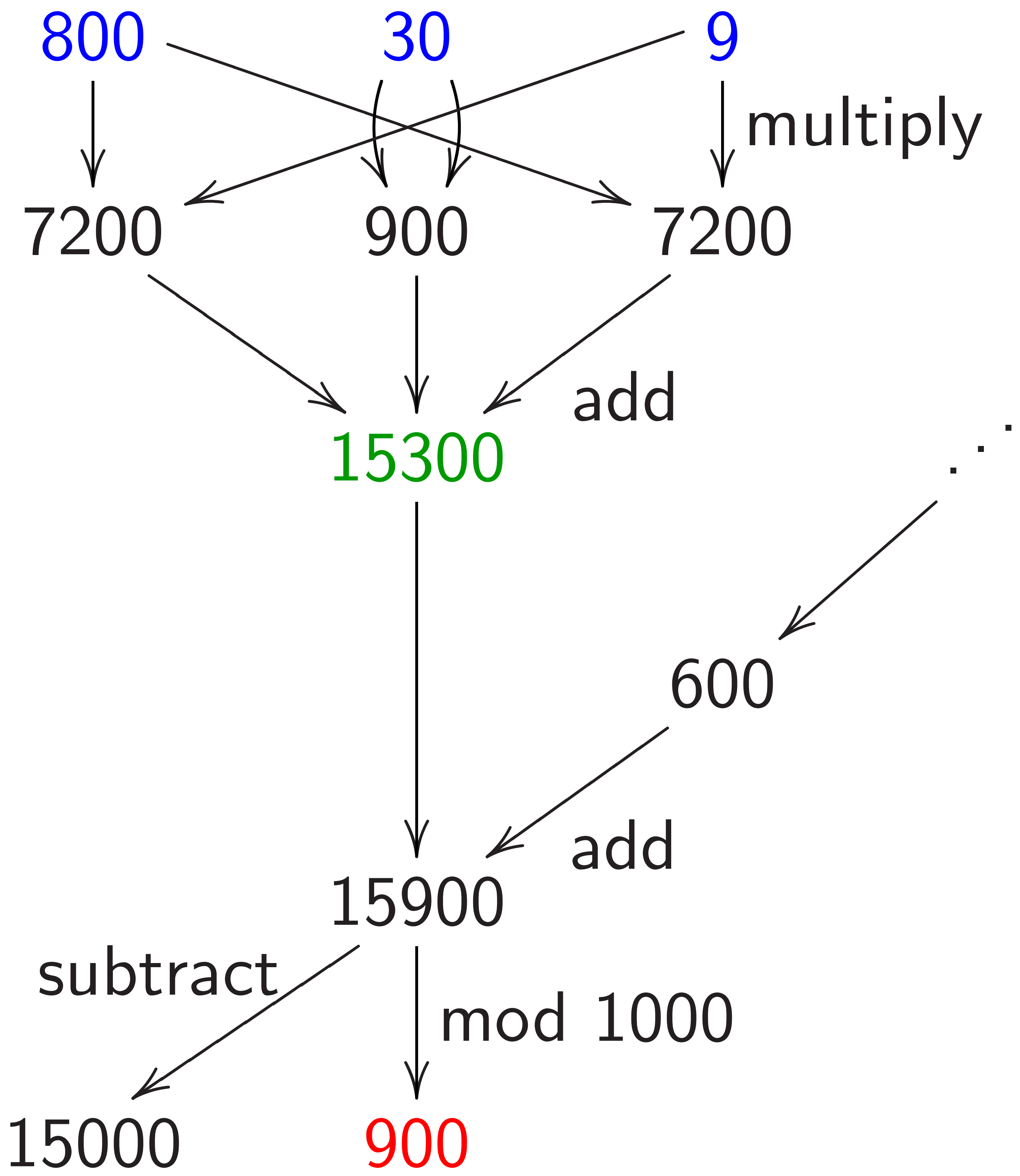
$640000t^4 + 48000t^3 + 15300t^2 +$

$620t^1 + 1t^0$; $\quad\ldots$

$700000t^5 + 0t^4 + 3000t^3 + 900t^2 +$

$20t^1 + 1t^0$.

What operations were used here?



Speedup: double inside squaring

$(\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0)^2$

has coefficients such as

$f_4 f_0 + f_3 f_1 + f_2 f_2 + f_1 f_3 + f_0 f_4$.

5 mults, 4 adds.

Speedup: double inside squaring

$(\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0)^2$
has coefficients such as
$f_4 f_0 + f_3 f_1 + f_2 f_2 + f_1 f_3 + f_0 f_4$.
5 mults, 4 adds.

Compute more efficiently as
$2 f_4 f_0 + 2 f_3 f_1 + f_2 f_2$.
3 mults, 2 adds, 2 doublings.

Save $\approx 1/2$ of the mults
if there are many coefficients.

Faster alternative:

$2(f_4 f_0 + f_3 f_1) + f_2 f_2$.

3 mults, 2 adds, 1 doubling.

Save $\approx 1/2$ of the adds
if there are many coefficients.

Faster alternative:

$2(f_4 f_0 + f_3 f_1) + f_2 f_2$.

3 mults, 2 adds, 1 doubling.

Save $\approx 1/2$ of the adds if there are many coefficients.

Even faster alternative:

$(2f_0)f_4 + (2f_1)f_3 + f_2 f_2$,

after precomputing $2f_0, 2f_1, \ldots$.

3 mults, 2 adds, 0 doublings.

Precomputation ≈ 0.5 doublings.

Speedup: allow negative coeffs

Recall $159 \mapsto 15, 9$.

Scaled: $15900 \mapsto 15000, 900$.

Alternative: $159 \mapsto 16, -1$.

Scaled: $15900 \mapsto 16000, -100$.

Use digits $\{-5, -4, \ldots, 4, 5\}$ instead of $\{0, 1, \ldots, 9\}$.

Small disadvantage: need $-$.

Several small advantages:
easily handle negative integers;
easily handle subtraction;
reduce products a bit.

Speedup: delay carries

Computing (e.g.) big $ab + c^2$: multiply a, b polynomials, carry, square c poly, carry, add, carry.

e.g. $a = 314$, $b = 271$, $c = 839$:
$(3t^2+1t^1+4t^0)(2t^2+7t^1+1t^0) =$
$6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0$;
carry: $8t^4 + 5t^3 + 0t^2 + 9t^1 + 4t^0$.

As before $(8t^2 + 3t^1 + 9t^0)^2 =$
$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0$;
$7t^5 + 0t^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$.

$+$: $7t^5+8t^4+8t^3+9t^2+11t^1+5t^0$;
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) + (64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0)$$
$$= 70t^4 + 71t^3 + 171t^2 + 83t^1 + 85t^0;$$
$$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0.$$

Eliminate intermediate carries. Outweighs cost of handling slightly larger coefficients.

Important to carry between multiplications (and squarings) to reduce coefficient size; but carries are usually a bad idea before additions, subtractions, etc.

Speedup: polynomial Karatsuba

How much work to multiply polys
$f = f_0 + f_1 t + \cdots + f_{19} t^{19}$,
$g = g_0 + g_1 t + \cdots + g_{19} t^{19}$?

Using the obvious method:
400 coeff mults, 361 coeff adds.

Faster: Write f as $F_0 + F_1 t^{10}$;
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
$F_1 = f_{10} + f_{11} t + \cdots + f_{19} t^9$.
Similarly write g as $G_0 + G_1 t^{10}$.

Then $fg = (F_0 + F_1)(G_0 + G_1)t^{10}$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

20 adds for $F_0 + F_1$, $G_0 + G_1$.
300 mults for three products
$F_0 G_0$, $F_1 G_1$, $(F_0 + F_1)(G_0 + G_1)$.
243 adds for those products.
9 adds for $F_0 G_0 - F_1 G_1 t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots (1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
Larger coefficients, slight expense;
still saves time.

Can apply idea recursively
as poly degree grows.

Many other algebraic speedups in polynomial multiplication: "Toom," "FFT," etc.

Increasingly important as polynomial degree grows. $O(n \lg n \lg \lg n)$ coeff operations to compute n-coeff product.

Useful for sizes of n that occur in cryptography? In some cases, yes! But Karatsuba is the limit for prime-field ECC/ECDLP on most current CPUs.

Modular reduction

How to compute $f \bmod p$?

Can use definition:
$f \bmod p = f - p \lfloor f/p \rfloor$.
Can multiply f by a precomputed $1/p$ approximation; easily adjust to obtain $\lfloor f/p \rfloor$.

Slight speedup: "2-adic inverse"; "Montgomery reduction."

e.g. 314159265358 mod 271828:

Precompute

$\lfloor 1000000000000/271828 \rfloor$

$= 3678796$.

Compute

$314159 \cdot 3678796$

$= 1155726872564$.

Compute

$314159265358 - 1155726 \cdot 271828$

$= 578230$.

Oops, too big:

$578230 - 271828 = 306402$.

$306402 - 271828 = 34574$.

We can do better: normally p is chosen with a special form to make f mod p much faster.

Special primes hurt security for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$), etc., but not for elliptic curves!

Curve25519: $p = 2^{255} - 19$.

NIST P-224: $p = 2^{224} - 2^{96} + 1$.

secp112r1: $p = (2^{128} - 3)/76439$. *Divides* special form.

gls1271: $p = 2^{127} - 1$, with degree-2 extension (a bit scary).

Small example: $p = 1000003$.
Then $1000000a + b \equiv b - 3a$.

e.g. $314159265358 =$
$314159 \cdot 1000000 + 265358 \equiv$
$314159(-3) + 265358 =$
$-942477 + 265358 =$
-677119.

Easily adjust $b - 3a$
to the range $\{0, 1, \ldots, p - 1\}$
by adding/subtracting a few p's:
e.g. $-677119 \equiv 322884$.

Hmmm, is adjustment so easy?

Conditional branches are slow and leak secrets through timing. Can eliminate the branches, but adjustment isn't free.

Speedup: Skip the adjustment for intermediate results. "Lazy reduction." Adjust only for output.

$b - 3a$ is small enough to continue computations.

Can delay carries until after multiplication by 3.

e.g. To square 314159 in $\mathbf{Z}/1000003$: Square poly $3t^5 + 1t^4 + 4t^3 + 1t^2 + 5t^1 + 9t^0$, obtaining $9t^{10} + 6t^9 + 25t^8 + 14t^7 + 48t^6 + 72t^5 + 59t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce: replace $(c_i)t^{6+i}$ by $(-3c_i)t^i$, obtaining $72t^5 + 32t^4 + 64t^3 - 32t^2 + 48t^1 - 63t^0$.

Carry: $8t^6 - 4t^5 - 2t^4 + 1t^3 + 2t^2 + 2t^1 - 3t^0$.

To minimize poly degree, mix reduction and carrying, carrying the top sooner.

e.g. Start from square $9t^{10}+6t^9+25t^8+14t^7+48t^6+72t^5+59t^4+82t^3+43t^2+90t^1+81t^0$.

Reduce $t^{10} \rightarrow t^4$ and carry $t^4 \rightarrow t^5 \rightarrow t^6$: $6t^9+25t^8+14t^7+56t^6-5t^5+2t^4+82t^3+43t^2+90t^1+81t^0$.

Finish reduction: $-5t^5+2t^4+64t^3-32t^2+48t^1-87t^0$. Carry $t^0 \rightarrow t^1 \rightarrow t^2 \rightarrow t^3 \rightarrow t^4 \rightarrow t^5$: $-4t^5-2t^4+1t^3+2t^2-1t^1+3t^0$.

Speedup: non-integer radix

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}?
$f_4 t^4 + f_3 t^3 + f_2 t^2 + f_1 t^1 + f_0 t^0$.
Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4 f_1 + f_3 f_2) t^5 + \cdots$.
Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61} - 1)$;
$\cdots + (2^5(f_4 f_1 + f_3 f_2) + f_0^2) t^0$.
Coeff could be $> 2^{29}$.
Very little room for
additions, delayed carries, etc.
on 32-bit platforms.

Scaled: Evaluate at $t = 1$.

f_4 is multiple of 2^{52};

f_3 is multiple of 2^{39};

f_2 is multiple of 2^{26};

f_1 is multiple of 2^{13};

f_0 is multiple of 2^0. Reduce:

$\cdots + (2^{-60}(f_4 f_1 + f_3 f_2) + f_0^2)t^0$.

Better: Non-integer radix $2^{12.2}$.

f_4 is multiple of 2^{49};

f_3 is multiple of 2^{37};

f_2 is multiple of 2^{25};

f_1 is multiple of 2^{13};

f_0 is multiple of 2^0.

Saves a few bits in coeffs.

More bad choices from NIST

NIST P-256 prime:
$2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$.
i.e. $t^8 - t^7 + t^6 + t^3 - 1$
evaluated at $t = 2^{32}$.

More bad choices from NIST

NIST P-256 prime:
$2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$.
i.e. $t^8 - t^7 + t^6 + t^3 - 1$
evaluated at $t = 2^{32}$.

Reduction: replace $c_i t^{8+i}$ with
$c_i t^{7+i} - c_i t^{6+i} - c_i t^{3+i} + c_i t^i$.
Minor problem: often slower than
small const mult and one add.

More bad choices from NIST

NIST P-256 prime:
$2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$.
i.e. $t^8 - t^7 + t^6 + t^3 - 1$
evaluated at $t = 2^{32}$.

Reduction: replace $c_i t^{8+i}$ with
$c_i t^{7+i} - c_i t^{6+i} - c_i t^{3+i} + c_i t^i$.
Minor problem: often slower than small const mult and one add.

Major problem: With radix 2^{32}, products are almost 2^{64}.
Sums are slightly above 2^{64}: bad for every common CPU.
Need very frequent carries.