Twisted Hessian curves

cr.yp.to/papers.html#hessian

Daniel J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Joint work with:

Chitchanok Chuengsatiansup
Technische Universiteit Eindhoven

David Kohel
Aix-Marseille Université

Tanja Lange
Technische Universiteit Eindhoven

1986 Chudnovsky–Chudnovsky, "Sequences of numbers generated by addition in formal groups and new primality and factorization tests":

"The crucial problem becomes the choice of the model of an algebraic group variety, where computations mod p are the least time consuming."

Most important computations:
ADD is $P, Q \mapsto P + Q$.
DBL is $P \mapsto 2P$.

"It is preferable to use models of elliptic curves lying in low-dimensional spaces, for otherwise the number of coordinates and operations is increasing. This limits us ... to 4 basic models of elliptic curves."

Short Weierstrass:
$y^2 = x^3 + ax + b$.

Jacobi intersection:
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

Jacobi quartic: $y^2 = x^4 + 2ax^2 + 1$.

Hessian: $x^3 + y^3 + 1 = 3dxy$.

"Our experience shows that the expression of the law of addition on the cubic Hessian form (d) of an elliptic curve is by far the best and the prettiest."

$$X_3 = Y_1X_2 \cdot Y_1Z_2 - Z_1Y_2 \cdot X_1Y_2,$$
$$Y_3 = X_1Z_2 \cdot X_1Y_2 - Y_1X_2 \cdot Z_1X_2,$$
$$Z_3 = Z_1Y_2 \cdot Z_1X_2 - X_1Z_2 \cdot Y_1Z_2.$$

12**M** for ADD,
where **M** is the cost
of multiplication in the field.

8.4**M** for DBL,
assuming 0.8**M** for the cost
of squaring in the field.

1990s: ECC standards instead use short Weierstrass curves in Jacobian coordinates for "the fastest arithmetic".

15.2**M** for ADD, much slower than Hessian.

Why is this a good idea?

1990s: ECC standards instead
use short Weierstrass curves
in Jacobian coordinates
for “the fastest arithmetic”.

15.2**M** for ADD,
much slower than Hessian.

Why is this a good idea?
Answer: Only 7.2**M** for DBL with
Chudnovsky–Chudnovsky formula.

1990s: ECC standards instead use short Weierstrass curves in Jacobian coordinates for “the fastest arithmetic”.

15.2**M** for ADD, much slower than Hessian.

Why is this a good idea? Answer: Only 7.2**M** for DBL with Chudnovsky–Chudnovsky formula.

2001 Bernstein: 15**M**, 7**M**.

1990s: ECC standards instead use short Weierstrass curves in Jacobian coordinates for "the fastest arithmetic".

15.2**M** for ADD, much slower than Hessian.

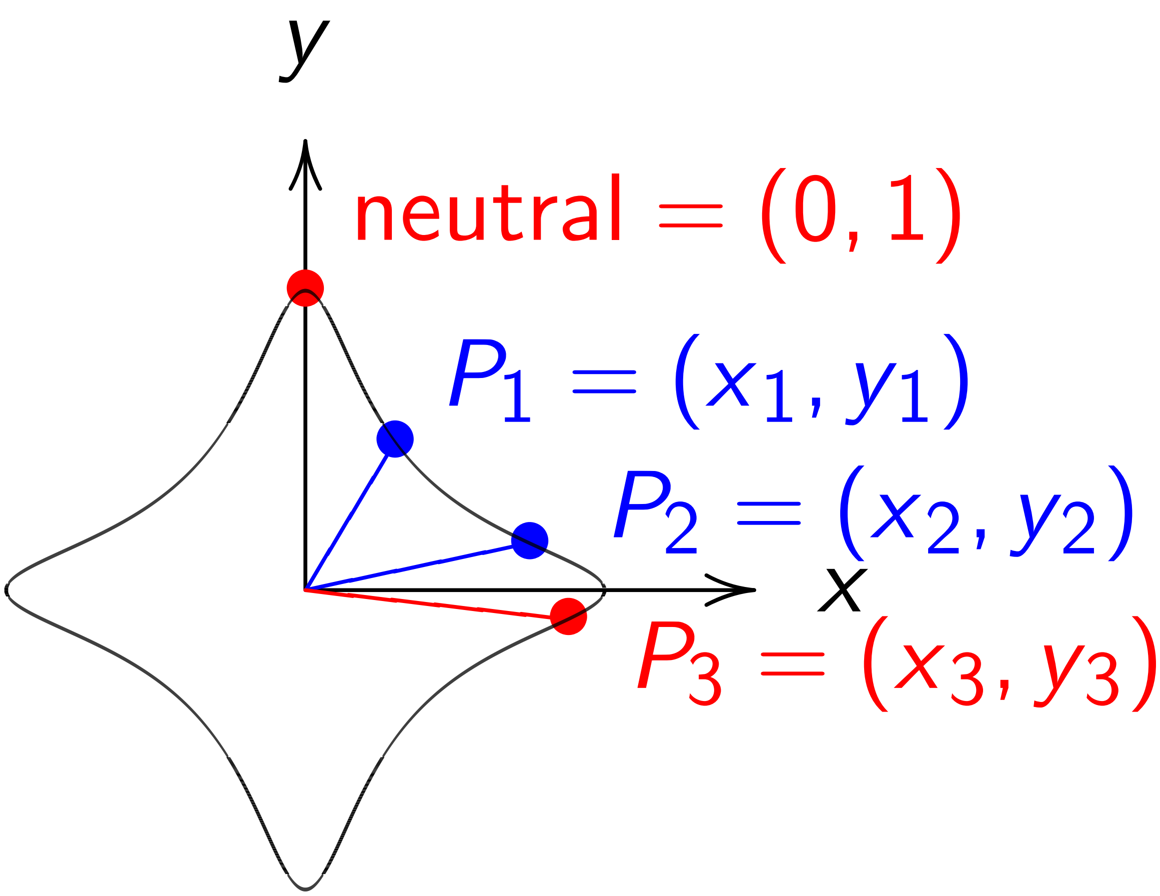
Why is this a good idea? Answer: Only 7.2**M** for DBL with Chudnovsky–Chudnovsky formula.

2001 Bernstein: 15**M**, 7**M**.

Compared to Hessian, Weierstrass saves 4**M** in typical DBL-DBL-DBL-DBL-DBL-ADD.

2007 Edwards: new curve shape.

2007 Bernstein–Lange: generalize, analyze speed, completeness.



Example: $x^2 + y^2 = 1 - 30x^2y^2$.

Sum of (x_1, y_1) and (x_2, y_2) is

$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$
$\quad(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2)).$

2007 Bernstein–Lange:

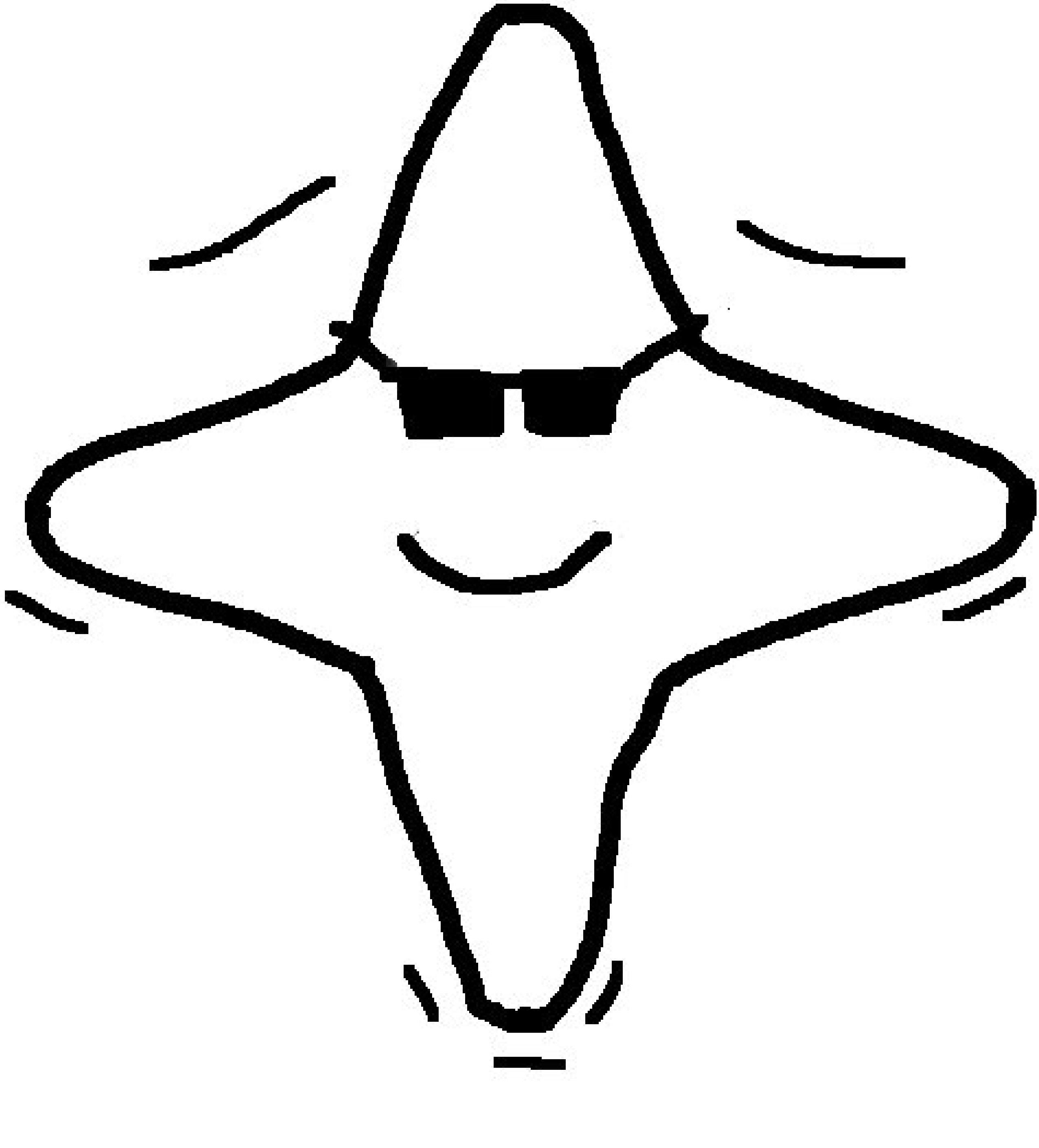
10.8**M** for ADD, 6.2**M** for DBL.

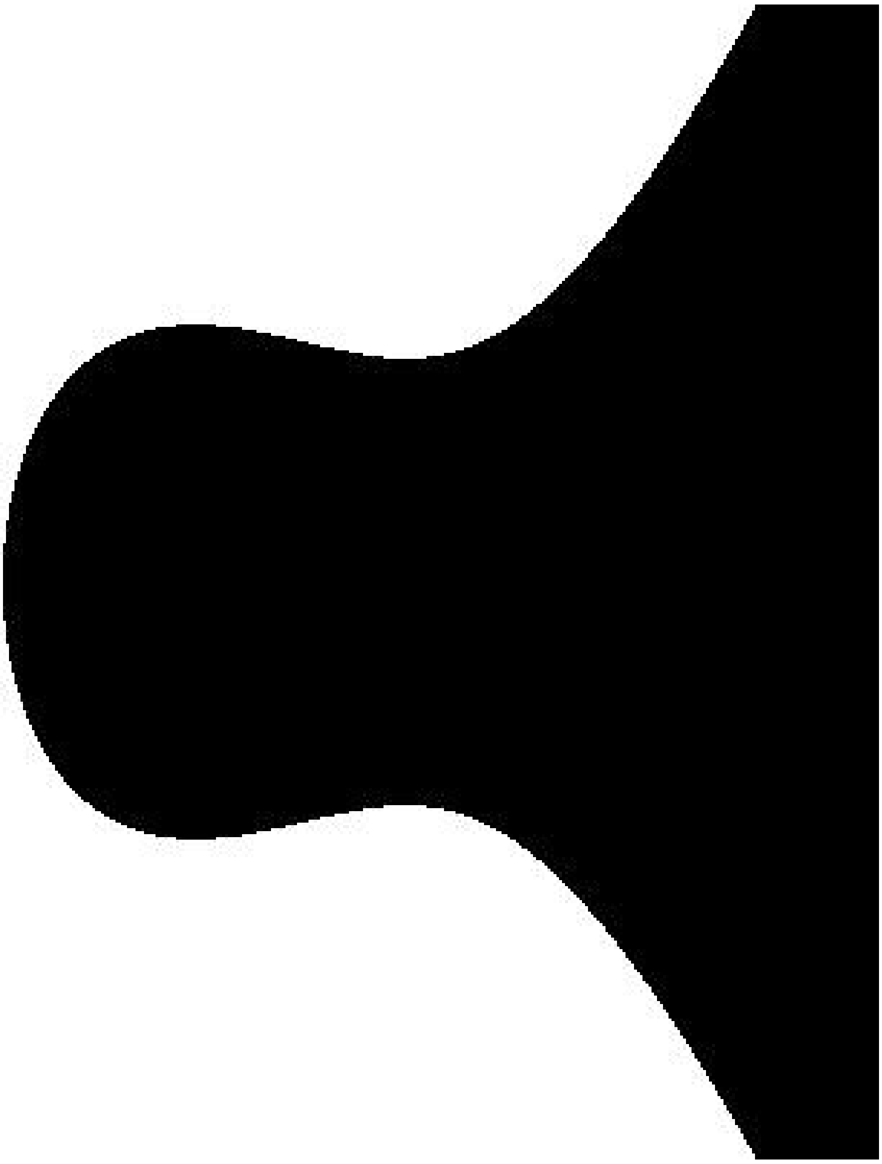
2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

2008 Hisil–Wong–Carter–Dawson:
just 8**M** for ADD.

2007 Bernstein–Lange:
10.8**M** for ADD, 6.2**M** for DBL.

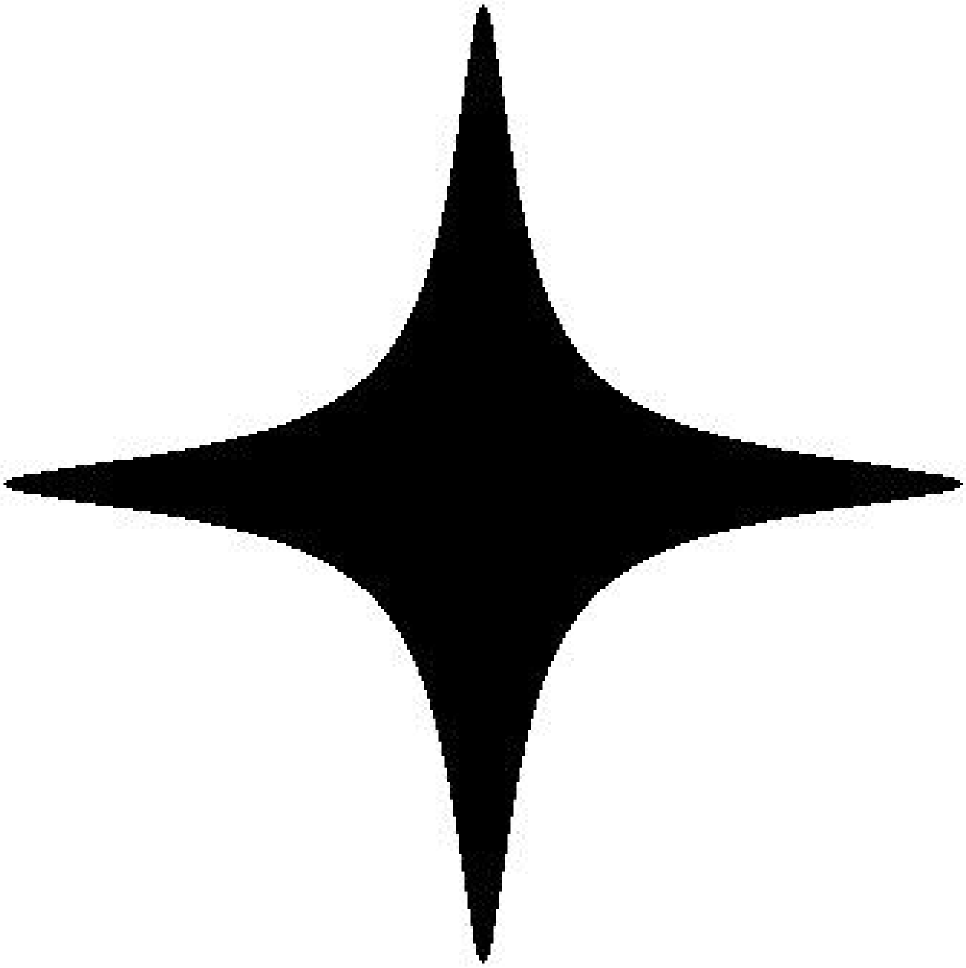
2008 Hisil–Wong–Carter–Dawson:
just 8**M** for ADD.



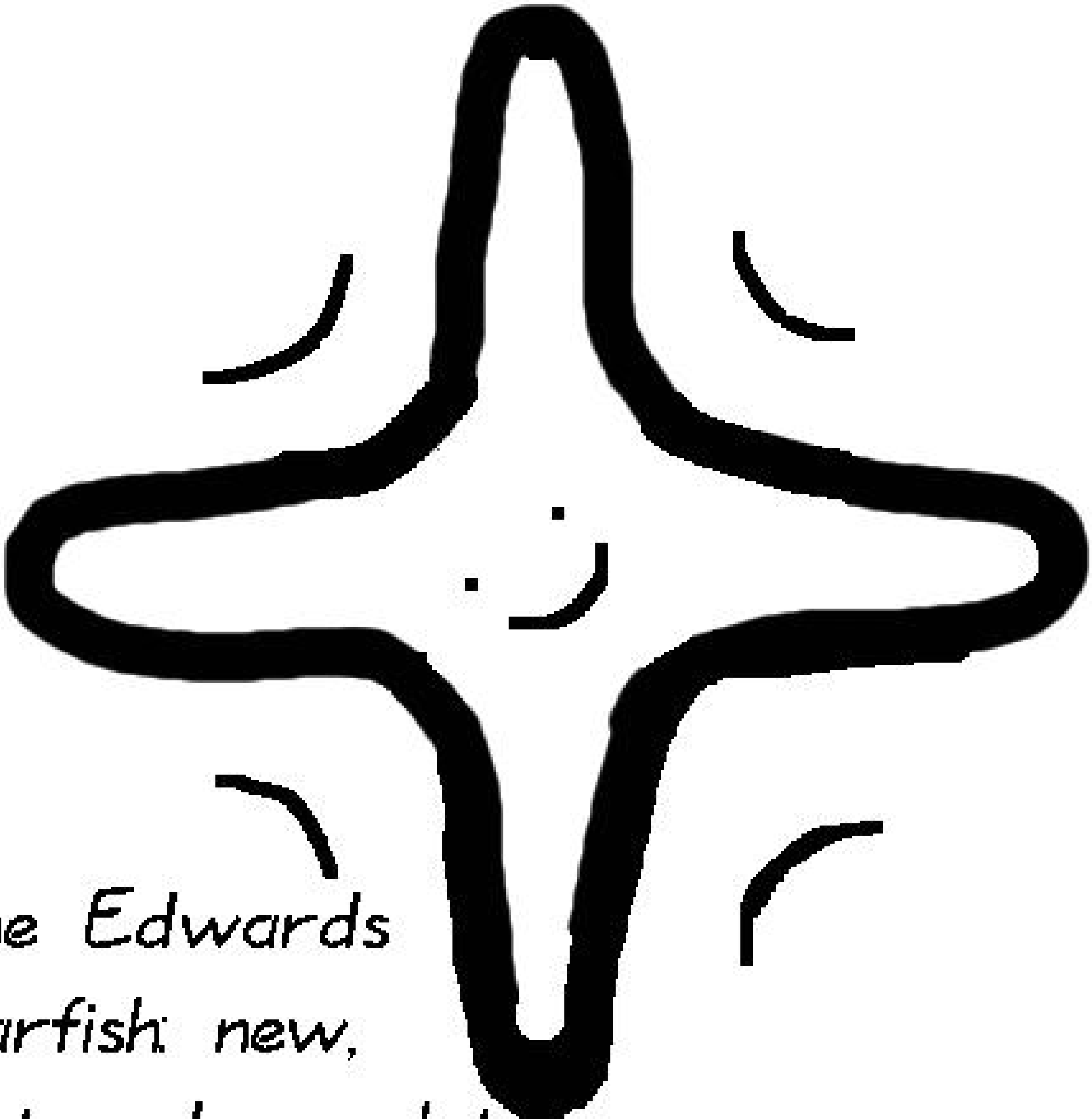


$$y^2 = x^3 - 0.4x + 0.7$$

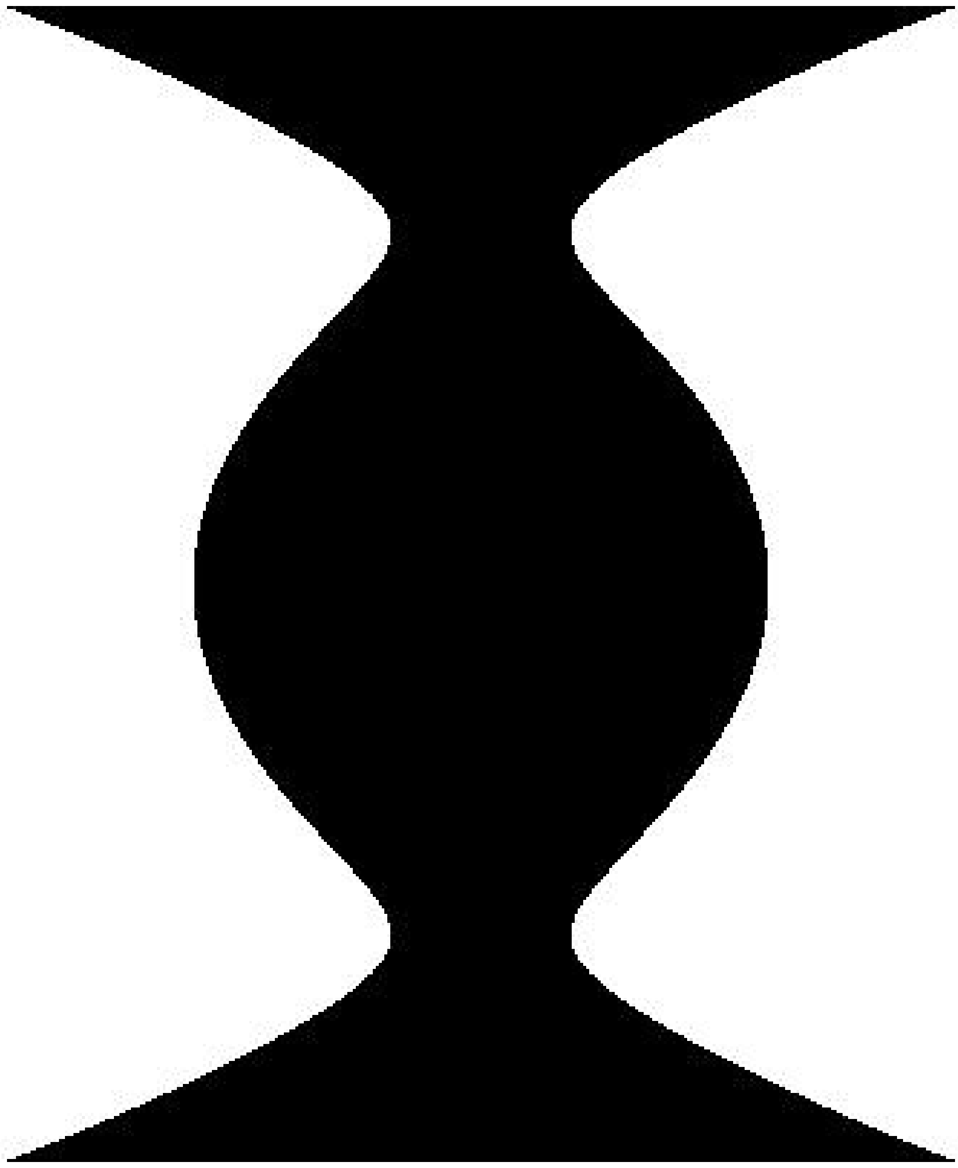
The Weierstrass-turtle: old, trusted and slow. Warning: (picture) incomplete!



$$x^2 + y^2 = 1 - 300x^2y^2$$

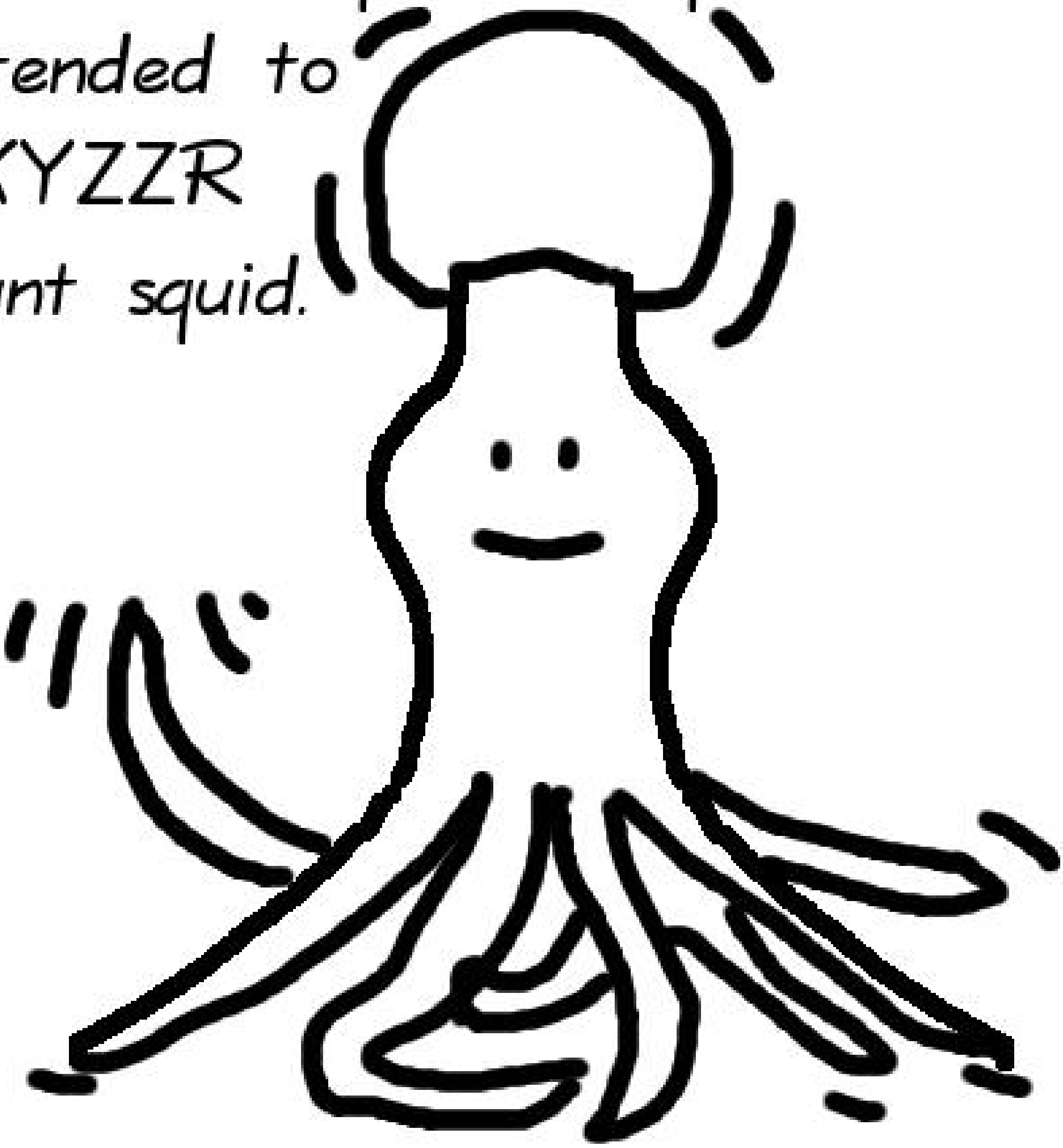


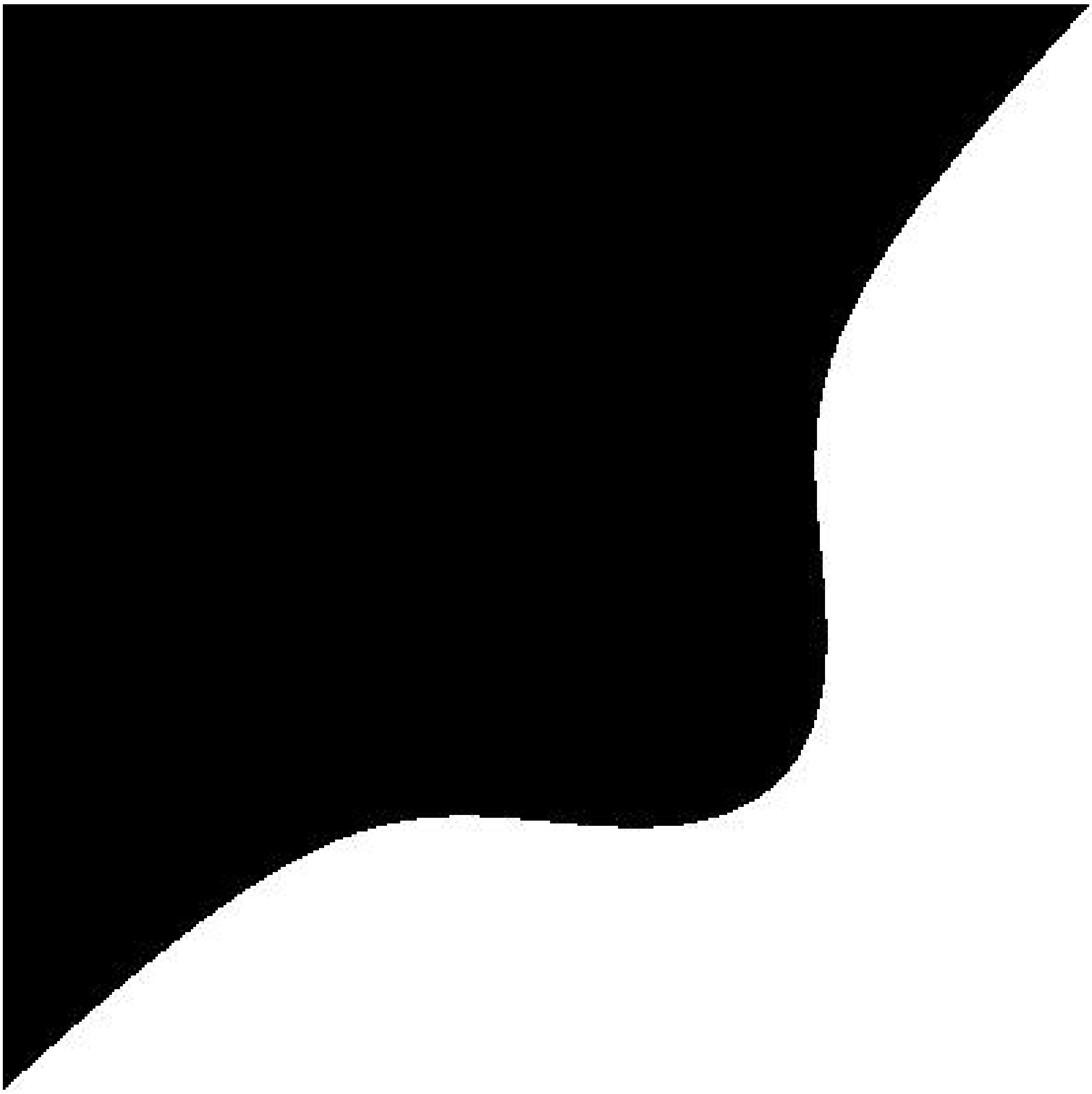
The Edwards
starfish: new,
fast and complete!



$$x^2 = y^4 - 1.9y^2 + 1$$

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.



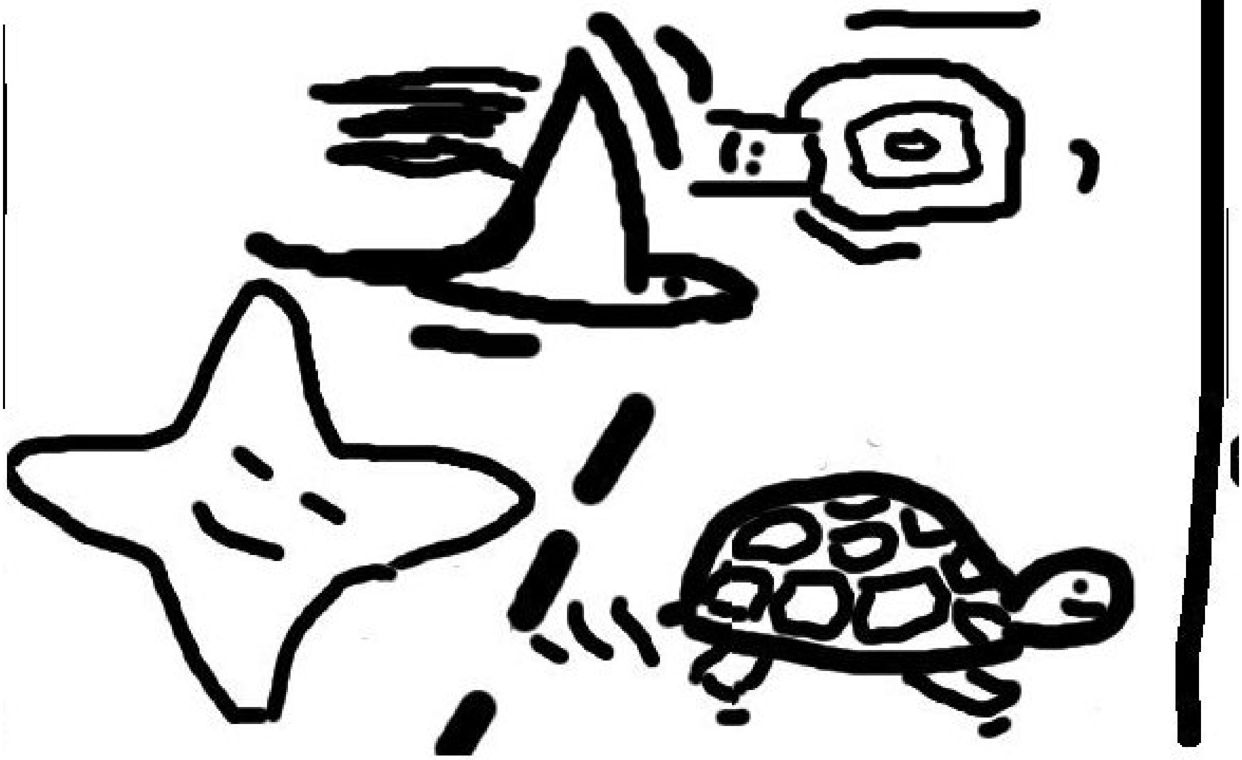


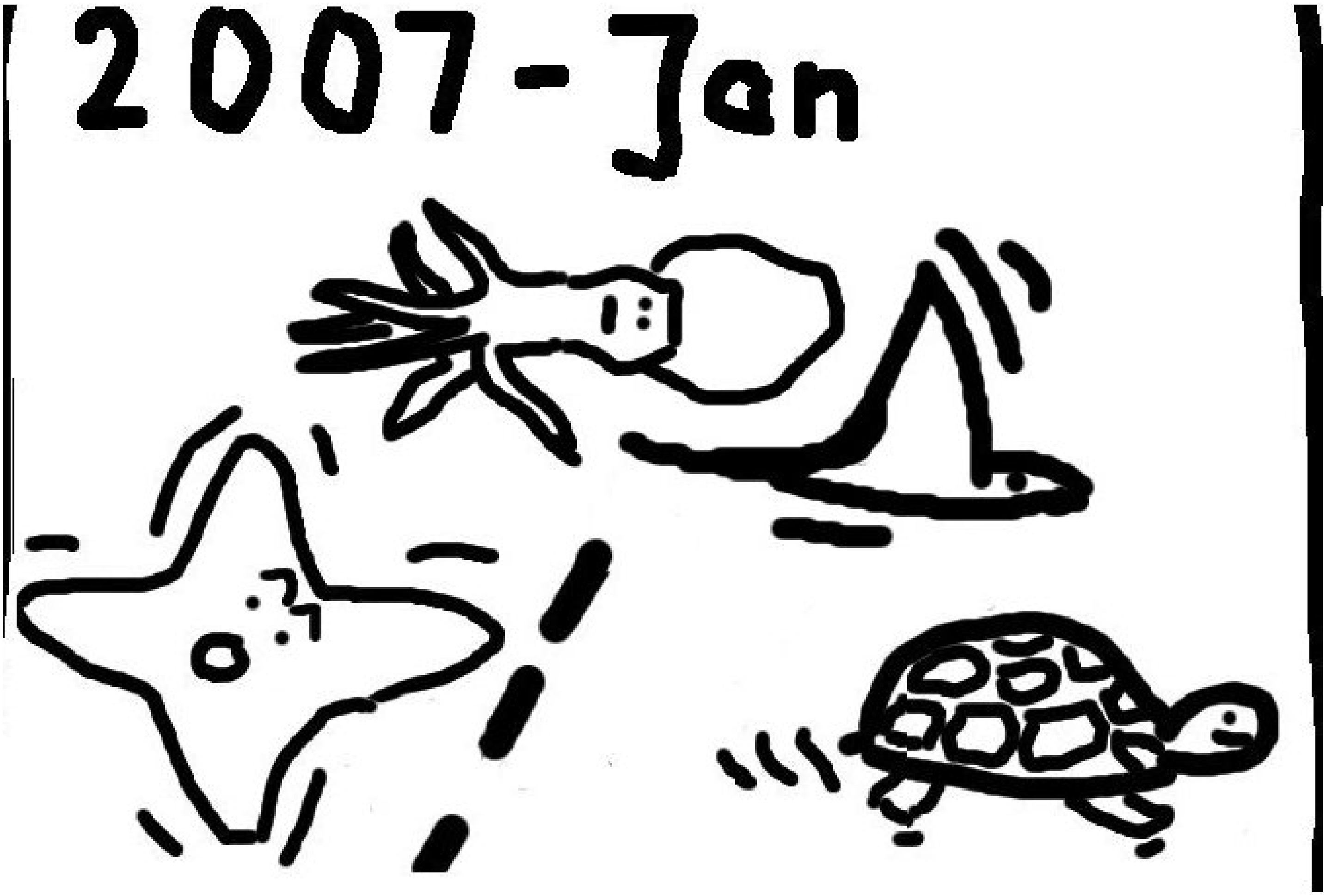
$$x^3 - y^3 + 1 = 0.3xy$$

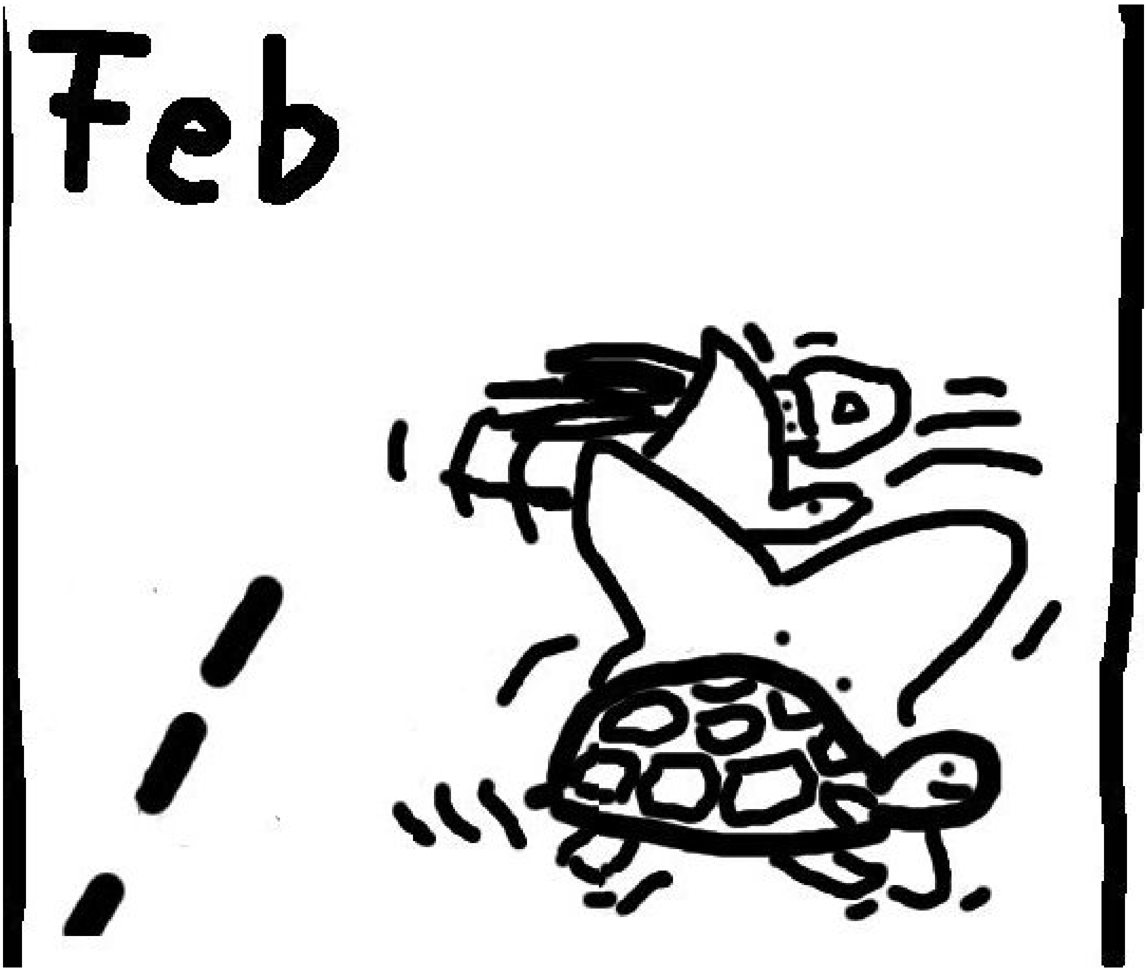

The Hessian-ray: uniform
but
not strongly so


START

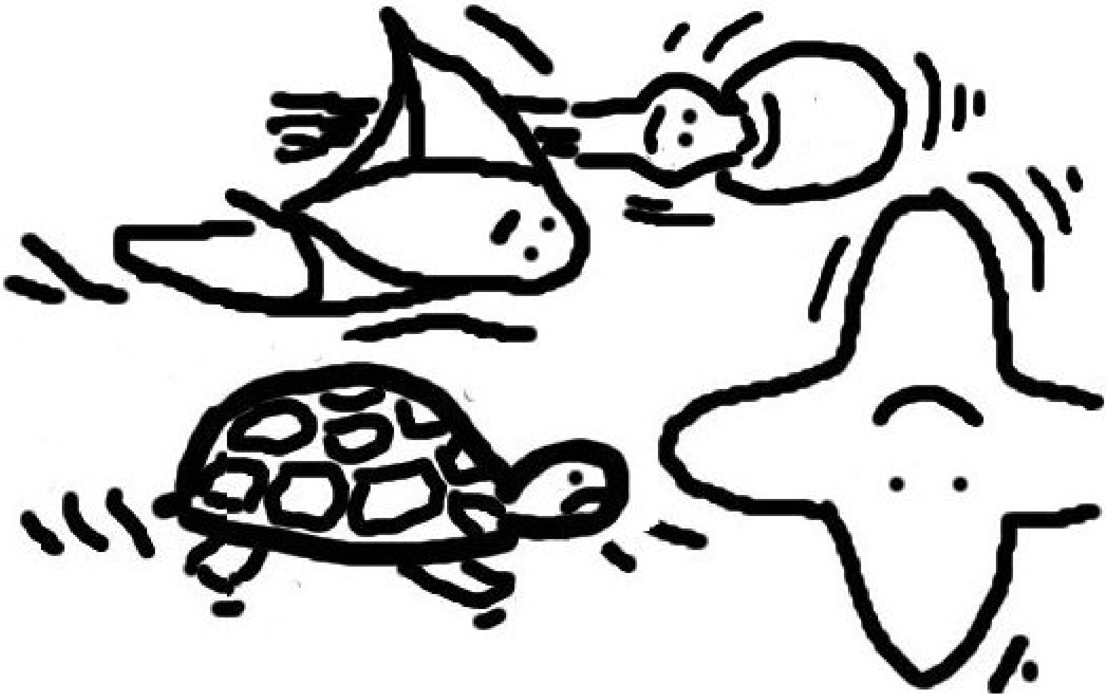
1985



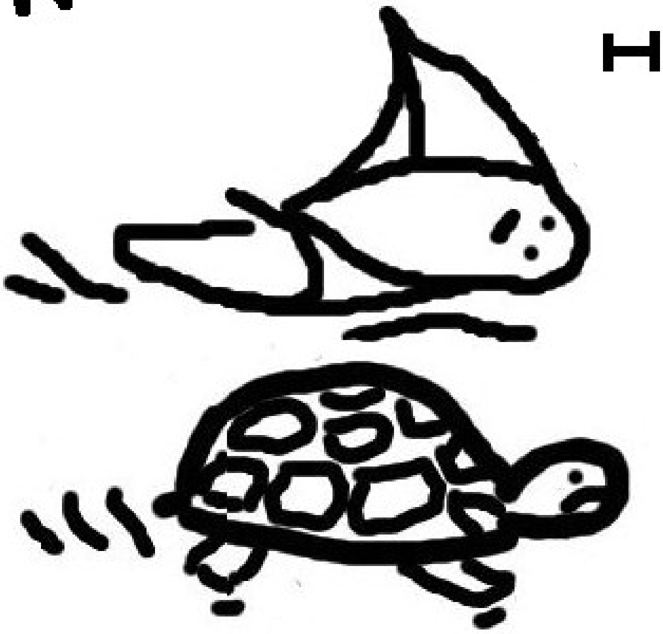

2007-Jan


Feb

Mar



Zoom



Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

Hessian tied with Weierstrass for
DBL-DBL-DBL-DBL-DBL-ADD.

Need to zoom in closer:
analyze exact **S**/**M**, overhead
for checking for special cases,
extra DBL, extra ADD, etc.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

Hessian tied with Weierstrass for
DBL-DBL-DBL-DBL-DBL-ADD.

Need to zoom in closer:
analyze exact **S**/**M**, overhead
for checking for special cases,
extra DBL, extra ADD, etc.

Or speed up Hessian more.

Faster Hessian arithmetic

2007 Hisil–Carter–Dawson:
7.8**M** for DBL.

2010 Hisil: 11**M** for ADD.

Hessian tied with Weierstrass for
DBL-DBL-DBL-DBL-DBL-ADD.

Need to zoom in closer:
analyze exact **S**/**M**, overhead
for checking for special cases,
extra DBL, extra ADD, etc.

Or speed up Hessian more.

New: 7.6**M** for DBL.

New (announced July 2009):

Generalize to more curves:
twisted Hessian curves
$aX^3 + Y^3 + Z^3 = dXYZ$
with $a(27a - d^3) \neq 0$.

2007 7.8**M** DBL idea fails, but
2010 11**M** ADD generalizes,
new 7.6**M** DBL generalizes.

New (announced July 2009):

Generalize to more curves:
twisted Hessian curves
$aX^3 + Y^3 + Z^3 = dXYZ$
with $a(27a - d^3) \neq 0$.

2007 7.8**M** DBL idea fails, but
2010 11**M** ADD generalizes,
new 7.6**M** DBL generalizes.

Rotate addition law
so that it also works for DBL;
complete if a is not a cube.
Eliminates special-case overhead,
helps stop side-channel attacks.

Triplings (assuming $d \neq 0$)

TPL is $P \mapsto 3P$.

2007 Hisil–Carter–Dawson:
12.8**M** for Hessian TPL.

Generalizes to twisted Hessian.

Triplings (assuming $d \neq 0$)

TPL is $P \mapsto 3P$.

2007 Hisil–Carter–Dawson:
12.8**M** for Hessian TPL.

Generalizes to twisted Hessian.

2015 Kohel: 11.2**M**.

Triplings (assuming $d \neq 0$)

TPL is $P \mapsto 3P$.

2007 Hisil–Carter–Dawson:
12.8**M** for Hessian TPL.

Generalizes to twisted Hessian.

2015 Kohel: 11.2**M**.

New: 10.8**M** assuming
field with fast primitive $\sqrt[3]{1}$;
e.g., $\mathbf{F}_q[\omega]/(\omega^2+\omega+1)$, or
$\mathbf{F}_p$ with $7p = 2^{298}+2^{149}+1$.

(More history in small char.
See paper for details.)

If $aX^3 + Y^3 + Z^3 = dXYZ$
then $VW(V + dU + aW) = U^3$
where
$U = -XYZ, V = Y^3, W = X^3.$

If $VW(V + dU + aW) = U^3$
then $aX_3^3 + Y_3^3 + Z_3^3 = dX_3Y_3Z_3$
where $Q = dU$, $R = aW$,
$S = -(V + Q + R)$,
$dX_3 = R^3 + S^3 + V^3 - 3RSV$,
$Y_3 = RS^2 + SV^2 + VR^2 - 3RSV$,
$Z_3 = RV^2 + SR^2 + VS^2 - 3RSV$.

Compose these 3-isogenies:
$(X_3 : Y_3 : Z_3) = 3(X : Y : Z)$.

To quickly triple $(X : Y : Z)$:

Three cubings for R, S, V.

For three choices of constants (α, β, γ) compute

$$(\alpha R + \beta S + \gamma V) \cdot$$
$$(\alpha S + \beta V + \gamma R) \cdot$$
$$(\alpha V + \beta R + \gamma S)$$
$$= \alpha\beta\gamma d X_3$$
$$+ (\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2) Y_3$$
$$+ (\beta\alpha^2 + \gamma\beta^2 + \alpha\gamma^2) Z_3$$
$$+ (\alpha + \beta + \gamma)^3 RSV.$$

Also use $a(R + S + V)^3 = d^3 RSV$.

Solve for dX_3, Y_3, Z_3.

2015 Kohel's 11.2**M**

(4 cubings + 4 mults)

introduced this TPL idea with

$(\alpha, \beta, \gamma) = (1, 1, 1)$,

$(\alpha, \beta, \gamma) = (1, -1, 0)$,

$(\alpha, \beta, \gamma) = (1, 1, 0)$.

2015 Kohel's 11.2**M**
(4 cubings + 4 mults)
introduced this TPL idea with
$(\alpha, \beta, \gamma) = (1, 1, 1)$,
$(\alpha, \beta, \gamma) = (1, -1, 0)$,
$(\alpha, \beta, \gamma) = (1, 1, 0)$.

New 10.8**M** (6 cubings)
makes faster choices
assuming fast primitive $\omega = \sqrt[3]{1}$:
$(\alpha, \beta, \gamma) = (1, 1, 1)$,
$(\alpha, \beta, \gamma) = (1, \omega, \omega^2)$,
$(\alpha, \beta, \gamma) = (1, \omega^2, \omega)$.

Are triplings useful?

2005 Dimitrov–Imbert–Mishra "double-base chains": e.g., compute $314159P$ as

$$2^{15}3^2P + 2^{11}3^2P + 2^8 3^1 P + 2^4 3^1 P - 2^0 3^0 P.$$

2TPL, 15DBL, 4ADD.

2006 Doche–Imbert generalized double-base chains: e.g., compute $314159P$ as

$$2^{12}3^3 3P - 2^7 3^3 5P - 2^4 3^1 7P - 2^0 3^0 P$$

after precomputing $3P, 5P, 7P$. 3TPL, 13DBL, 6ADD.

Not good for constant time.
Good for signature verification, factorization, math, etc.

Also need time to compute chain.
Good for scalars used many times.

Not good for constant time.
Good for signature verification,
factorization, math, etc.

Also need time to compute chain.
Good for scalars used many times.

Analysis+optimization from 2007
Bernstein–Birkner–Lange–Peters:

Double-base chains speed up
Weierstrass curves slightly:
9.29**M**/bit for 256-bit scalars.

More savings for, e.g., Hessian:
9.65**M**/bit. Still not competitive.

Revisit conclusions
using latest Hessian formulas,
latest double-base techniques.

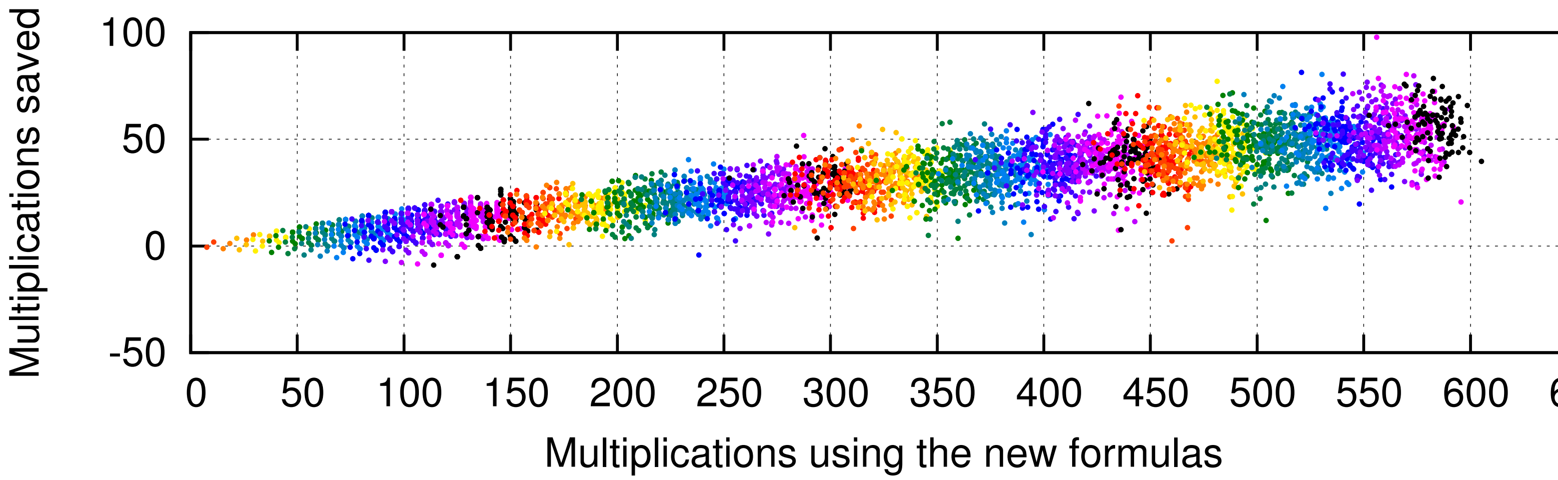
Revisit conclusions
using latest Hessian formulas,
latest double-base techniques.

New: 8.77**M**/bit for 256 bits.

Revisit conclusions
using latest Hessian formulas,
latest double-base techniques.

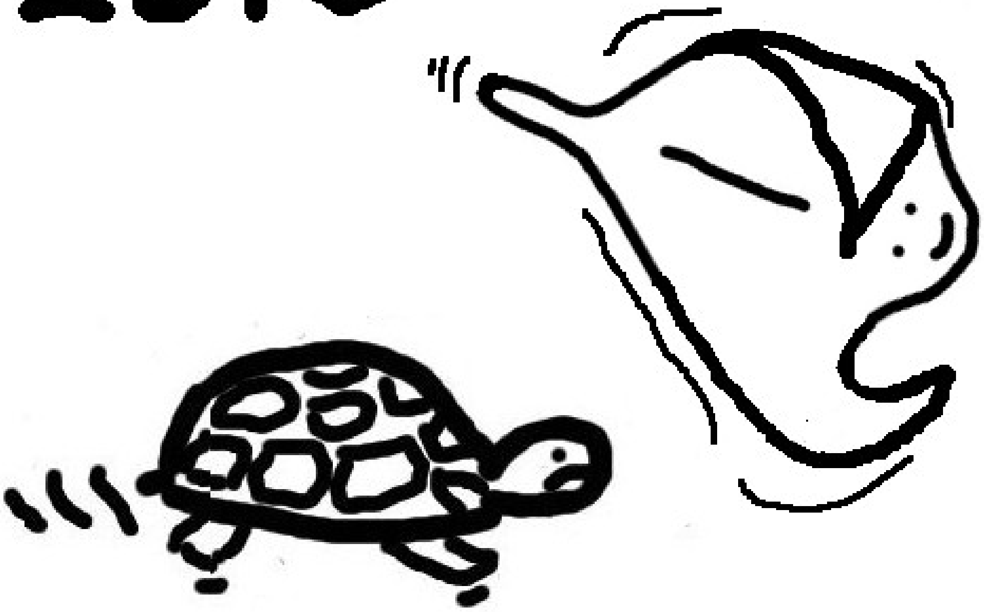
New: 8.77**M**/bit for 256 bits.

Comparison to Weierstrass for
1-bit, 2-bit, . . . , 64-bit scalars:



Uses 2008 Doche–Habsieger
"tree search" and some new
improvements: e.g., account for
costs of ADD, DBL, TPL.

Mar2015



Summary:
Twisted Hessian curves
solidly beat Weierstrass.

Chuengsatiansup talk tomorrow:
even better double-base chains
from shortest paths in DAG—
and also new Edwards speeds!