

Introduction to  
quantum algorithms  
and introduction to  
code-based cryptography

Daniel J. Bernstein

University of Illinois at Chicago &  
Technische Universiteit Eindhoven

Data (“state”) stored in  $n$  bits:  
an element of  $\{0, 1\}^n$ ,  
often viewed as representing  
an element of  $\{0, 1, \dots, 2^n - 1\}$ .

Data (“state”) stored in  $n$  bits:  
an element of  $\{0, 1\}^n$ ,  
often viewed as representing  
an element of  $\{0, 1, \dots, 2^n - 1\}$ .

State stored in  $n$  qubits:  
a nonzero element of  $\mathbf{C}^{2^n}$ .

Retrieving this vector is tough!

Data (“state”) stored in  $n$  bits:  
an element of  $\{0, 1\}^n$ ,  
often viewed as representing  
an element of  $\{0, 1, \dots, 2^n - 1\}$ .

State stored in  $n$  qubits:  
a nonzero element of  $\mathbf{C}^{2^n}$ .

Retrieving this vector is tough!

If  $n$  qubits have state

$(a_0, a_1, \dots, a_{2^n-1})$  then

**measuring** the qubits produces  
an element of  $\{0, 1, \dots, 2^n - 1\}$   
and destroys the state.

Measurement produces element  $q$   
with probability  $|a_q|^2 / \sum_r |a_r|^2$ .

Some examples of 3-qubit states:

$(1, 0, 0, 0, 0, 0, 0, 0)$  is

“ $|0\rangle$ ” in standard notation.

Measurement produces 0.

Some examples of 3-qubit states:

$(1, 0, 0, 0, 0, 0, 0, 0)$  is

“ $|0\rangle$ ” in standard notation.

Measurement produces 0.

$(0, 0, 0, 0, 0, 0, 1, 0)$  is

“ $|6\rangle$ ” in standard notation.

Measurement produces 6.

Some examples of 3-qubit states:

$(1, 0, 0, 0, 0, 0, 0, 0)$  is

“ $|0\rangle$ ” in standard notation.

Measurement produces 0.

$(0, 0, 0, 0, 0, 0, 1, 0)$  is

“ $|6\rangle$ ” in standard notation.

Measurement produces 6.

$(0, 0, 0, 0, 0, 0, -7i, 0) = -7i|6\rangle$ :

Measurement produces 6.

Some examples of 3-qubit states:

$(1, 0, 0, 0, 0, 0, 0, 0)$  is  
“ $|0\rangle$ ” in standard notation.

Measurement produces 0.

$(0, 0, 0, 0, 0, 0, 1, 0)$  is  
“ $|6\rangle$ ” in standard notation.

Measurement produces 6.

$(0, 0, 0, 0, 0, 0, -7i, 0) = -7i|6\rangle$ :

Measurement produces 6.

$(0, 0, 4, 0, 0, 0, 8, 0) = 4|2\rangle + 8|6\rangle$ :

Measurement produces

2 with probability 20%,

6 with probability 80%.



# Fast quantum operations, part 1

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$

$(a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6)$

is complementing index bit 0,

hence “complementing qubit 0”.

# Fast quantum operations, part 1

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$

$(a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6)$

is complementing index bit 0,

hence “complementing qubit 0”.

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$

is measured as  $(q_0, q_1, q_2)$ ,

representing  $q = q_0 + 2q_1 + 4q_2$ ,

with probability  $|a_q|^2 / \sum_r |a_r|^2$ .

$(a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6)$

is measured as  $(q_0 \oplus 1, q_1, q_2)$ ,

representing  $q \oplus 1$ ,

with probability  $|a_q|^2 / \sum_r |a_r|^2$ .

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$$

$$(a_4, a_5, a_6, a_7, a_0, a_1, a_2, a_3)$$

is “complementing qubit 2”:

$$(q_0, q_1, q_2) \mapsto (q_0, q_1, q_2 \oplus 1).$$

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$$

$$(a_4, a_5, a_6, a_7, a_0, a_1, a_2, a_3)$$

is “complementing qubit 2”:

$$(q_0, q_1, q_2) \mapsto (q_0, q_1, q_2 \oplus 1).$$

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$$

$$(a_0, a_4, a_2, a_6, a_1, a_5, a_3, a_7)$$

is “swapping qubits 0 and 2”:

$$(q_0, q_1, q_2) \mapsto (q_2, q_1, q_0).$$

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$$

$$(a_4, a_5, a_6, a_7, a_0, a_1, a_2, a_3)$$

is “complementing qubit 2”:

$$(q_0, q_1, q_2) \mapsto (q_0, q_1, q_2 \oplus 1).$$

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$$

$$(a_0, a_4, a_2, a_6, a_1, a_5, a_3, a_7)$$

is “swapping qubits 0 and 2”:

$$(q_0, q_1, q_2) \mapsto (q_2, q_1, q_0).$$

Complementing qubit 2

= swapping qubits 0 and 2

- complementing qubit 0
- swapping qubits 0 and 2.

Similarly: swapping qubits  $i, j$ .

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$

$(a_0, a_1, a_3, a_2, a_4, a_5, a_7, a_6)$

is a “reversible XOR gate” =

“controlled NOT gate”:

$(q_0, q_1, q_2) \mapsto (q_0 \oplus q_1, q_1, q_2)$ .

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$$

$$(a_0, a_1, a_3, a_2, a_4, a_5, a_7, a_6)$$

is a “reversible XOR gate” =

“controlled NOT gate”:

$$(q_0, q_1, q_2) \mapsto (q_0 \oplus q_1, q_1, q_2).$$

Example with more qubits:

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7,$$

$$a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15},$$

$$a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23},$$

$$a_{24}, a_{25}, a_{26}, a_{27}, a_{28}, a_{29}, a_{30}, a_{31})$$

$$\mapsto (a_0, a_1, a_3, a_2, a_4, a_5, a_7, a_6,$$

$$a_8, a_9, a_{11}, a_{10}, a_{12}, a_{13}, a_{15}, a_{14},$$

$$a_{16}, a_{17}, a_{19}, a_{18}, a_{20}, a_{21}, a_{23}, a_{22},$$

$$a_{24}, a_{25}, a_{27}, a_{26}, a_{28}, a_{29}, a_{31}, a_{30}).$$

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$$

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_7, a_6)$$

is a “Toffoli gate” =

“controlled controlled NOT gate”:

$$(q_0, q_1, q_2) \mapsto (q_0 \oplus q_1 q_2, q_1, q_2).$$



$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$$

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_7, a_6)$$

is a “Toffoli gate” =

“controlled controlled NOT gate”:

$$(q_0, q_1, q_2) \mapsto (q_0 \oplus q_1 q_2, q_1, q_2).$$

Example with more qubits:

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7,$$

$$a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15},$$

$$a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23},$$

$$a_{24}, a_{25}, a_{26}, a_{27}, a_{28}, a_{29}, a_{30}, a_{31})$$

$$\mapsto (a_0, a_1, a_2, a_3, a_4, a_5, a_7, a_6,$$

$$a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{15}, a_{14},$$

$$a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{23}, a_{22},$$

$$a_{24}, a_{25}, a_{26}, a_{27}, a_{28}, a_{29}, a_{31}, a_{30}).$$

# Reversible computation

Say  $p$  is a permutation of  $\{0, 1, \dots, 2^n - 1\}$ .

General strategy to compose these fast quantum operations to obtain index permutation

$(a_0, a_1, \dots, a_{2^n-1}) \mapsto$

$(a_{p^{-1}(0)}, a_{p^{-1}(1)}, \dots, a_{p^{-1}(2^n-1)})$ :

# Reversible computation

Say  $p$  is a permutation of  $\{0, 1, \dots, 2^n - 1\}$ .

General strategy to compose these fast quantum operations to obtain index permutation

$$(a_0, a_1, \dots, a_{2^n-1}) \mapsto (a_{p^{-1}(0)}, a_{p^{-1}(1)}, \dots, a_{p^{-1}(2^n-1)}):$$

1. Build a traditional circuit to compute  $j \mapsto p(j)$  using NOT/XOR/AND gates.
2. Convert into reversible gates: e.g., convert AND into Toffoli.

Example: Let's compute

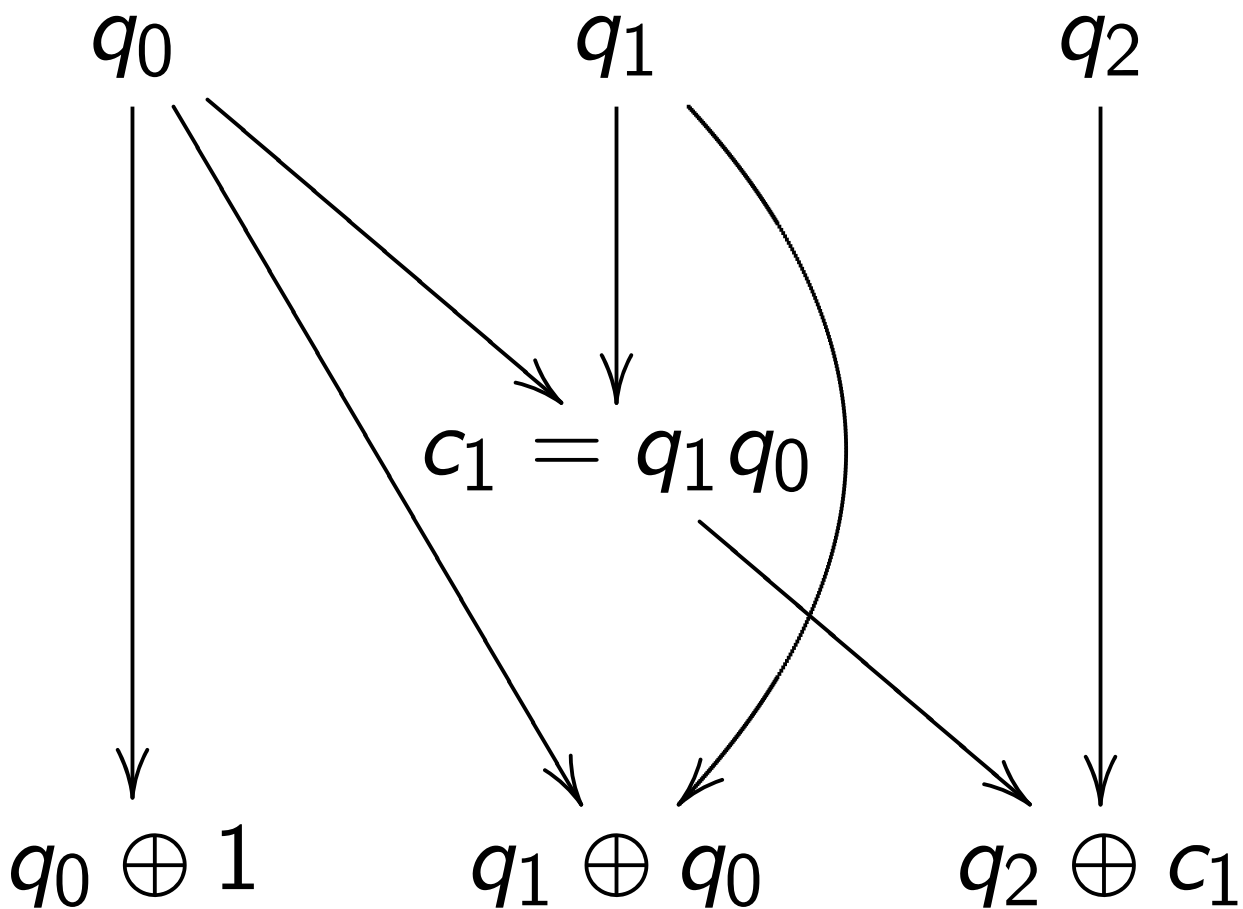
$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$

$(a_7, a_0, a_1, a_2, a_3, a_4, a_5, a_6)$ ;

permutation  $q \mapsto q + 1 \pmod 8$ .

1. Build a traditional circuit

to compute  $q \mapsto q + 1 \pmod 8$ .



## 2. Convert into reversible gates.

Toffoli for  $q_2 \leftarrow q_2 \oplus q_1 q_0$ :

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$

$(a_0, a_1, a_2, a_7, a_4, a_5, a_6, a_3)$ .

## 2. Convert into reversible gates.

Toffoli for  $q_2 \leftarrow q_2 \oplus q_1 q_0$ :

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto \\ (a_0, a_1, a_2, a_7, a_4, a_5, a_6, a_3).$$

Controlled NOT for  $q_1 \leftarrow q_1 \oplus q_0$ :

$$(a_0, a_1, a_2, a_7, a_4, a_5, a_6, a_3) \mapsto \\ (a_0, a_7, a_2, a_1, a_4, a_3, a_6, a_5).$$

## 2. Convert into reversible gates.

Toffoli for  $q_2 \leftarrow q_2 \oplus q_1 q_0$ :

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto (a_0, a_1, a_2, a_7, a_4, a_5, a_6, a_3).$$

Controlled NOT for  $q_1 \leftarrow q_1 \oplus q_0$ :

$$(a_0, a_1, a_2, a_7, a_4, a_5, a_6, a_3) \mapsto (a_0, a_7, a_2, a_1, a_4, a_3, a_6, a_5).$$

NOT for  $q_0 \leftarrow q_0 \oplus 1$ :

$$(a_0, a_7, a_2, a_1, a_4, a_3, a_6, a_5) \mapsto (a_7, a_0, a_1, a_2, a_3, a_4, a_5, a_6).$$

This permutation example  
was deceptively easy.

It didn't need many operations.

For large  $n$ , most permutations  $p$   
need many operations  $\Rightarrow$  slow.

Really want *fast* circuits.



This permutation example  
was deceptively easy.

It didn't need many operations.

For large  $n$ , most permutations  $p$   
need many operations  $\Rightarrow$  slow.

Really want *fast* circuits.

Also, it didn't need extra storage:  
circuit operated "in place" after  
computation  $c_1 \leftarrow q_1 q_0$  was  
merged into  $q_2 \leftarrow q_2 \oplus c_1$ .

Typical circuits aren't in-place.

Start from any circuit:

inputs  $b_1, b_2, \dots, b_i$ ;

$$b_{i+1} = 1 \oplus b_{f(i+1)} b_{g(i+1)};$$

$$b_{i+2} = 1 \oplus b_{f(i+2)} b_{g(i+2)};$$

...

$$b_T = 1 \oplus b_{f(T)} b_{g(T)};$$

specified outputs.

Start from any circuit:

inputs  $b_1, b_2, \dots, b_i$ ;

$$b_{i+1} = 1 \oplus b_{f(i+1)} b_{g(i+1)};$$

$$b_{i+2} = 1 \oplus b_{f(i+2)} b_{g(i+2)};$$

...

$$b_T = 1 \oplus b_{f(T)} b_{g(T)};$$

specified outputs.

Reversible but dirty:

inputs  $b_1, b_2, \dots, b_T$ ;

$$b_{i+1} \leftarrow 1 \oplus b_{i+1} \oplus b_{f(i+1)} b_{g(i+1)};$$

$$b_{i+2} \leftarrow 1 \oplus b_{i+2} \oplus b_{f(i+2)} b_{g(i+2)};$$

...

$$b_T \leftarrow 1 \oplus b_T \oplus b_{f(T)} b_{g(T)}.$$

Same outputs if all of

$b_{i+1}, \dots, b_T$  started as 0.

Reversible and clean:  
after finishing dirty computation,  
set non-outputs back to 0,  
by repeating same operations  
on non-outputs in reverse order.

Original computation:  
 $(\text{inputs}) \mapsto$   
 $(\text{inputs}, \text{dirt}, \text{outputs})$ .

Dirty reversible computation:  
 $(\text{inputs}, \text{zeros}, \text{zeros}) \mapsto$   
 $(\text{inputs}, \text{dirt}, \text{outputs})$ .

Clean reversible computation:  
 $(\text{inputs}, \text{zeros}, \text{zeros}) \mapsto$   
 $(\text{inputs}, \text{zeros}, \text{outputs})$ .

Given fast circuit for  $p$   
and fast circuit for  $p^{-1}$ ,  
build fast reversible circuit for  
 $(x, \text{zeros}) \mapsto (p(x), \text{zeros})$ .

Given fast circuit for  $p$   
and fast circuit for  $p^{-1}$ ,  
build fast reversible circuit for  
 $(x, \text{zeros}) \mapsto (p(x), \text{zeros})$ .

Replace reversible bit operations  
with Toffoli gates etc.

permuting  $\mathbf{C}^{2^{n+z}} \rightarrow \mathbf{C}^{2^{n+z}}$ .

Permutation on first  $2^n$  entries is  
 $(a_0, a_1, \dots, a_{2^n-1}) \mapsto$   
 $(a_{p^{-1}(0)}, a_{p^{-1}(1)}, \dots, a_{p^{-1}(2^n-1)})$ .

Typically prepare vectors  
supported on first  $2^n$  entries  
so don't care how permutation  
acts on last  $2^{n+z} - 2^n$  entries.

Warning: Number of **qubits**  
 $\approx$  number of **bit operations**  
in original  $p, p^{-1}$  circuits.

This can be much larger  
than number of **bits stored**  
in the original circuits.

Warning: Number of **qubits**  
 $\approx$  number of **bit operations**  
in original  $p, p^{-1}$  circuits.

This can be much larger  
than number of **bits stored**  
in the original circuits.

Many useful techniques  
to compress into fewer qubits,  
but often these lose time.

Many subtle tradeoffs.



Warning: Number of **qubits**  
 $\approx$  number of **bit operations**  
in original  $p, p^{-1}$  circuits.

This can be much larger  
than number of **bits stored**  
in the original circuits.

Many useful techniques  
to compress into fewer qubits,  
but often these lose time.

Many subtle tradeoffs.

Crude “poly-time” analyses  
don't care about this,  
but serious cryptanalysis  
is much more precise.

## Fast quantum operations, part 2

“Hadamard” :

$$(a_0, a_1) \mapsto (a_0 + a_1, a_0 - a_1).$$

## Fast quantum operations, part 2

“Hadamard”:

$$(a_0, a_1) \mapsto (a_0 + a_1, a_0 - a_1).$$

$$(a_0, a_1, a_2, a_3) \mapsto$$

$$(a_0 + a_1, a_0 - a_1, a_2 + a_3, a_2 - a_3).$$

## Fast quantum operations, part 2

“Hadamard”:

$$(a_0, a_1) \mapsto (a_0 + a_1, a_0 - a_1).$$

$$(a_0, a_1, a_2, a_3) \mapsto$$

$$(a_0 + a_1, a_0 - a_1, a_2 + a_3, a_2 - a_3).$$

Same for qubit 1:

$$(a_0, a_1, a_2, a_3) \mapsto$$

$$(a_0 + a_2, a_1 + a_3, a_0 - a_2, a_1 - a_3).$$

## Fast quantum operations, part 2

“Hadamard” :

$$(a_0, a_1) \mapsto (a_0 + a_1, a_0 - a_1).$$

$$(a_0, a_1, a_2, a_3) \mapsto$$

$$(a_0 + a_1, a_0 - a_1, a_2 + a_3, a_2 - a_3).$$

Same for qubit 1:

$$(a_0, a_1, a_2, a_3) \mapsto$$

$$(a_0 + a_2, a_1 + a_3, a_0 - a_2, a_1 - a_3).$$

Qubit 0 and then qubit 1:

$$(a_0, a_1, a_2, a_3) \mapsto$$

$$(a_0 + a_1, a_0 - a_1, a_2 + a_3, a_2 - a_3) \mapsto$$

$$(a_0 + a_1 + a_2 + a_3, a_0 - a_1 + a_2 - a_3, \\ a_0 + a_1 - a_2 - a_3, a_0 - a_1 - a_2 + a_3).$$

Repeat  $n$  times: e.g.,

$(1, 0, 0, \dots, 0) \mapsto (1, 1, 1, \dots, 1)$ .

Measuring  $(1, 0, 0, \dots, 0)$

always produces 0.

Measuring  $(1, 1, 1, \dots, 1)$

can produce any output:

$\Pr[\text{output} = q] = 1/2^n$ .

Repeat  $n$  times: e.g.,

$(1, 0, 0, \dots, 0) \mapsto (1, 1, 1, \dots, 1)$ .

Measuring  $(1, 0, 0, \dots, 0)$

always produces 0.

Measuring  $(1, 1, 1, \dots, 1)$

can produce any output:

$\Pr[\text{output} = q] = 1/2^n$ .

Aside from “normalization”

(irrelevant to measurement),

have Hadamard = Hadamard<sup>-1</sup>,

so easily work backwards

from “uniform superposition”

$(1, 1, 1, \dots, 1)$  to “pure state”

$(1, 0, 0, \dots, 0)$ .

## Simon's algorithm

Assume: nonzero  $s \in \{0, 1\}^n$

satisfies  $f(x) = f(x \oplus s)$

for every  $x \in \{0, 1\}^n$ .

Can we find this period  $s$ ,  
given a fast circuit for  $f$ ?



## Simon's algorithm

Assume: nonzero  $s \in \{0, 1\}^n$   
satisfies  $f(x) = f(x \oplus s)$   
for every  $x \in \{0, 1\}^n$ .

Can we find this period  $s$ ,  
given a fast circuit for  $f$ ?

We don't have enough data  
if  $f$  has many periods.

Assume:  $\{\text{periods}\} = \{0, s\}$ .

## Simon's algorithm

Assume: nonzero  $s \in \{0, 1\}^n$   
satisfies  $f(x) = f(x \oplus s)$   
for every  $x \in \{0, 1\}^n$ .

Can we find this period  $s$ ,  
given a fast circuit for  $f$ ?

We don't have enough data  
if  $f$  has many periods.

Assume:  $\{\text{periods}\} = \{0, s\}$ .

Traditional solution:

Compute  $f$  for many inputs,  
sort, analyze collisions.

Success probability is very low  
until  $\#$ inputs approaches  $2^{n/2}$ .

Simon's algorithm uses  
far fewer qubit operations  
if  $n$  is large and  
reversibility overhead is low.

Simon's algorithm uses  
far fewer qubit operations  
if  $n$  is large and  
reversibility overhead is low.

Say  $f$  maps  $n$  bits to  $m$  bits using  
 $z$  "ancilla" bits for reversibility.

Prepare  $n + m + z$  qubits  
in pure zero state:  
vector  $(1, 0, 0, \dots)$ .

Simon's algorithm uses  
far fewer qubit operations  
if  $n$  is large and  
reversibility overhead is low.

Say  $f$  maps  $n$  bits to  $m$  bits using  
 $z$  "ancilla" bits for reversibility.

Prepare  $n + m + z$  qubits  
in pure zero state:  
vector  $(1, 0, 0, \dots)$ .

Use  $n$ -fold Hadamard  
to move first  $n$  qubits  
into uniform superposition:  
 $(1, 1, 1, \dots, 1, 0, 0, \dots)$   
with  $2^n$  entries 1, others 0.

Apply fast vector permutation  
for reversible  $f$  computation:

1 in position  $(q, 0, 0)$

moves to position  $(q, f(q), 0)$ .

Note symmetry between

1 at  $(q, f(q), 0)$  and

1 at  $(q \oplus s, f(q), 0)$ .

Apply fast vector permutation  
for reversible  $f$  computation:  
1 in position  $(q, 0, 0)$   
moves to position  $(q, f(q), 0)$ .

Note symmetry between  
1 at  $(q, f(q), 0)$  and  
1 at  $(q \oplus s, f(q), 0)$ .

Apply  $n$ -fold Hadamard.

Apply fast vector permutation  
for reversible  $f$  computation:  
1 in position  $(q, 0, 0)$   
moves to position  $(q, f(q), 0)$ .

Note symmetry between  
1 at  $(q, f(q), 0)$  and  
1 at  $(q \oplus s, f(q), 0)$ .

Apply  $n$ -fold Hadamard.

Measure. By symmetry,  
output is orthogonal to  $s$ .



Apply fast vector permutation  
for reversible  $f$  computation:  
1 in position  $(q, 0, 0)$   
moves to position  $(q, f(q), 0)$ .

Note symmetry between  
1 at  $(q, f(q), 0)$  and  
1 at  $(q \oplus s, f(q), 0)$ .

Apply  $n$ -fold Hadamard.

Measure. By symmetry,  
output is orthogonal to  $s$ .

Repeat  $n + 10$  times.

Use Gaussian elimination  
to (probably) find  $s$ .

Example, 3 bits to 3 bits:

$$f(0) = 4.$$

$$f(1) = 7.$$

$$f(2) = 2.$$

$$f(3) = 3.$$

$$f(4) = 7.$$

$$f(5) = 4.$$

$$f(6) = 3.$$

$$f(7) = 2.$$

Example, 3 bits to 3 bits:

$$f(0) = 4.$$

$$f(1) = 7.$$

$$f(2) = 2.$$

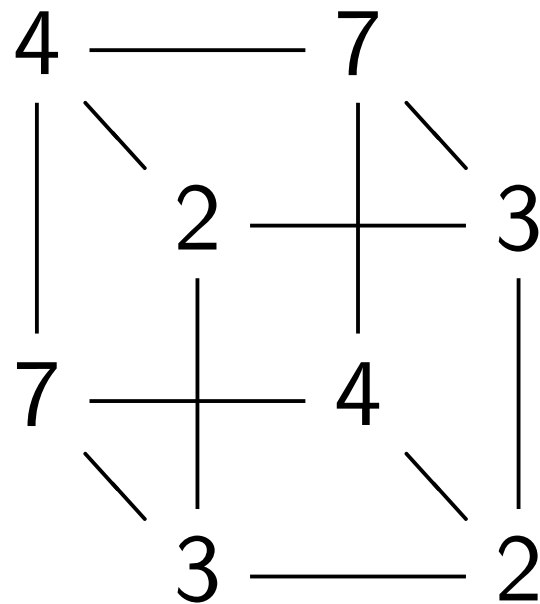
$$f(3) = 3.$$

$$f(4) = 7.$$

$$f(5) = 4.$$

$$f(6) = 3.$$

$$f(7) = 2.$$



Example, 3 bits to 3 bits:

$$f(0) = 4.$$

$$f(1) = 7.$$

$$f(2) = 2.$$

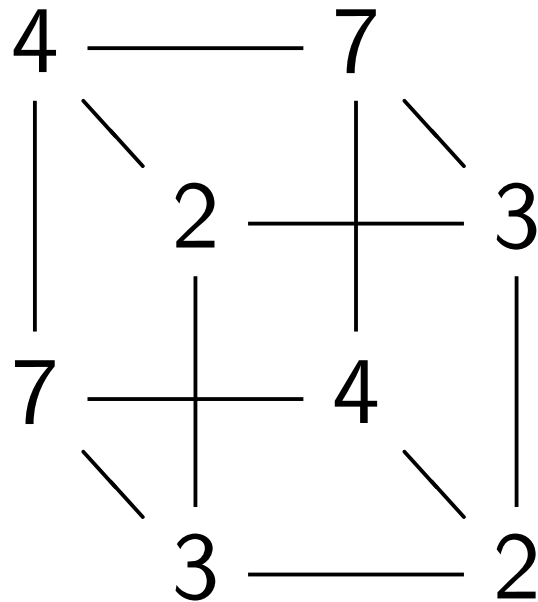
$$f(3) = 3.$$

$$f(4) = 7.$$

$$f(5) = 4.$$

$$f(6) = 3.$$

$$f(7) = 2.$$



Complete table shows that

$$f(x) = f(x \oplus 5) \text{ for all } x.$$

Let's watch Simon's algorithm for  $f$ , using 6 qubits.











Step 5.  $(q, 0) \mapsto (q, f(q))$ :

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 1,

0, 0, 0, 1, 0, 0, 1, 0,

1, 0, 0, 0, 0, 1, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 1, 0, 0, 1, 0, 0, 0.

Step 6. Hadamard on qubit 0:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 1, 0, 0, 1,  $\bar{1}$ ,

0, 0, 1,  $\bar{1}$ , 0, 0, 1, 1,

1, 1, 0, 0, 1,  $\bar{1}$ , 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

1,  $\bar{1}$ , 0, 0, 1, 1, 0, 0.

Notation:  $\bar{1} = -1$ .

Step 7. Hadamard on qubit 1:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

1, 1,  $\bar{1}$ ,  $\bar{1}$ , 1,  $\bar{1}$ ,  $\bar{1}$ , 1,

1,  $\bar{1}$ ,  $\bar{1}$ , 1, 1, 1,  $\bar{1}$ ,  $\bar{1}$ ,

1, 1, 1, 1, 1,  $\bar{1}$ , 1,  $\bar{1}$ ,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

1,  $\bar{1}$ , 1,  $\bar{1}$ , 1, 1, 1, 1.

Step 8. Hadamard on qubit 2:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0,  $\bar{2}$ , 0, 0,  $\bar{2}$ , 0,  $\bar{2}$ ,

2, 0,  $\bar{2}$ , 0, 0,  $\bar{2}$ , 0, 2,

2, 0, 2, 0, 0, 2, 0, 2,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, 2, 0, 0,  $\bar{2}$ , 0,  $\bar{2}$ .

Step 8. Hadamard on qubit 2:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0,  $\bar{2}$ , 0, 0,  $\bar{2}$ , 0,  $\bar{2}$ ,

2, 0,  $\bar{2}$ , 0, 0,  $\bar{2}$ , 0, 2,

2, 0, 2, 0, 0, 2, 0, 2,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, 2, 0, 0,  $\bar{2}$ , 0,  $\bar{2}$ .

Step 9. Measure.

First 3 qubits are uniform random vector orthogonal to 101: i.e., 000, 010, 101, or 111.

## Grover's algorithm

Assume: unique  $s \in \{0, 1\}^n$   
has  $f(s) = 0$ .

Traditional algorithm to find  $s$ :  
compute  $f$  for many inputs,  
hope to find output 0.

Success probability is very low  
until #inputs approaches  $2^n$ .

## Grover's algorithm

Assume: unique  $s \in \{0, 1\}^n$   
has  $f(s) = 0$ .

Traditional algorithm to find  $s$ :  
compute  $f$  for many inputs,  
hope to find output 0.

Success probability is very low  
until #inputs approaches  $2^n$ .

Grover's algorithm takes only  $2^{n/2}$   
reversible computations of  $f$ .

Typically: reversibility overhead  
is small enough that this  
easily beats traditional algorithm.

Start from uniform superposition over all  $n$ -bit strings  $q$ .

Step 1: Set  $a \leftarrow b$  where

$$b_q = -a_q \text{ if } f(q) = 0,$$

$$b_q = a_q \text{ otherwise.}$$

This is fast.

Step 2: “Grover diffusion”.

Negate  $a$  around its average.

This is also fast.

Repeat Step 1 + Step 2

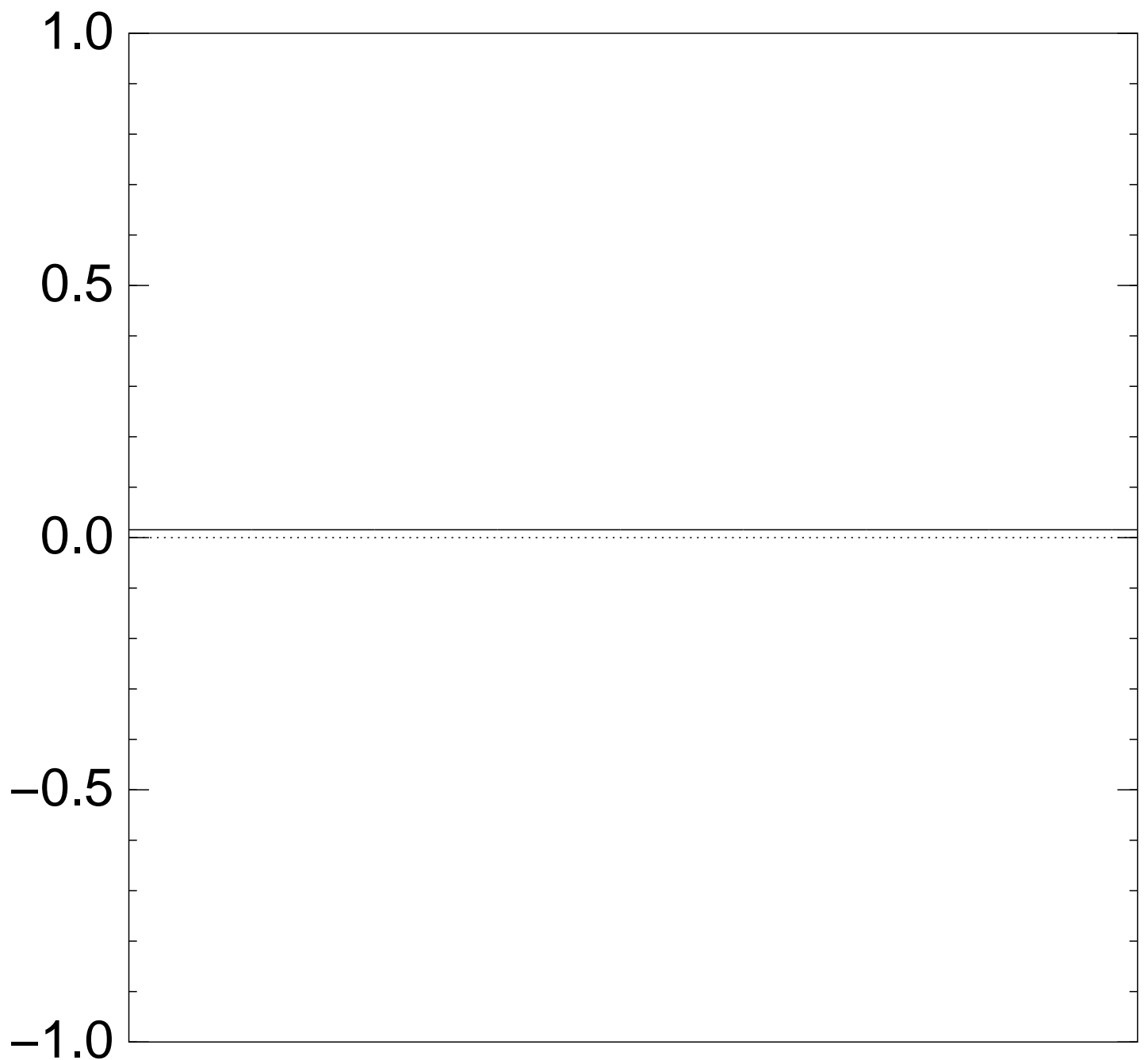
about  $0.58 \cdot 2^{0.5n}$  times.

Measure the  $n$  qubits.

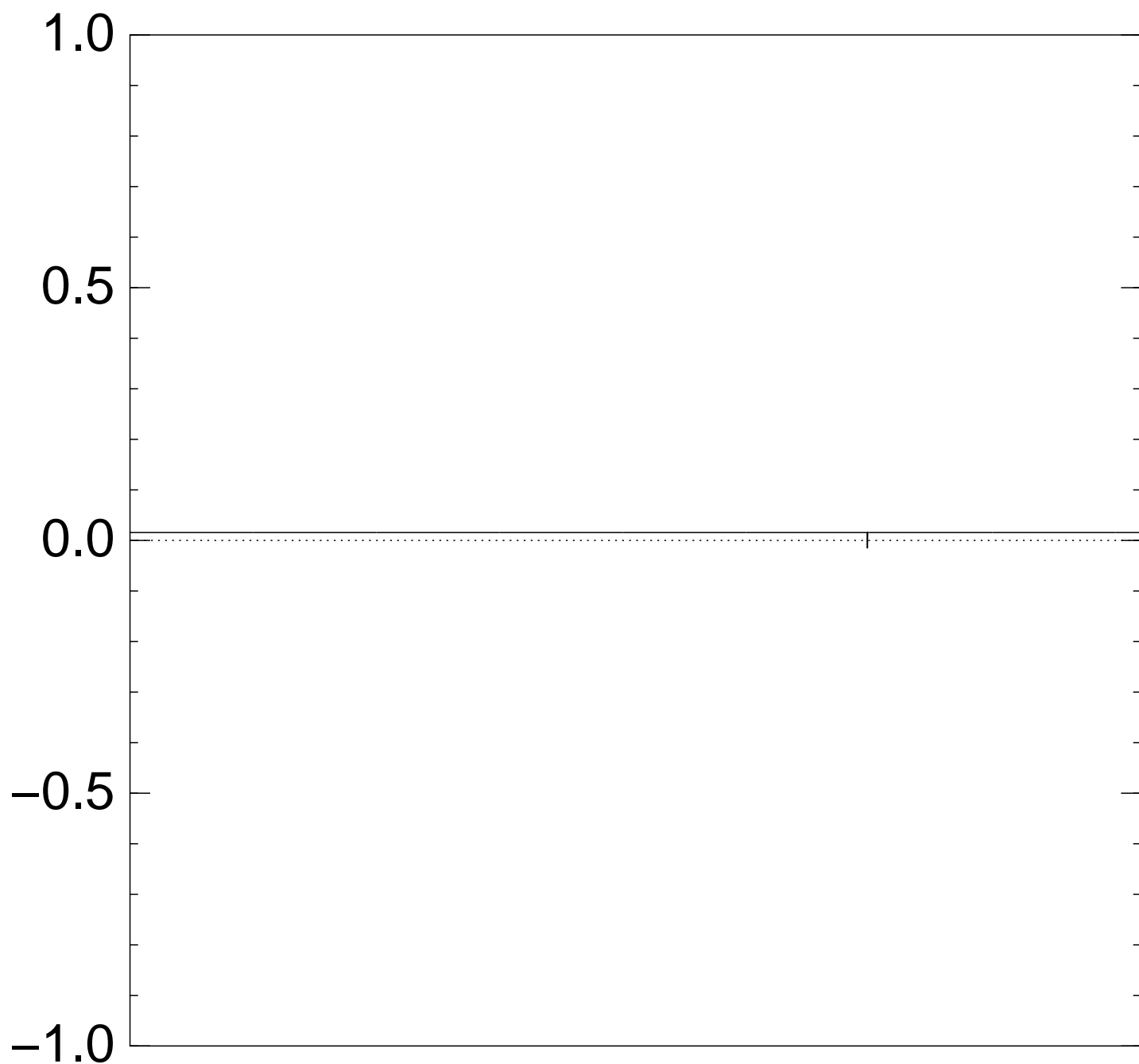
With high probability this finds  $s$ .



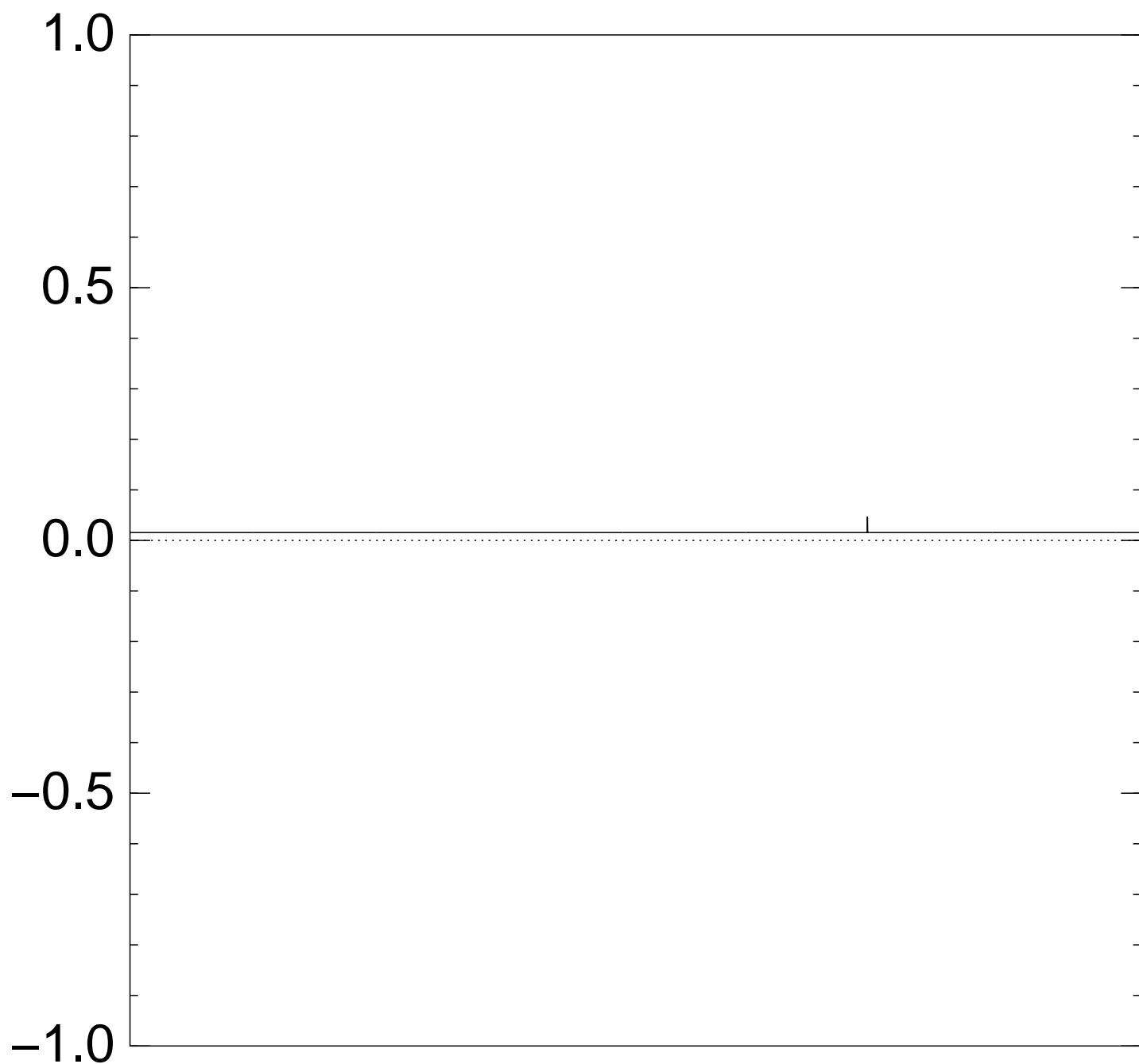
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after 0 steps:



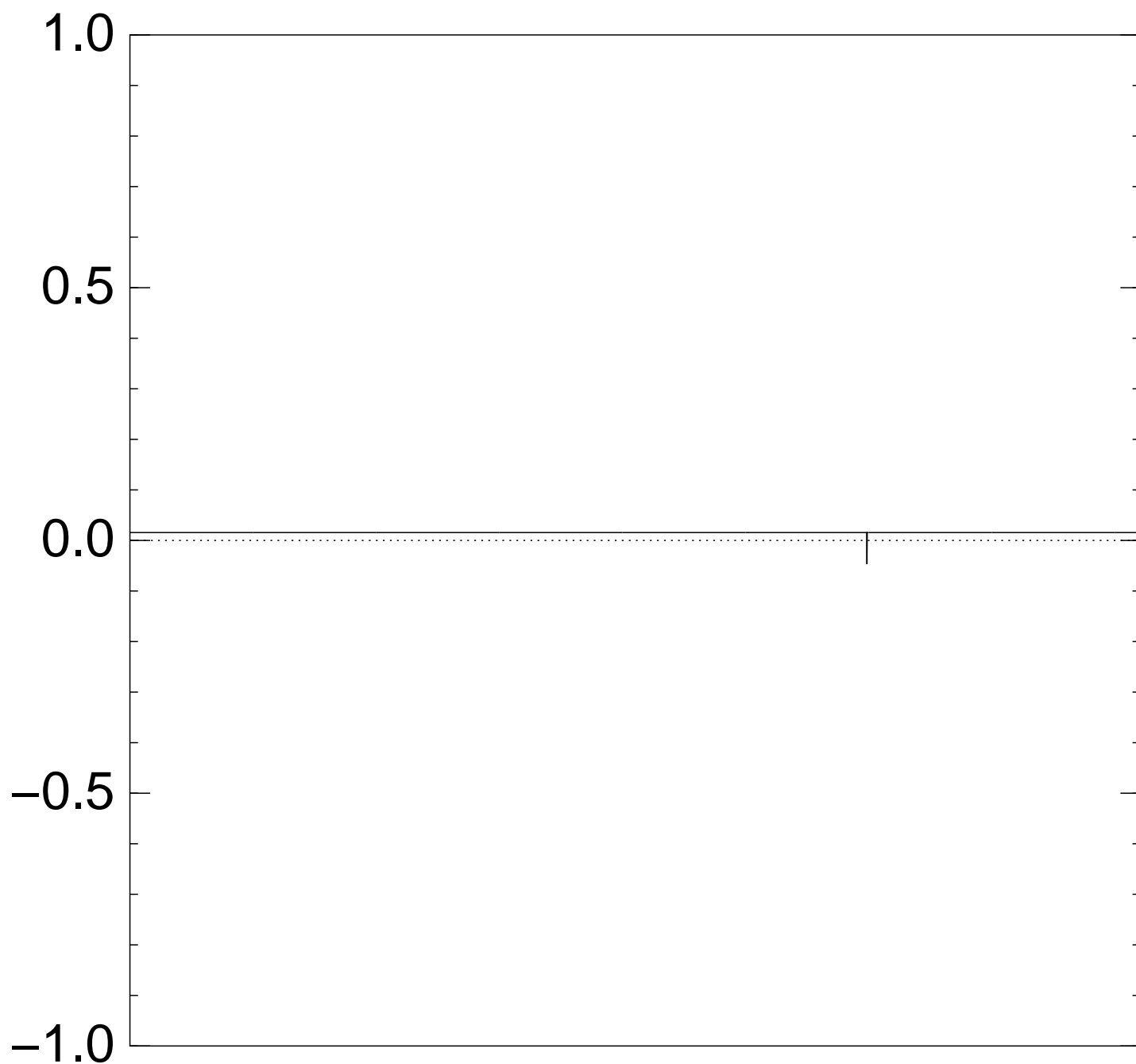
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after Step 1:



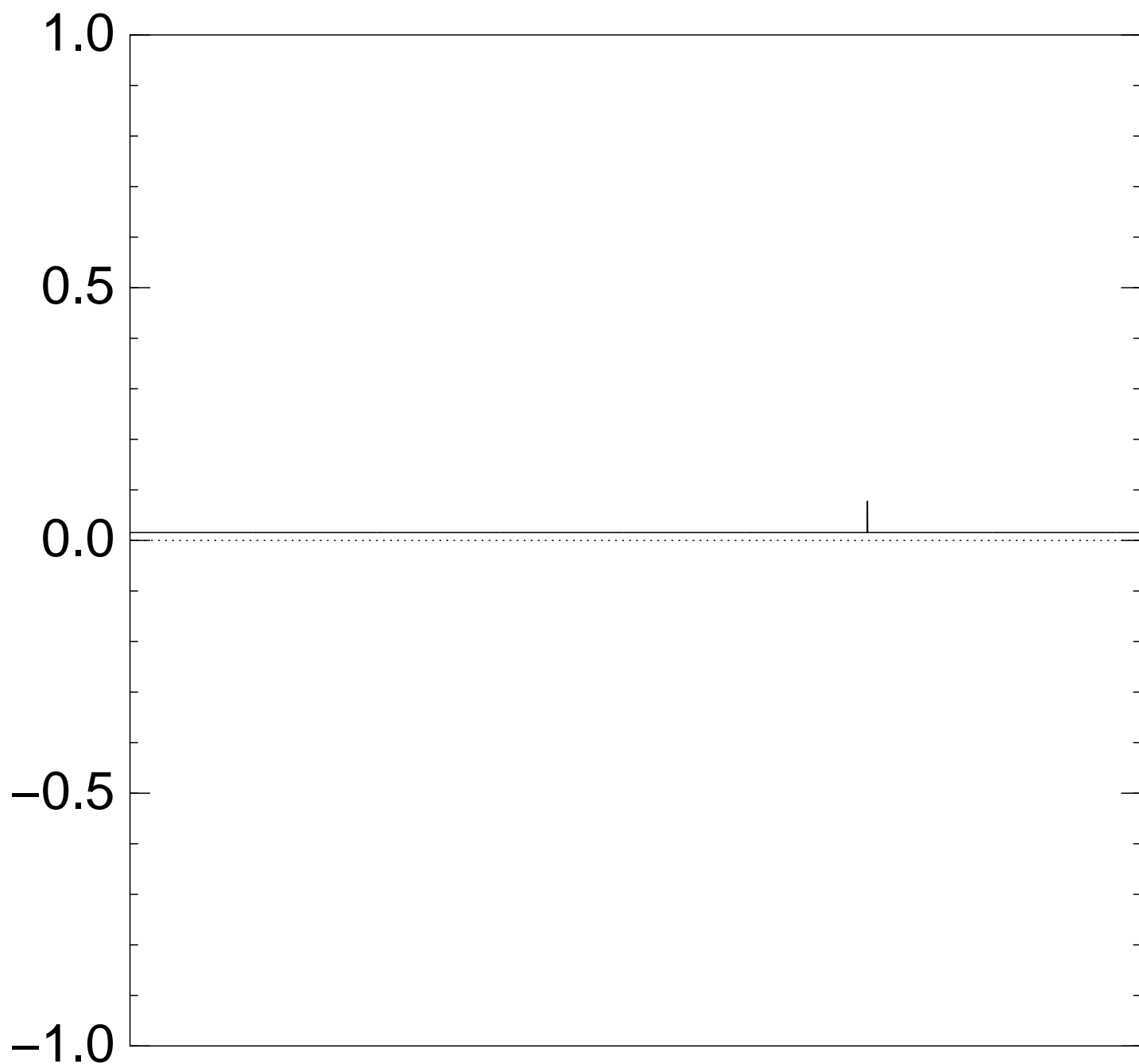
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after Step 1 + Step 2:



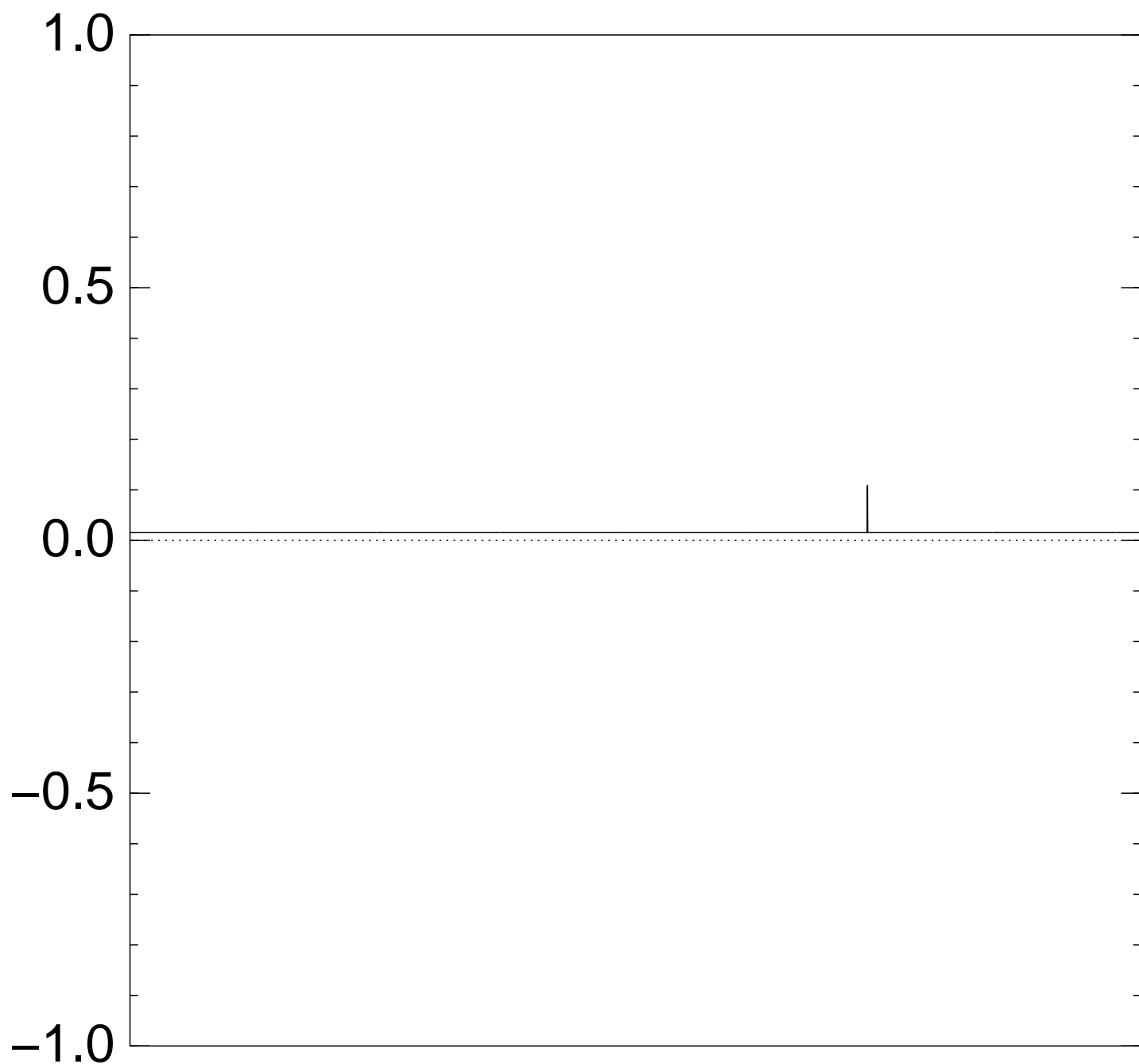
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after Step 1 + Step 2 + Step 1:



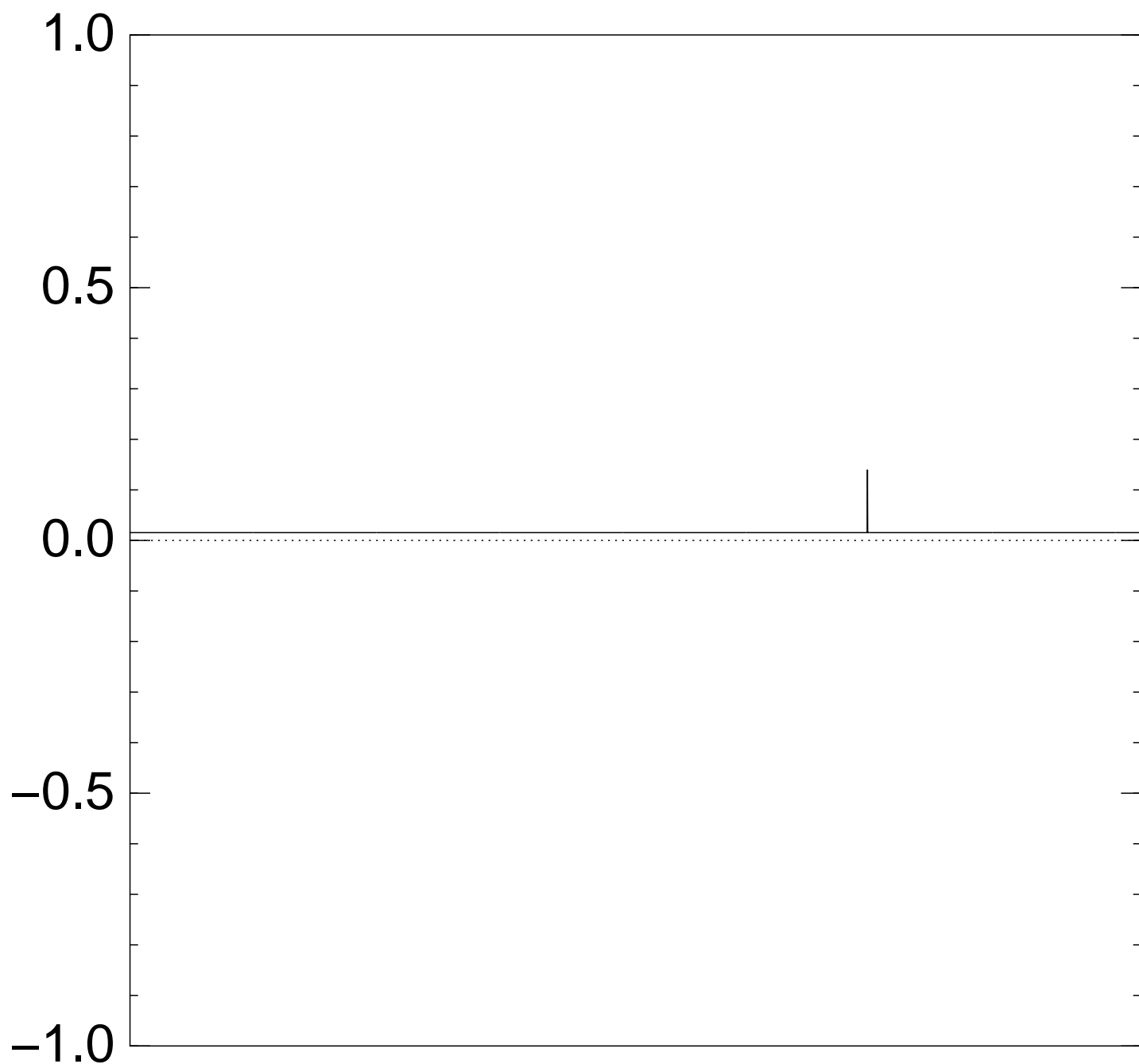
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $2 \times$  (Step 1 + Step 2):



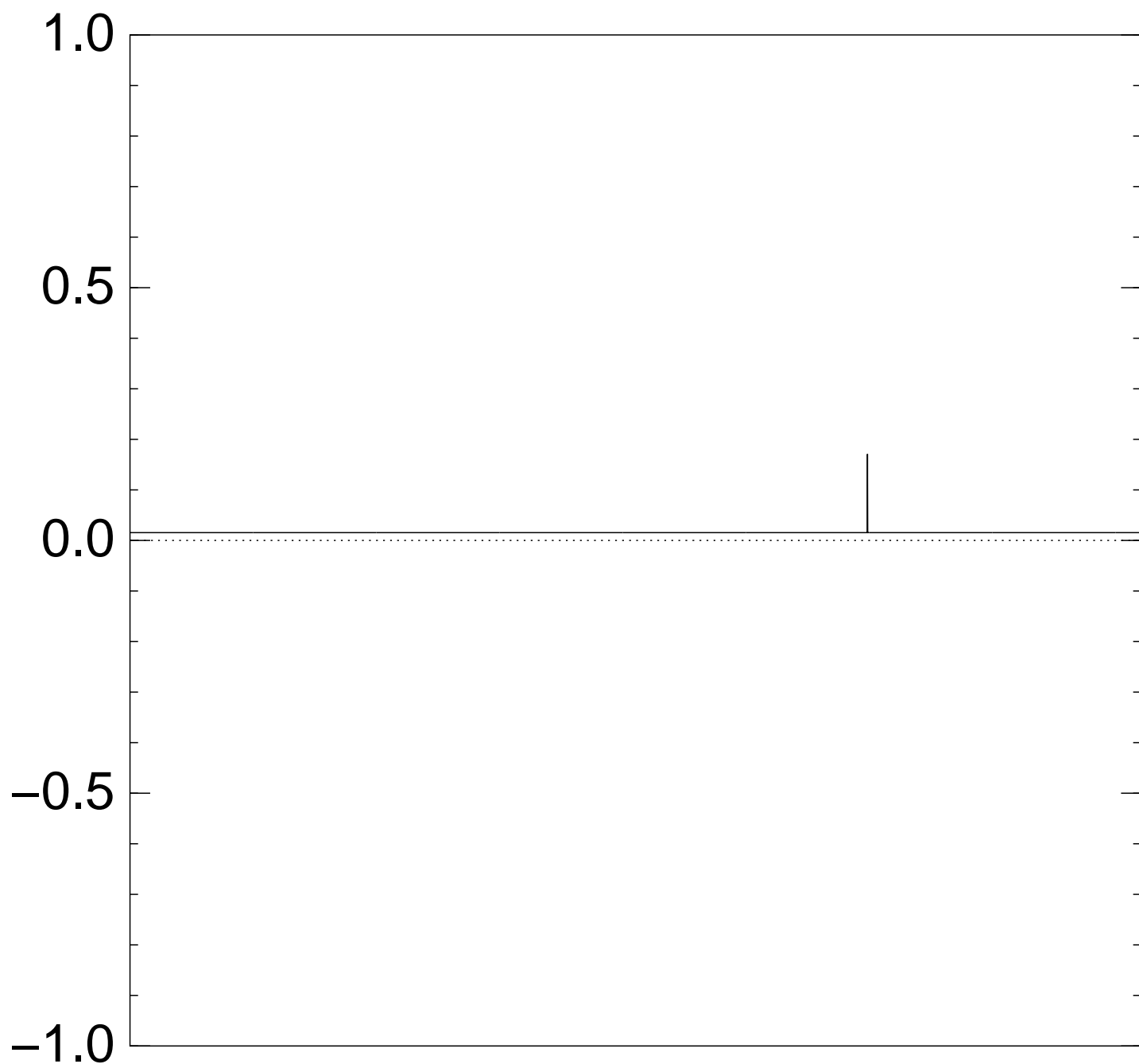
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $3 \times$  (Step 1 + Step 2):



Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $4 \times$  (Step 1 + Step 2):

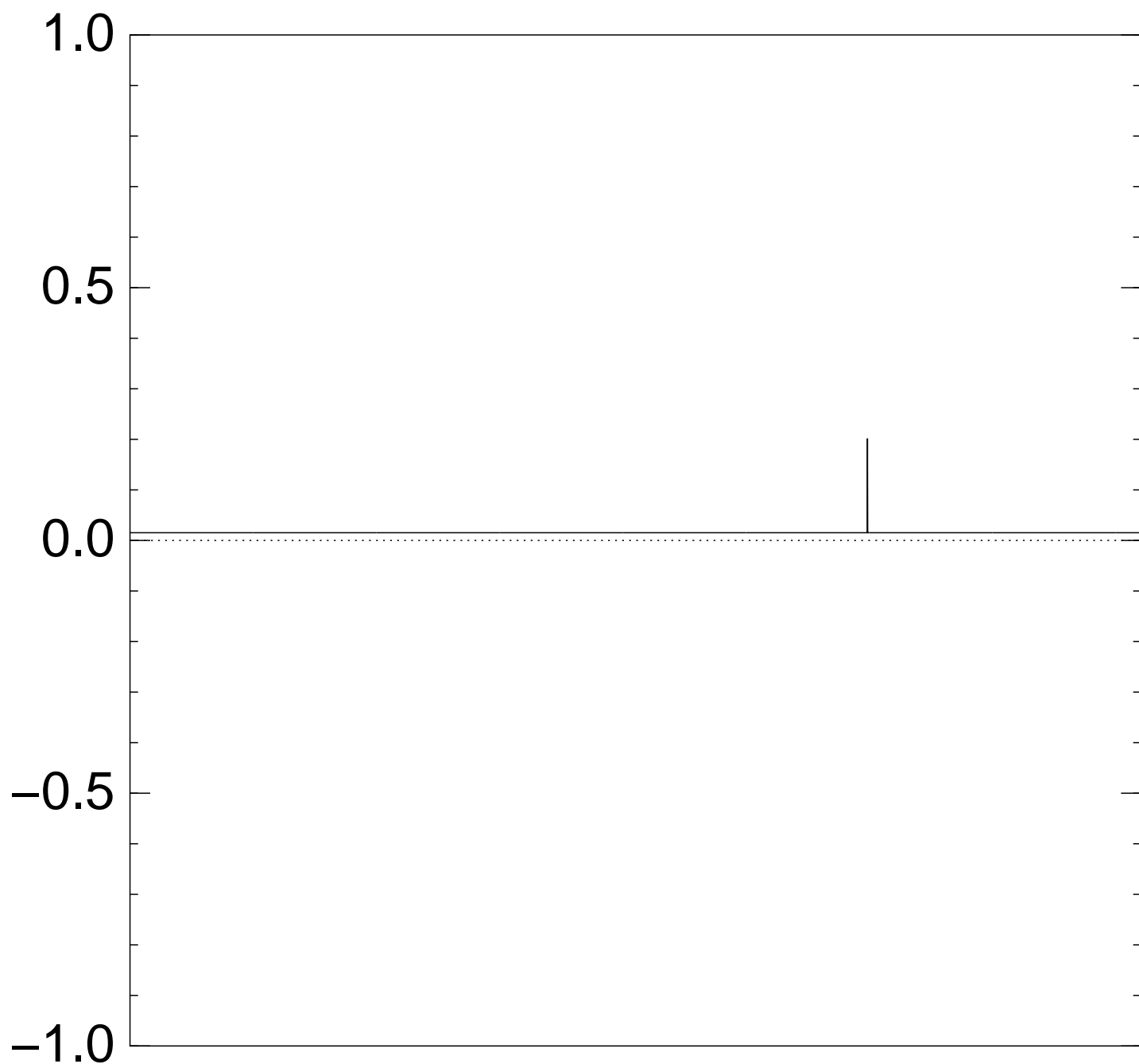


Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $5 \times$  (Step 1 + Step 2):

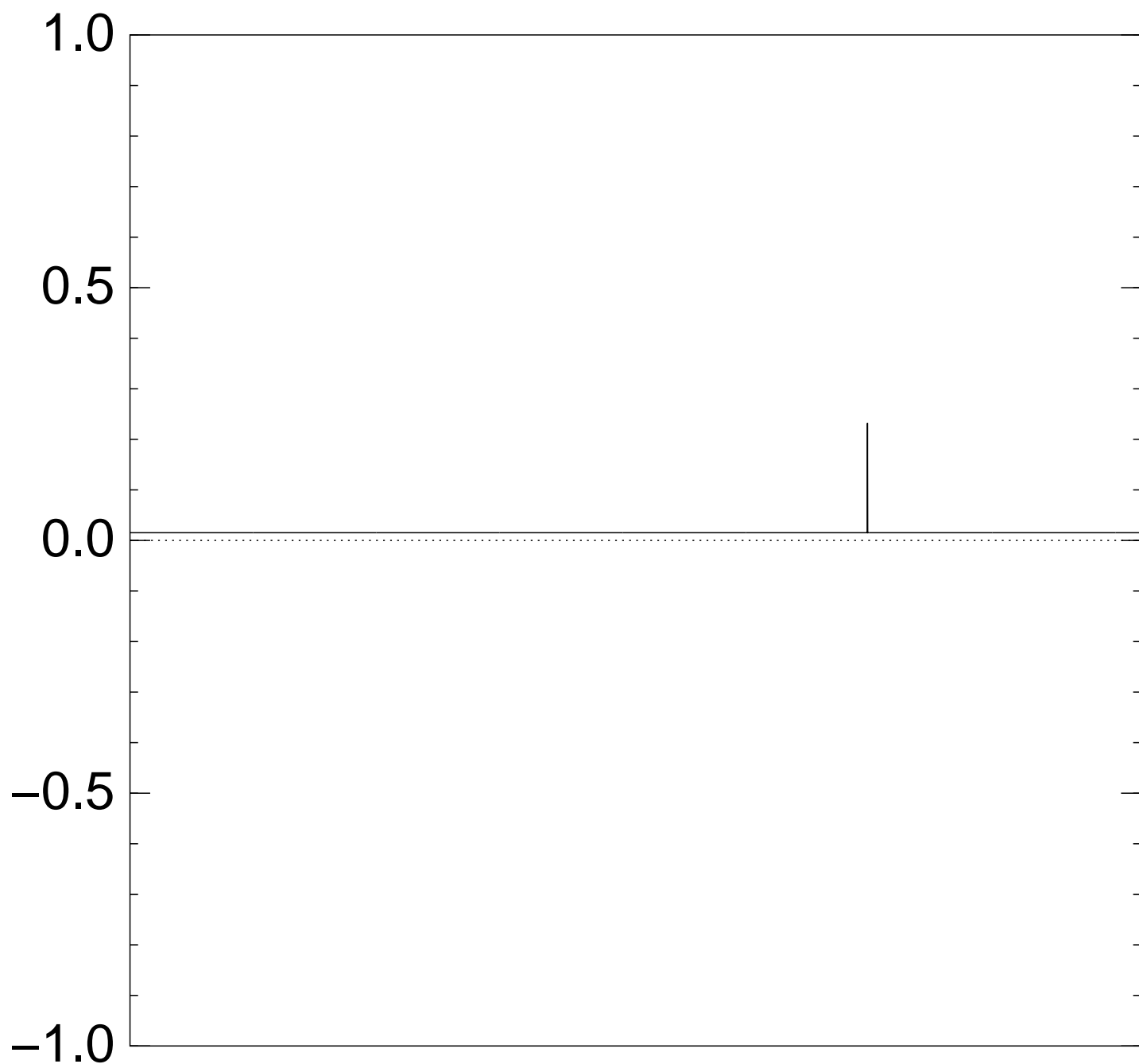




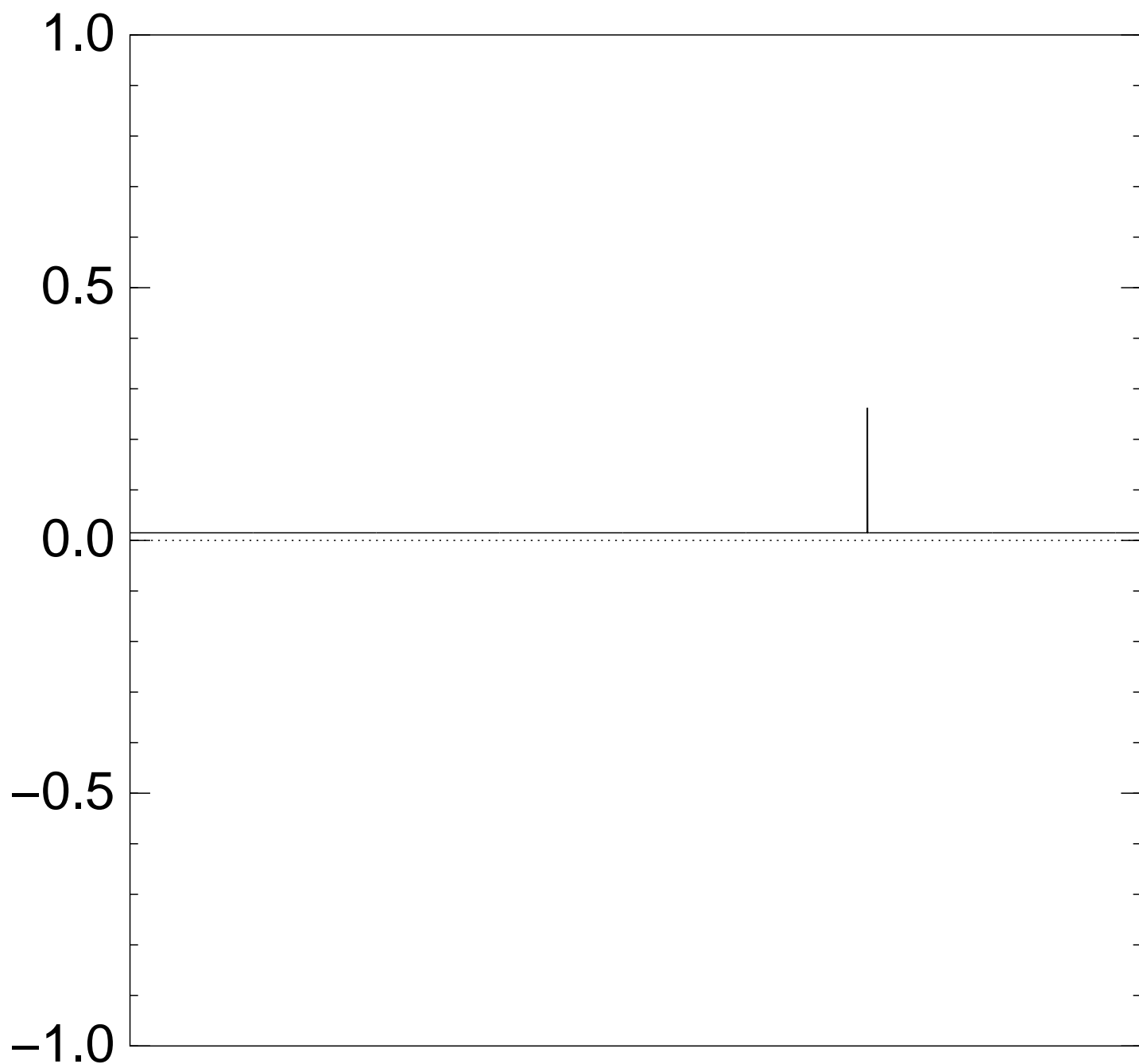
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $6 \times$  (Step 1 + Step 2):



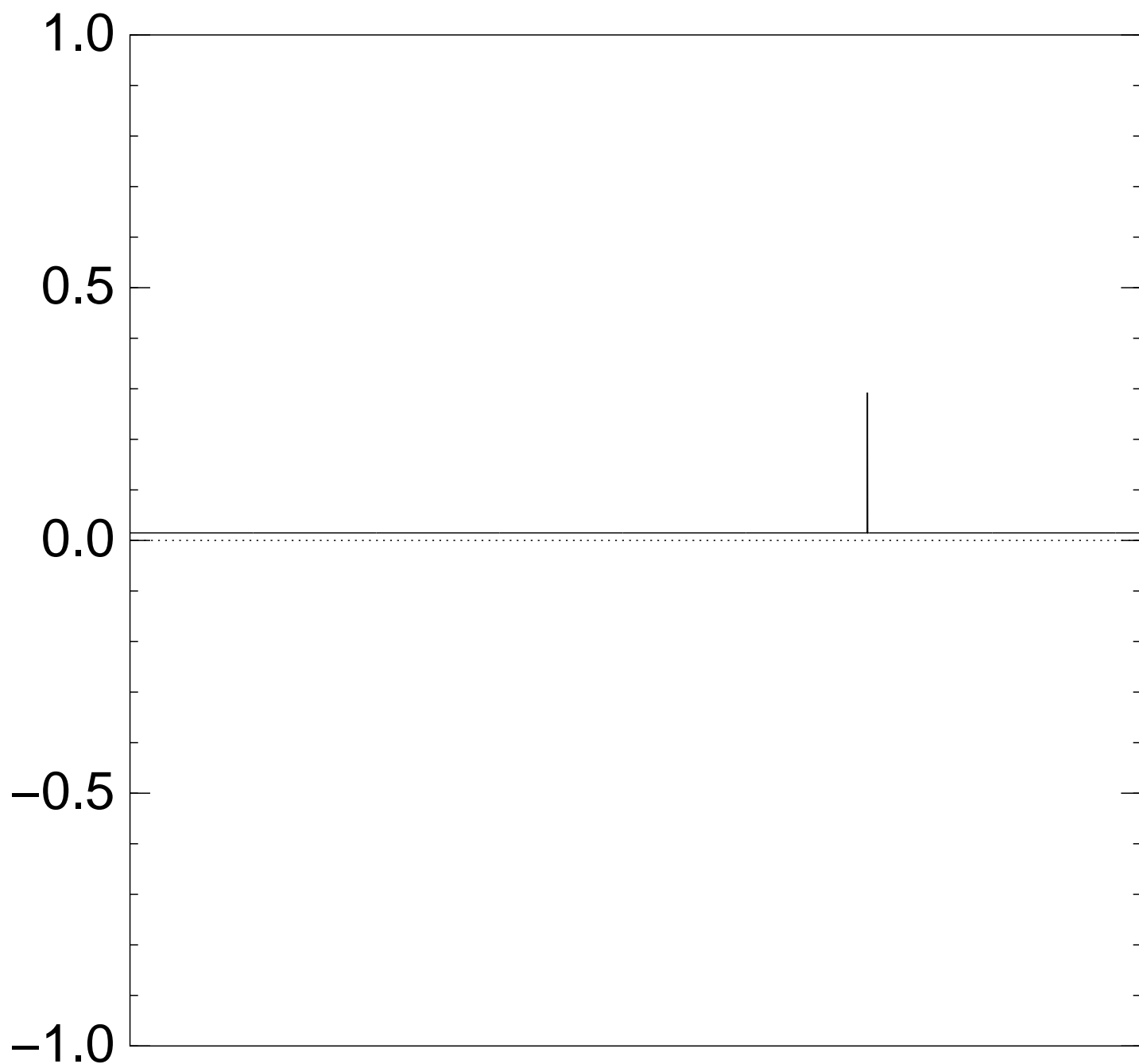
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $7 \times$  (Step 1 + Step 2):



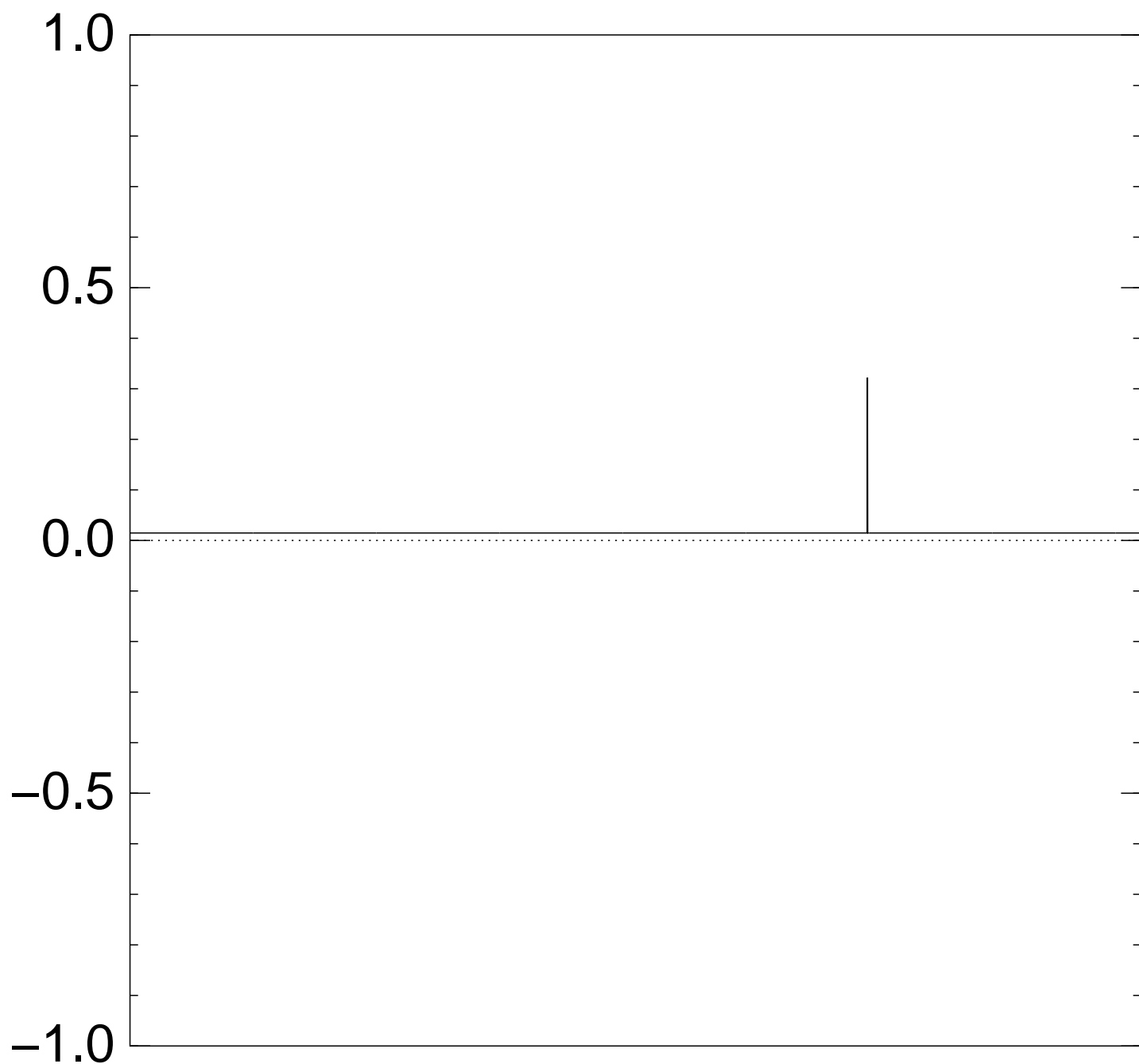
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $8 \times$  (Step 1 + Step 2):



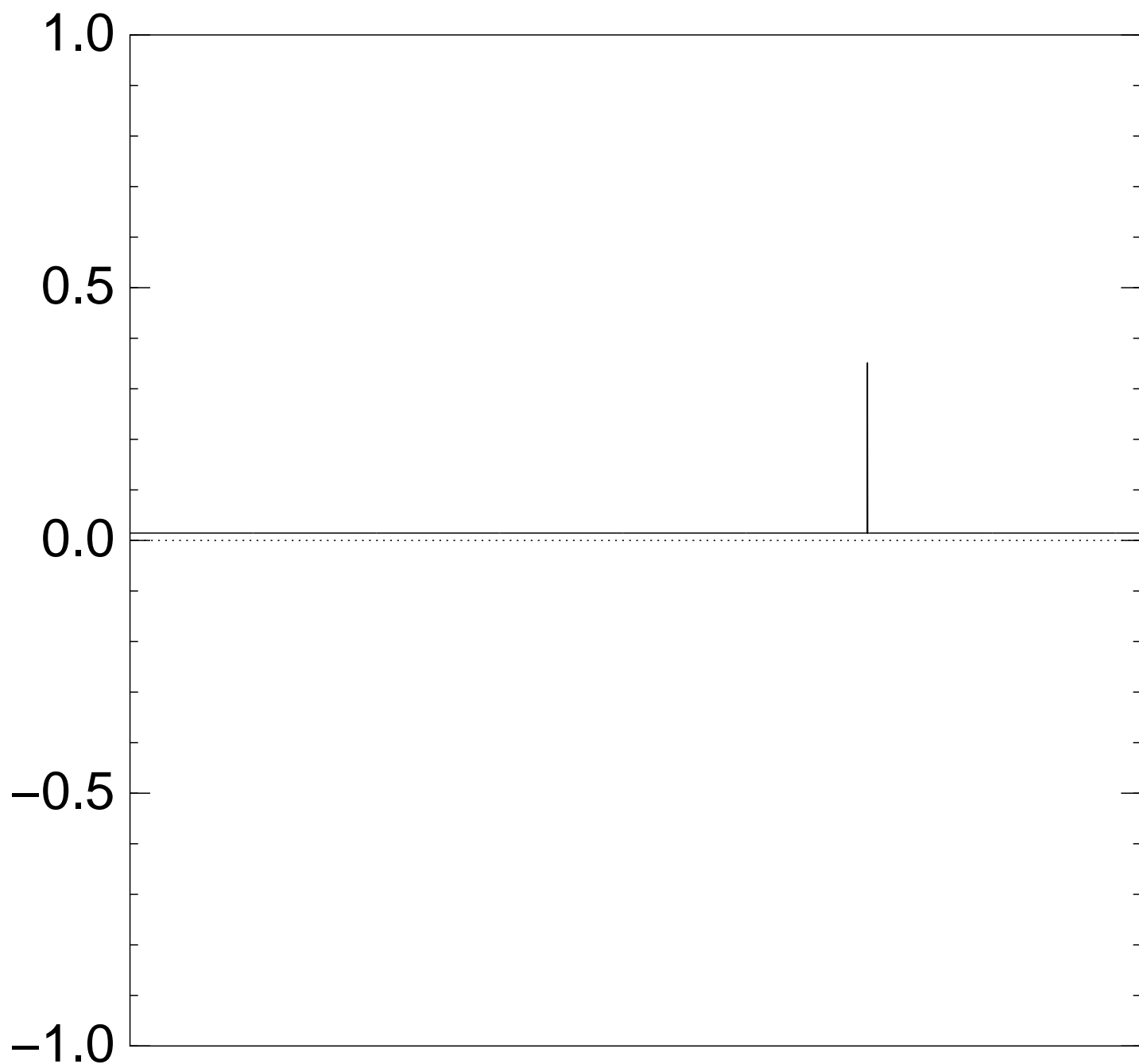
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $9 \times$  (Step 1 + Step 2):



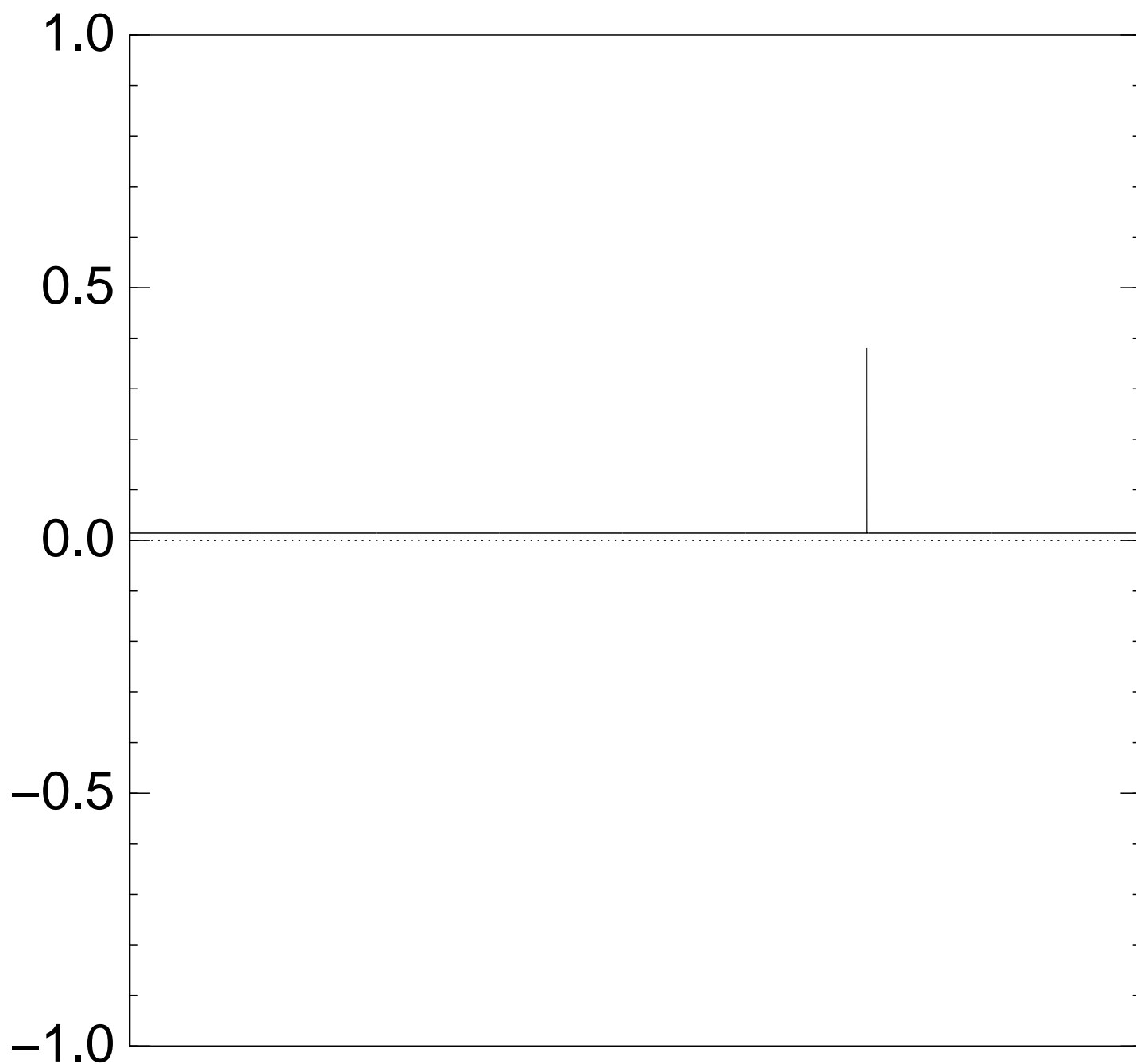
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $10 \times$  (Step 1 + Step 2):



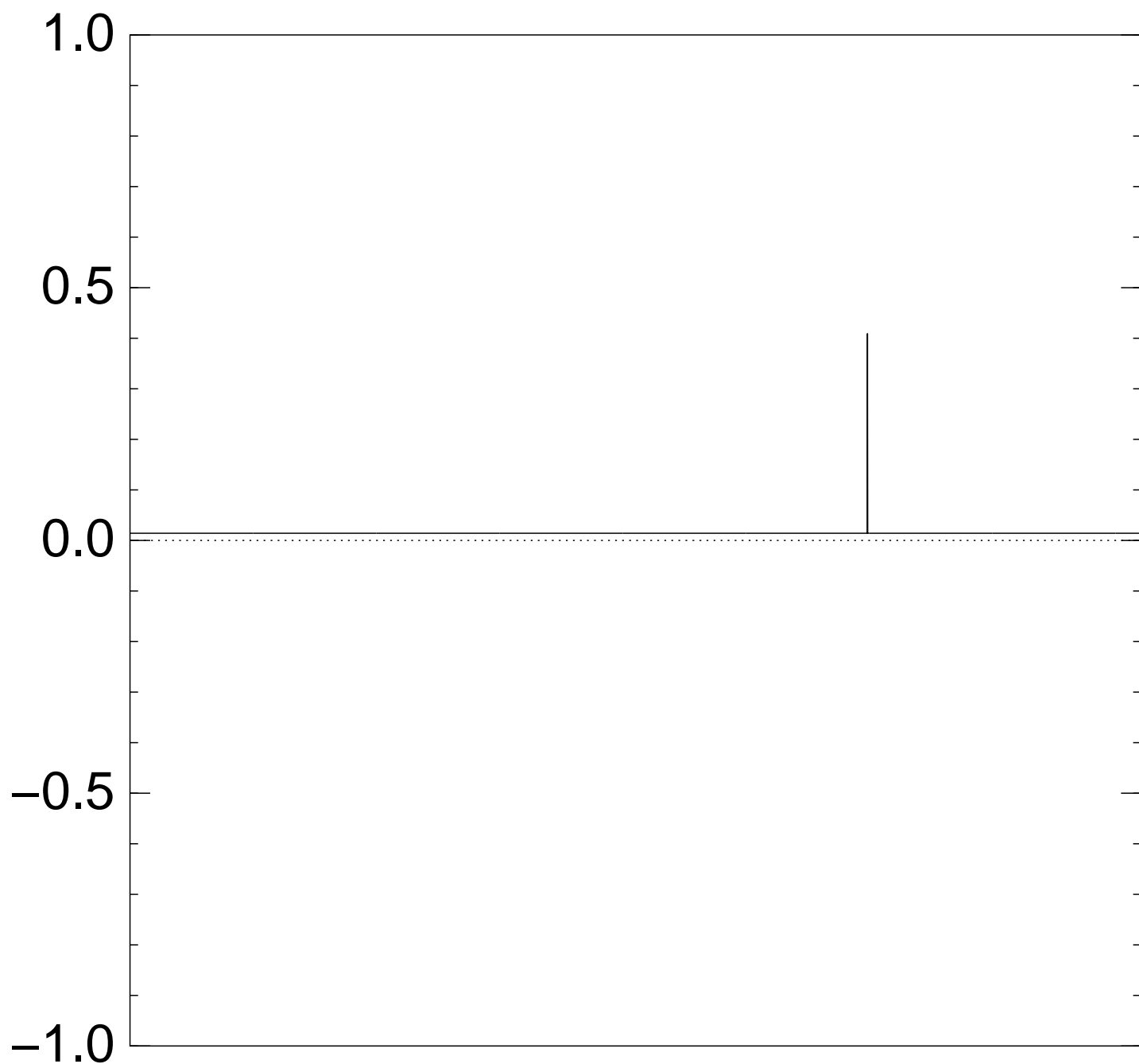
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $11 \times$  (Step 1 + Step 2):



Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $12 \times$  (Step 1 + Step 2):

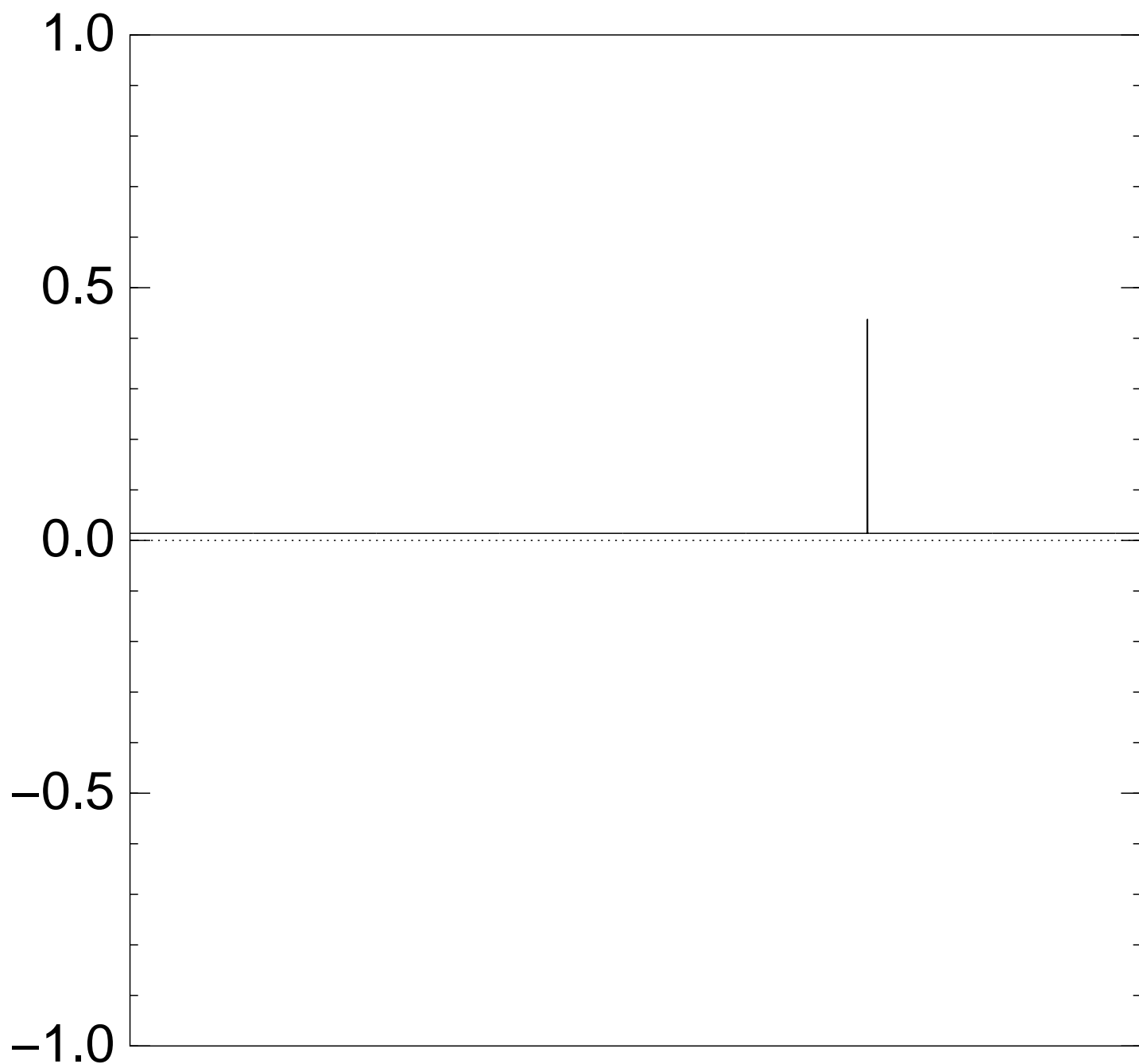


Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $13 \times$  (Step 1 + Step 2):

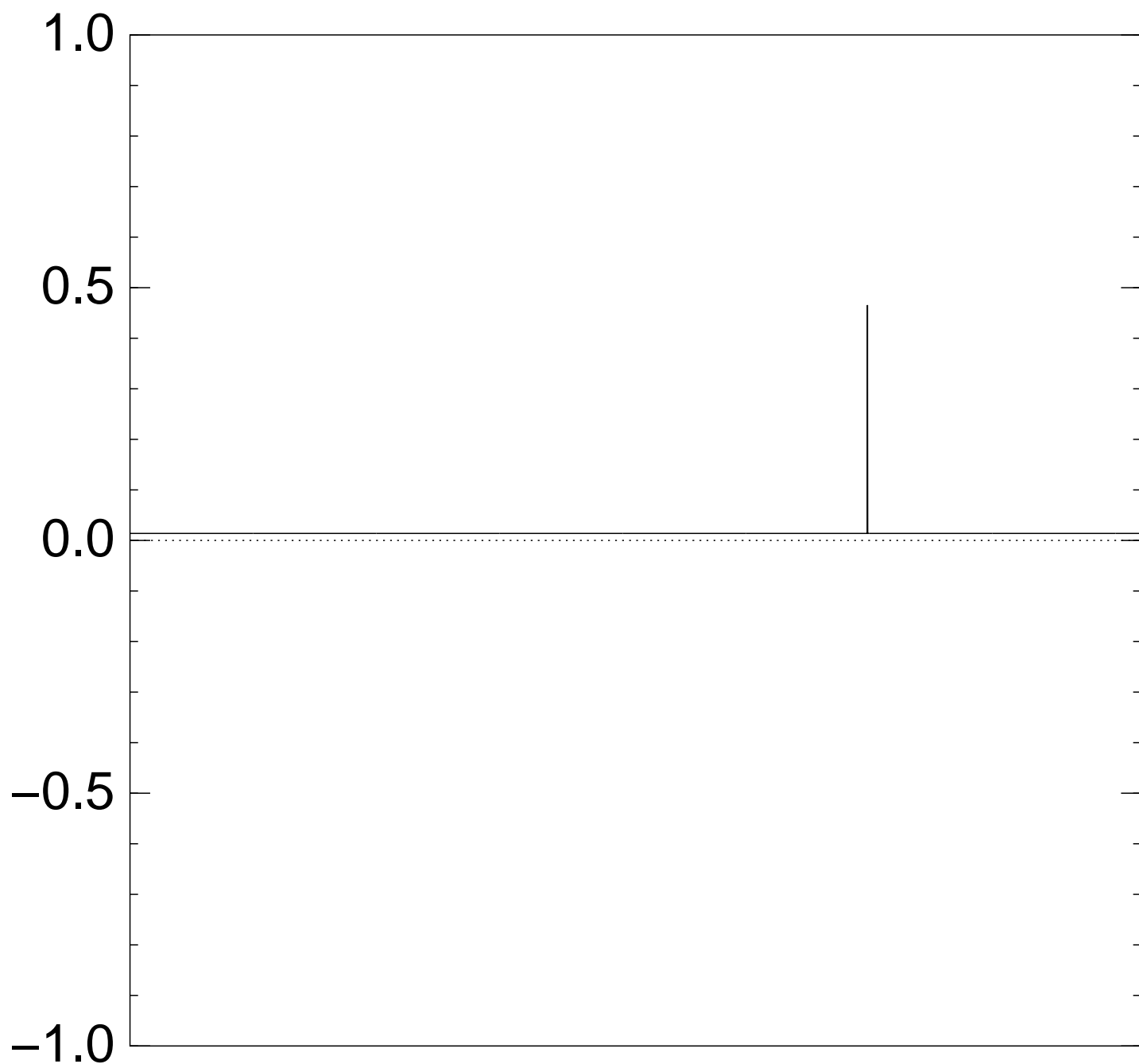




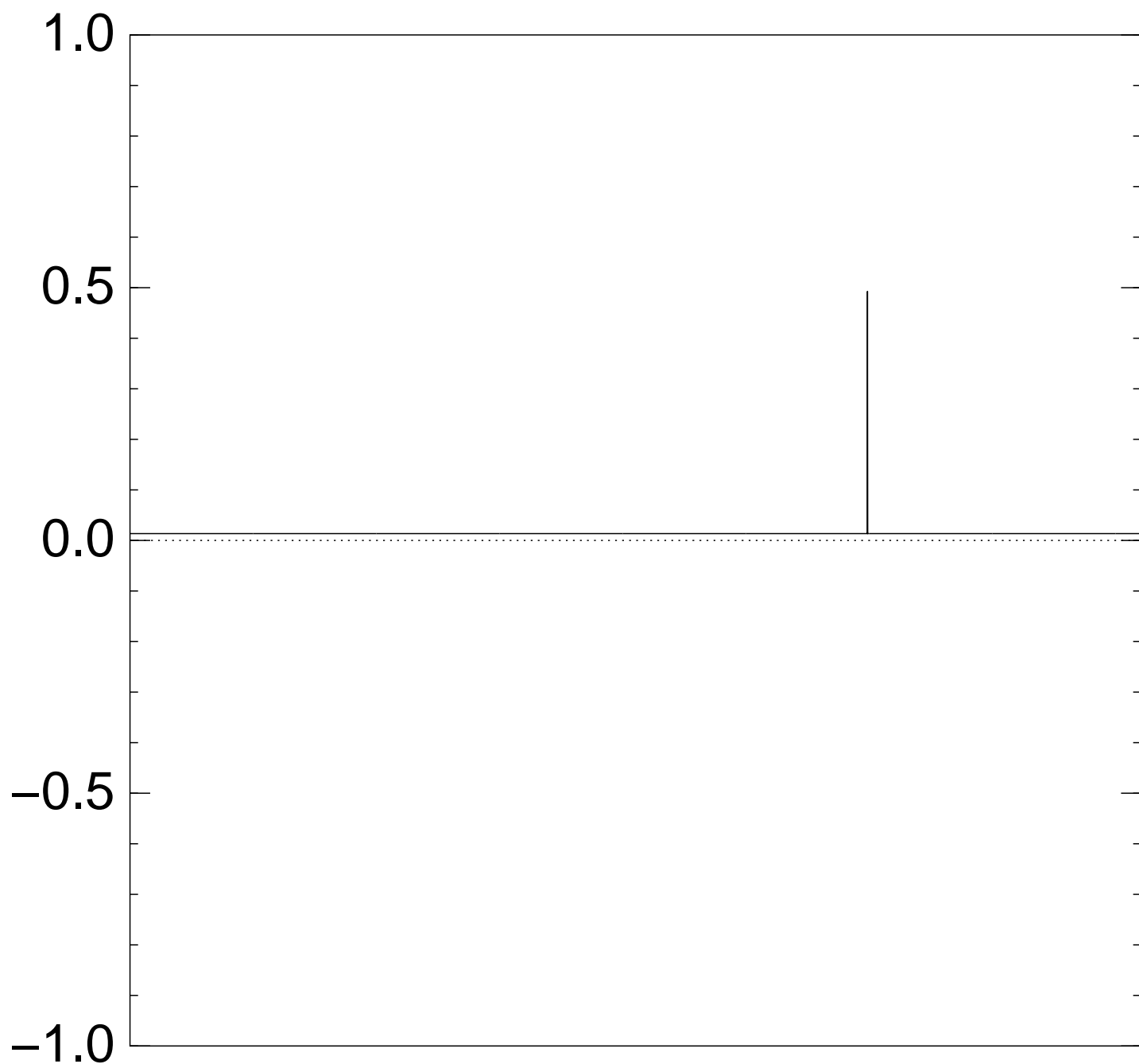
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $14 \times$  (Step 1 + Step 2):



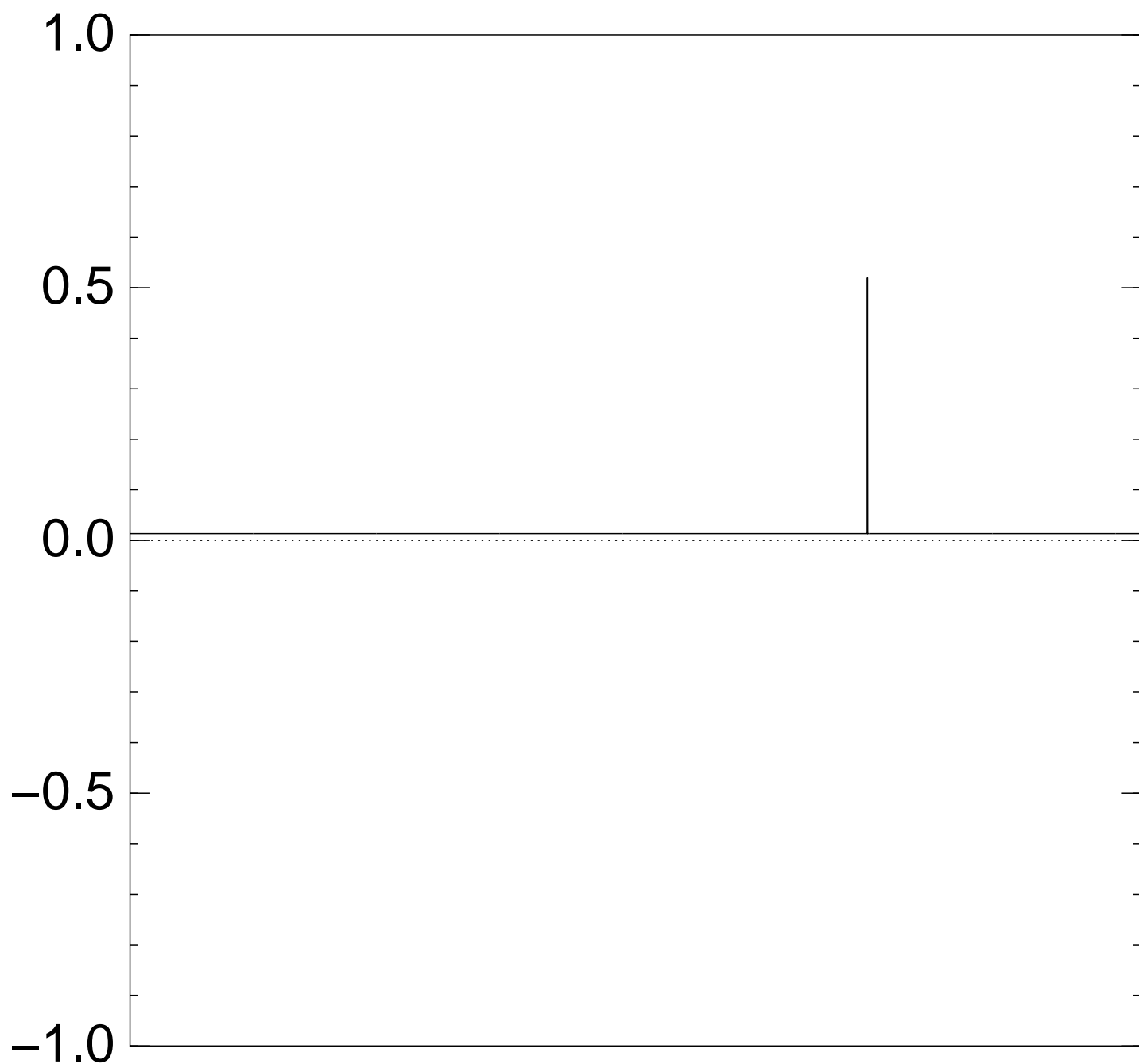
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $15 \times$  (Step 1 + Step 2):



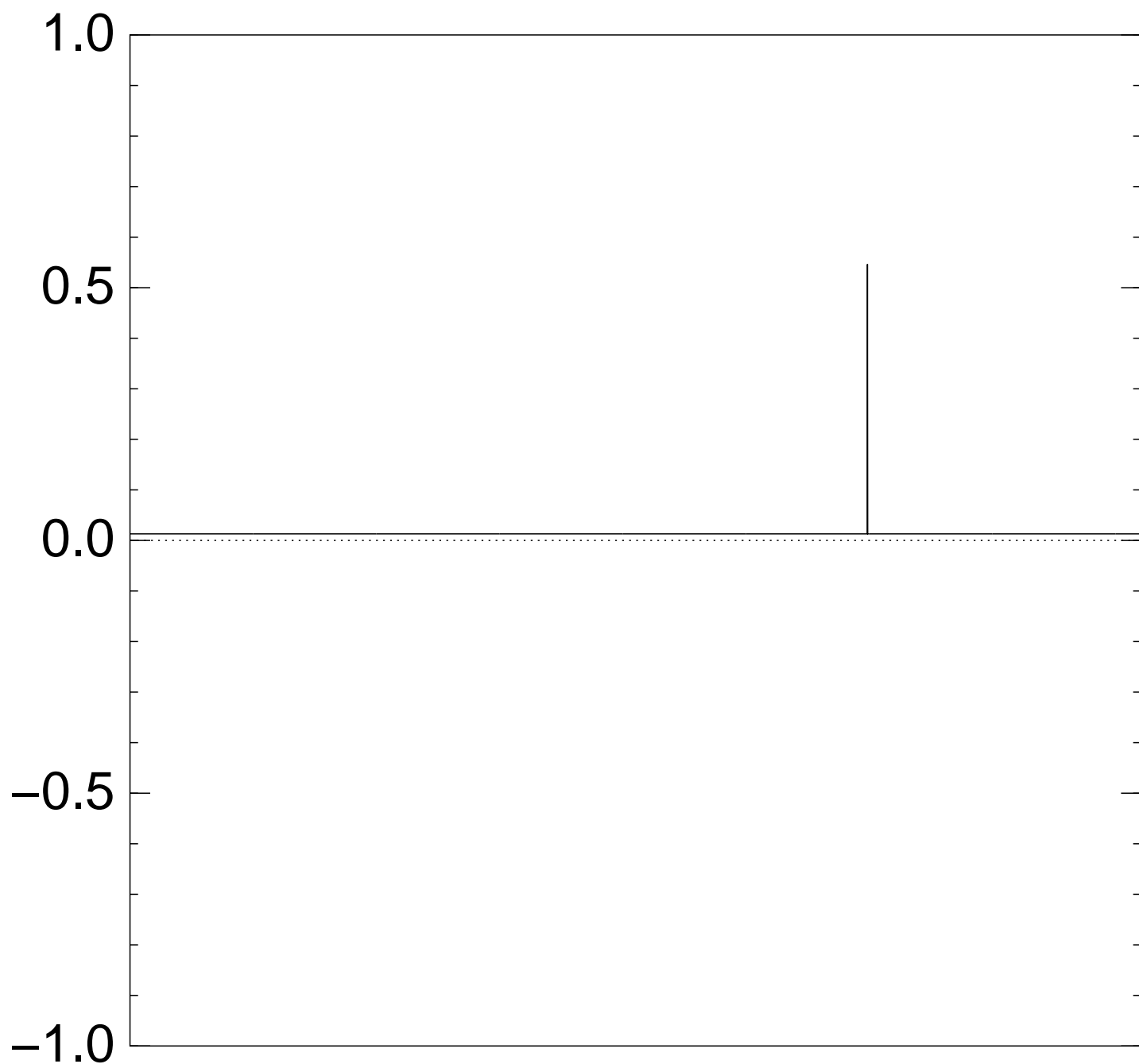
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $16 \times$  (Step 1 + Step 2):



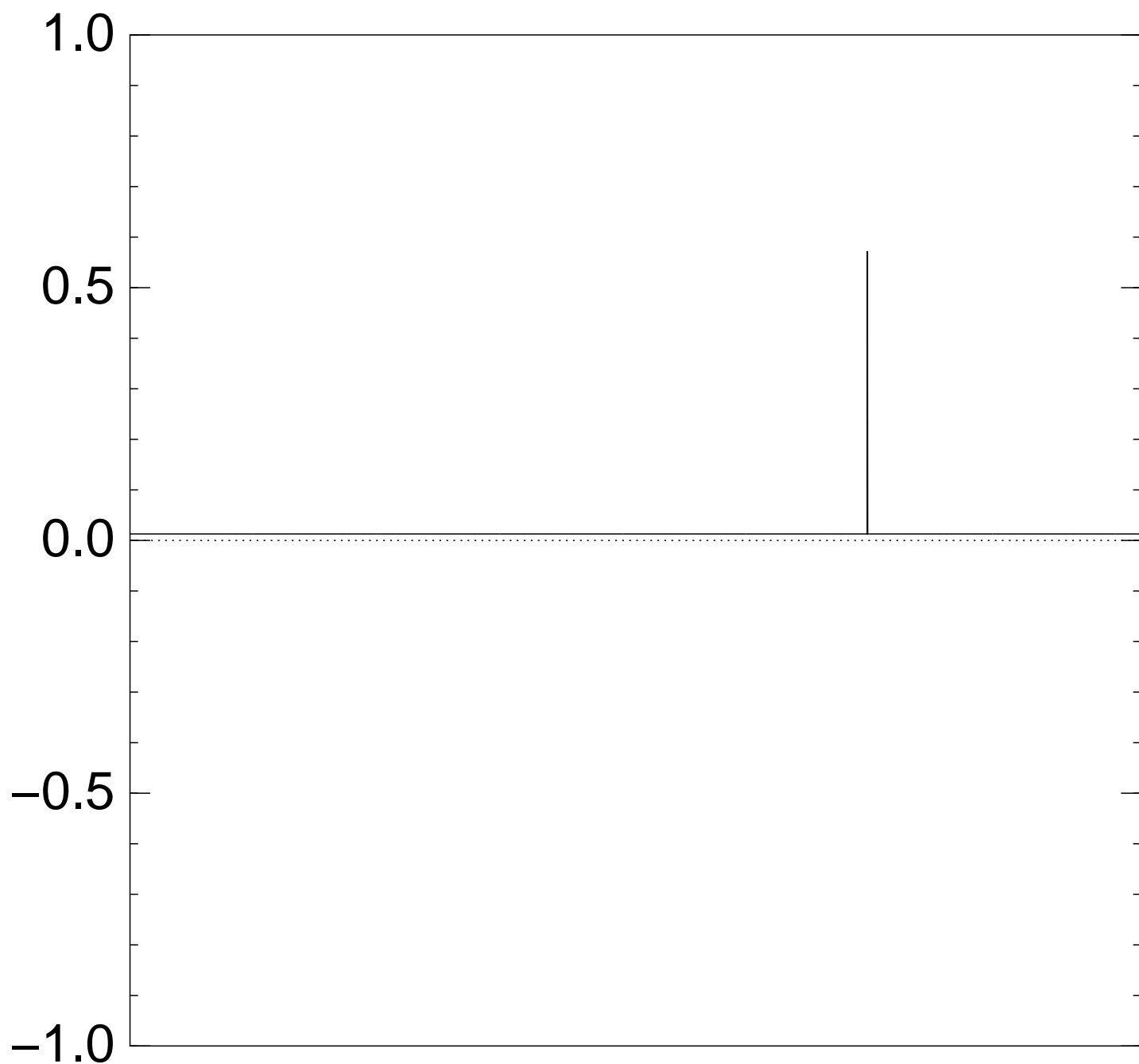
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $17 \times$  (Step 1 + Step 2):



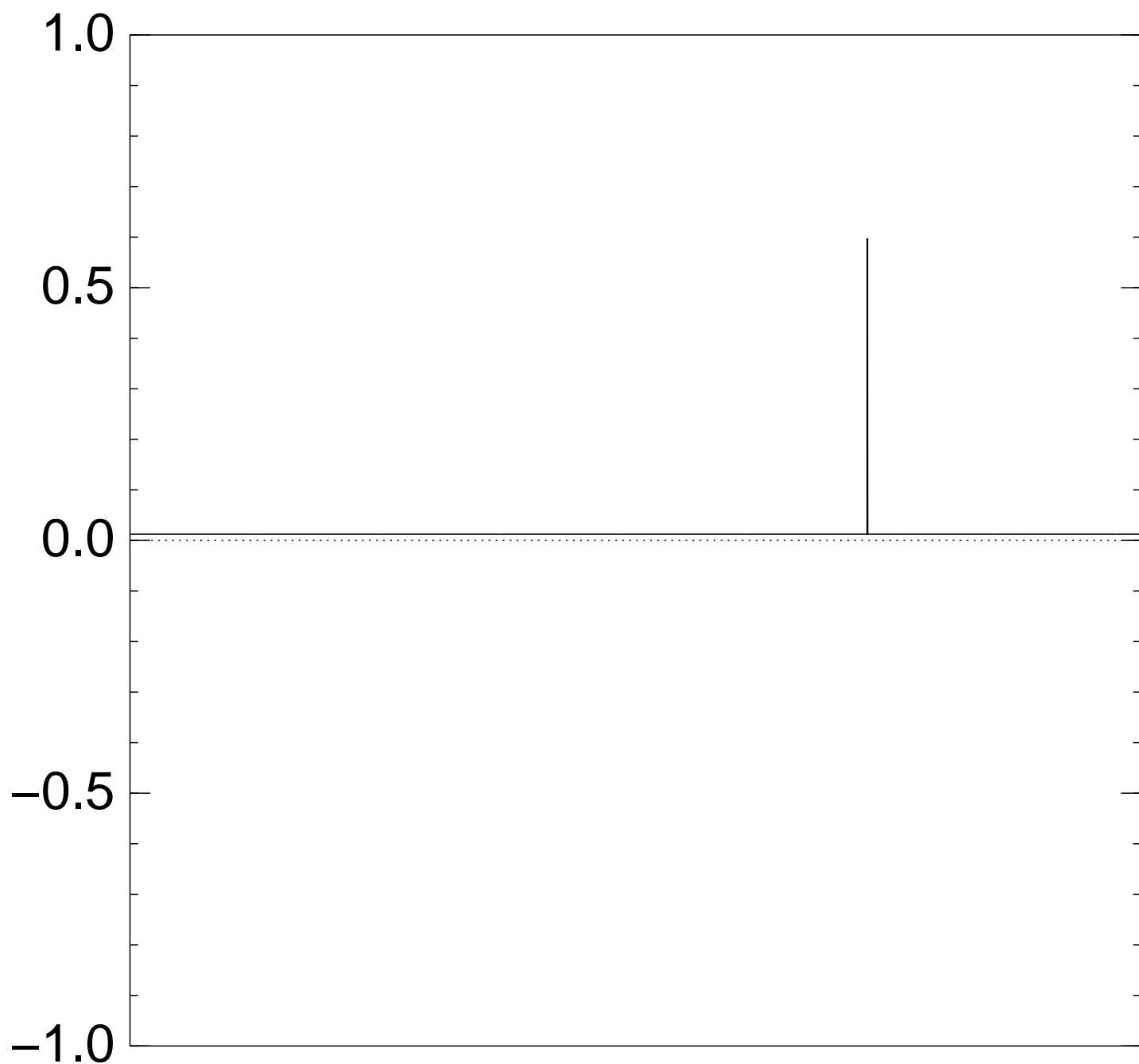
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $18 \times$  (Step 1 + Step 2):



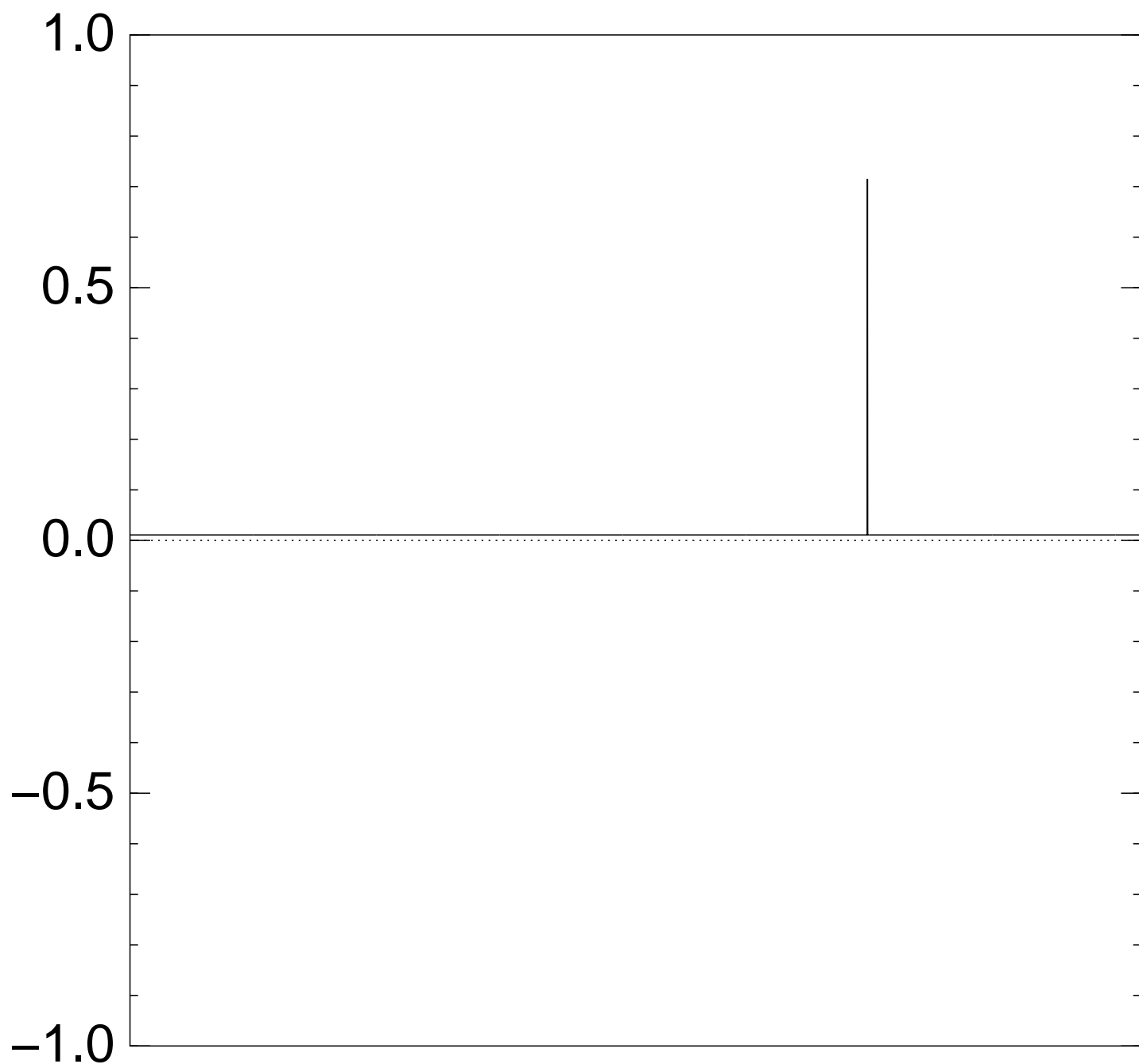
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $19 \times$  (Step 1 + Step 2):



Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $20 \times$  (Step 1 + Step 2):

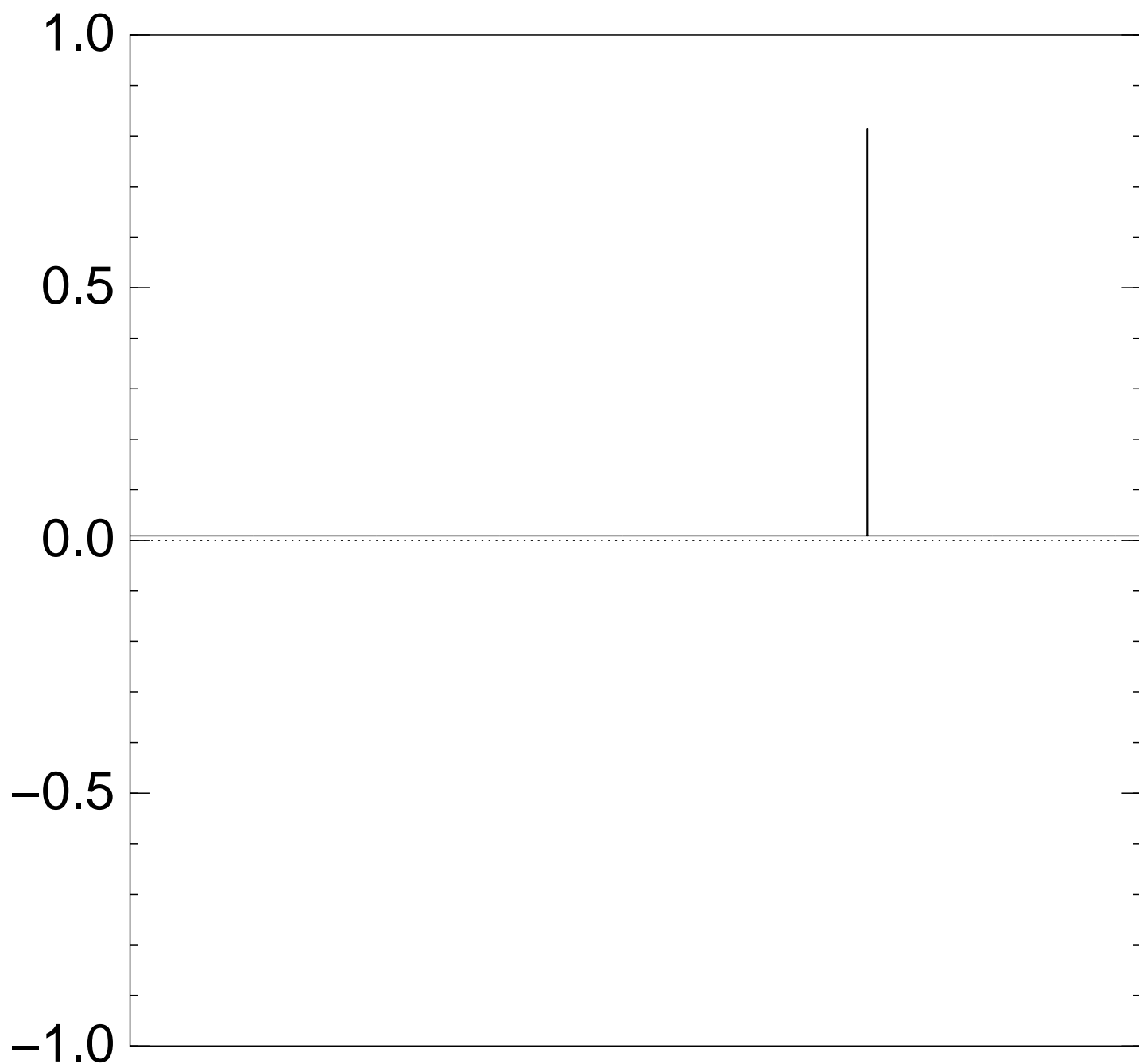


Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $25 \times$  (Step 1 + Step 2):

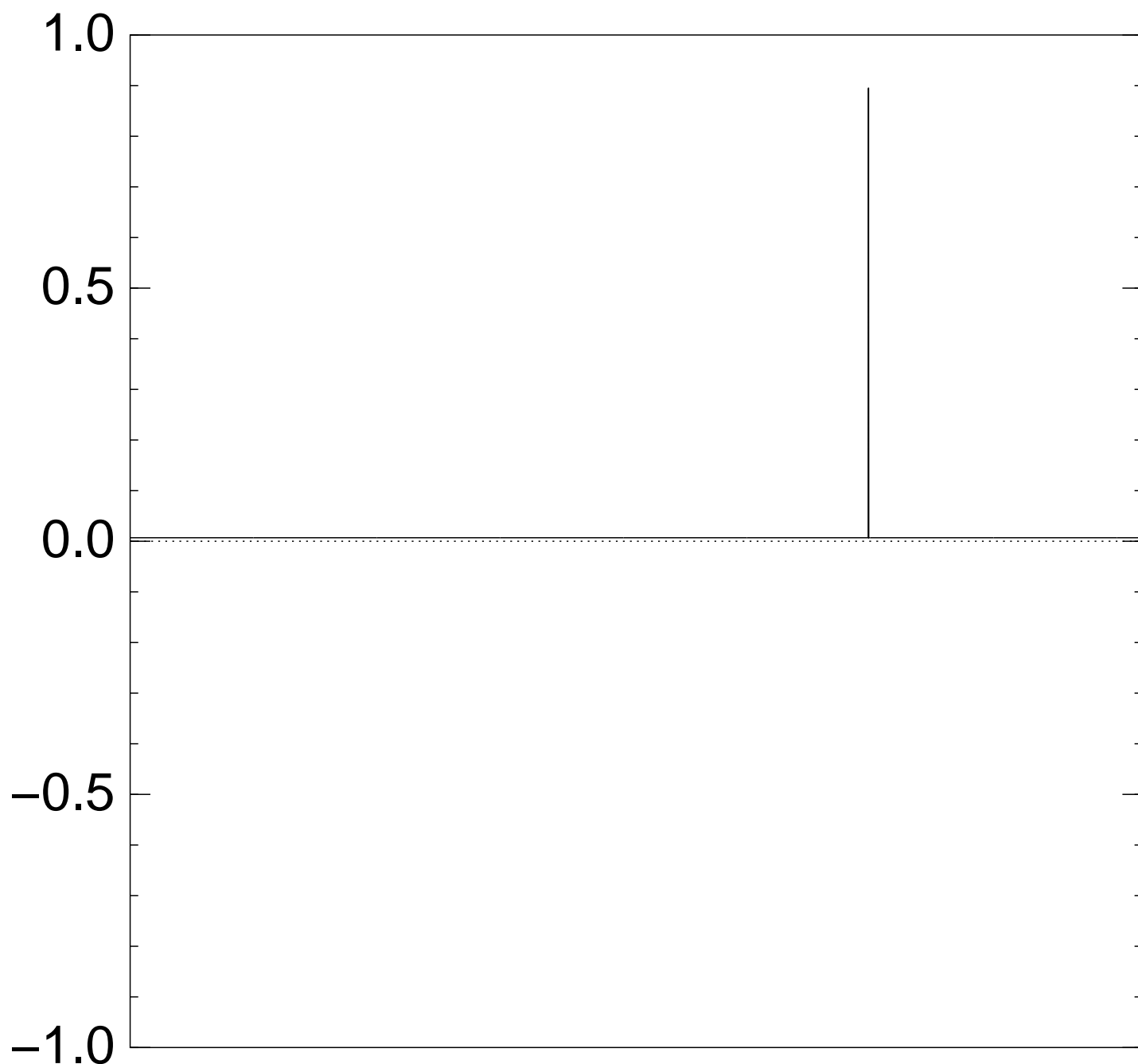




Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $30 \times$  (Step 1 + Step 2):

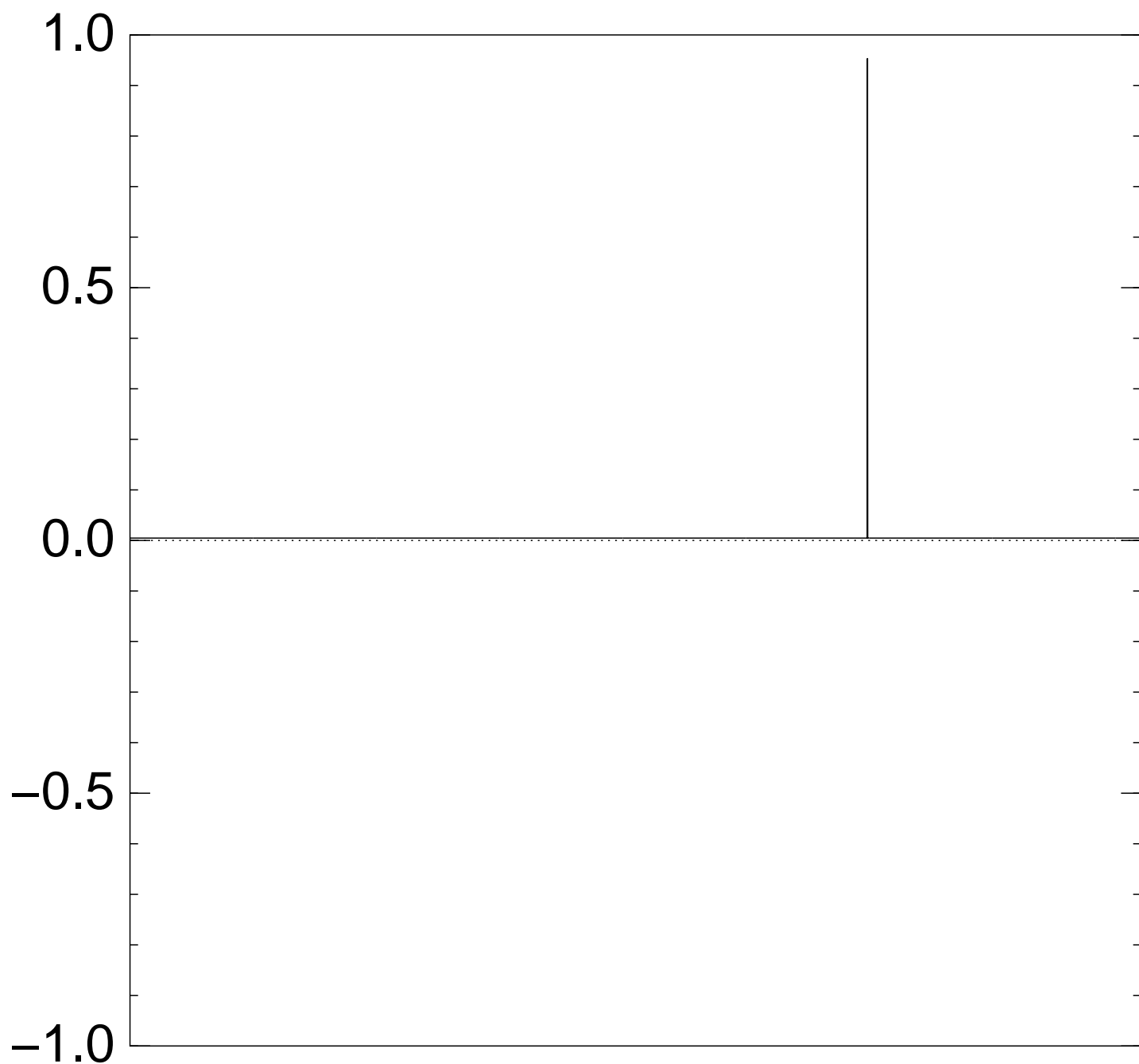


Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $35 \times$  (Step 1 + Step 2):

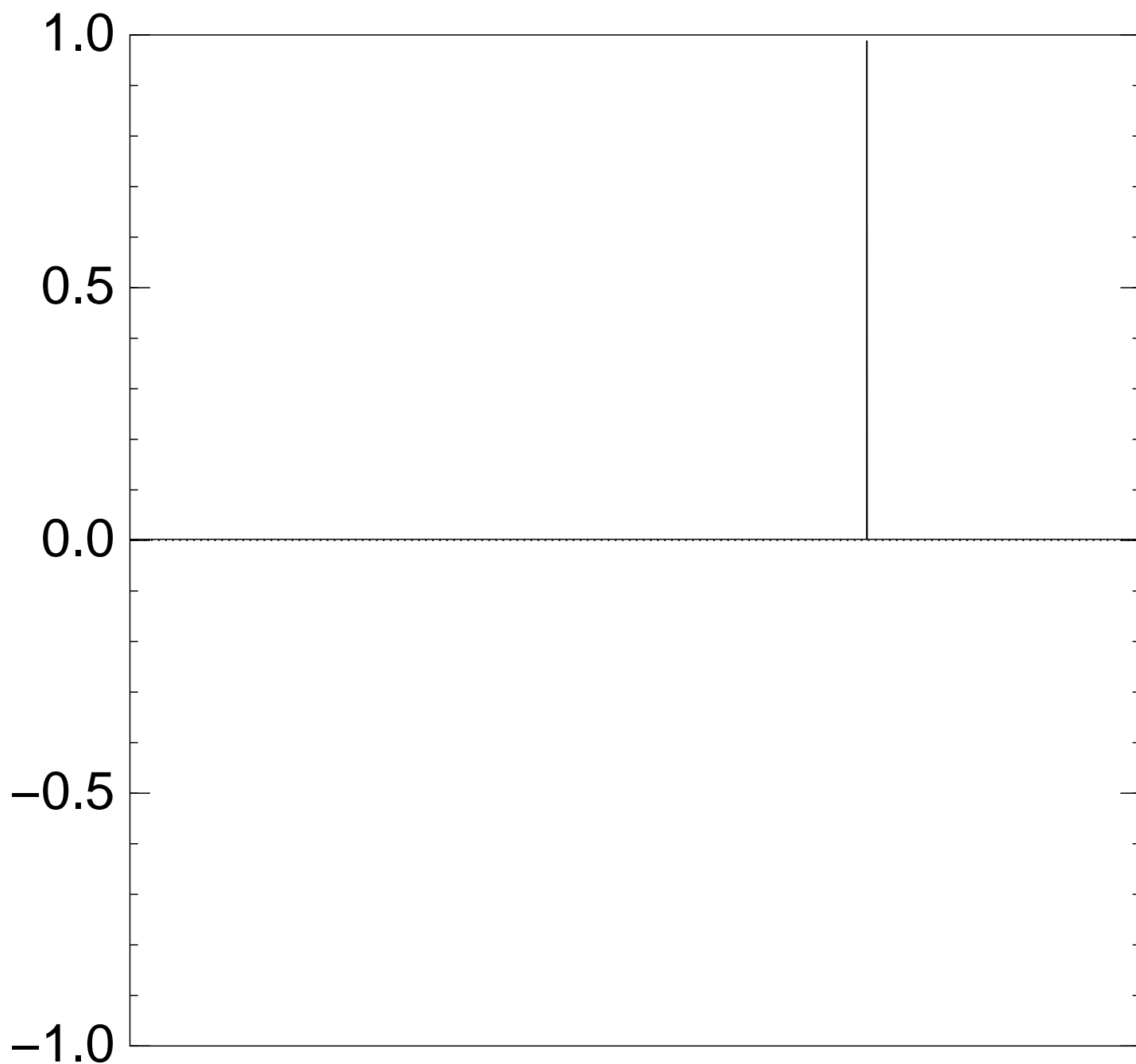


Good moment to stop, measure.

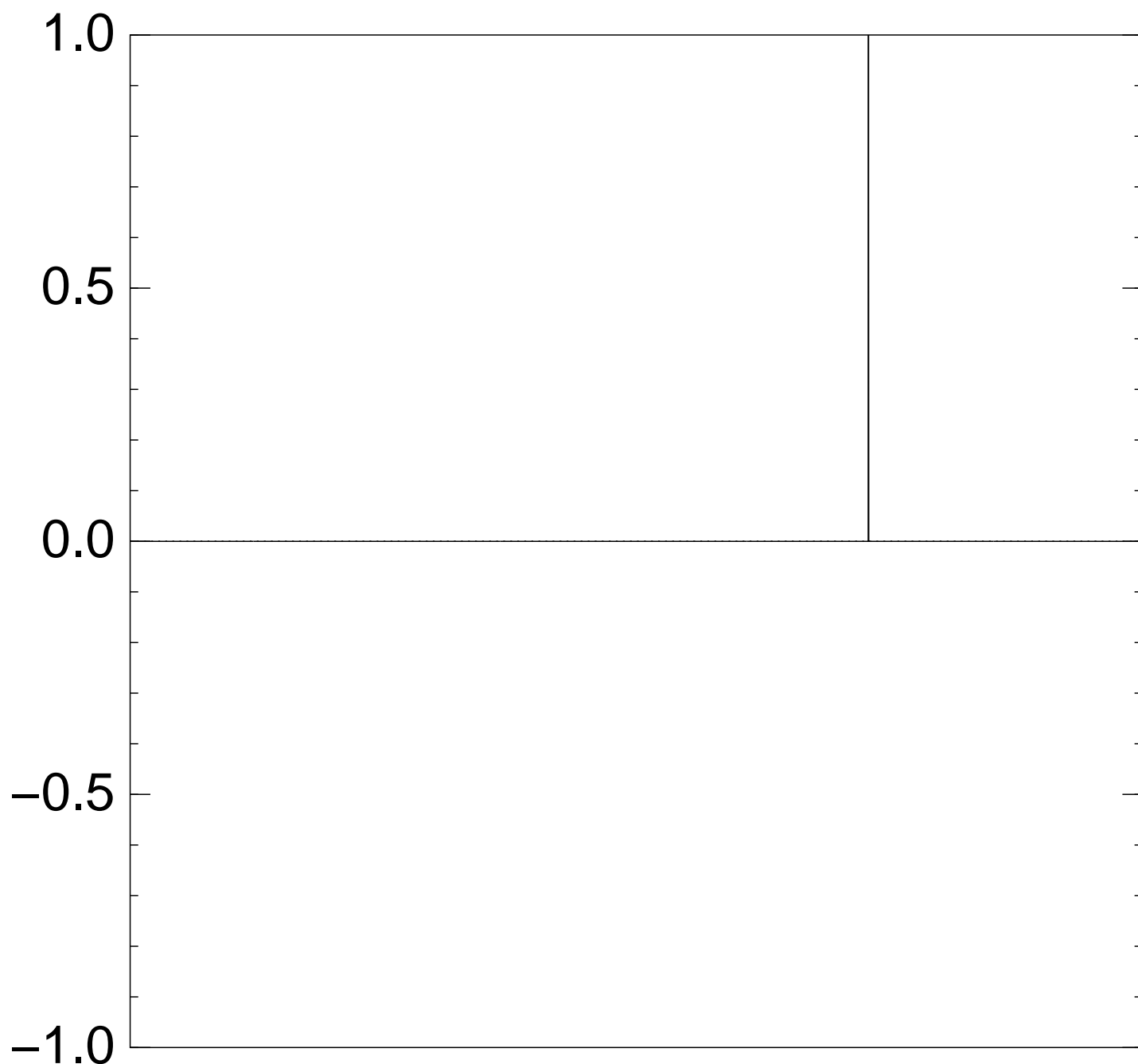
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $40 \times$  (Step 1 + Step 2):



Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $45 \times$  (Step 1 + Step 2):

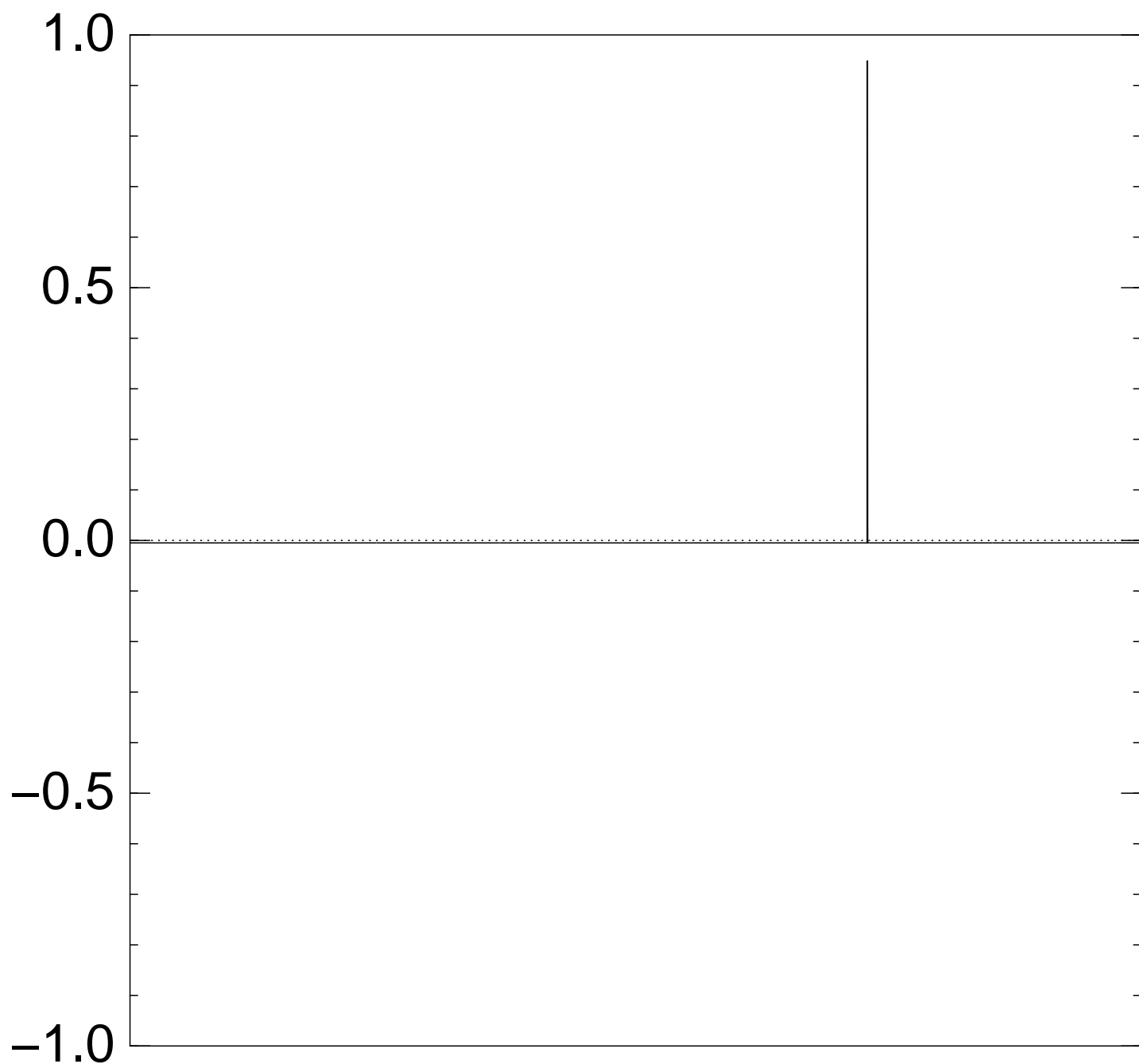


Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $50 \times$  (Step 1 + Step 2):

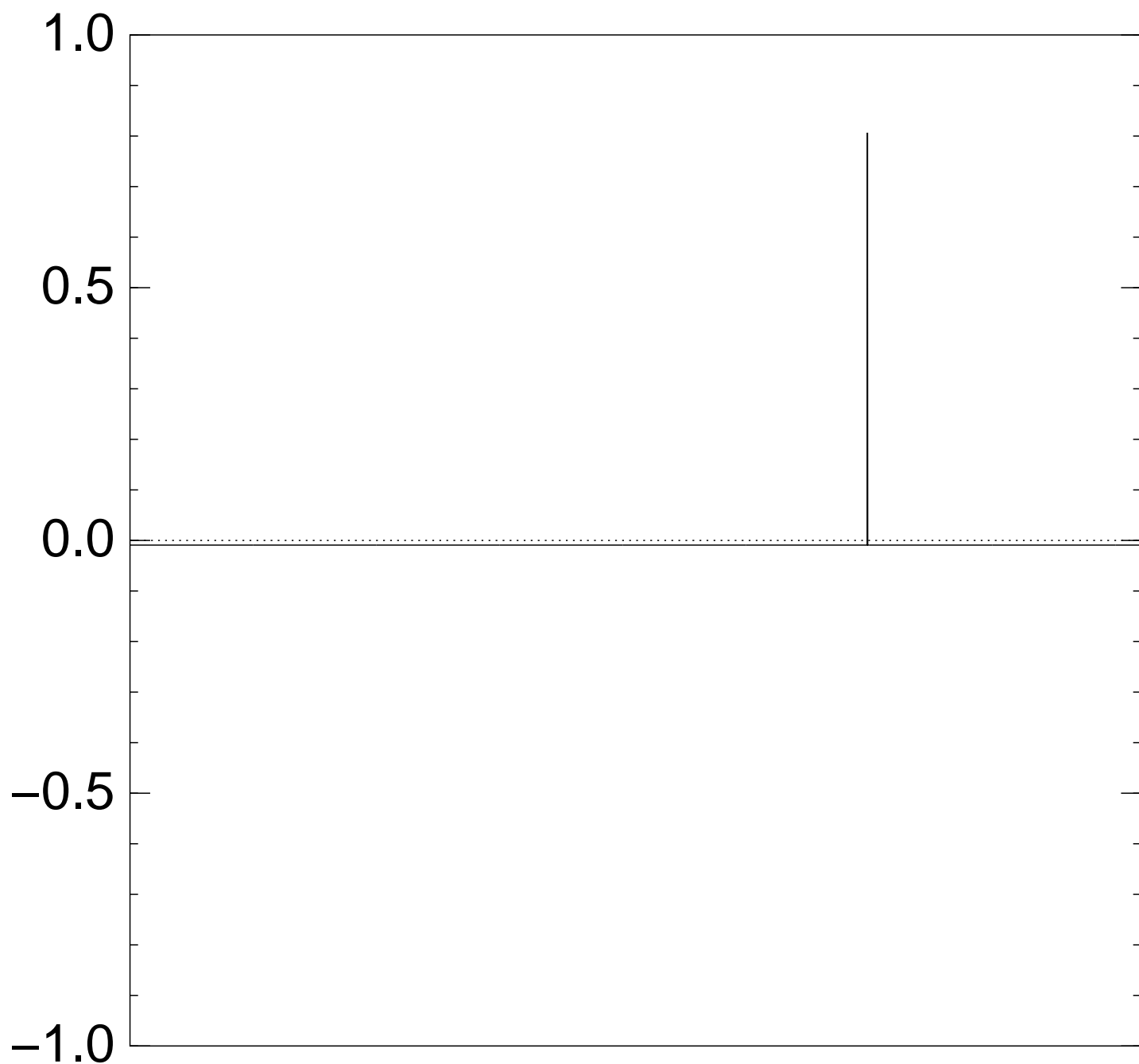


Traditional stopping point.

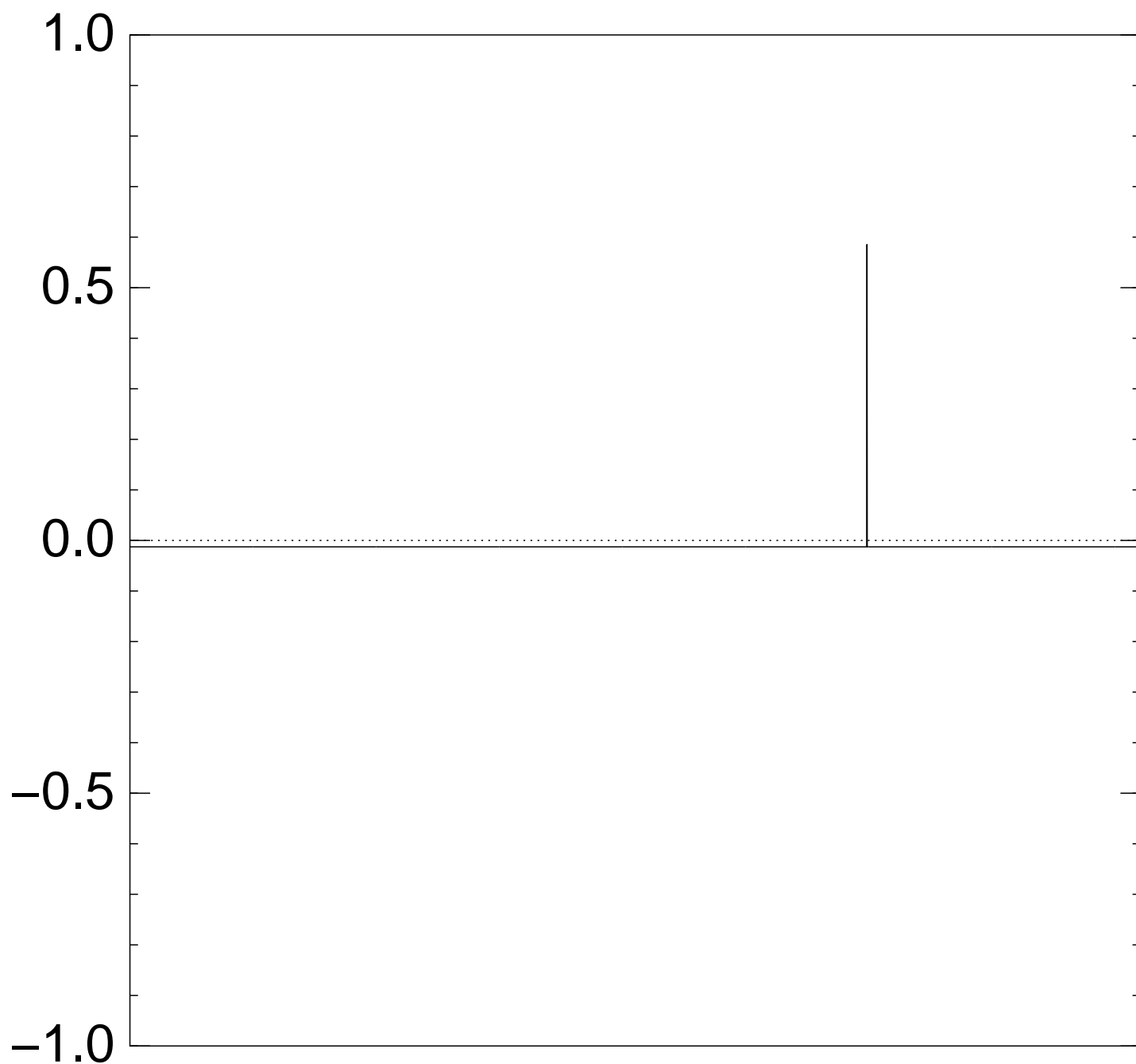
Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $60 \times (\text{Step 1} + \text{Step 2})$ :



Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $70 \times$  (Step 1 + Step 2):

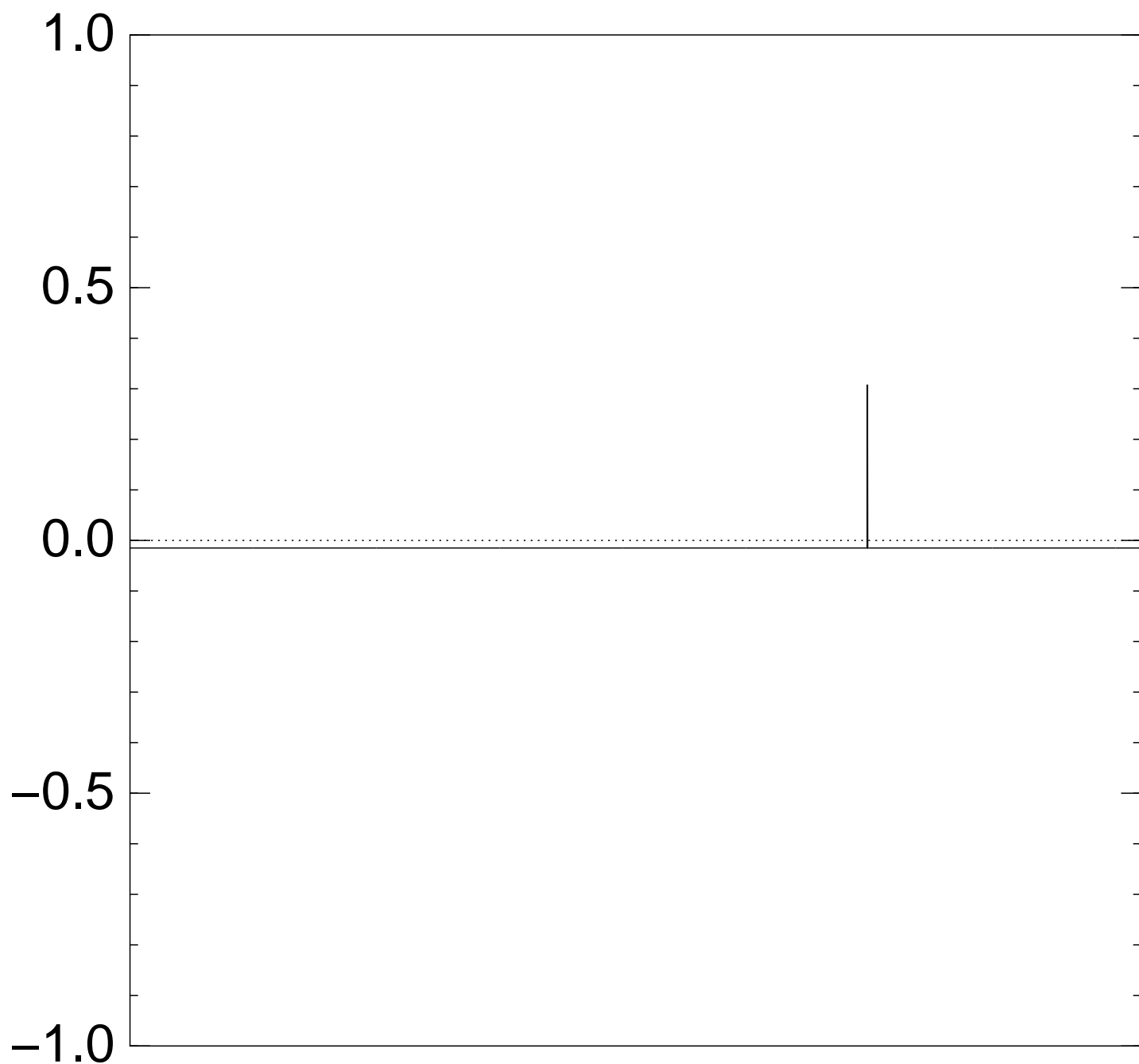


Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $80 \times$  (Step 1 + Step 2):

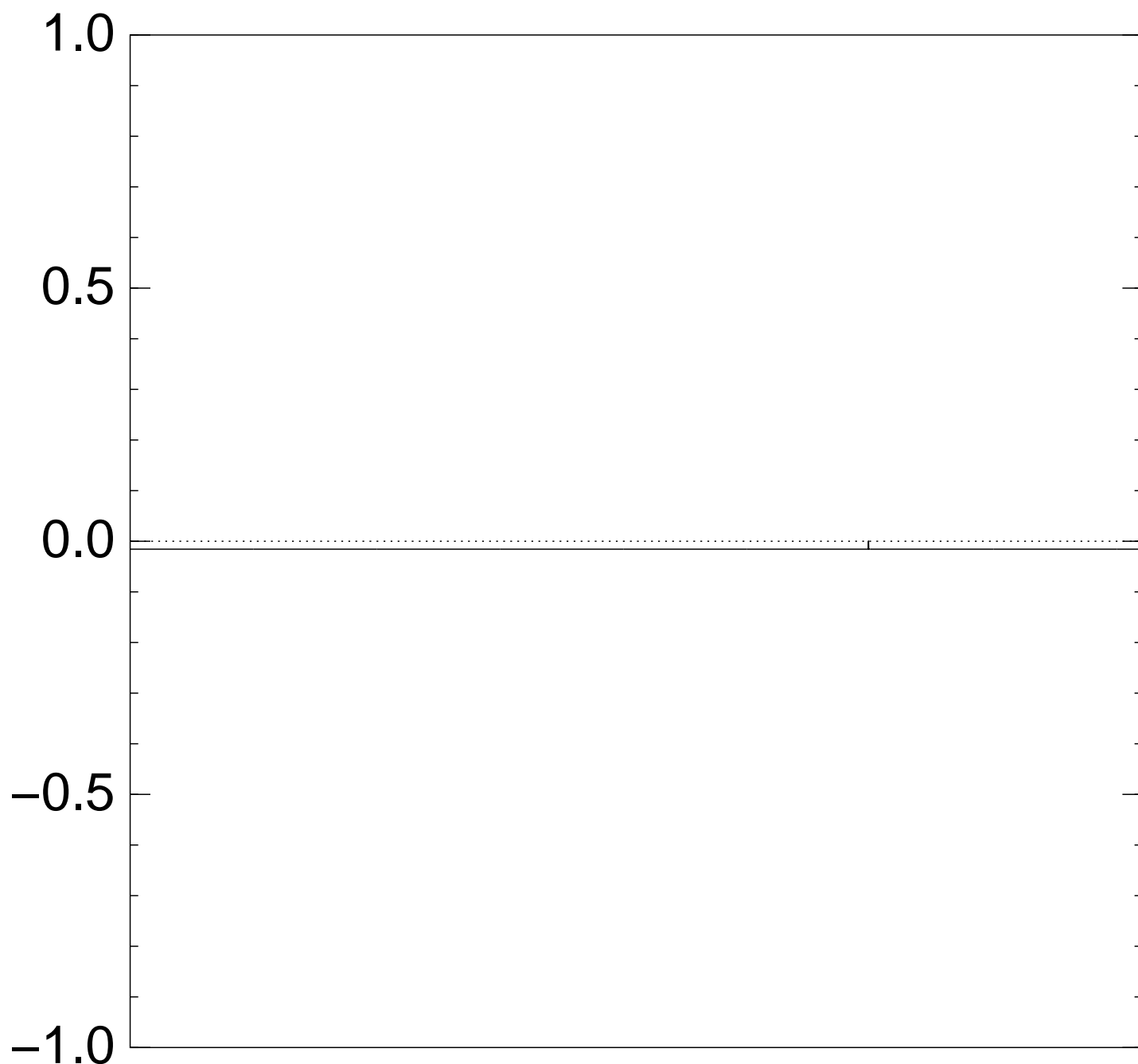




Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $90 \times$  (Step 1 + Step 2):



Normalized graph of  $q \mapsto a_q$   
for an example with  $n = 12$   
after  $100 \times$  (Step 1 + Step 2):



Very bad stopping point.

$q \mapsto a_q$  is completely described by a vector of two numbers (with fixed multiplicities):

(1)  $a_q$  for roots  $q$ ;

(2)  $a_q$  for non-roots  $q$ .

Step 1 + Step 2

act linearly on this vector.

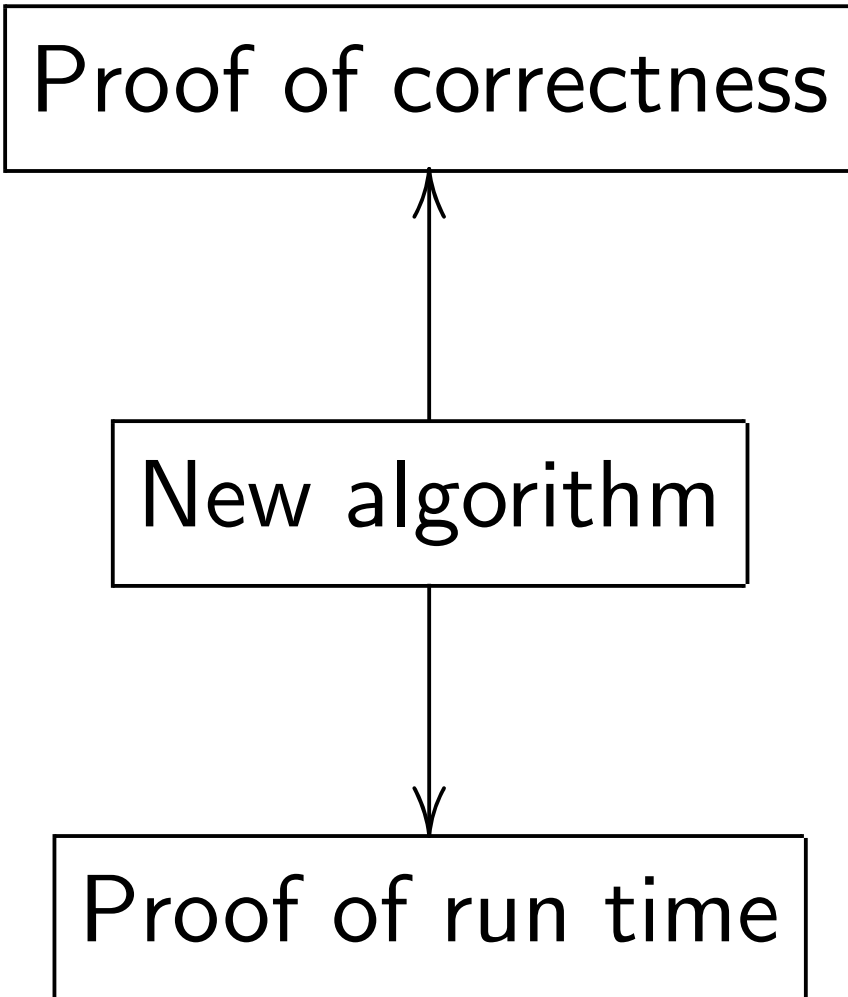
Easily compute eigenvalues and powers of this linear map to understand evolution of state of Grover's algorithm.

$\Rightarrow$  Probability is  $\approx 1$

after  $\approx (\pi/4)2^{0.5n}$  iterations.

# Notes on provability

Textbook algorithm analysis:



Mislead students into thinking that best algorithm = best *proven* algorithm.

Reality: state-of-the-art  
cryptanalytic algorithms  
are almost never proven.

Reality: state-of-the-art  
cryptanalytic algorithms  
are almost never proven.

Ignorant response:

“Work harder, find proofs!”

Reality: state-of-the-art  
cryptanalytic algorithms  
are almost never proven.

Ignorant response:

“Work harder, find proofs!”

Consensus of the experts:  
proofs probably do not *exist*  
for most of these algorithms.  
So demanding proofs is silly.

Reality: state-of-the-art cryptanalytic algorithms are almost never proven.

Ignorant response:

“Work harder, find proofs!”

Consensus of the experts: proofs probably do not *exist* for most of these algorithms. So demanding proofs is silly.

Without proofs, how do we analyze correctness+speed?

Answer: Real algorithm analysis relies critically on heuristics and **computer experiments.**



What about quantum algorithms?  
Want to analyze, optimize  
quantum algorithms *today*  
to figure out safe crypto  
against *future* quantum attack.

What about quantum algorithms?

Want to analyze, optimize

quantum algorithms *today*

to figure out safe crypto

against *future* quantum attack.

1. Simulate *tiny* q. computer?

⇒ Huge extrapolation errors.

What about quantum algorithms?

Want to analyze, optimize  
quantum algorithms *today*

to figure out safe crypto

against *future* quantum attack.

1. Simulate *tiny* q. computer?

⇒ Huge extrapolation errors.

2. Faster algorithm-specific

simulation? Yes, sometimes.

What about quantum algorithms?  
Want to analyze, optimize  
quantum algorithms *today*  
to figure out safe crypto  
against *future* quantum attack.

1. Simulate *tiny* q. computer?  
⇒ Huge extrapolation errors.

2. Faster algorithm-specific  
simulation? Yes, sometimes.

3. Fast **trapdoor simulation**.

Simulator (like prover) knows  
more than the algorithm does.

Tung Chou has implemented this,  
found errors in two publications.

# Post-quantum cryptography

Grover's algorithm finds  
128-bit AES key using  
 $2^{64}$  quantum AES evaluations.

# Post-quantum cryptography

Grover's algorithm finds  
128-bit AES key using  
 $2^{64}$  quantum AES evaluations.

Sensible risk management:  
Assume that this is feasible—  
or will be feasible in, e.g., 2025.

# Post-quantum cryptography

Grover's algorithm finds  
128-bit AES key using  
 $2^{64}$  quantum AES evaluations.

Sensible risk management:  
Assume that this is feasible—  
or will be feasible in, e.g., 2025.  
“AES-128 is dead.”

# Post-quantum cryptography

Grover's algorithm finds  
128-bit AES key using  
 $2^{64}$  quantum AES evaluations.

Sensible risk management:  
Assume that this is feasible—  
or will be feasible in, e.g., 2025.  
“AES-128 is dead.”

Fix: Switch to AES-256.



## Post-quantum cryptography

Grover's algorithm finds  
128-bit AES key using  
 $2^{64}$  quantum AES evaluations.

Sensible risk management:  
Assume that this is feasible—  
or will be feasible in, e.g., 2025.  
“AES-128 is dead.”

Fix: Switch to AES-256.

AES-256 has 14 rounds.

Maybe 12 rounds are enough  
for  $2^{128}$  post-quantum security?

Maybe 10 rounds are enough?

Shor's algorithm

(similar to Simon's algorithm)

factors RSA modulus  $N$  by

finding period of  $x \mapsto 2^x \bmod N$ .

Number of qubit operations

$\approx$  number of bit operations

to compute  $2^x \bmod N$ .

Shor's algorithm

(similar to Simon's algorithm)

factors RSA modulus  $N$  by

finding period of  $x \mapsto 2^x \bmod N$ .

Number of qubit operations

$\approx$  number of bit operations

to compute  $2^x \bmod N$ .

$\approx 2^{64}$  qubit operations

when  $N$  is around 1 gigabyte.

Shor's algorithm

(similar to Simon's algorithm)

factors RSA modulus  $N$  by

finding period of  $x \mapsto 2^x \bmod N$ .

Number of qubit operations

$\approx$  number of bit operations

to compute  $2^x \bmod N$ .

$\approx 2^{64}$  qubit operations

when  $N$  is around 1 gigabyte.

Shor also finds  $\log_g h$  by

finding period of  $(x, y) \mapsto g^x h^y$ .

Shor's algorithm

(similar to Simon's algorithm)

factors RSA modulus  $N$  by

finding period of  $x \mapsto 2^x \bmod N$ .

Number of qubit operations

$\approx$  number of bit operations

to compute  $2^x \bmod N$ .

$\approx 2^{64}$  qubit operations

when  $N$  is around 1 gigabyte.

Shor also finds  $\log_g h$  by

finding period of  $(x, y) \mapsto g^x h^y$ .

“RSA is dead. ECC is dead.”

Shor's algorithm

(similar to Simon's algorithm)

factors RSA modulus  $N$  by

finding period of  $x \mapsto 2^x \bmod N$ .

Number of qubit operations

$\approx$  number of bit operations

to compute  $2^x \bmod N$ .

$\approx 2^{64}$  qubit operations

when  $N$  is around 1 gigabyte.

Shor also finds  $\log_g h$  by

finding period of  $(x, y) \mapsto g^x h^y$ .

“RSA is dead. ECC is dead.”

But some systems seem safe.

## **Hash-based signatures.**

Example: 1979 Merkle hash-tree public-key signature system.

## **Code-based cryptography.**

Example: 1978 McEliece hidden-Goppa-code public-key encryption system.

## **Lattice-based cryptography.**

Example: 1998 “NTRU”.

## **Multivariate-quadratic-equations cryptography.**

Example:

1996 Patarin “HFE<sup>v-</sup>”

public-key signature system.

Daniel J. Bernstein  
Johannes Buchmann  
Erik Dahmen  
*Editors*

# Post-Quantum Cryptography

 Springer



# The 1978 McEliece cryptosystem

(with 1986 Niederreiter speedup)

Receiver's public key: "random"

$500 \times 1024$  matrix  $K$  over  $\mathbf{F}_2$ .

Specifies linear  $\mathbf{F}_2^{1024} \rightarrow \mathbf{F}_2^{500}$ .

# The 1978 McEliece cryptosystem

(with 1986 Niederreiter speedup)

Receiver's public key: "random"

$500 \times 1024$  matrix  $K$  over  $\mathbf{F}_2$ .

Specifies linear  $\mathbf{F}_2^{1024} \rightarrow \mathbf{F}_2^{500}$ .

Messages suitable for encryption:

1024-bit strings of weight 50.

$\{e \in \mathbf{F}_2^{1024} : \#\{i : e_i = 1\} = 50\}$ .

# The 1978 McEliece cryptosystem

(with 1986 Niederreiter speedup)

Receiver's public key: "random"

$500 \times 1024$  matrix  $K$  over  $\mathbf{F}_2$ .

Specifies linear  $\mathbf{F}_2^{1024} \rightarrow \mathbf{F}_2^{500}$ .

Messages suitable for encryption:

1024-bit strings of weight 50.

$\{e \in \mathbf{F}_2^{1024} : \#\{i : e_i = 1\} = 50\}$ .

Encryption of  $e$  is  $Ke \in \mathbf{F}_2^{500}$ .

# The 1978 McEliece cryptosystem

(with 1986 Niederreiter speedup)

Receiver's public key: "random"

$500 \times 1024$  matrix  $K$  over  $\mathbf{F}_2$ .

Specifies linear  $\mathbf{F}_2^{1024} \rightarrow \mathbf{F}_2^{500}$ .

Messages suitable for encryption:

1024-bit strings of weight 50.

$\{e \in \mathbf{F}_2^{1024} : \#\{i : e_i = 1\} = 50\}$ .

Encryption of  $e$  is  $Ke \in \mathbf{F}_2^{500}$ .

"Padding": Choose random  $e$ ;  
send  $Ke$ ; use  $\text{SHA-256}(e, Ke)$  as  
AES-256-GCM key to encrypt  
actual message of any length.

Attacker, by linear algebra,  
easily works backwards  
from  $Ke$  to *some*  $v \in \mathbf{F}_2^{1024}$   
such that  $Kv = Ke$ .

Attacker, by linear algebra,  
easily works backwards  
from  $Ke$  to *some*  $v \in \mathbf{F}_2^{1024}$   
such that  $Kv = Ke$ .

i.e. Attacker finds *some*  
element  $v \in e + \text{Ker } K$ .

Note that  $\# \text{Ker } K \geq 2^{524}$ .

Attacker wants to decode  $v$ :  
to find element of  $\text{Ker } K$   
at distance only 50 from  $v$ .

Presumably unique, revealing  $e$ .

Attacker, by linear algebra,  
easily works backwards  
from  $Ke$  to *some*  $v \in \mathbf{F}_2^{1024}$   
such that  $Kv = Ke$ .

i.e. Attacker finds *some*  
element  $v \in e + \text{Ker } K$ .

Note that  $\# \text{Ker } K \geq 2^{524}$ .

Attacker wants to decode  $v$ :  
to find element of  $\text{Ker } K$   
at distance only 50 from  $v$ .

Presumably unique, revealing  $e$ .

But decoding isn't easy!

## Information-set decoding

Choose random size-500 subset

$$S \subseteq \{1, 2, 3, \dots, 1024\}.$$

For typical  $K$ : Good chance

$$\text{that } \mathbf{F}_2^S \hookrightarrow \mathbf{F}_2^{1024} \xrightarrow{K} \mathbf{F}_2^{500}$$

is invertible.



## Information-set decoding

Choose random size-500 subset  $S \subseteq \{1, 2, 3, \dots, 1024\}$ .

For typical  $K$ : Good chance that  $\mathbf{F}_2^S \hookrightarrow \mathbf{F}_2^{1024} \xrightarrow{K} \mathbf{F}_2^{500}$  is invertible.

Hope  $e \in \mathbf{F}_2^S$ ; chance  $\approx 2^{-53}$ .

Apply inverse map to  $Ke$ , revealing  $e$  if  $e \in \mathbf{F}_2^S$ .

## Information-set decoding

Choose random size-500 subset  $S \subseteq \{1, 2, 3, \dots, 1024\}$ .

For typical  $K$ : Good chance that  $\mathbf{F}_2^S \hookrightarrow \mathbf{F}_2^{1024} \xrightarrow{K} \mathbf{F}_2^{500}$  is invertible.

Hope  $e \in \mathbf{F}_2^S$ ; chance  $\approx 2^{-53}$ .

Apply inverse map to  $Ke$ , revealing  $e$  if  $e \in \mathbf{F}_2^S$ .

If  $e \notin \mathbf{F}_2^S$ , try again.

$\approx 2^{80}$  bit operations in total.

## Information-set decoding

Choose random size-500 subset  $S \subseteq \{1, 2, 3, \dots, 1024\}$ .

For typical  $K$ : Good chance that  $\mathbf{F}_2^S \hookrightarrow \mathbf{F}_2^{1024} \xrightarrow{K} \mathbf{F}_2^{500}$  is invertible.

Hope  $e \in \mathbf{F}_2^S$ ; chance  $\approx 2^{-53}$ .

Apply inverse map to  $Ke$ , revealing  $e$  if  $e \in \mathbf{F}_2^S$ .

If  $e \notin \mathbf{F}_2^S$ , try again.

$\approx 2^{80}$  bit operations in total.

Bad estimate by McEliece:  $\approx 2^{64}$ .

Analyzing and optimizing attacks:  
1962 Prange. 1981 Omura.  
1988 Lee–Brickell. 1988 Leon.  
1989 Krouk. 1989 Stern.  
1989 Dumer.  
1990 Coffey–Goodman.  
1990 van Tilburg. 1991 Dumer.  
1991 Coffey–Goodman–Farrell.  
1993 Chabanne–Courteau.  
1993 Chabaud.  
1994 van Tilburg.  
1994 Canteaut–Chabanne.  
1998 Canteaut–Chabaud.  
1998 Canteaut–Sendrier.

- 2008 Bernstein–Lange–Peters:  
more speedups;  $\approx 2^{60}$  cycles;  
attack **actually carried out**.
- 2009 Bernstein–Lange–  
Peters–van Tilborg.
- 2009 Bernstein: post-quantum.
- 2009 Finiasz–Sendrier.
- 2010 Bernstein–Lange–Peters.
- 2011 May–Meurer–Thomae.
- 2011 Becker–Coron–Joux.
- 2012 Becker–Joux–May–Meurer.
- 2013 Bernstein–Jeffery–Lange–  
Meurer: post-quantum.
- 2015 May–Ozerov.

## Modern McEliece

Easily rescue system by using  
a larger public key: “random”  
 $(n/2) \times n$  matrix  $K$  over  $\mathbf{F}_2$ .  
e.g.,  $1800 \times 3600$ .

## Modern McEliece

Easily rescue system by using  
a larger public key: “random”  
 $(n/2) \times n$  matrix  $K$  over  $\mathbf{F}_2$ .  
e.g.,  $1800 \times 3600$ .

Larger weight  $w \approx n/(2 \lg n)$ .  
e.g.  $e \in \mathbf{F}_2^{3600}$  of weight 150.

## Modern McEliece

Easily rescue system by using  
a larger public key: “random”  
 $(n/2) \times n$  matrix  $K$  over  $\mathbf{F}_2$ .  
e.g.,  $1800 \times 3600$ .

Larger weight  $w \approx n/(2 \lg n)$ .  
e.g.  $e \in \mathbf{F}_2^{3600}$  of weight 150.

1962 attack cost:  $2^{(1+o(1))w}$ .



## Modern McEliece

Easily rescue system by using a larger public key: “random”  $(n/2) \times n$  matrix  $K$  over  $\mathbf{F}_2$ .  
e.g.,  $1800 \times 3600$ .

Larger weight  $w \approx n/(2 \lg n)$ .  
e.g.  $e \in \mathbf{F}_2^{3600}$  of weight 150.

1962 attack cost:  $2^{(1+o(1))w}$ .

After extensive research,

2015 attack cost:  $2^{(1+o(1))w}$ .

## Modern McEliece

Easily rescue system by using a larger public key: “random”  $(n/2) \times n$  matrix  $K$  over  $\mathbf{F}_2$ .  
e.g.,  $1800 \times 3600$ .

Larger weight  $w \approx n/(2 \lg n)$ .  
e.g.  $e \in \mathbf{F}_2^{3600}$  of weight 150.

1962 attack cost:  $2^{(1+o(1))w}$ .

After extensive research,  
2015 attack cost:  $2^{(1+o(1))w}$ .

Post-quantum:  $2^{(0.5+o(1))w}$ .  
e.g.  $\approx 2^{26}$  Grover iterations  
to search  $2^{53}$  choices of  $S$ .