Computational algebraic number theory tackles lattice-based cryptography

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Moving to the left
Moving to the right
Big generator
Moving through the night
—Yes, "Big Generator", 1987

2013.07 talk slide online:
"I think NTRU should switch to random prime-degree extensions with big Galois groups."

2014.02 blog post:
"Here's a concrete suggestion, which I'll call NTRU Prime, for eliminating the structures that I find worrisome in existing ideal-lattice-based encryption systems."

NTRU Prime uses primes p, q with field $(\mathbf{Z}/q)[x]/(x^p - x - 1)$.

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Disadvantage of cyclotomics: many more symmetries feed a scary attack strategy. Already serious damage to some lattice-based systems, concerns about other systems.

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No. Dangerous exaggeration!
There are many obvious gaps between lattice-based systems and the classic lattice problems: e.g., the systems use ideals.
Important to study these gaps.

2009 Smart–Vercauteren "Fully homomorphic encryption with relatively small key and ciphertext sizes": "Recovering the private key given the public key is therefore an instance of the small principal ideal problem: . . . Given a principal ideal . . . compute a 'small' generator of the ideal. This is one of the core problems in computational number theory and has formed the basis of previous cryptographic proposals, see for example [3]."

Smart–Vercauteren, continued:

"There are currently two approaches to the problem. . . . In conclusion determining the private key given only the public key is an instance of a classical and well studied problem in algorithmic number theory. In particular there are no efficient solutions for this problem, and the only sub-exponential method does not find a solution which is equivalent to our private key."

In fact, the classical studies
focus on small dimensions:
e.g., make table of class numbers
for many quadratic fields,
make table of class numbers
for many cubic fields.

Highlights multiplicative issues.
Low-dim lattice issues are easy.

Far fewer papers
consider scalability
of the algorithmic ideas
to much larger dimensions.

The short-generator problem

Take degree-n number field K.
i.e. field $K \subseteq \mathbf{C}$ with $\text{len}_{\mathbf{Q}} K = n$.

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e.g. $K = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots, \sqrt{29})$.

Define $\mathcal{O} = \overline{\mathbf{Z}} \cap K$; subring of K.
$\mathcal{O} \hookrightarrow\!\!\!\!\!\twoheadrightarrow \mathbf{Z}^n$ as $\mathbf{Z}$-modules.

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e.g. $K = \mathbf{Q}(\sqrt{5}) \Rightarrow \mathcal{O} =$
$\mathbf{Z}[(1+\sqrt{5})/2] \hookrightarrow\!\!\!\!\!\!\twoheadrightarrow \mathbf{Z}[x]/(x^2-x-1)$.

The short-generator problem:
Find "short" nonzero $g \in \mathcal{O}$ given the principal ideal $g\mathcal{O}$.

e.g. $\zeta = \exp(\pi i/4)$; $K = \mathbf{Q}(\zeta)$;
$\mathcal{O} = \mathbf{Z}[\zeta] \hookrightarrow \mathbf{Z}[x]/(x^4 + 1)$.
The $\mathbf{Z}$-submodule of $\mathcal{O}$ gen by
$201 - 233\zeta - 430\zeta^2 - 712\zeta^3$,
$935 - 1063\zeta - 1986\zeta^2 - 3299\zeta^3$,
$979 - 1119\zeta - 2092\zeta^2 - 3470\zeta^3$,
$718 - 829\zeta - 1537\zeta^2 - 2546\zeta^3$
is an ideal I of $\mathcal{O}$.
Can you find a short $g \in \mathcal{O}$ such that $I = g\mathcal{O}$?

The lattice perspective

Use LLL to quickly find short elements of lattice $\mathbf{Z}A + \mathbf{Z}B + \mathbf{Z}C + \mathbf{Z}D$ where

$A = (201, -233, -430, -712)$,

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Also find, e.g., $(-4, -1, 3, 1)$.
Multiplying by root of unity (here ζ^2) preserves shortness.

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Fancier lattice algorithms:
Under reasonable assumptions,
2015 Laarhoven–de Weger
finds g in time $\approx 1.23^n$.
Big progress compared to, e.g.,
2008 Nguyen–Vidick ($\approx 1.33^n$)
but still exponential time.

Exploiting factorization

Use LLL, BKZ, etc. to
generate rather short $\alpha \in g\mathcal{O}$.
What happens if $\alpha\mathcal{O} \neq g\mathcal{O}$?

Pure lattice approach: Discard α.
Work much harder, find shorter α.

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$P = \alpha_1\alpha_3^{-1}\mathcal{O}$ and $Q = \alpha_2\alpha_3^{-1}\mathcal{O}$
and $g\mathcal{O} = \alpha_1^{-1}\alpha_2^{-2}\alpha_3^4\mathcal{O}$.

General strategy: For many α's, factor $\alpha\mathcal{O}$ into products of powers of some primes and $g\mathcal{O}$.

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Familiar issue from “index calculus” DL methods, CFRAC, LS, QS, NFS, etc.
Model the norm of $(\alpha/g)\mathcal{O}$ as “random” integer in $[1, x]$;
y-smoothness chance $\approx 1/y$ if $\log y \approx \sqrt{(1/2) \log x \log \log x}$.

Variation: Ignore $g\mathcal{O}$.
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After enough α's,
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"Do all primes have generators?"

— Standard heuristics:
For many (most?) number fields,
yes; but for big cyclotomics, no!
Modulo a few small primes, yes.

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Factoring many small $\alpha\mathcal{O}$ is a standard textbook method of computing class group and generators of ideals.

Also compute unit group $\mathcal{O}^*$ via ratios of generators.

A note on time analysis

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$\exp(\Theta(N \log N))$ factor
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Silly: BKZ works just fine.
The whole algorithm will be subexponential unless norms are much worse than exponential.

Big generator

Smart–Vercauteren: "However this method is likely to produce a generator of large height, i.e., with large coefficients. Indeed so large, that writing the obtained generator down as a polynomial in θ may take exponential time."

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How do we find g from gu?

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$\operatorname{Log} \mathcal{O}^*$ is a lattice of rank $r_1 + r_2 - 1$ where

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e.g. $\zeta = \exp(\pi i/256)$, $K = \mathbf{Q}(\zeta)$: images of ζ under ring maps are $\zeta, \zeta^3, \zeta^5, \ldots, \zeta^{511}$.
$r_1 = 0$; $r_2 = 128$; rank 127.

Compute Log $g u$
as sum of multiples of Log α
for the original α's.

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This is a close-vector problem
("bounded-distance decoding").
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This finds $\operatorname{Log} u$.
Easily reconstruct g
up to a root of unity.
#{roots of unity} is small.

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Find elements close to $\operatorname{Log} g u$. Lower-dimension lattice problem, if unit rank of F is positive.

Start by recursively computing $\operatorname{Log}\operatorname{norm}_{K:F} g$ via norm of $g\mathcal{O}$ for *each* $F \subset K$.

Various constraints on $\operatorname{Log} u$, depending on subfield structure.

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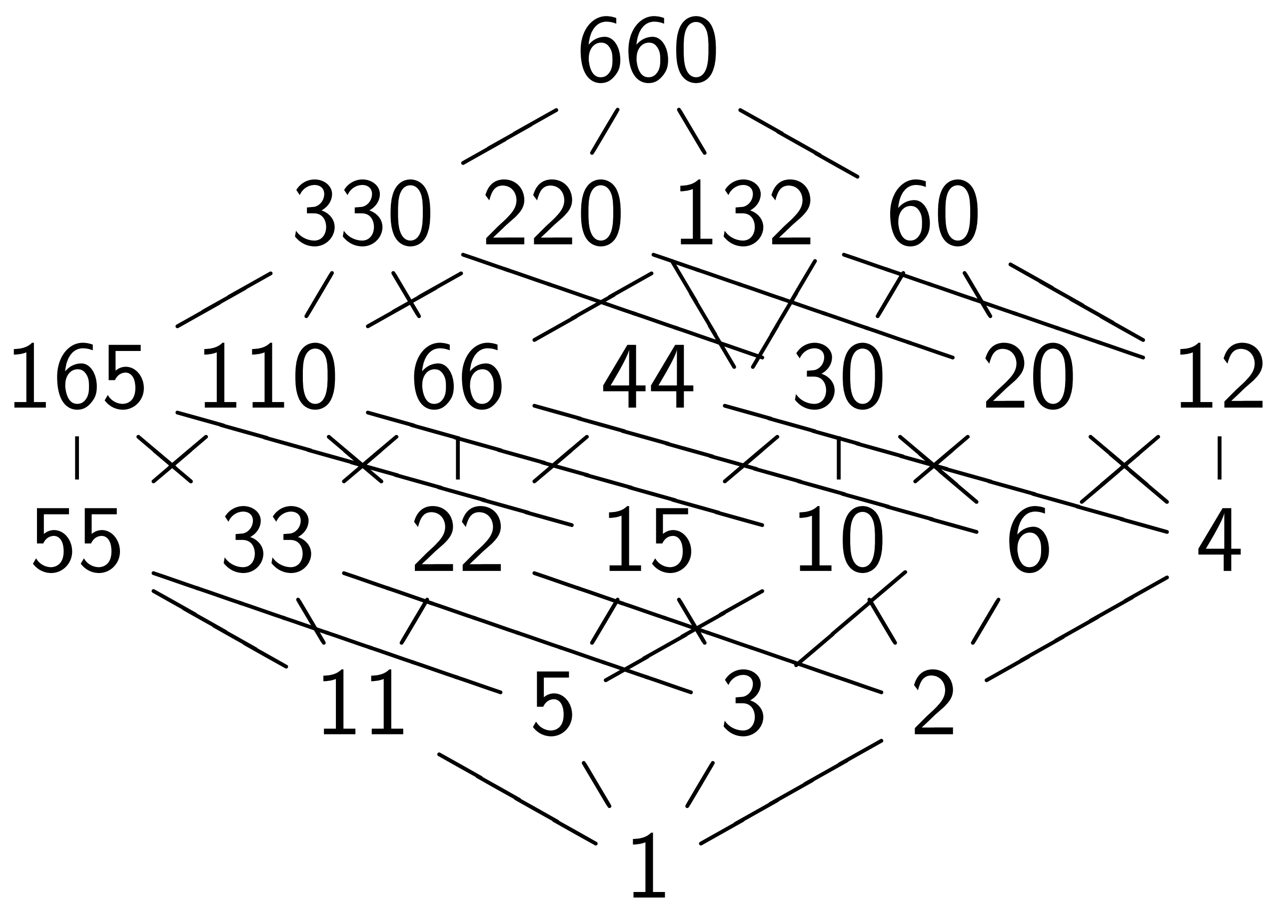
Various constraints on $\text{Log}\,u$, depending on subfield structure.

e.g. $\zeta = \exp(2\pi i/661)$, $K = \mathbf{Q}(\zeta)$. Degrees of subfields of K:



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For cyclotomics this approach is superseded by subsequent Campbell–Groves–Shepherd algorithm, using known (good) basis for cyclotomic units.