Batch NFS

D. J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Tanja Lange
Technische Universiteit Eindhoven

In this talk $\log L$ means $(1 + o(1))(\log N)^{1/3}(\log \log N)^{2/3}$.
L is often written "$L_N(1/3)$" or "$L_N(1/3)^{1+o(1)}$".

Exponents of L in this talk are limited to $10^{-6}\mathbf{Z}$.

Rigorously proven? Ha ha ha.

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using chip area $L^{0.790420}$;
AT is $L^{1.976052}$.

Our main result for
a batch of $L^{0.5}$ keys:
time $L^{1.022400}$
using chip area $L^{1.181600}$;
AT per key is $L^{1.704000}$.

This paper also looks more closely
at $L^{o(1)}$, analyzing asymptotic
speedup from early-abort ECM.
Results are not what one would
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etc.

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Eratosthenes for s

Sieving small integ

using primes $2, 3, 5$

1								
2	2							
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7								7
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<u>Eratosthenes for smoothness</u>

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1								
2	2							
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4	2	2						
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14	2							7
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16	2	2	2	2				
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18	2				3	3		
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etc.

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1								
2	2							
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14	2							7
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18	2				3	3		
19								
20	2	2					5	

etc.

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2	2							
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14	2							7
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18	2				3	3		
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20	2	2					5	

etc.

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Eratosthenes for smoothness

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

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2	2							
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6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
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14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

etc.

The **Q** sieve

Sieving i and 611
using primes $2, 3, 5$

1								
2	2							
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4	2	2						
5							5	
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7								7
8	2	2	2					
9					3	3		
10	2						5	
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14	2							7
15					3		5	
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17								
18	2				3	3		
19								
20	2	2					5	

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626	2
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628	2
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630	2
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etc.

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Eratosthenes for smoothness

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

etc.

The **Q** sieve

Sieving i and $611 + i$ for sm
using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

612	2	2			3	3		
613								
614	2							
615					3			5
616	2	2	2					
617								
618	2				3			
619								
620	2	2						5
621					3	3	3	
622	2							
623								
624	2	2	2	2	3			
625								5
626	2							
627					3			
628	2	2						
629								
630	2				3	3		5
631								

etc.

Eratosthenes for smoothness

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

i	2	3	5	7
1				
2	2			
3		3		
4	2 2			
5			5	
6	2	3		
7				7
8	2 2 2			
9		3 3		
10	2		5	
11				
12	2 2	3		
13				
14	2			7
15		3	5	
16	2 2 2 2			
17				
18	2	3 3		
19				
20	2 2		5	

etc.

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

i	2	3	5	7
1				
2	2			
3		3		
4	2 2			
5			5	
6	2	3		
7				7
8	2 2 2			
9		3 3		
10	2		5	
11				
12	2 2	3		
13				
14	2			7
15		3	5	
16	2 2 2 2			
17				
18	2	3 3		
19				
20	2 2		5	

$611+i$	2	3	5	7
612	2 2	3 3		
613				
614	2			
615		3	5	
616	2 2 2			7
617				
618	2	3		
619				
620	2 2		5	
621		3 3 3		
622	2			
623				7
624	2 2 2 2	3		
625			5 5 5 5	
626	2			
627		3		
628	2 2			
629				
630	2	3 3	5	7
631				

etc.

nenes for smoothness

small integers $i > 0$

imes $2, 3, 5, 7$:

	5	
		7
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	5	
		7
	5	
3		
	5	

The $\mathbf{Q}$ sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
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20	2	2					5	

612	2	2			3	3						
613												
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616	2	2	2									7
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618	2				3							
619												
620	2	2						5				
621					3	3	3					
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626	2											
627					3							
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etc.

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$14 \cdot 625$

$64 \cdot 675$

$75 \cdot 686$

$14 \cdot 64 \cdot$

$= 2^8 3^4 5$

$\gcd\{611$

$= 47.$

$611 = 4$

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gers $i > 0$

$5, 7$:

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2 2 3 3
613	
614	2
615	3 5
616	2 2 2 7
617	
618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	7
624	2 2 2 2 3
625	5 5 5 5
626	2
627	3
628	2 2
629	
630	2 3 3 5 7
631	

etc.

Have complete fac

the congruences i

for some i's.

$14 \cdot 625 = 2^1 3^0 5^4$

$64 \cdot 675 = 2^6 3^3 5^2$

$75 \cdot 686 = 2^1 3^1 5^2$

$14 \cdot 64 \cdot 75 \cdot 625 \cdot$

$= 2^8 3^4 5^8 7^4 = (2^4$

$\gcd\{611, 14 \cdot 64 \cdot$

$= 47.$

$611 = 47 \cdot 13.$

s

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1				
2	2			
3		3		
4	2 2			
5			5	
6	2	3		
7				7
8	2 2 2			
9		3 3		
10	2		5	
11				
12	2 2	3		
13				
14	2			7
15		3	5	
16	2 2 2 2			
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18	2	3 3		
19				
20	2 2		5	

612	2 2	3 3		
613				
614	2			
615		3	5	
616	2 2 2			7
617				
618	2	3		
619				
620	2 2		5	
621		3 3 3		
622	2			
623				7
624	2 2 2 2	3		
625			5 5 5 5	
626	2			
627		3		
628	2 2			
629				
630	2	3 3	5	7
631				

etc.

Have complete factorization
the congruences $i \equiv 611 + i$
for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5$
$= 47$.

$611 = 47 \cdot 13$.

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

i	2	3	5	7
1				
2	2			
3		3		
4	2 2			
5			5	
6	2	3		
7				7
8	2 2 2			
9		3 3		
10	2		5	
11				
12	2 2	3		
13				
14	2			7
15		3	5	
16	2 2 2 2			
17				
18	2	3 3		
19				
20	2 2		5	

$611+i$	2	3	5	7
612	2 2	3 3		
613				
614	2			
615		3	5	
616	2 2 2			7
617				
618	2	3		
619				
620	2 2		5	
621		3 3 3		
622	2			
623				7
624	2 2 2 2	3		
625			5 5 5 5	
626	2			
627		3		
628	2 2			
629				
630	2	3 3	5	7
631				

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$= 47$.

$611 = 47 \cdot 13$.

sieve

i and $611 + i$ for small i

imes $2, 3, 5, 7$:

	5	
		7
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612	2	2			3	3						
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630	2				3	3		5				7
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$= 47$.

$611 = 47 \cdot 13$.

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2 2 2	3		
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	3 3	5	7

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

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The number-field s

Generalize $i \equiv i +$
$\to a \equiv a + bN \quad ($
$\to a - bm \equiv a - b$
for root $\alpha \in \mathbf{C}$
of nonzero integer

For any m can fin
so that factoring n
produces factoriza

Optimal choice of
$(\mu + o(1))(\log N)^2$

all i

7

7

5 5 5

7

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

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$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \pmod$
$\to a \equiv a + bN \pmod N$
$\to a - bm \equiv a - b\alpha \pmod$
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
produces factorization of N.

Optimal choice of $\log m$ is
$(\mu + o(1))(\log N)^{2/3}(\log\log$

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

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$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

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$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \pmod N$
$\to a \equiv a + bN \pmod N$
$\to a - bm \equiv a - b\alpha \pmod{m - \alpha}$
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
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Optimal choice of $\log m$ is
$(\mu + o(1))(\log N)^{2/3}(\log\log N)^{1/3}$.

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gruences $i \equiv 611 + i$
e i's.

$= 2^1 3^0 5^4 7^1$.
$= 2^6 3^3 5^2 7^0$.
$= 2^1 3^1 5^2 7^3$.

$75 \cdot 625 \cdot 675 \cdot 686$
${}^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$., 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$

$7 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \pmod N$
$\rightarrow a \equiv a + bN \pmod N$
$\rightarrow a - bm \equiv a - b\alpha \pmod{m - \alpha}$
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
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2. Make many copies of each N,
close to each (a, b) generator.
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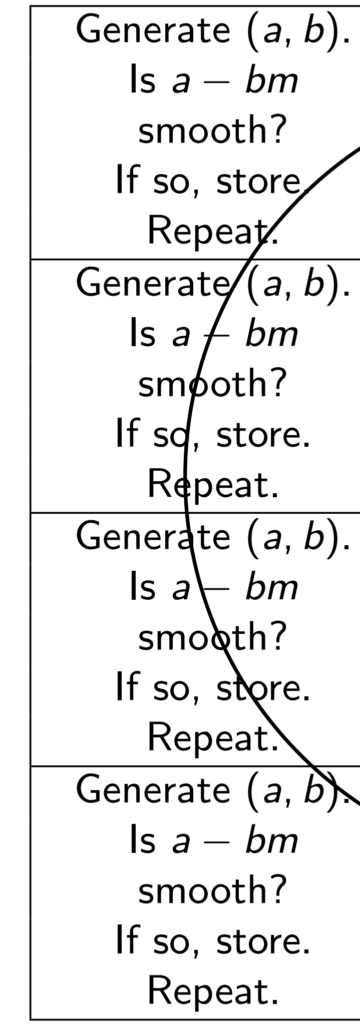

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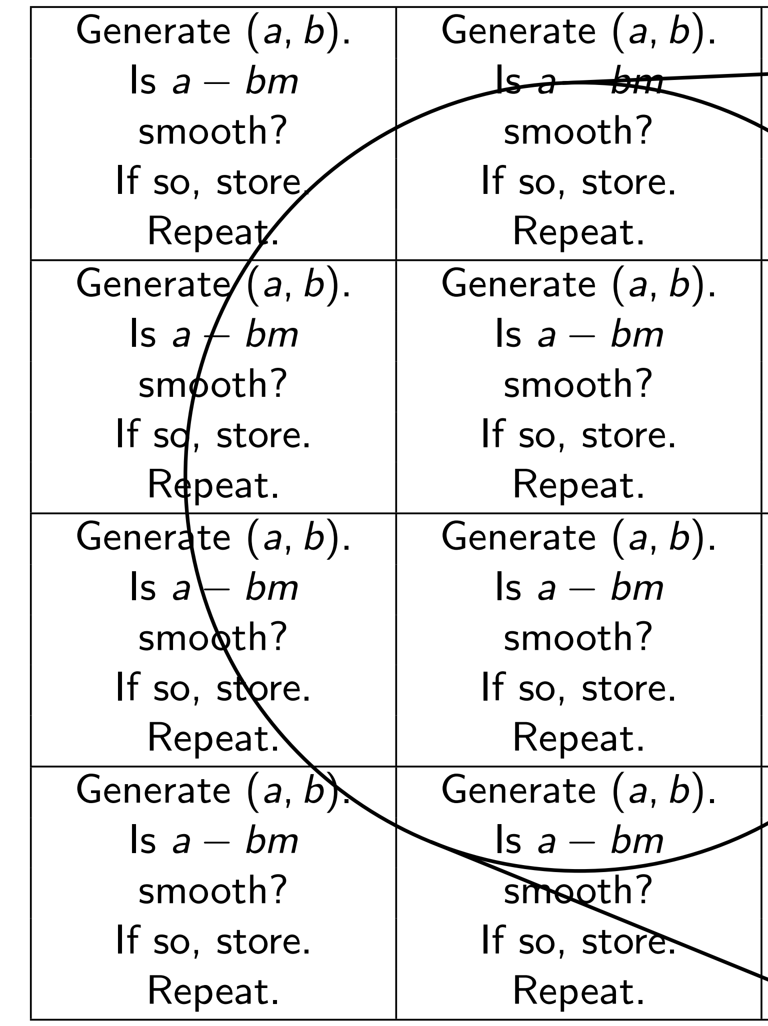

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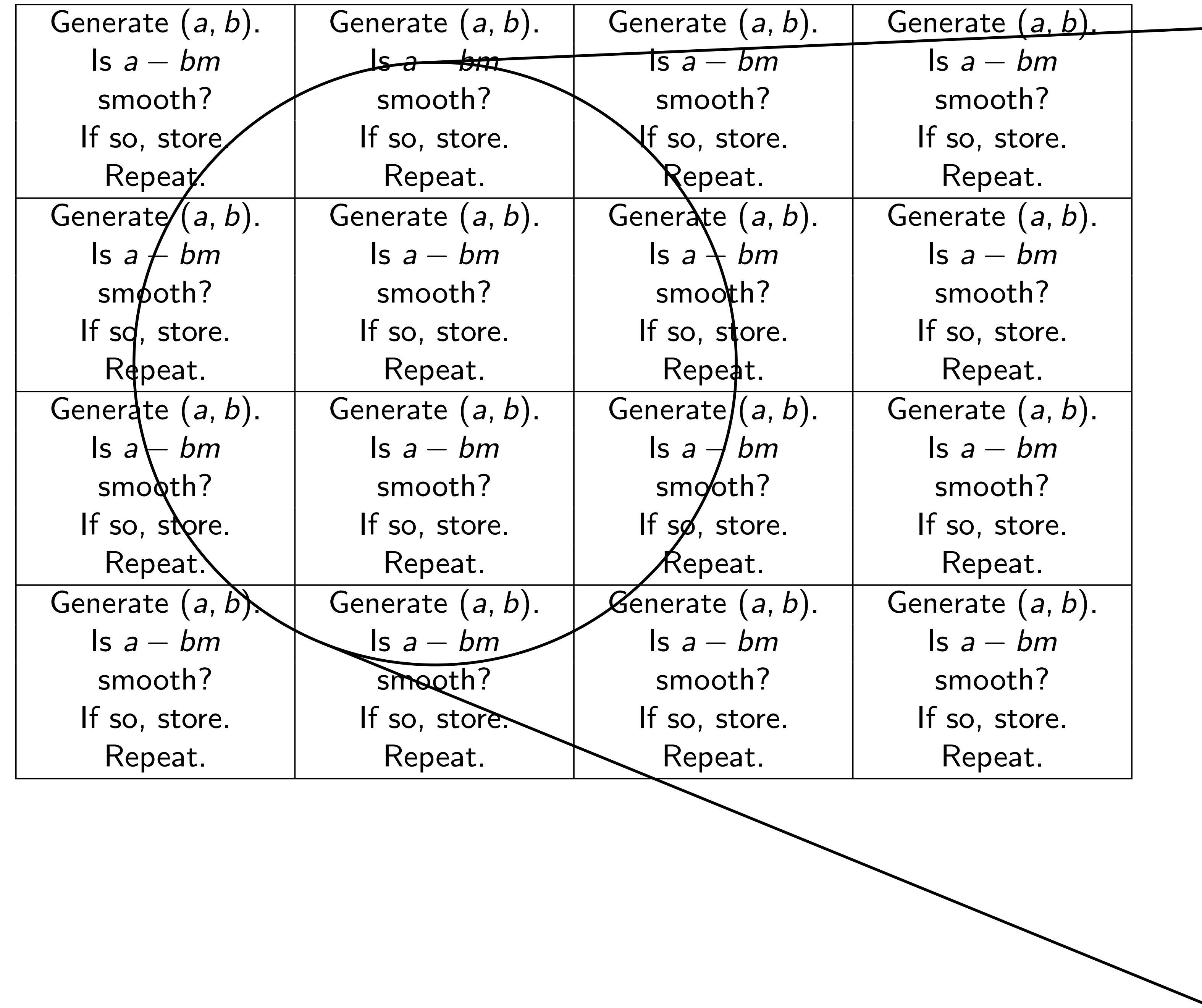

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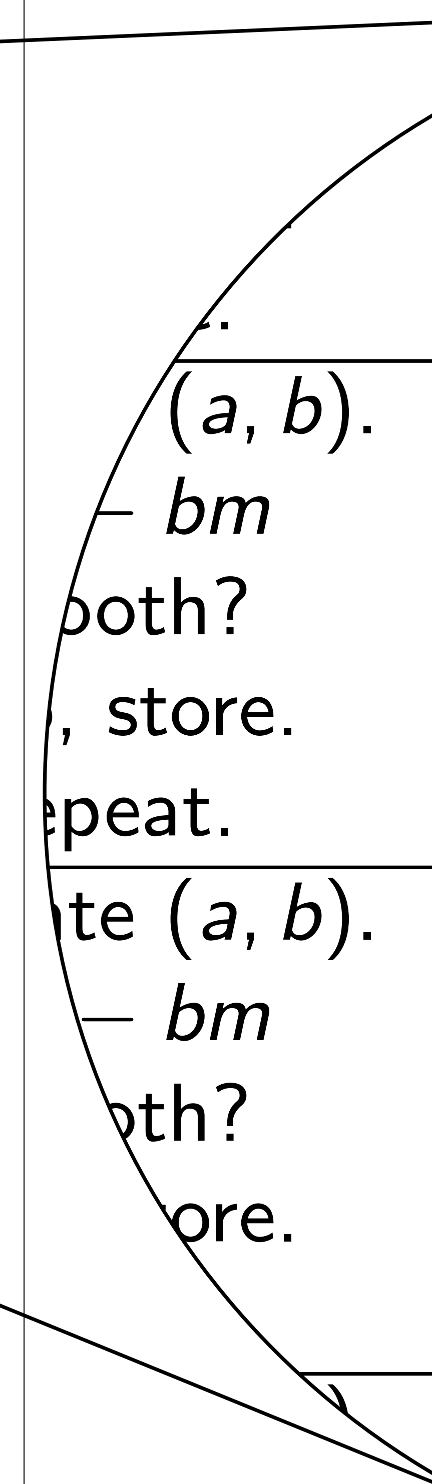

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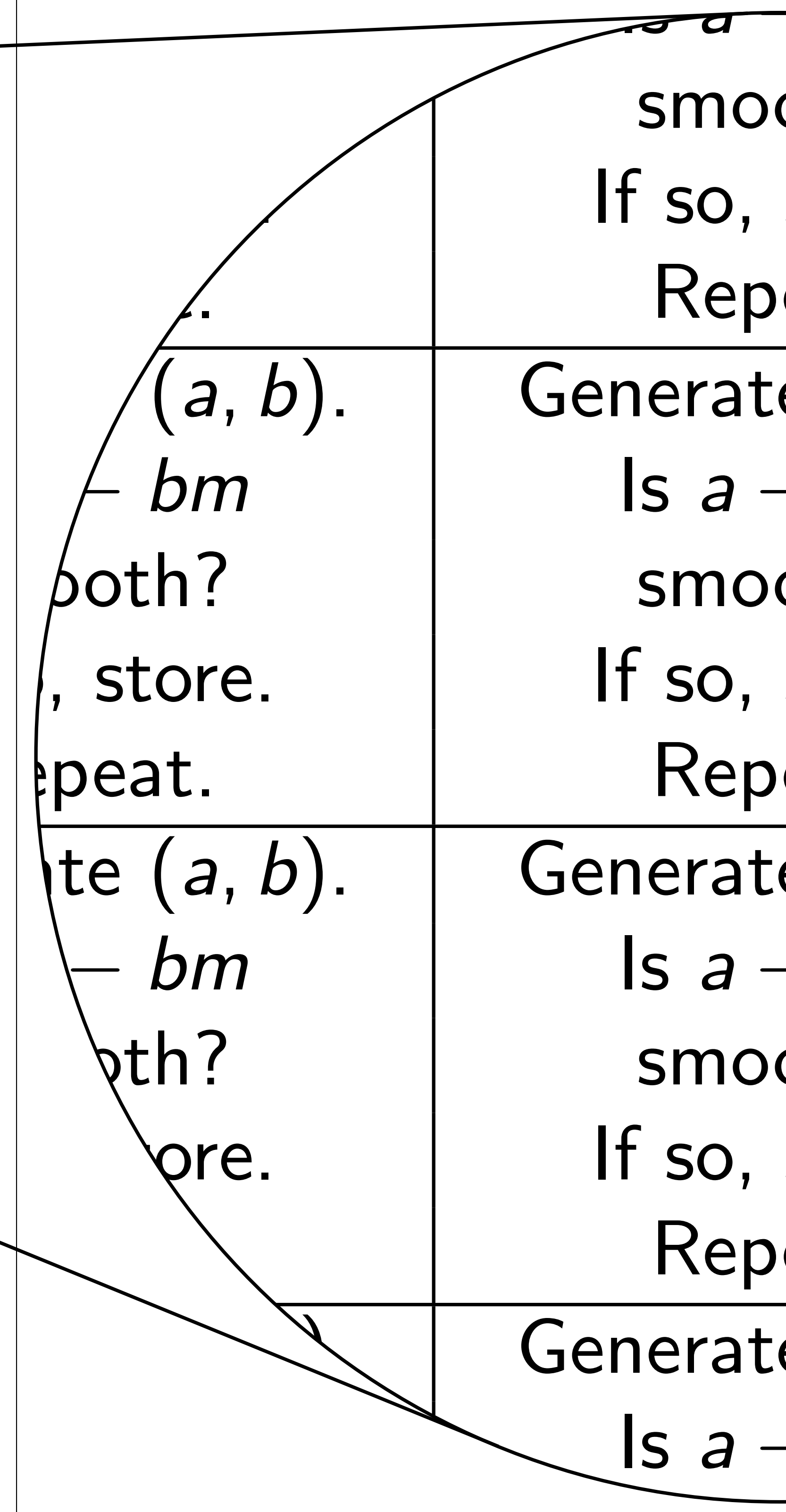
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.



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Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
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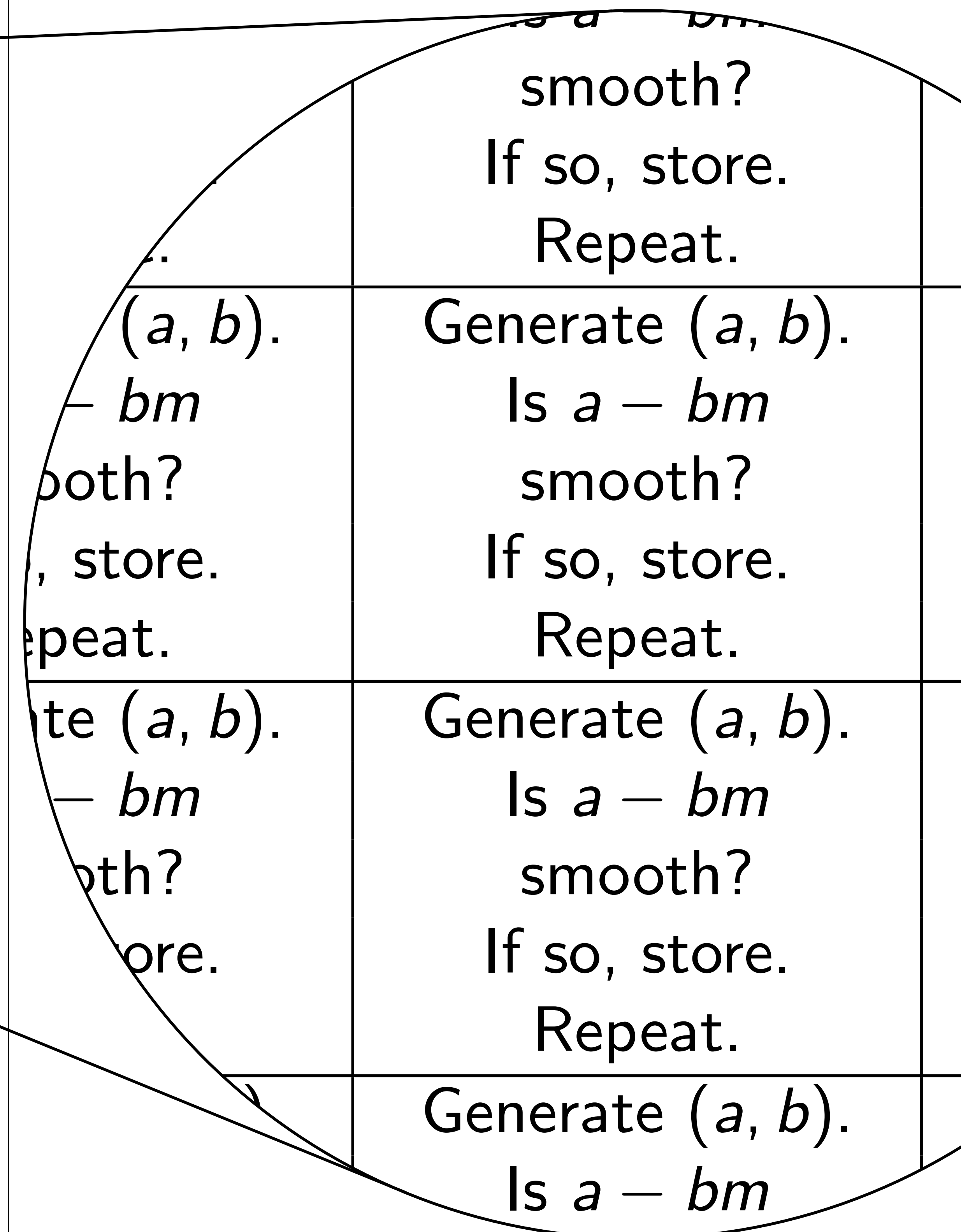
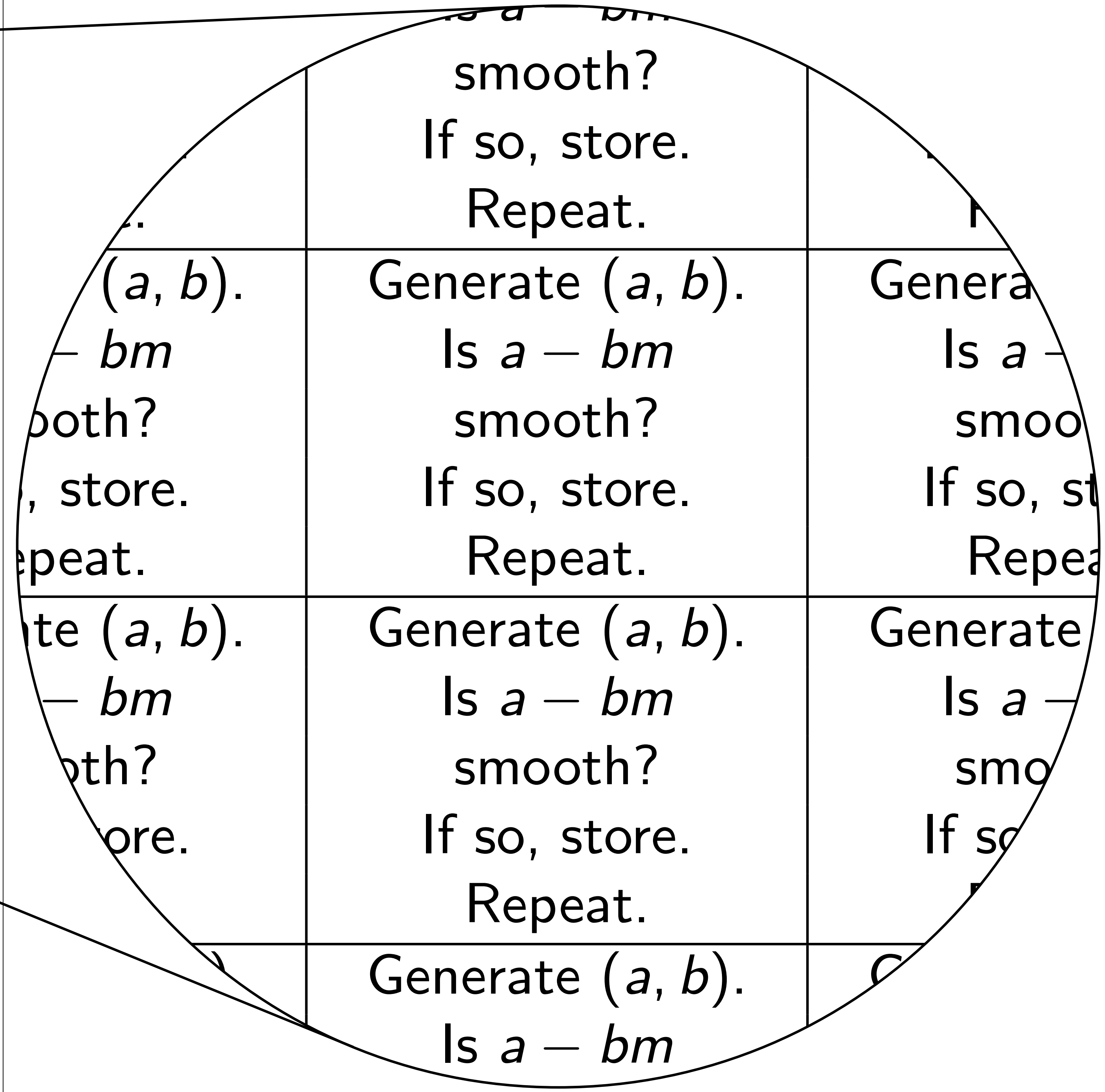
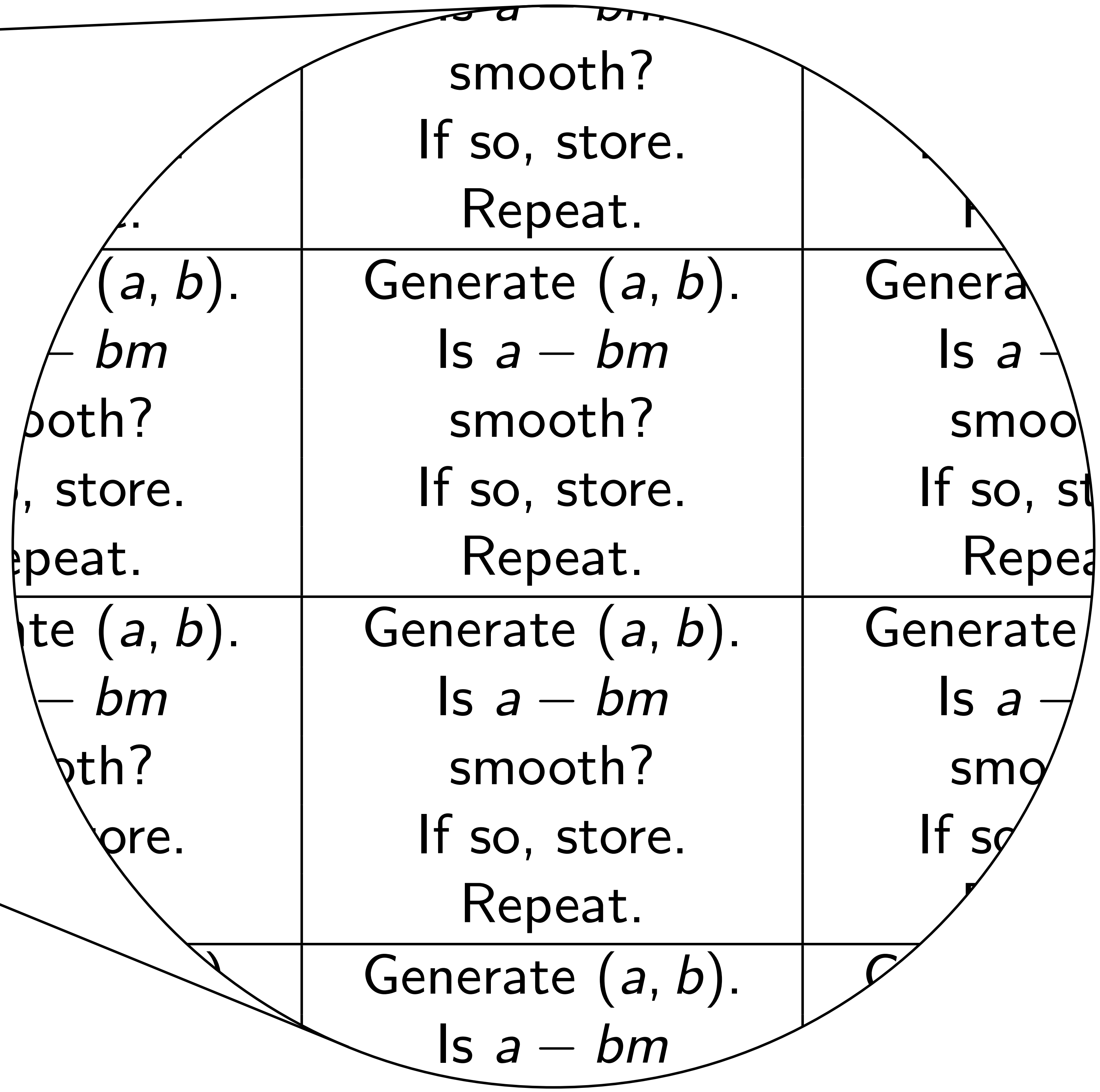
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.



Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.

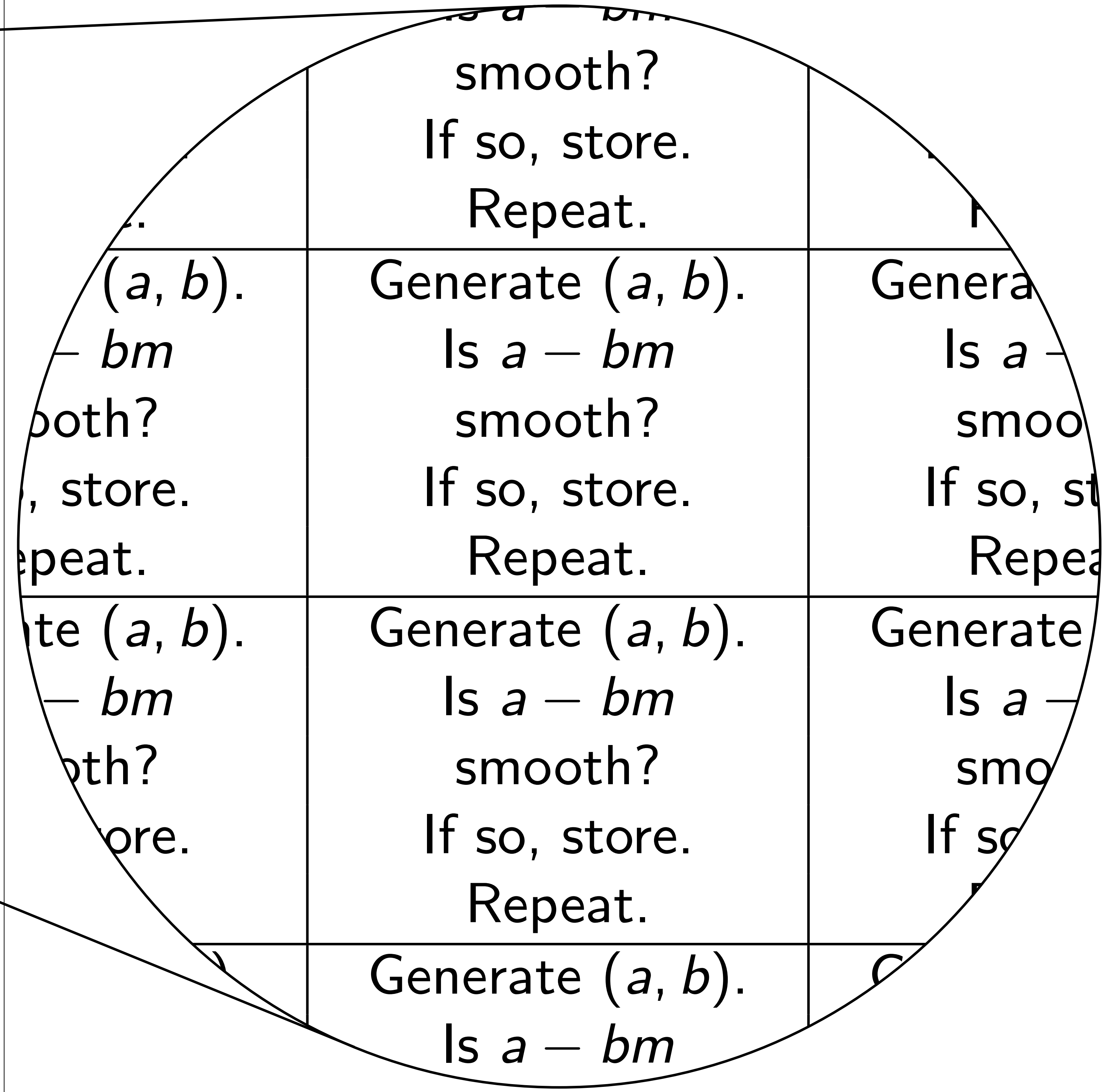


Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.

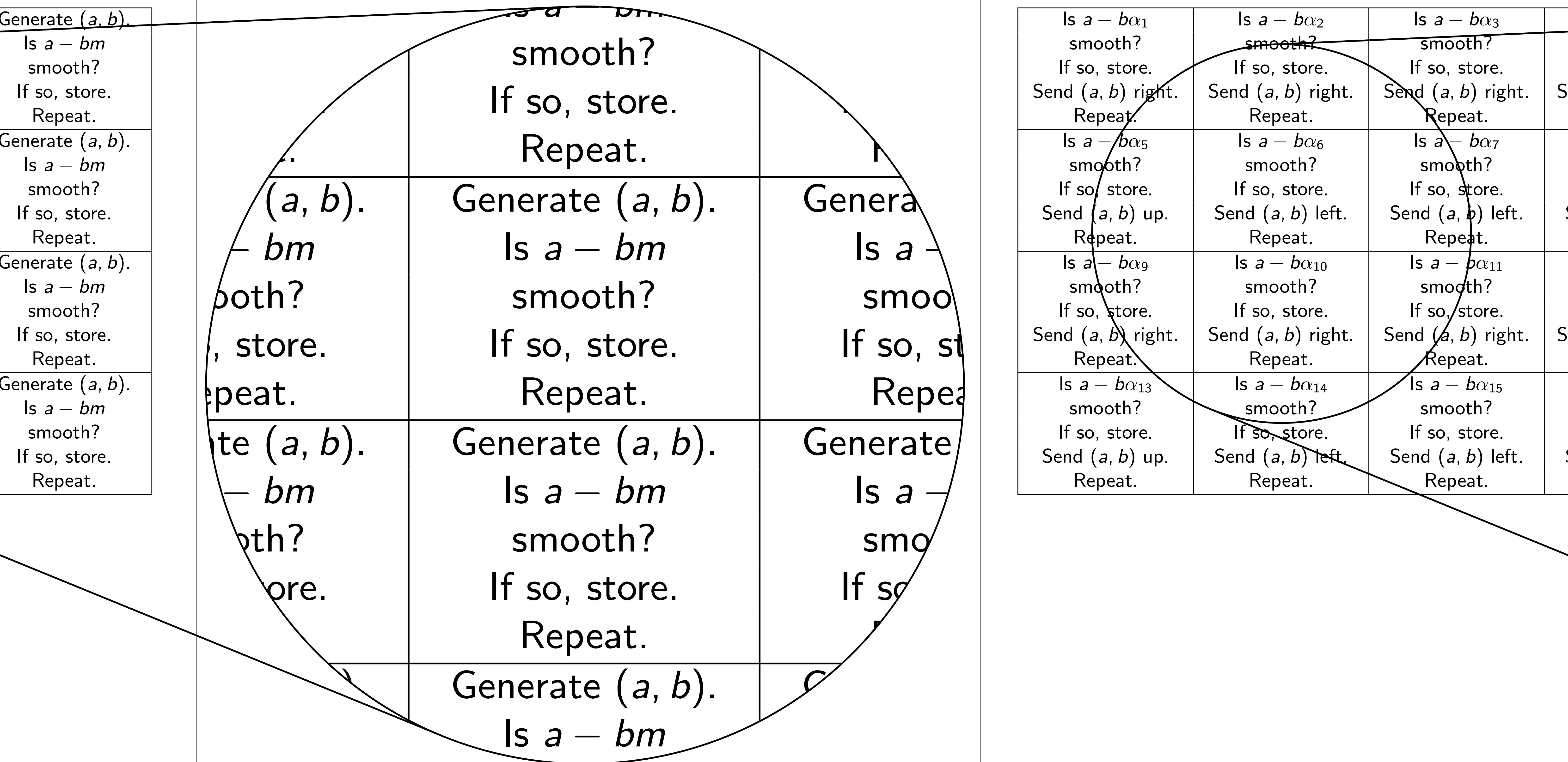


Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.

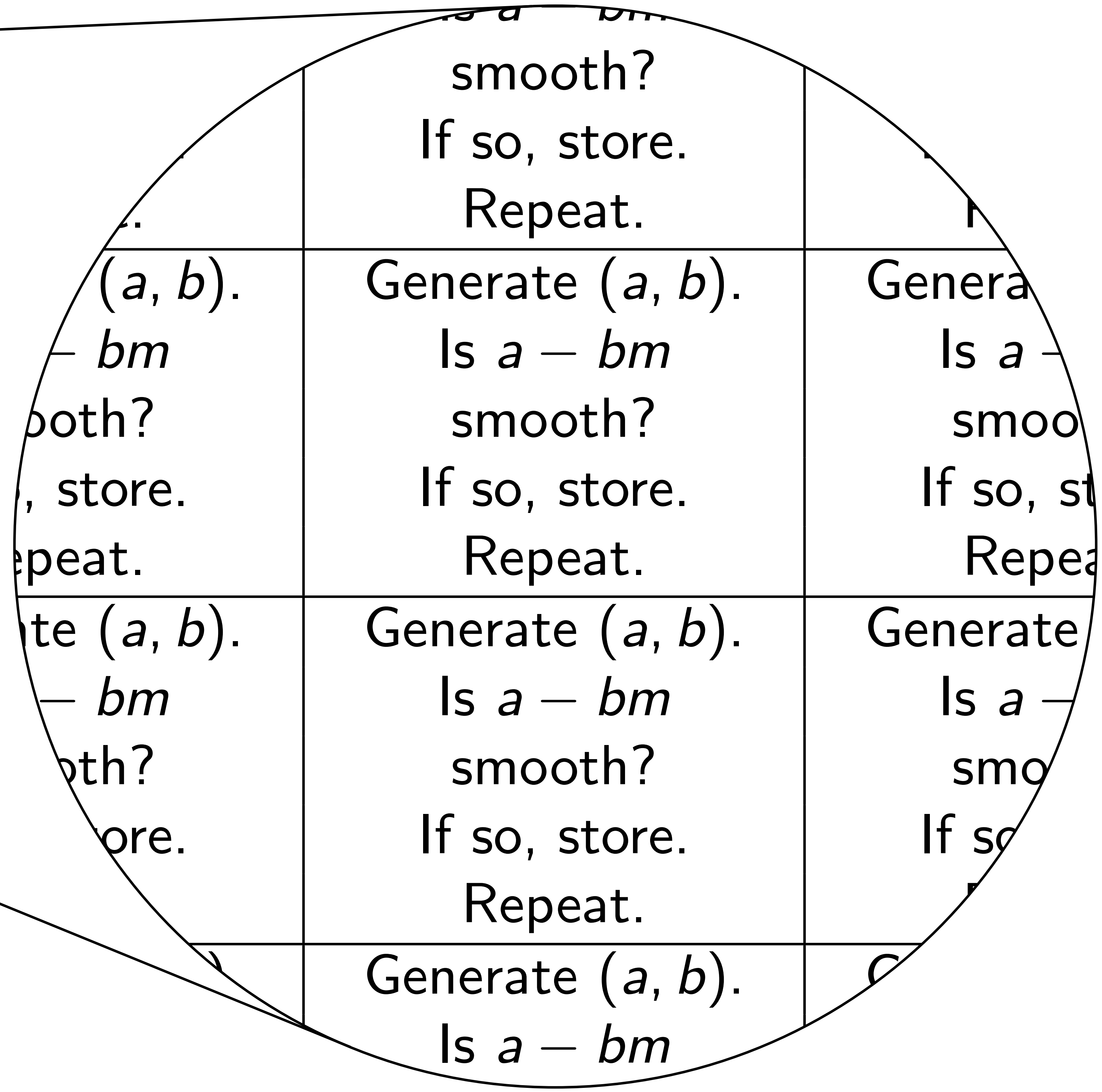
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.



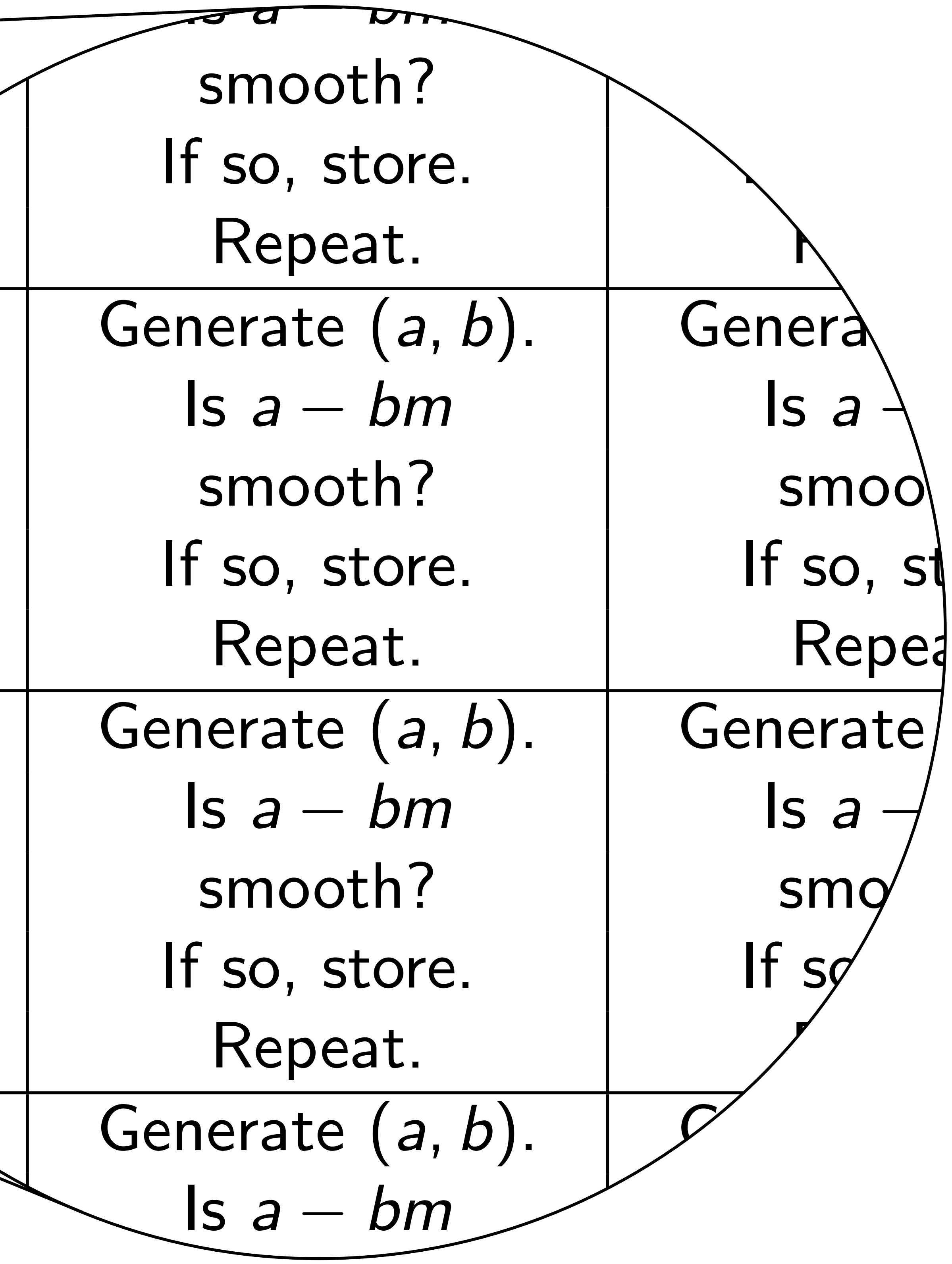
Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.



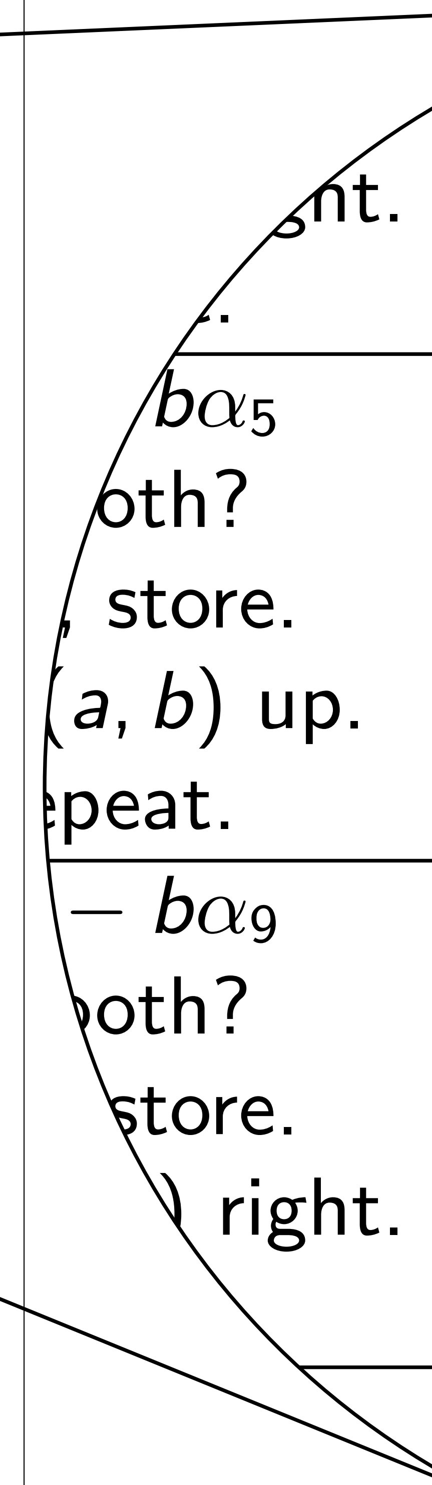
Generate (a, b).
Is $a - bm$
smooth?
If so, store.
Repeat.
Is $a - b\alpha_1$
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is $a - b\alpha_2$
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is $a - b\alpha_3$
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is $a - b\alpha_5$
smooth?
If so, store.
Send (a, b) up.
Repeat.
Is $a - b\alpha_6$
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is $a - b\alpha_7$
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is $a - b\alpha_9$
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is $a - b\alpha_{10}$
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is $a - b\alpha_{11}$
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is $a - b\alpha_{13}$
smooth?
If so, store.
Send (a, b) up.
Repeat.
Is $a - b\alpha_{14}$
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is $a - b\alpha_{15}$
smooth?
If so, store.
Send (a, b) left.
Repeat.

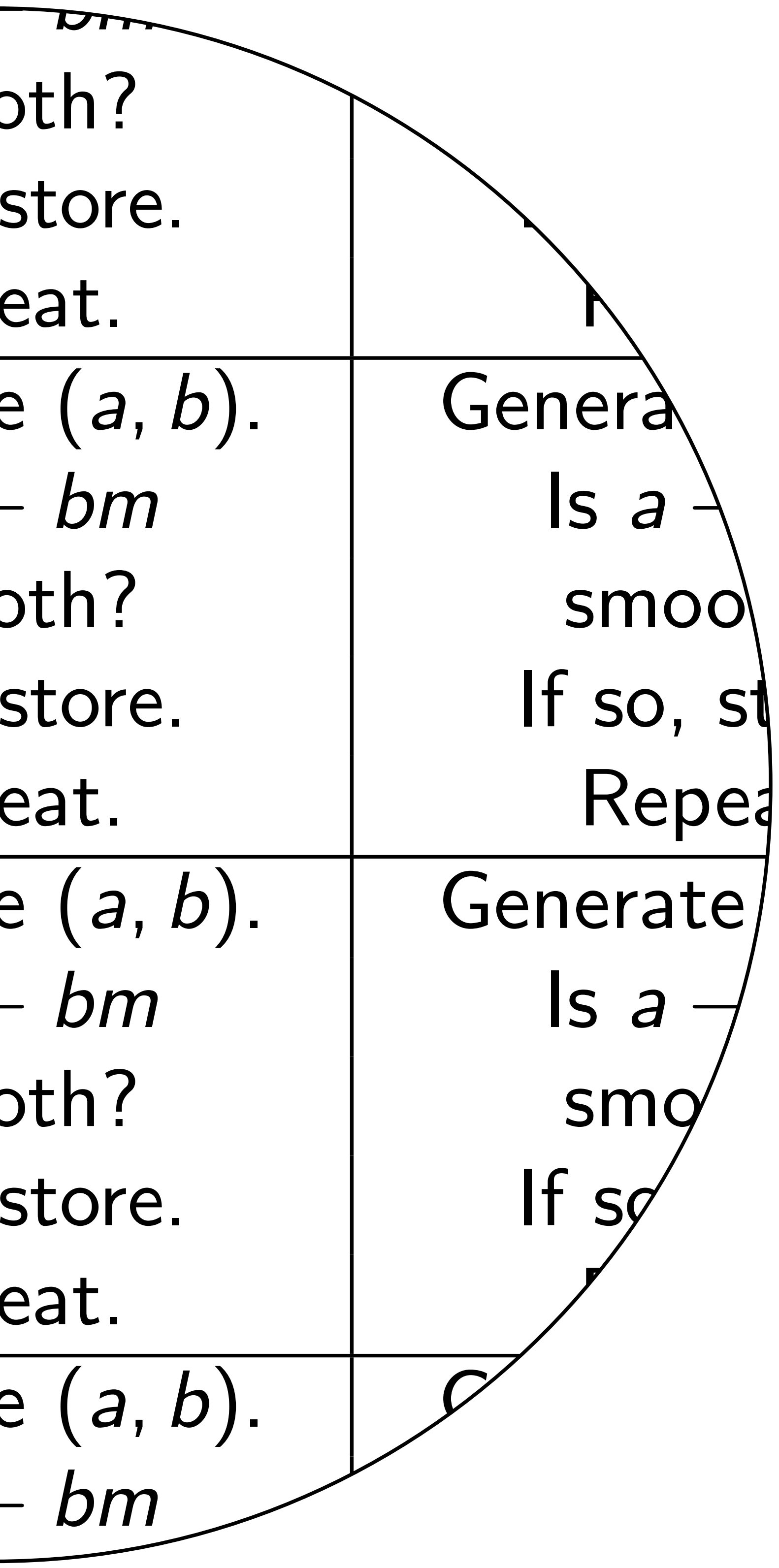


Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) left. Repeat.

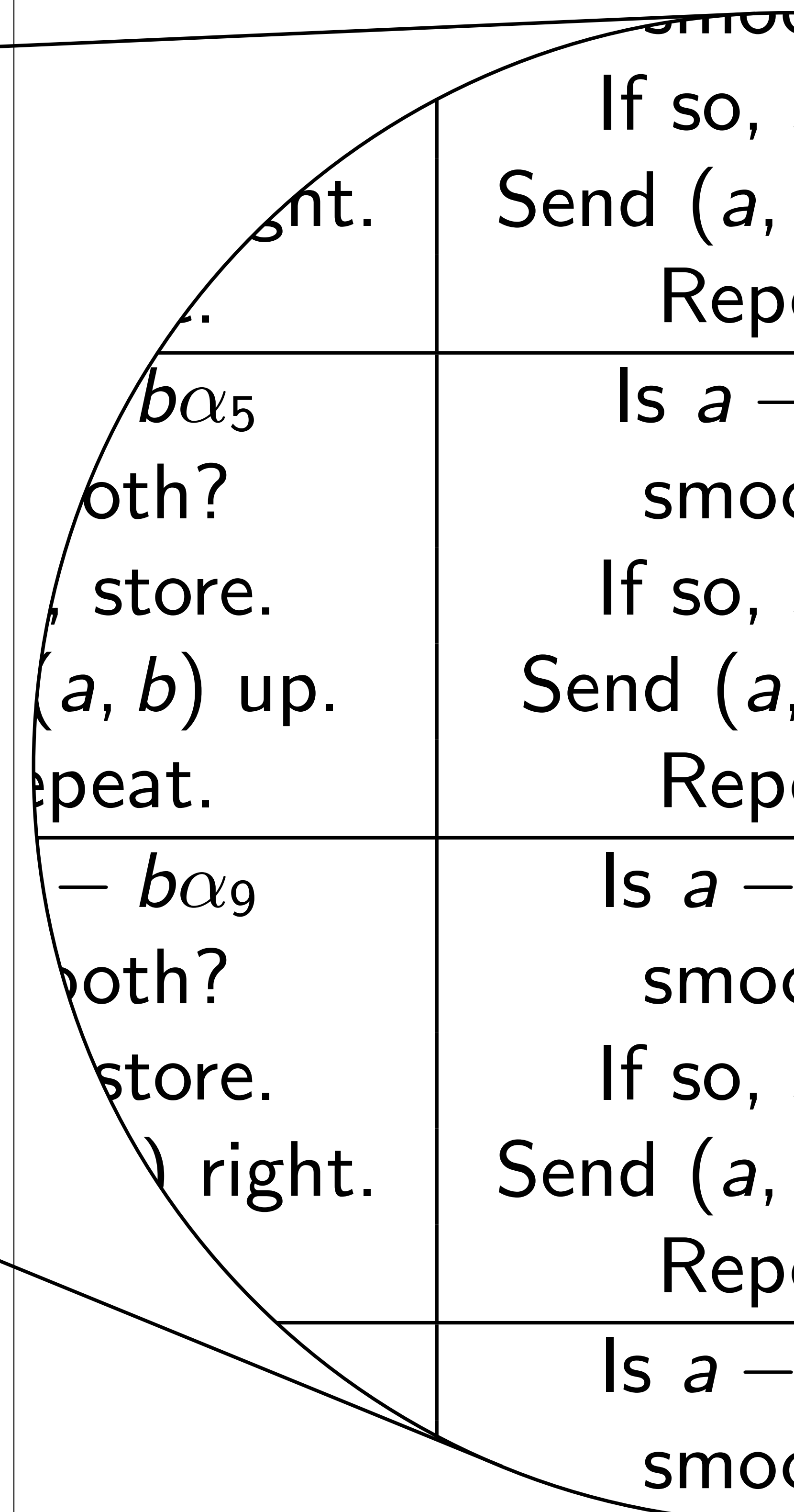


Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) left. Repeat.





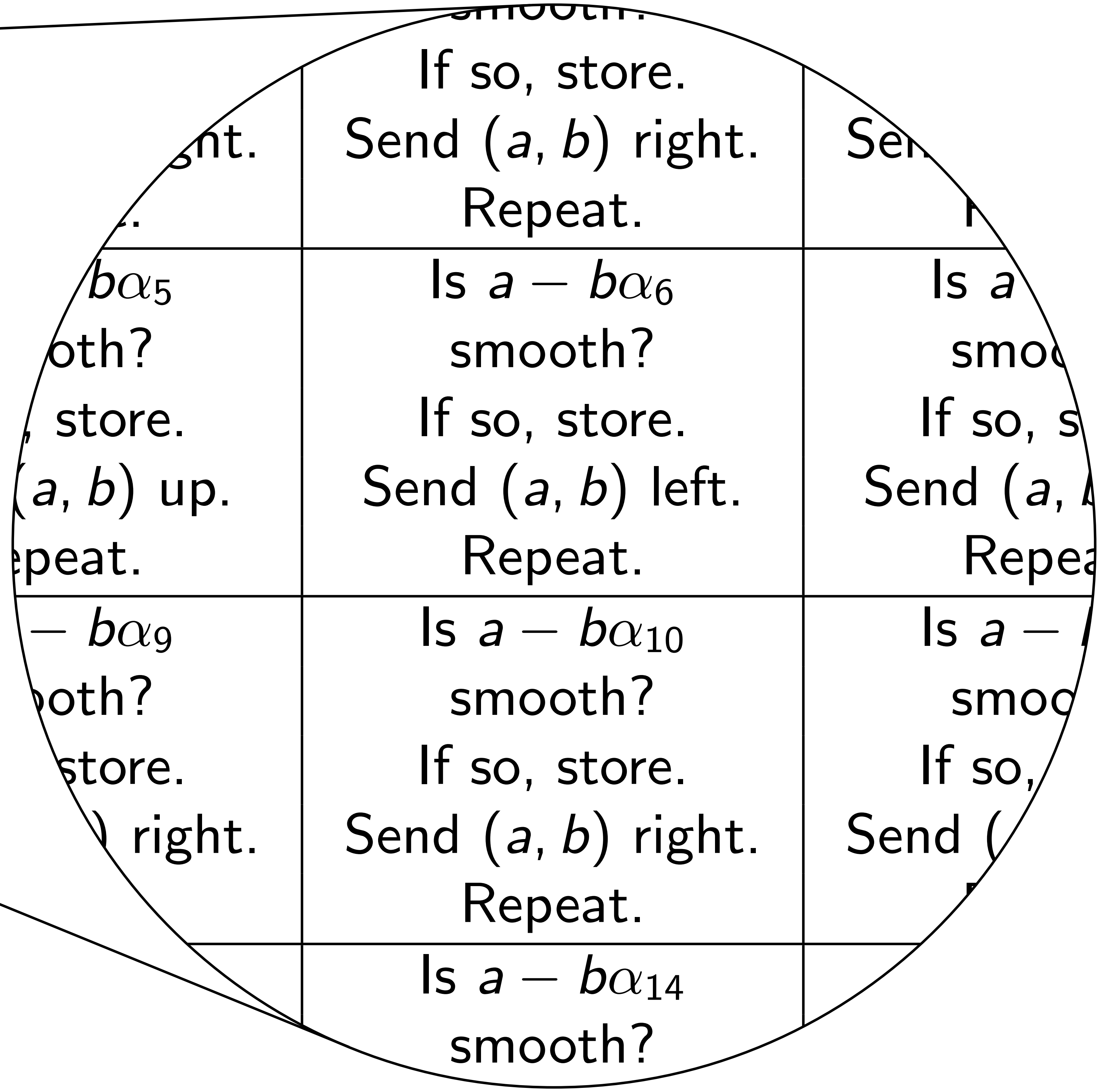
Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) left. Repeat.

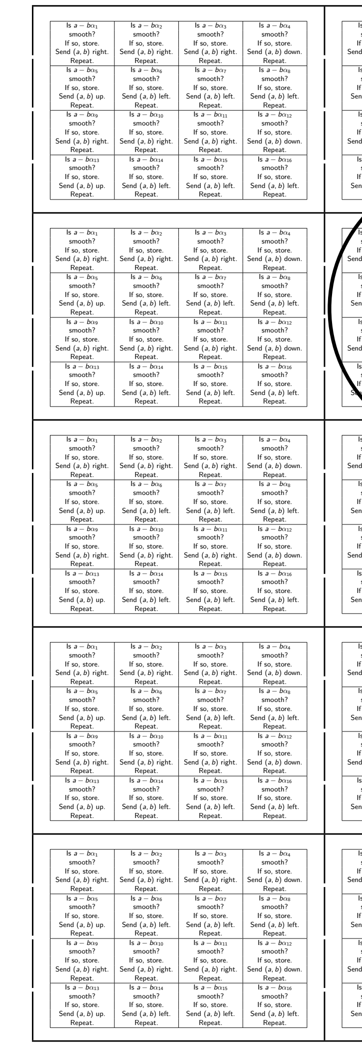


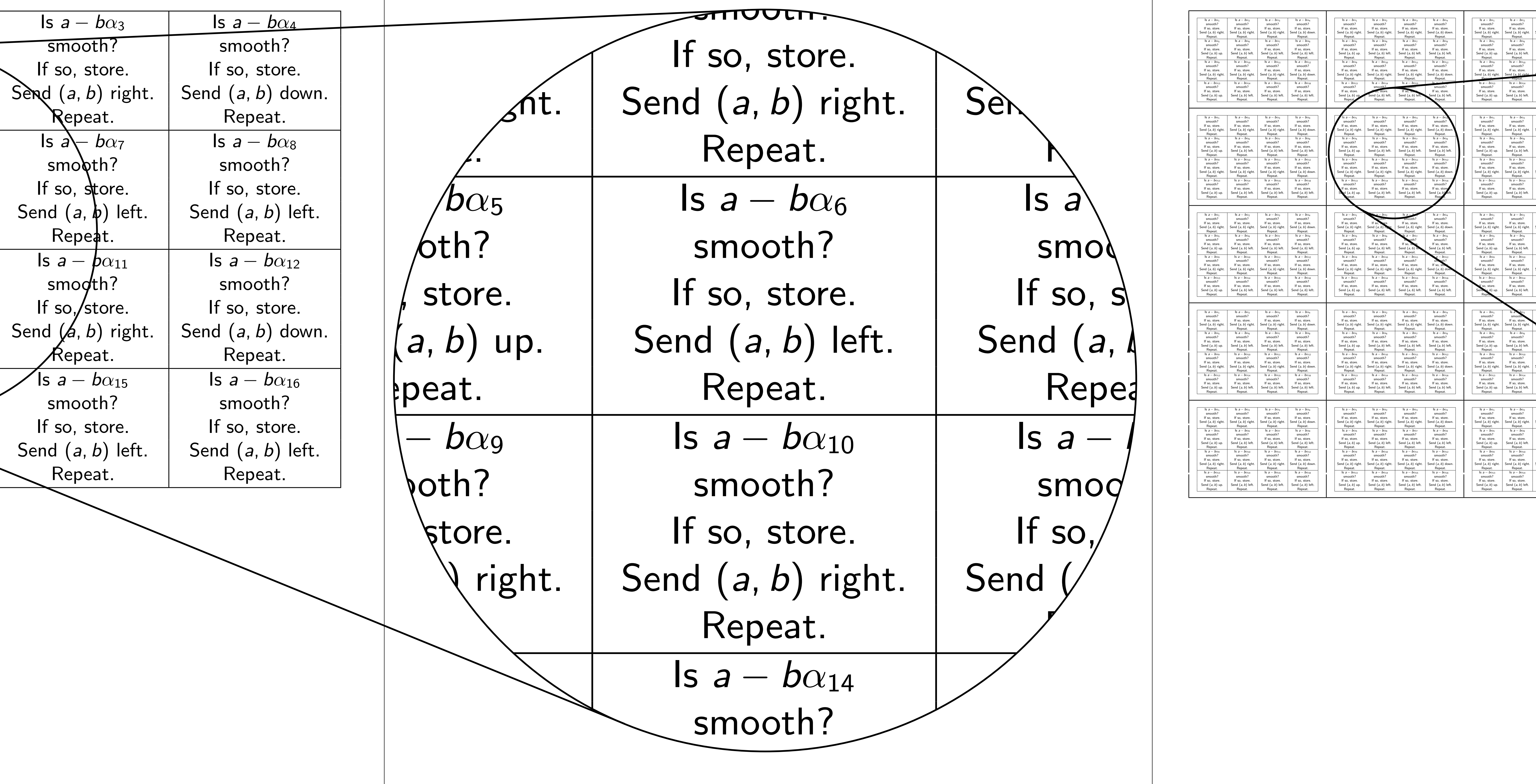
Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) left. Repeat.

Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) left. Repeat.

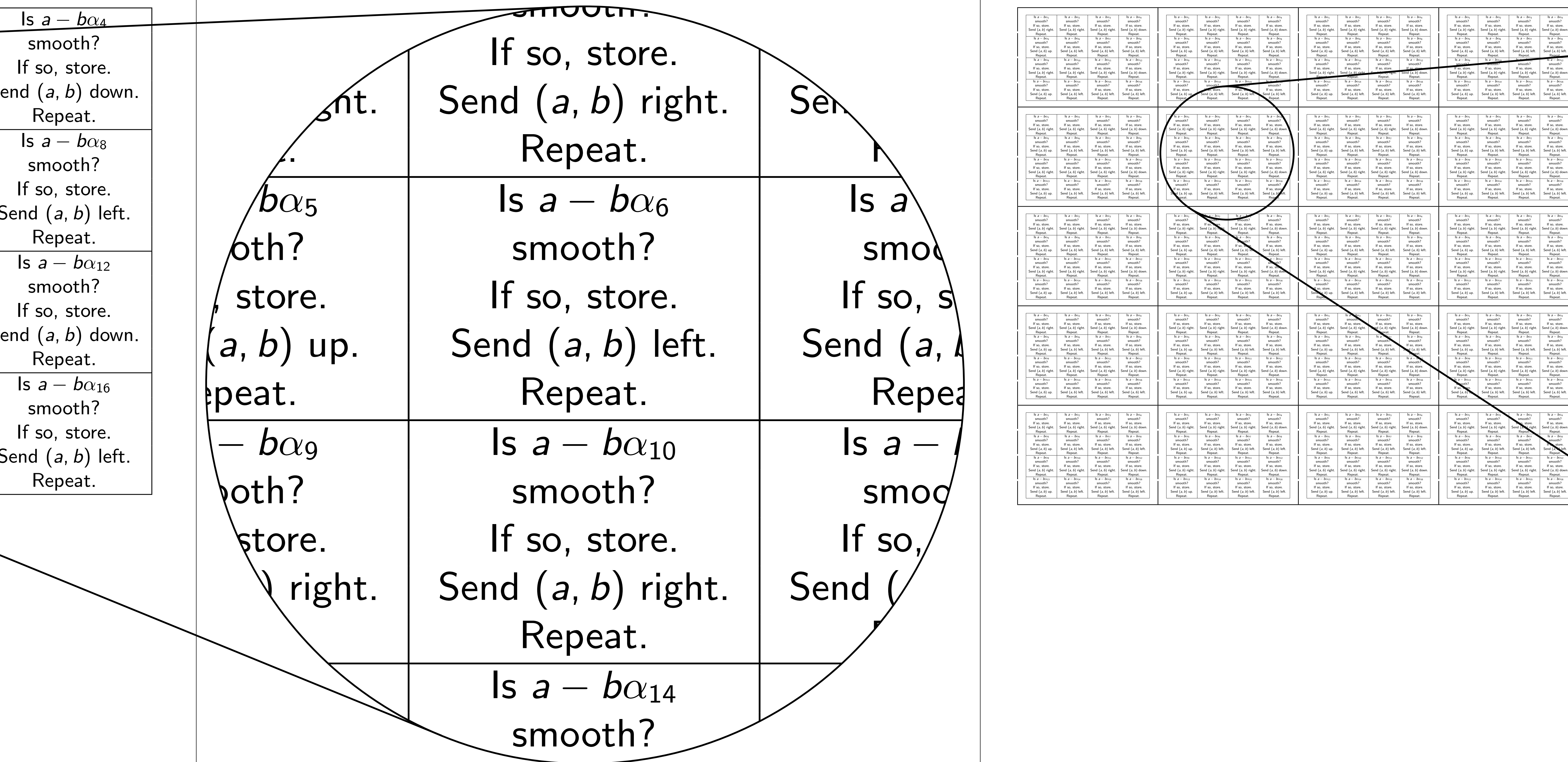
Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) left. Repeat.



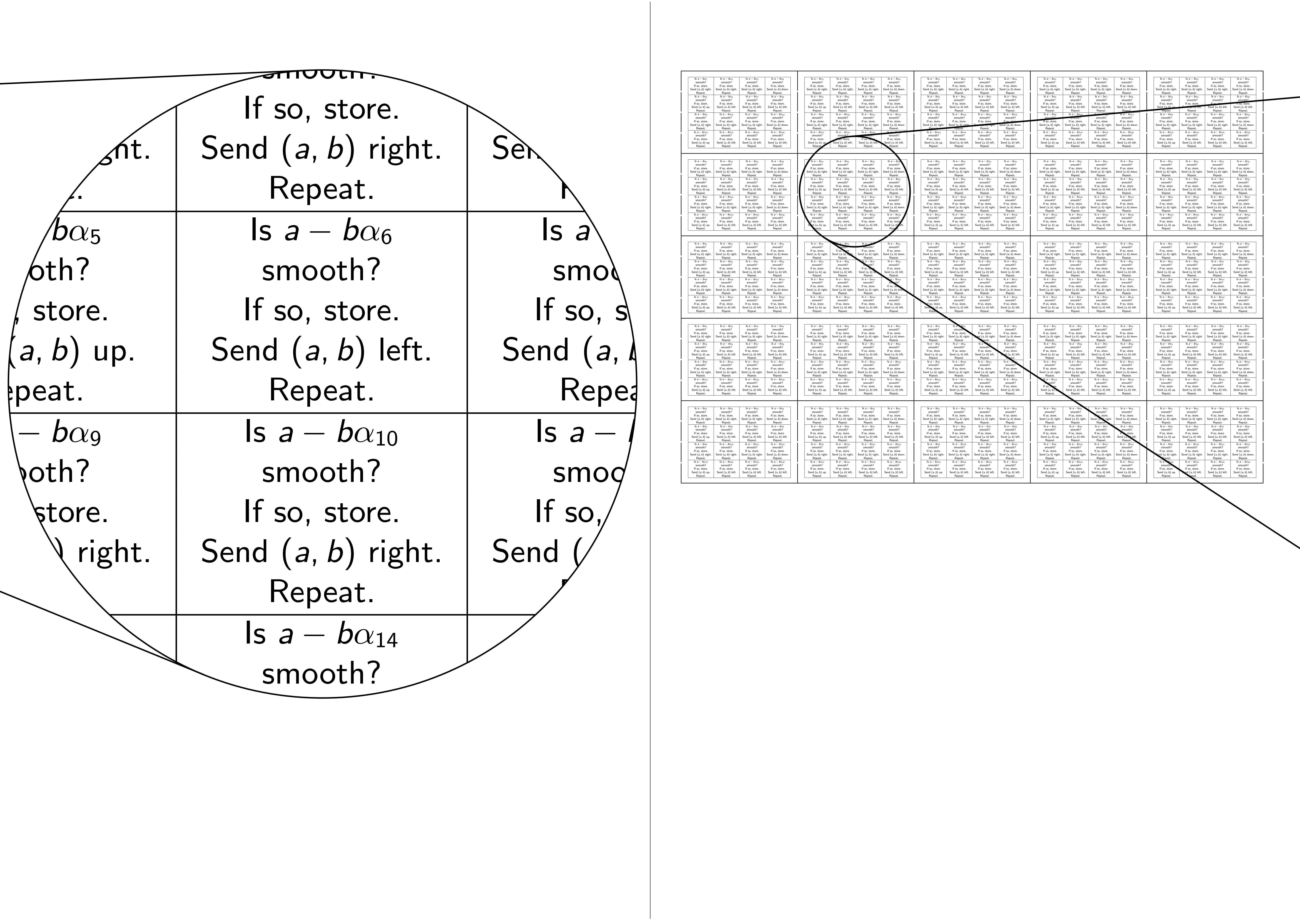


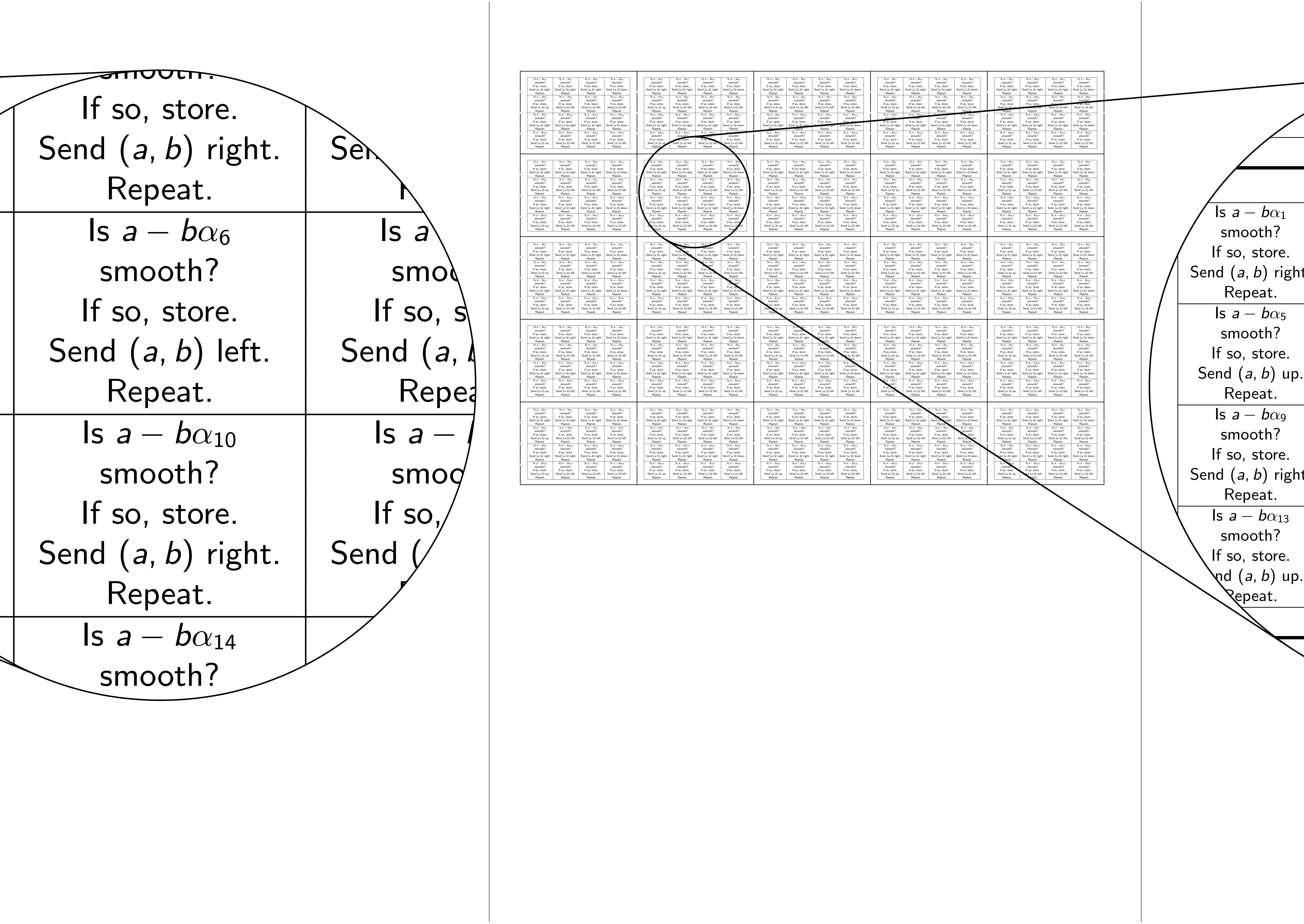


Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) left. Repeat.
If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{14}$ smooth?

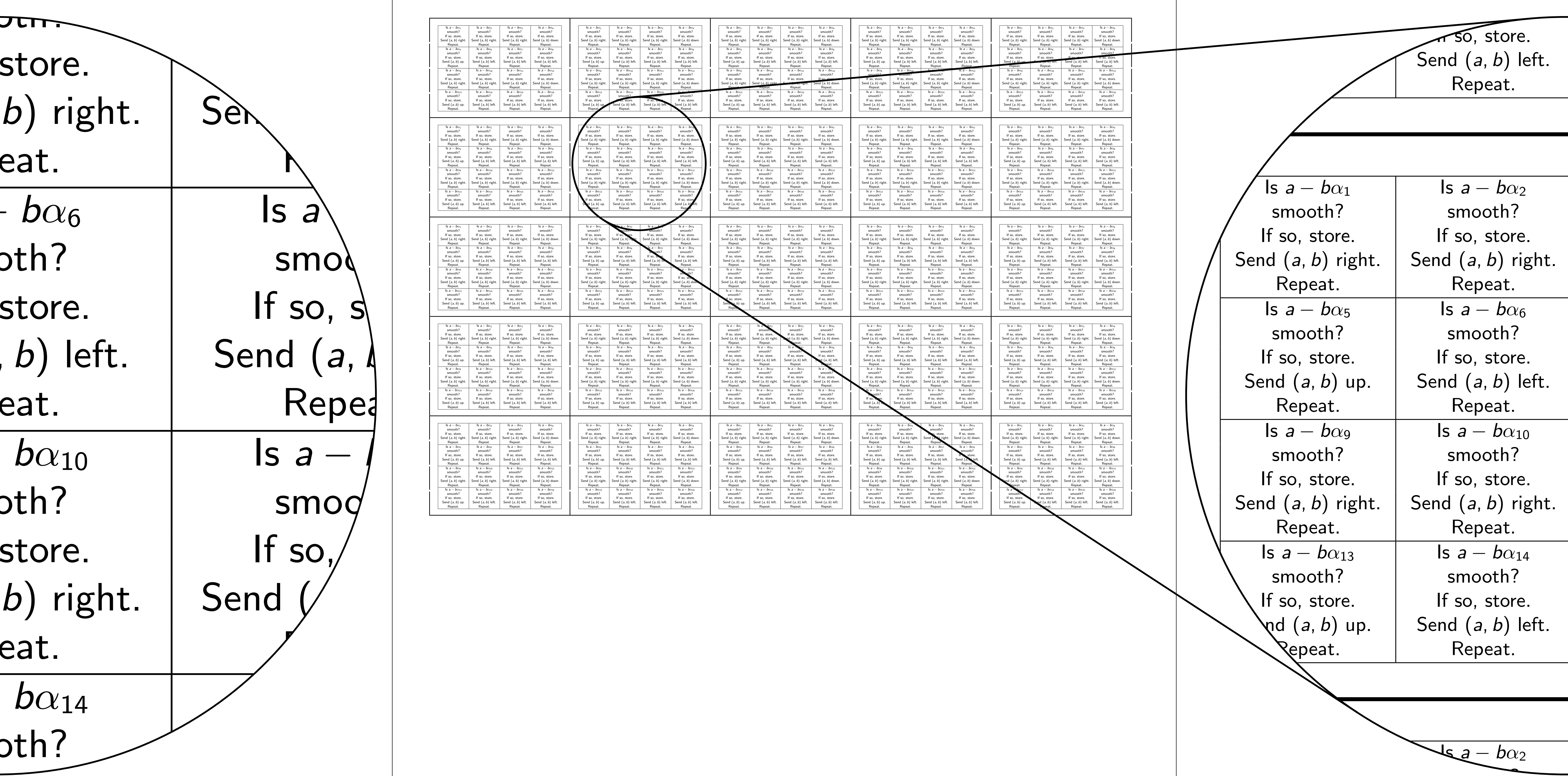


Is a − bα4
smooth?
If so, store.
end (a, b) down.
Repeat.
Is a − bα8
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα12
smooth?
If so, store.
end (a, b) down.
Repeat.
Is a − bα16
smooth?
If so, store.
Send (a, b) left.
Repeat.
If so, store.
Send (a, b) right.
Repeat.
bα5
, store.
(a, b) up.
Is a − bα6
smooth?
If so, store.
Send (a, b) left.
Repeat.
− bα9
store.
) right.
Is a − bα10
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα14
smooth?

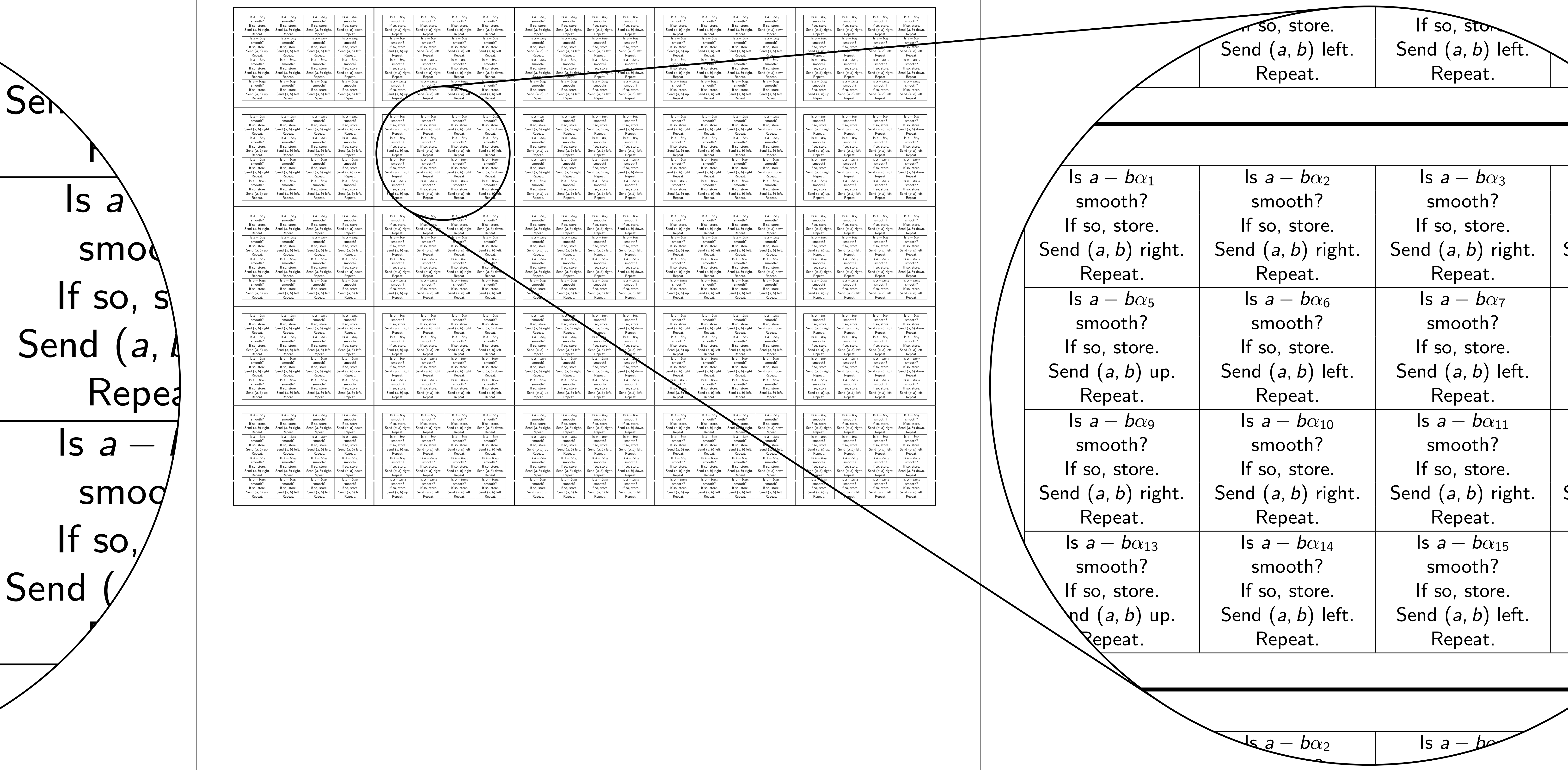

If so, store.
Send (a, b) right.
Repeat.
Is a − bα6
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα10
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα14
smooth?



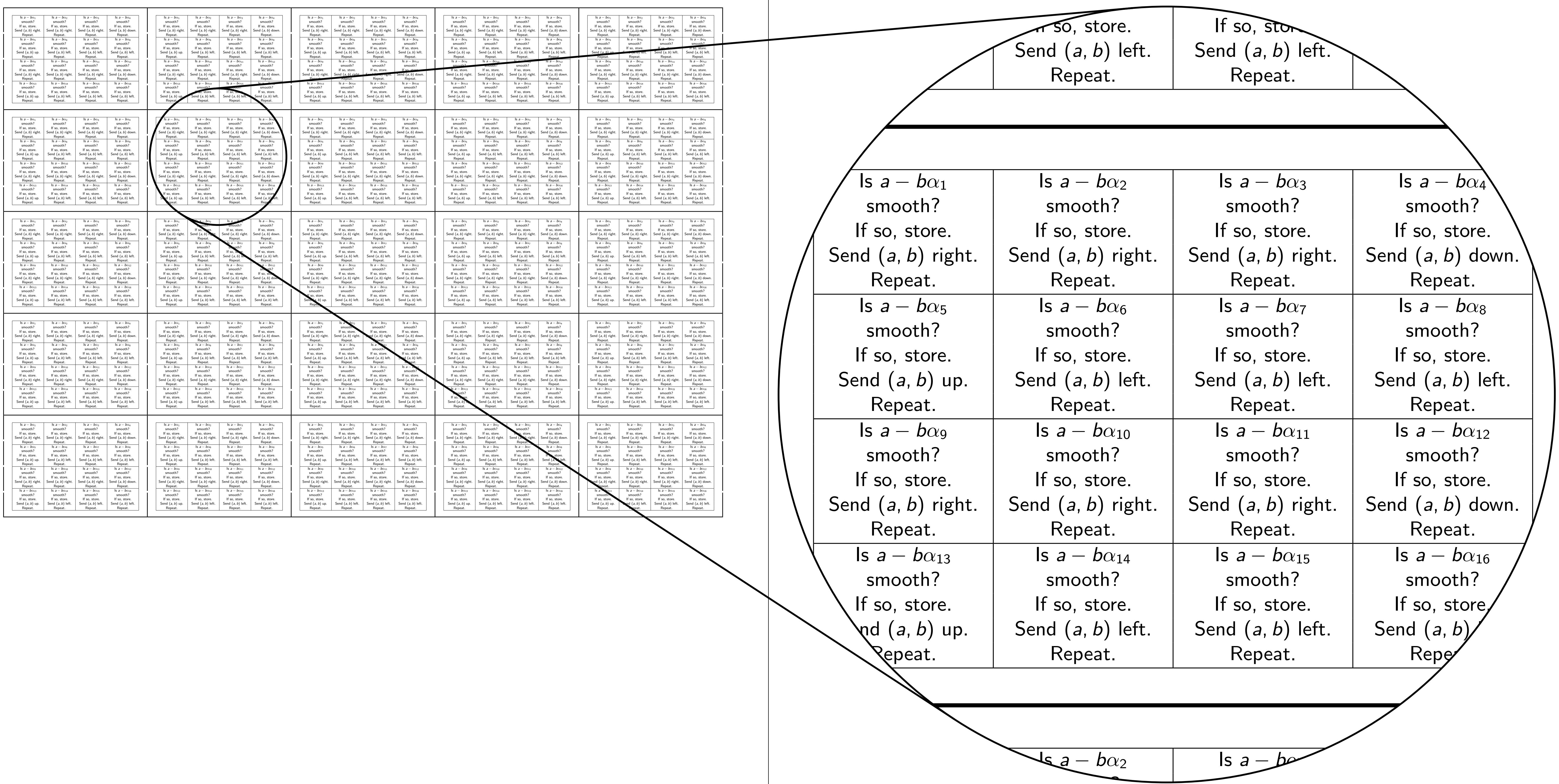
If so, store.
Send (a, b) right.
Repeat.
Is a − bα6
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα10
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα14
smooth?
Is a − bα1
smooth?
If so, store.
Repeat.
Is a − bα5
smooth?
If so, store.
Send (a, b) up.
Repeat.
Is a − bα9
smooth?
If so, store.
Repeat.
Is a − bα13
smooth?
If so, store.



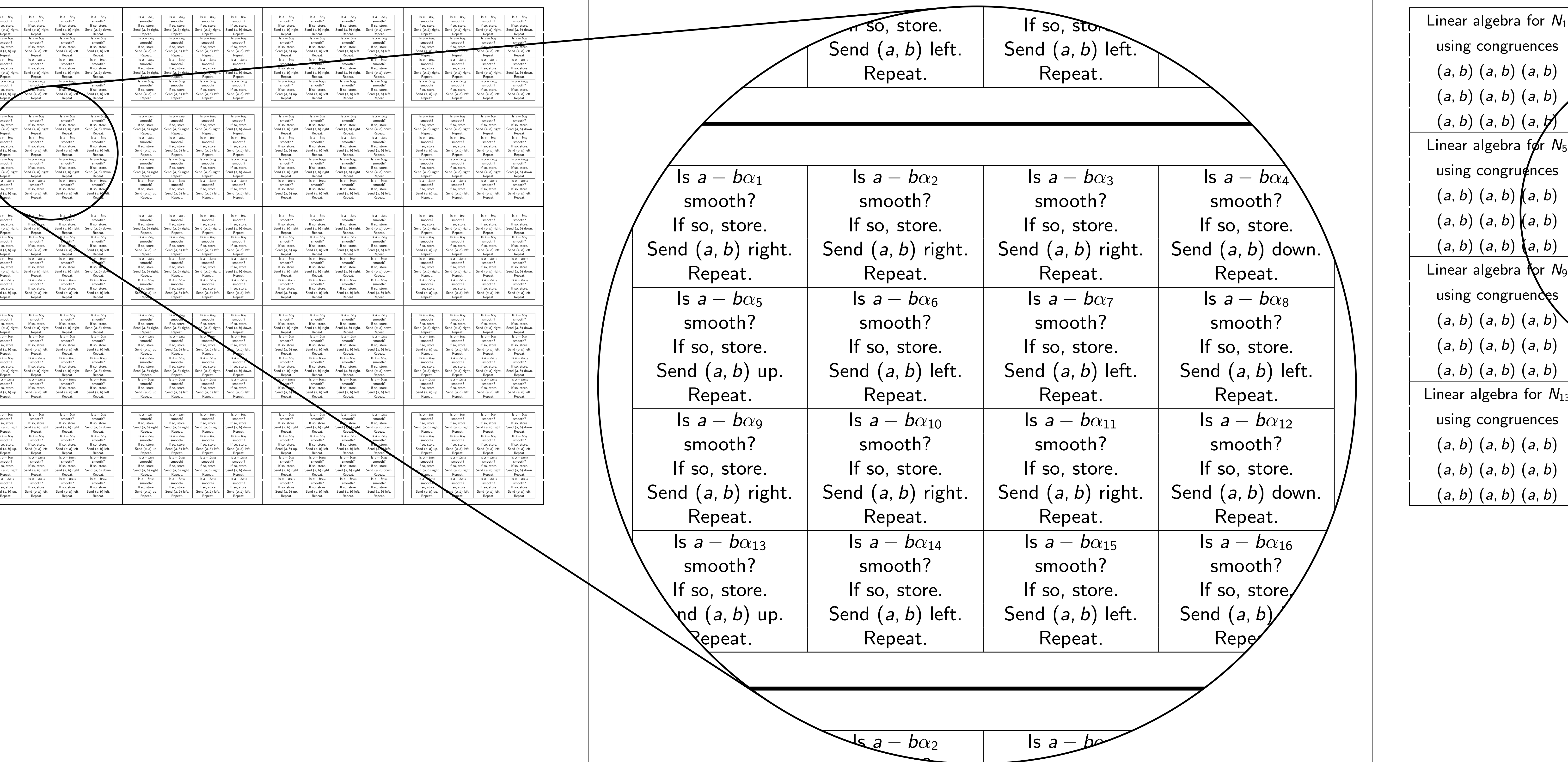
Send (a, b) left.
Repeat.
Is a − bα1
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα2
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα5
smooth?
If so, store.
Send (a, b) up.
Repeat.
Is a − bα6
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα9
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα10
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα13
smooth?
If so, store.
Is a − bα14
smooth?
If so, store.
Send (a, b) left.
Repeat.



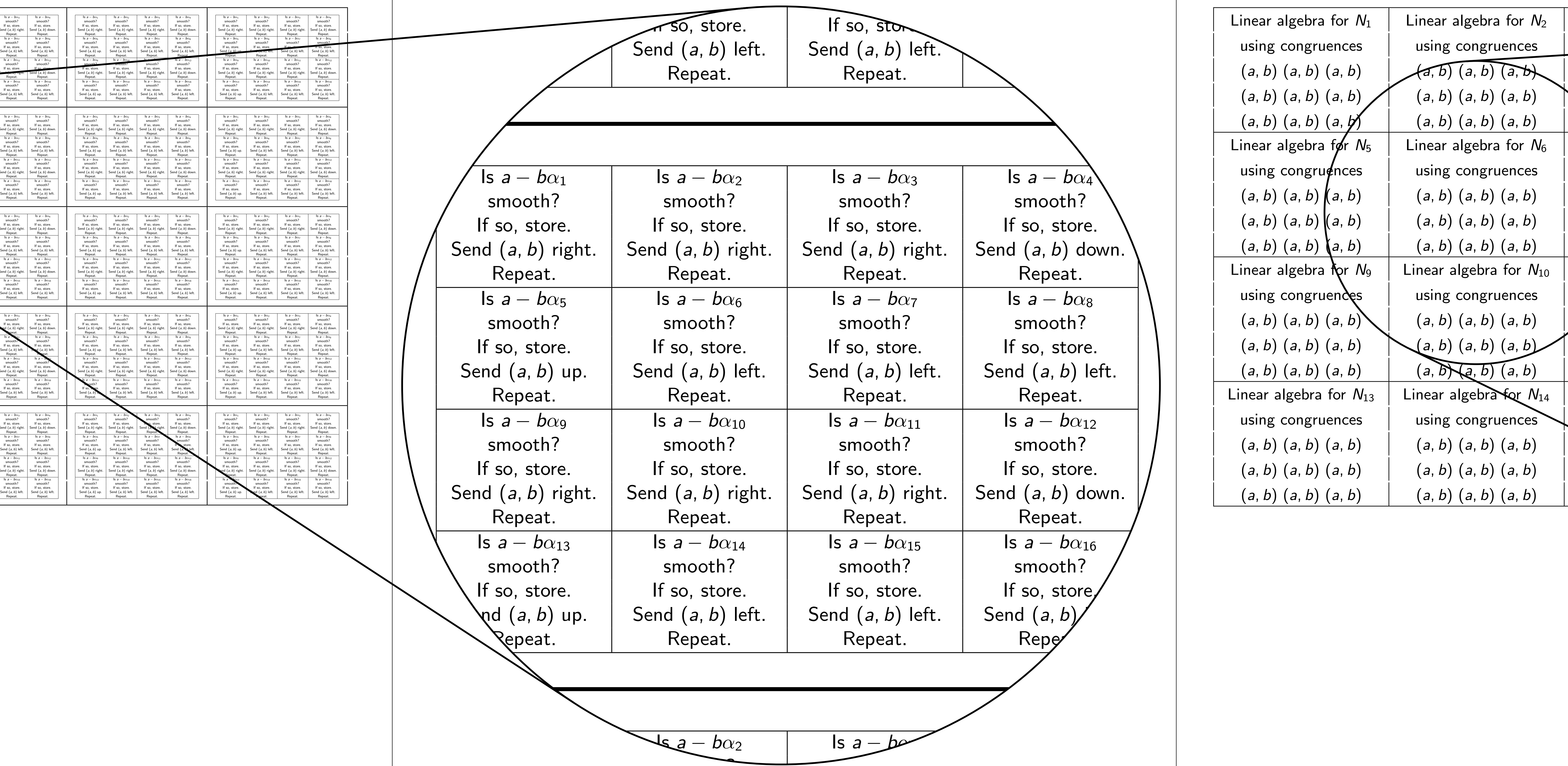
Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.
Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.
Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.



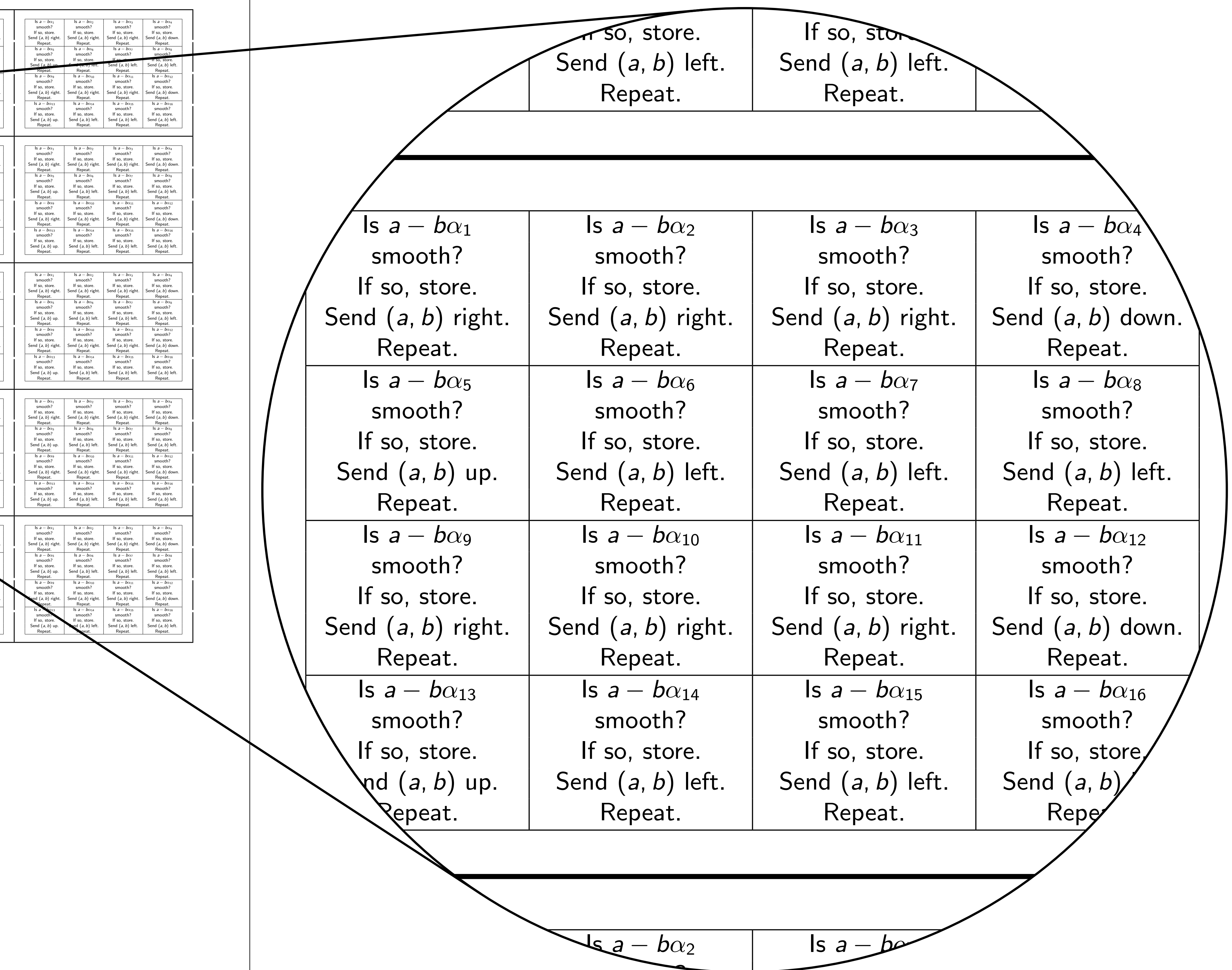
Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.
Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.
Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_{16}$ smooth? If so, store.



If so, store.
Send (a, b) left.
Repeat.
Is a − bα1
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα2
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα3
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα4
smooth?
If so, store.
Send (a, b) down.
Repeat.
Is a − bα5
smooth?
If so, store.
Send (a, b) up.
Repeat.
Is a − bα6
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα7
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα8
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα9
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα10
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα11
smooth?
If so, store.
Send (a, b) right.
Repeat.
Is a − bα12
smooth?
If so, store.
Send (a, b) down.
Repeat.
Is a − bα13
smooth?
If so, store.
Is a − bα14
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα15
smooth?
If so, store.
Send (a, b) left.
Repeat.
Is a − bα16
smooth?
If so, store.
Is a − bα2
Linear algebra for N1
using congruences
(a, b) (a, b) (a, b)
Linear algebra for N5
using congruences
Linear algebra for N9
using congruences
using congruences



Send (a, b) left.
Repeat.
If so, store.
Send (a, b) left.
Repeat.
Is a − bα1 smooth? If so, store. Send (a, b) right. Repeat.
Is a − bα2 smooth? If so, store. Send (a, b) right. Repeat.
Is a − bα3 smooth? If so, store. Send (a, b) right. Repeat.
Is a − bα4 smooth? If so, store. Send (a, b) down. Repeat.
Is a − bα5 smooth? If so, store. Send (a, b) up. Repeat.
Is a − bα6 smooth? If so, store. Send (a, b) left. Repeat.
Is a − bα7 smooth? If so, store. Send (a, b) left. Repeat.
Is a − bα8 smooth? If so, store. Send (a, b) left. Repeat.
Is a − bα9 smooth? If so, store. Send (a, b) right. Repeat.
Is a − bα10 smooth? If so, store. Send (a, b) right. Repeat.
Is a − bα11 smooth? If so, store. Send (a, b) right. Repeat.
Is a − bα12 smooth? If so, store. Send (a, b) down. Repeat.
Is a − bα13 smooth? If so, store.
Is a − bα14 smooth? If so, store. Send (a, b) left. Repeat.
Is a − bα15 smooth? If so, store. Send (a, b) left. Repeat.
Is a − bα16 smooth?
Is a − bα2
Linear algebra for N1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N10 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N13 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N14 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)



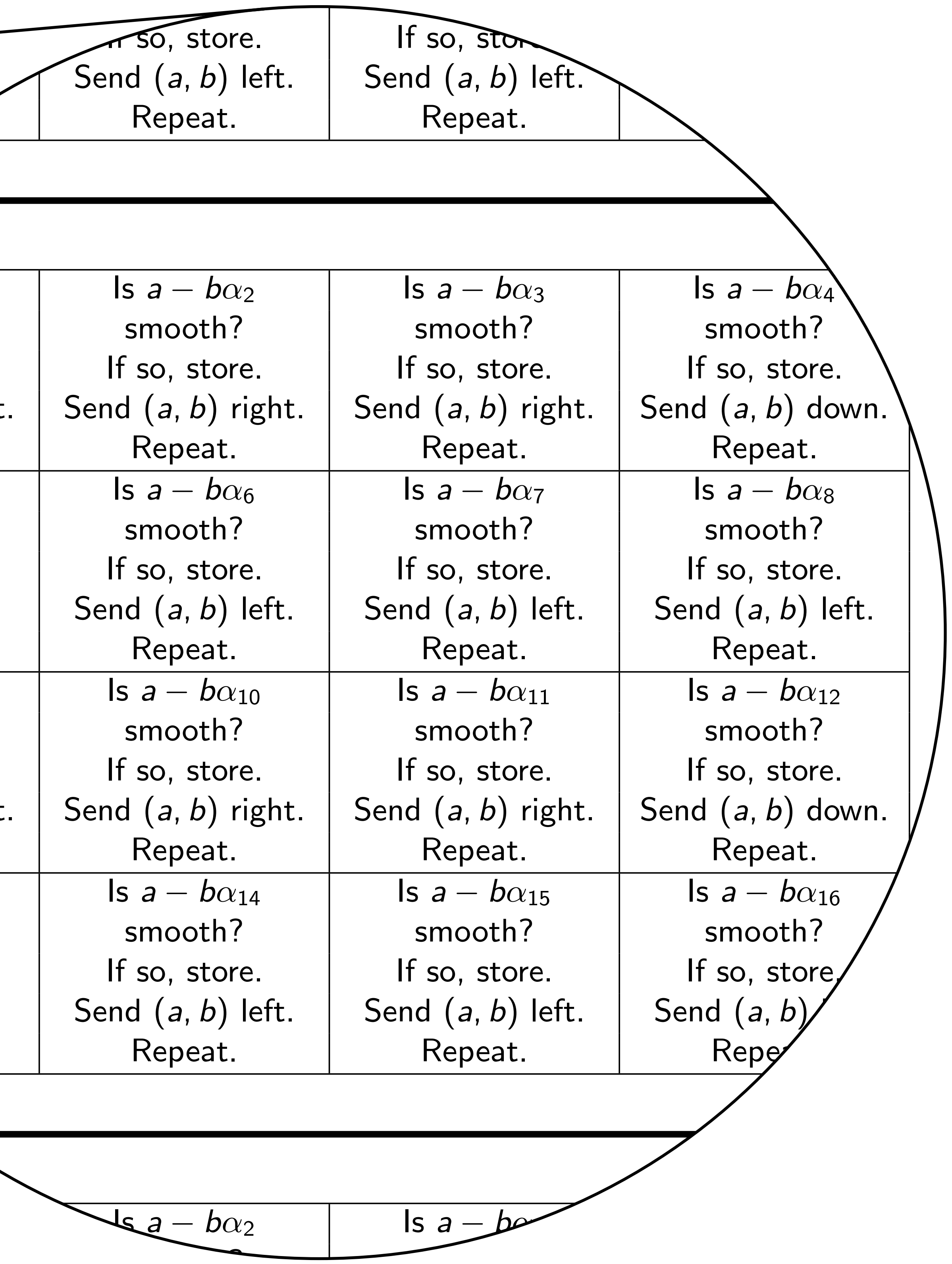
Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_3 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	L
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_7 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	L
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{10} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{11} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Li
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{14} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{15} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Li

… so, store. Send (a, b) left. Repeat. | If so, store… Send (a, b) left. Repeat.

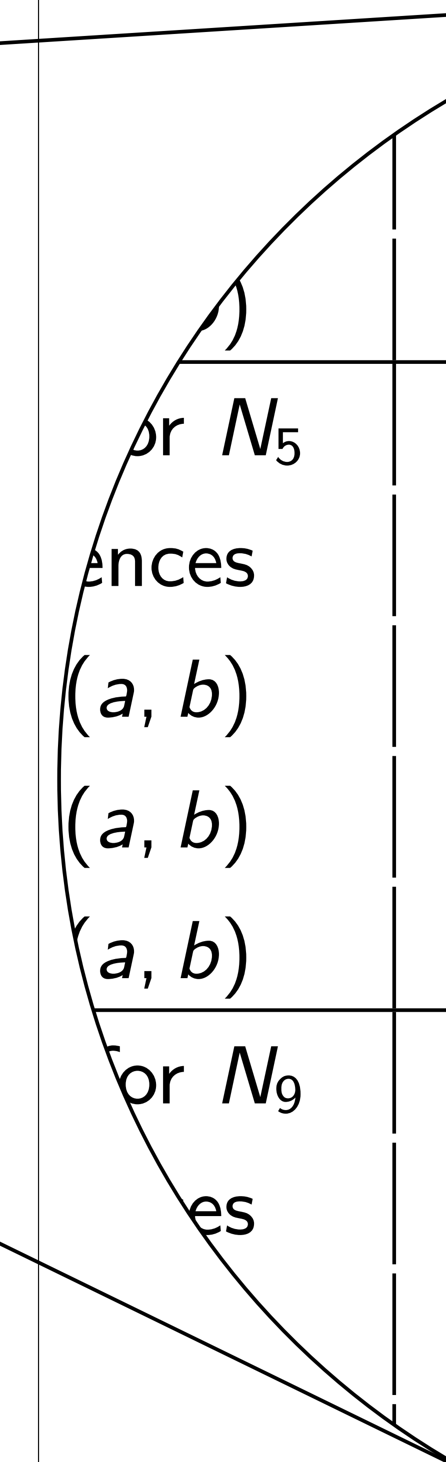
Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) … Repe…

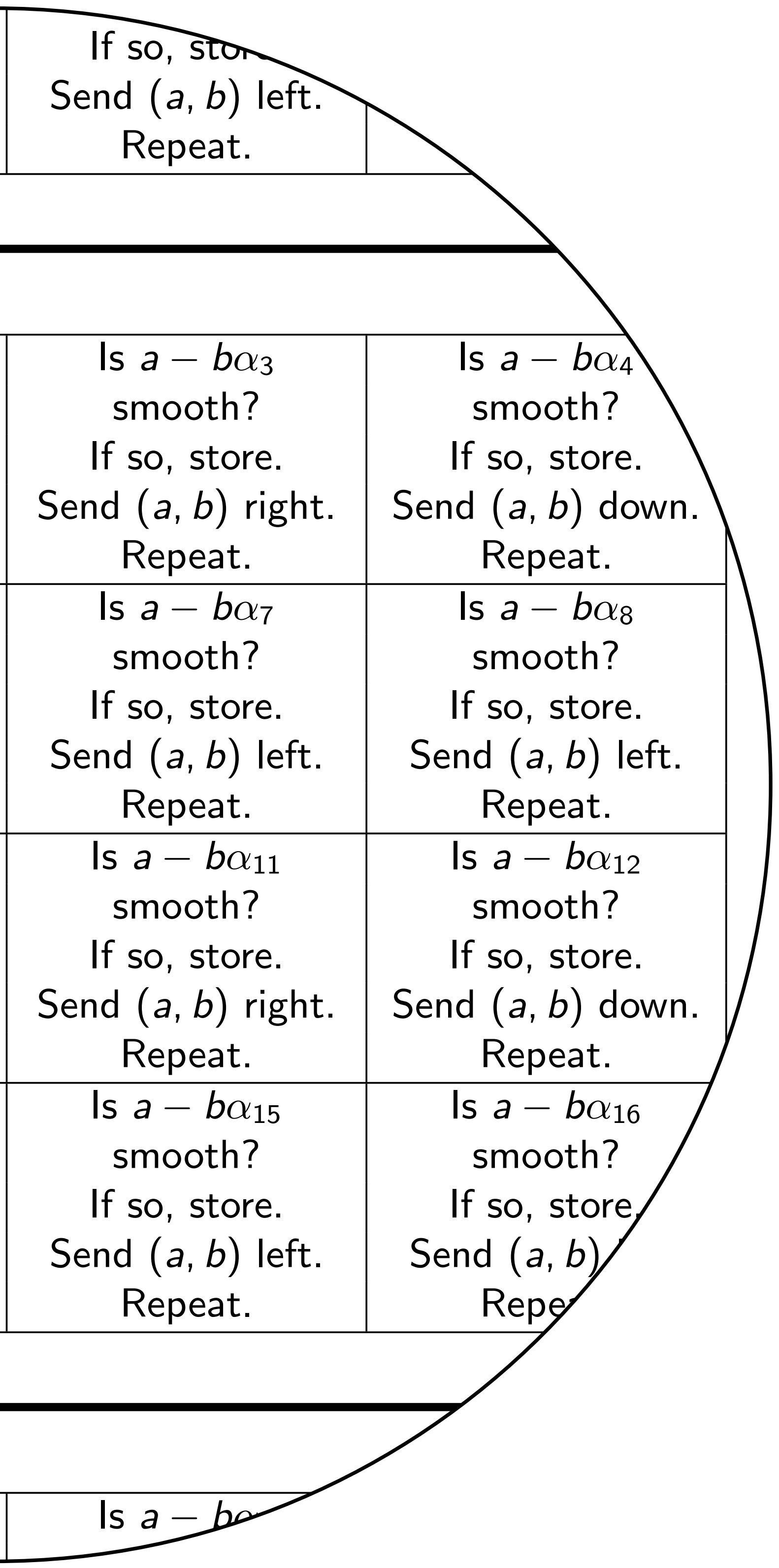
Is $a - b\alpha_2$ | Is $a - b$…

Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_3 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_4 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_7 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_8 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{10} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{11} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{12} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{14} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{15} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{16} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)

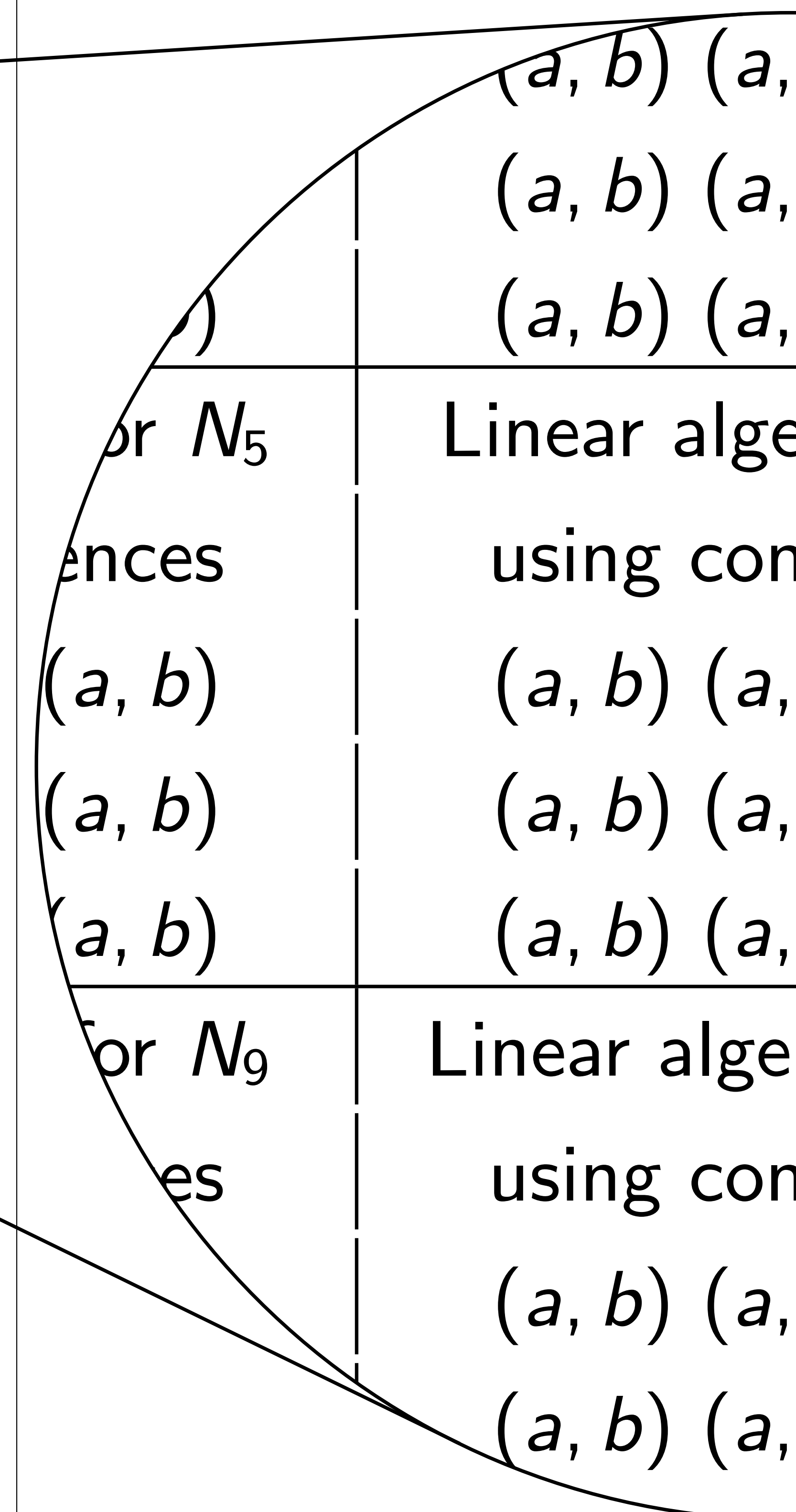


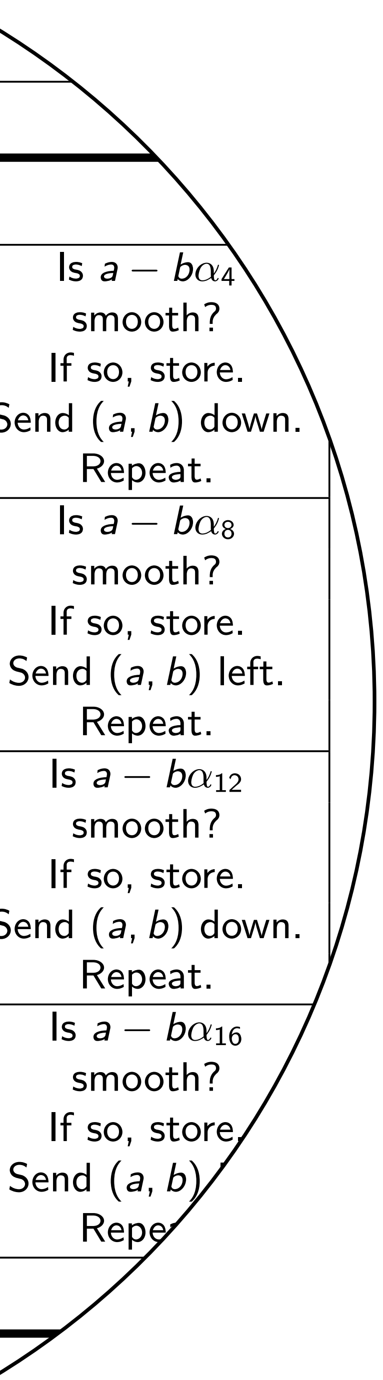
Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_3 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_4 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_7 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_8 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{10} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{11} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{12} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{14} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{15} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{16} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)



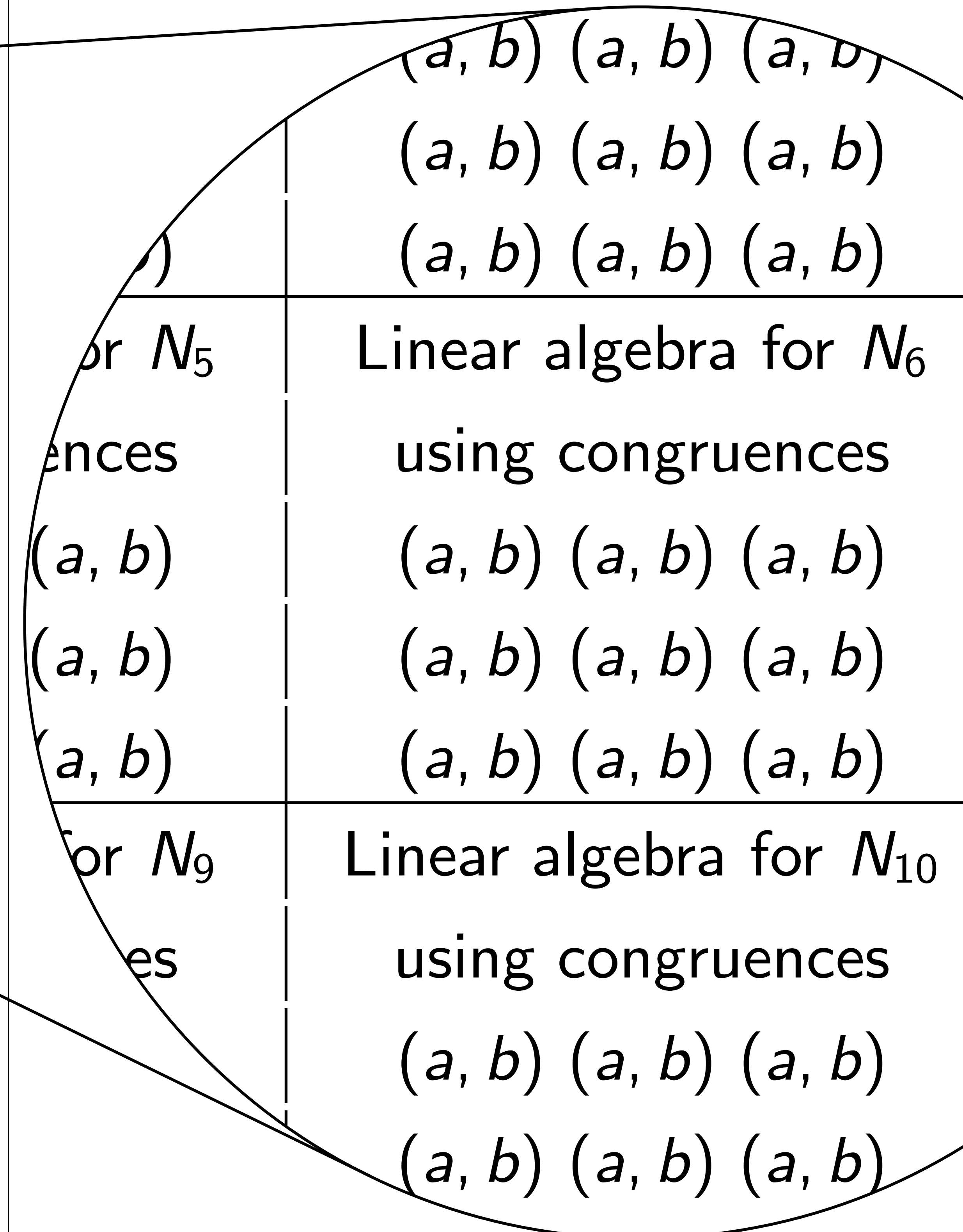


Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_3 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_4 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_7 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_8 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{10} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{11} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{12} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{14} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{15} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{16} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)





Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_3 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_4 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_7 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_8 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{10} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{11} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{12} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{14} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{15} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{16} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)



Linear algebra for N_1 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_2 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_3 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_4 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_5 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_6 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_7 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_8 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_9 using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{10} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{11} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{12} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)
Linear algebra for N_{13} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{14} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{15} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)	Linear algebra for N_{16} using congruences (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b) (a, b)