Introduction to quantum algorithms

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an element of $\{0, 1\}^n$,
often viewed as representing
an element of $\{0, 1, \ldots, 2^n - 1\}$.

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If n qubits have state $(a_0, a_1, \ldots, a_{2^n-1})$ then **measuring** the qubits produces an element of $\{0,1,\ldots,2^n-1\}$ and destroys the state. Measurement produces element q with probability $|a_q|^2/\sum_r |a_r|^2$.

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$(0, 0, 4, 0, 0, 0, 8, 0) = 4|2\rangle + 8|6\rangle$:
Measurement produces
2 with probability 20%,
6 with probability 80%.

Fast quantum operations, part 1

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$
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$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7)$
is measured as (q_0, q_1, q_2),
representing $q = q_0 + 2q_1 + 4q_2$,
with probability $|a_q|^2 / \sum_r |a_r|^2$.

$(a_1, a_0, a_3, a_2, a_5, a_4, a_7, a_6)$
is measured as $(q_0 \oplus 1, q_1, q_2)$,
representing $q \oplus 1$,
with probability $|a_q|^2 / \sum_r |a_r|^2$.

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$
$(a_4, a_5, a_6, a_7, a_0, a_1, a_2, a_3)$
is “complementing qubit 2”:
$(q_0, q_1, q_2) \mapsto (q_0, q_1, q_2 \oplus 1)$.

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Complementing qubit 2
= swapping qubits 0 and 2
 ∘ complementing qubit 0
 ∘ swapping qubits 0 and 2.

Similarly: swapping qubits i, j.

$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$
$(a_0, a_1, a_3, a_2, a_4, a_5, a_7, a_6)$
is a "reversible XOR gate" $=$
"controlled NOT gate":
$(q_0, q_1, q_2) \mapsto (q_0 \oplus q_1, q_1, q_2)$.

Example with more qubits:

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7,$$
$$a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15},$$
$$a_{16}, a_{17}, a_{18}, a_{19}, a_{20}, a_{21}, a_{22}, a_{23},$$
$$a_{24}, a_{25}, a_{26}, a_{27}, a_{28}, a_{29}, a_{30}, a_{31})$$
$$\mapsto (a_0, a_1, a_3, a_2, a_4, a_5, a_7, a_6,$$
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$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$
$(a_0, a_1, a_2, a_3, a_4, a_5, a_7, a_6)$
is a “Toffoli gate” =
“controlled controlled NOT gate”:
$(q_0, q_1, q_2) \mapsto (q_0 \oplus q_1 q_2, q_1, q_2)$.

Example with more qubits:
$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7,$
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Reversible computation

Say p is a permutation of $\{0, 1, \ldots, 2^n - 1\}$.

General strategy to compose these fast quantum operations to obtain index permutation $(a_{p(0)}, a_{p(1)}, \ldots, a_{p(2^n-1)}) \mapsto (a_0, a_1, \ldots, a_{2^n-1})$:

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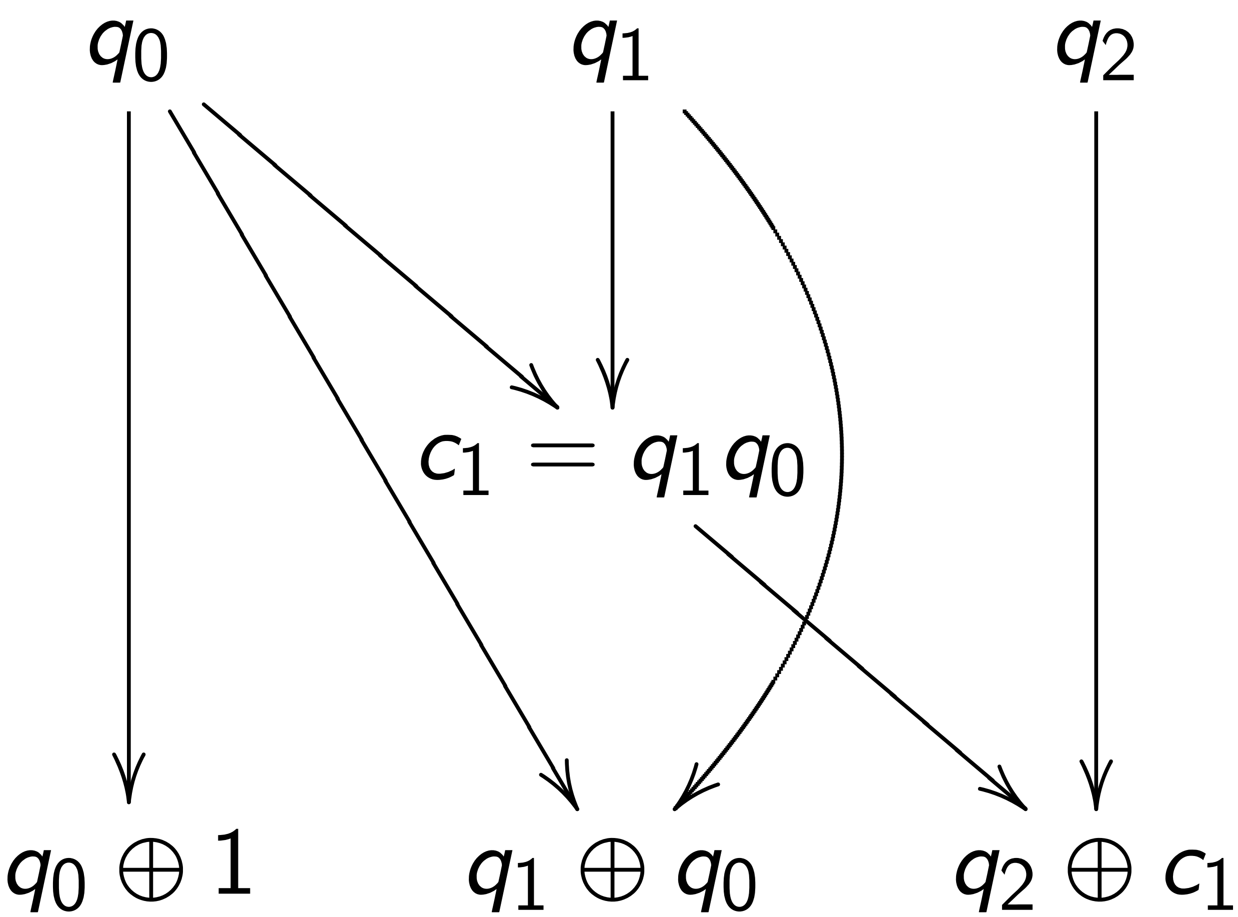
General strategy to compose these fast quantum operations to obtain index permutation $(a_{p(0)}, a_{p(1)}, \ldots, a_{p(2^n-1)})$ $\mapsto (a_0, a_1, \ldots, a_{2^n-1})$:

1. Build a traditional circuit to compute $j \mapsto p(j)$ using NOT/XOR/AND gates.

2. Convert into reversible gates: e.g., convert AND into Toffoli.

Example: Let's compute $(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto (a_7, a_0, a_1, a_2, a_3, a_4, a_5, a_6)$; permutation $q \mapsto q + 1 \bmod 8$.

1. Build a traditional circuit to compute $q \mapsto q + 1 \bmod 8$.



2. Convert into reversible gates.

Toffoli for $q_2 \leftarrow q_2 \oplus q_1 q_0$:
$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) \mapsto$
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Controlled NOT for $q_1 \leftarrow q_1 \oplus q_0$:
$(a_0, a_1, a_2, a_7, a_4, a_5, a_6, a_3) \mapsto$
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NOT for $q_0 \leftarrow q_0 \oplus 1$:
$(a_0, a_7, a_2, a_1, a_4, a_3, a_6, a_5) \mapsto$
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This permutation example
was deceptively easy.

It didn't need many operations.

For large n, most permutations p
need many operations $\Rightarrow$ slow.
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Also, it didn't need extra storage:
circuit operated "in place" after
computation $c_1 \leftarrow q_1 q_0$ was
merged into $q_2 \leftarrow q_2 \oplus c_1$.

Typical circuits aren't in-place.

Start from any circuit:

inputs $b_1, b_2, \ldots, b_i$;

$b_{i+1} = 1 \oplus b_{f(i+1)} b_{g(i+1)}$;

$b_{i+2} = 1 \oplus b_{f(i+2)} b_{g(i+2)}$;

$\ldots$

$b_T = 1 \oplus b_{f(T)} b_{g(T)}$;

specified outputs.

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specified outputs.

Reversible but dirty:

inputs $b_1, b_2, \ldots, b_T$;

$b_{i+1} \leftarrow 1 \oplus b_{i+1} \oplus b_{f(i+1)} b_{g(i+1)}$;

$b_{i+2} \leftarrow 1 \oplus b_{i+2} \oplus b_{f(i+2)} b_{g(i+2)}$;

$\ldots$

$b_T \leftarrow 1 \oplus b_T \oplus b_{f(T)} b_{g(T)}$.

Same outputs if all of

$b_{i+1}, \ldots, b_T$ started as 0.

Reversible and clean:
after finishing dirty computation,
set non-outputs back to 0,
by repeating same operations
on non-outputs in reverse order.

Original computation:
$(\text{inputs}) \mapsto$
$(\text{inputs}, \text{dirt}, \text{outputs})$.

Dirty reversible computation:
$(\text{inputs}, \text{zeros}, \text{zeros}) \mapsto$
$(\text{inputs}, \text{dirt}, \text{outputs})$.

Clean reversible computation:
$(\text{inputs}, \text{zeros}, \text{zeros}) \mapsto$
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Given fast circuit for p
and fast circuit for p^{-1},
build fast reversible circuit for
$(x, \text{zeros}) \mapsto (p(x), \text{zeros})$.

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Replace reversible bit operations
with Toffoli gates etc.
permuting $\mathbf{C}^{2^{n+z}} \to \mathbf{C}^{2^{n+z}}$.

Permutation on first 2^n entries is
$(a_{p(0)}, a_{p(1)}, \ldots, a_{p(2^n-1)})$
$\mapsto (a_0, a_1, \ldots, a_{2^n-1})$.

Typically prepare vectors
supported on first 2^n entries
so don't care how permutation
acts on last $2^{n+z} - 2^n$ entries.

Warning: Number of **qubits**
$\approx$ number of **bit operations**
in original p, p^{-1} circuits.

This can be much larger
than number of **bits stored**
in the original circuits.

Many useful techniques
to compress into fewer qubits,
but often these lose time.
Many subtle tradeoffs.

Crude "poly-time" analyses
don't care about this,
but serious cryptanalysis
is much more precise.

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Same for qubit 1:

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Qubit 0 and then qubit 1:

$(a_0, a_1, a_2, a_3) \mapsto$
$(a_0 + a_1, a_0 - a_1, a_2 + a_3, a_2 - a_3) \mapsto$
$(a_0 + a_1 + a_2 + a_3, a_0 - a_1 + a_2 - a_3,$
$a_0 + a_1 - a_2 - a_3, a_0 - a_1 - a_2 + a_3)$.

Repeat n times: e.g.,
$(1, 0, 0, \ldots, 0) \mapsto (1, 1, 1, \ldots, 1)$.

Measuring $(1, 0, 0, \ldots, 0)$
always produces 0.

Measuring $(1, 1, 1, \ldots, 1)$
can produce any output:
$\Pr[\text{output} = q] = 1/2^n$.

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Aside from "normalization" (irrelevant to measurement), have Hadamard = Hadamard^{-1}, so easily work backwards from "uniform superposition" $(1, 1, 1, \ldots, 1)$ to "pure state" $(1, 0, 0, \ldots, 0)$.

Simon's algorithm

Assume: nonzero $s \in \{0,1\}^n$
satisfies $f(x) = f(x \oplus s)$
for every $x \in \{0,1\}^n$.
Can we find this period s,
given a fast circuit for f?

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Traditional solution:
Compute f for many inputs,
sort, analyze collisions.
Success probability is very low
until #inputs approaches $2^{n/2}$.

Simon's algorithm
is much, much, much faster.

Say f maps n bits to m bits,
using z "ancilla" bits
for reversibility.

Prepare $n + m + z$ qubits
in pure zero state:
vector $(1, 0, 0, \ldots)$.

Use n-fold Hadamard
to move first n qubits
into uniform superposition:
$(1, 1, 1, \ldots, 1, 0, 0, \ldots)$
with 2^n entries 1, others 0.

Apply fast vector permutation
for reversible f computation:
1 in position $(q, 0, 0)$
moves to position $(q, f(q), 0)$.

Note symmetry between
1 at $(q, f(q), 0)$ and
1 at $(q \oplus s, f(q), 0)$.

Apply n-fold Hadamard.

Measure. By symmetry,
output is orthogonal to s.

Repeat $n + 10$ times.
Use Gaussian elimination
to (probably) find s.

Grover's algorithm

Assume: unique $s \in \{0,1\}^n$ has $f(s) = 0$.

Traditional algorithm to find s: compute f for many inputs, hope to find output 0. Success probability is very low until #inputs approaches 2^n.

Grover's algorithm takes only $2^{n/2}$ reversible computations of f. Typically: reversibility overhead is small enough that this easily beats traditional algorithm.

Start from uniform superposition over all n-bit strings q.

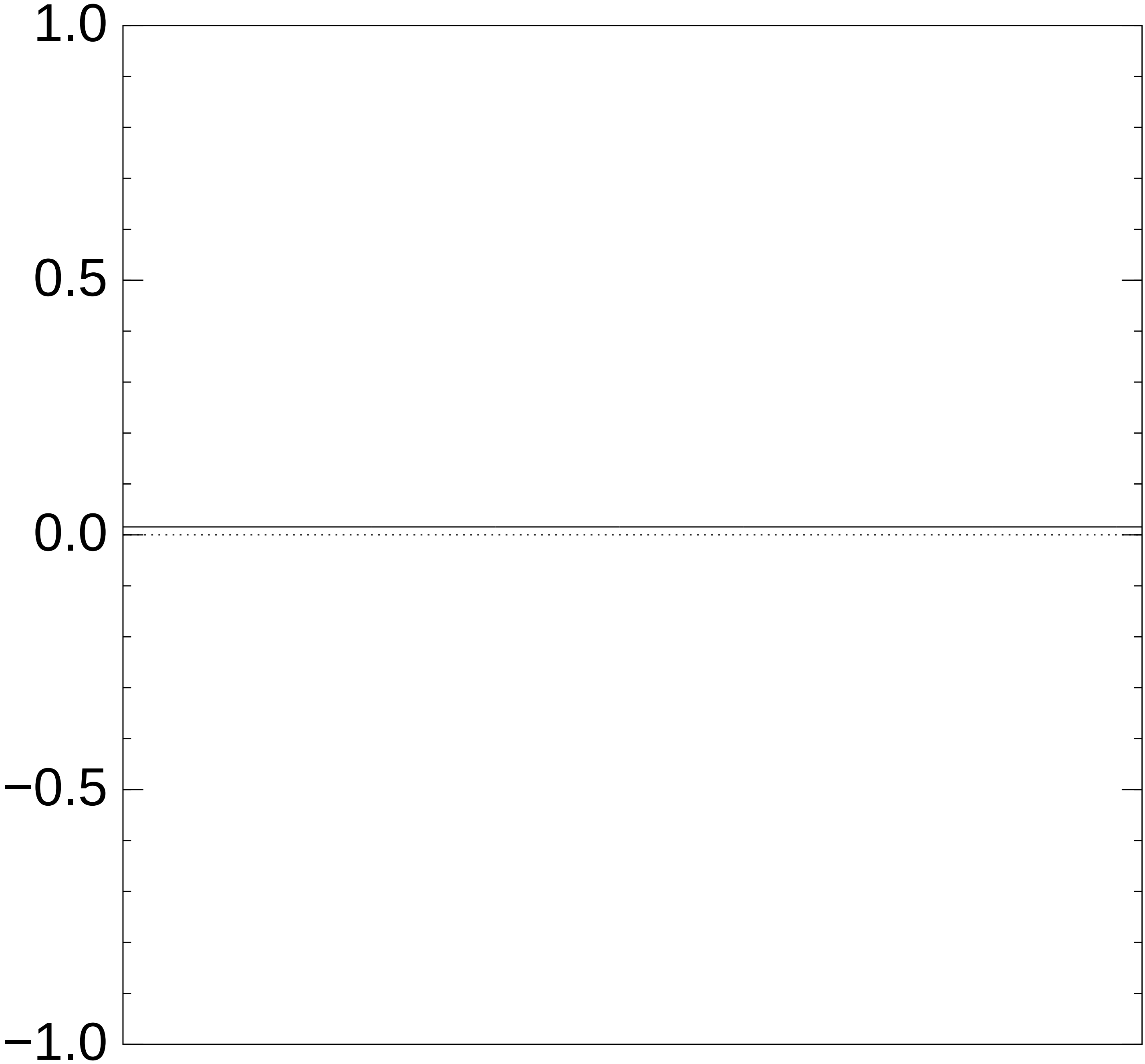
Step 1: Set $a \leftarrow b$ where
$b_q = -a_q$ if $f(q) = 0$,
$b_q = a_q$ otherwise.
This is fast.

Step 2: "Grover diffusion".
Negate a around its average.
This is also fast.

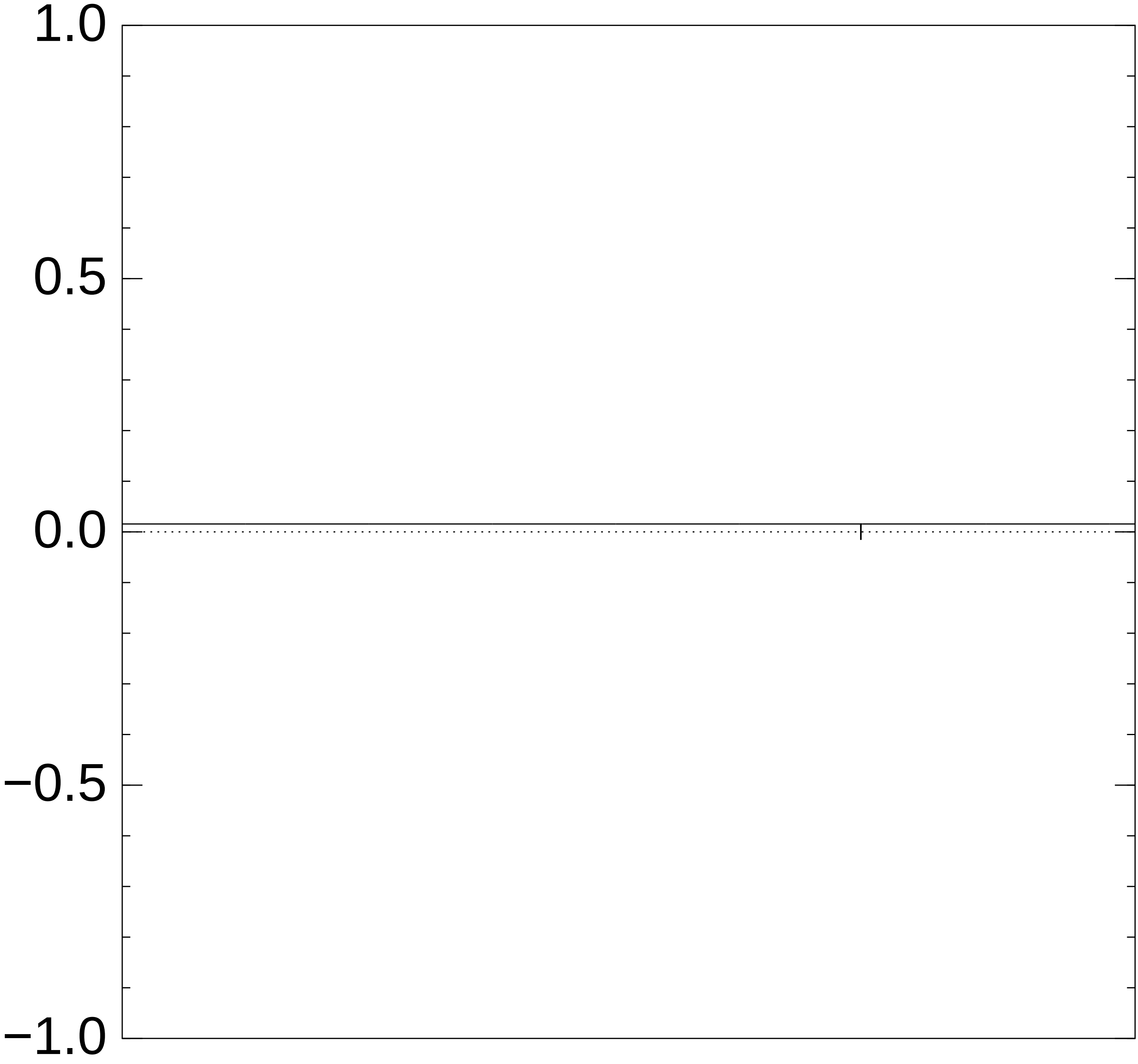
Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds s.

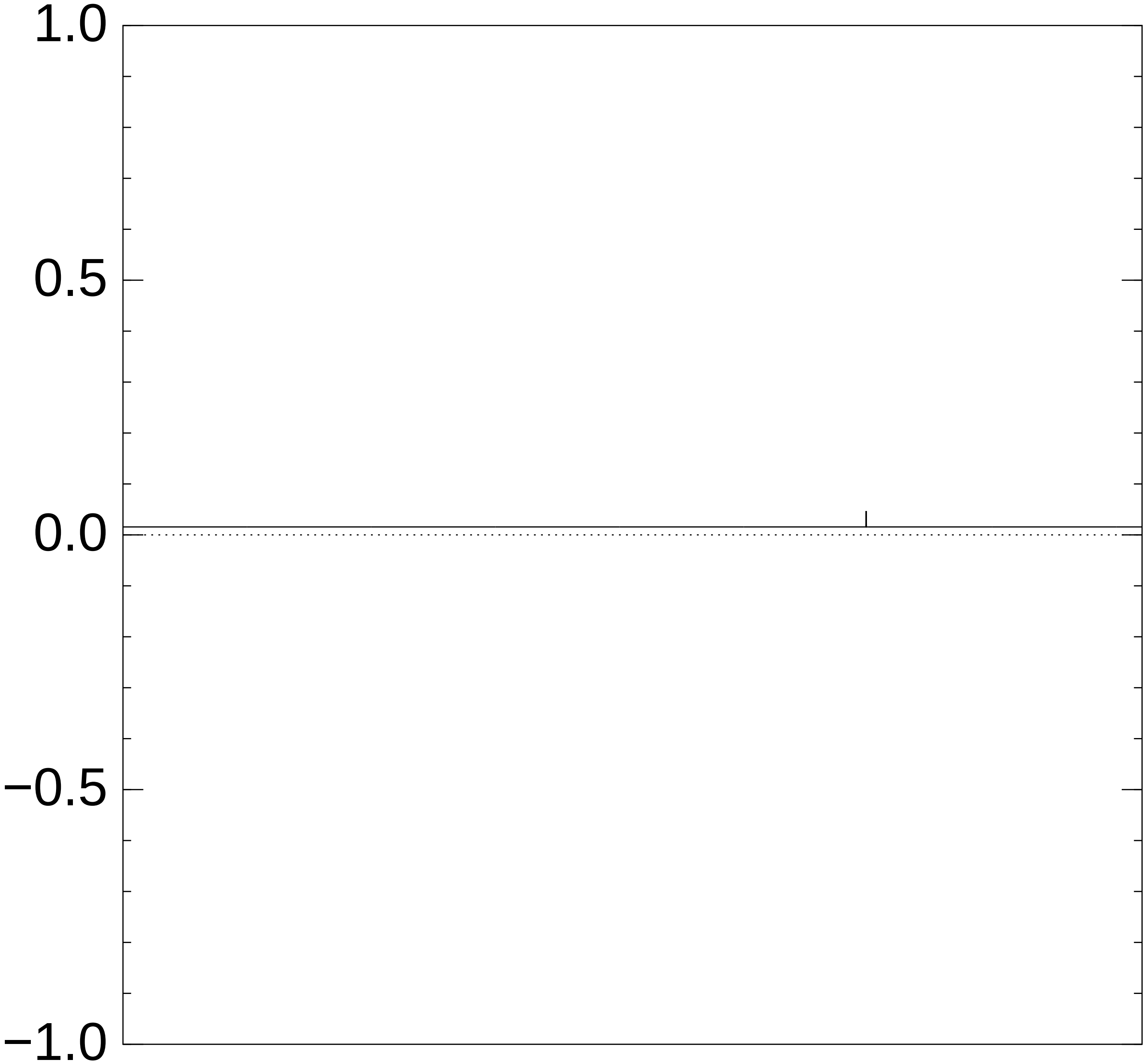
Graph of $q \mapsto a_q$ for an example with $n = 12$ after 0 steps:



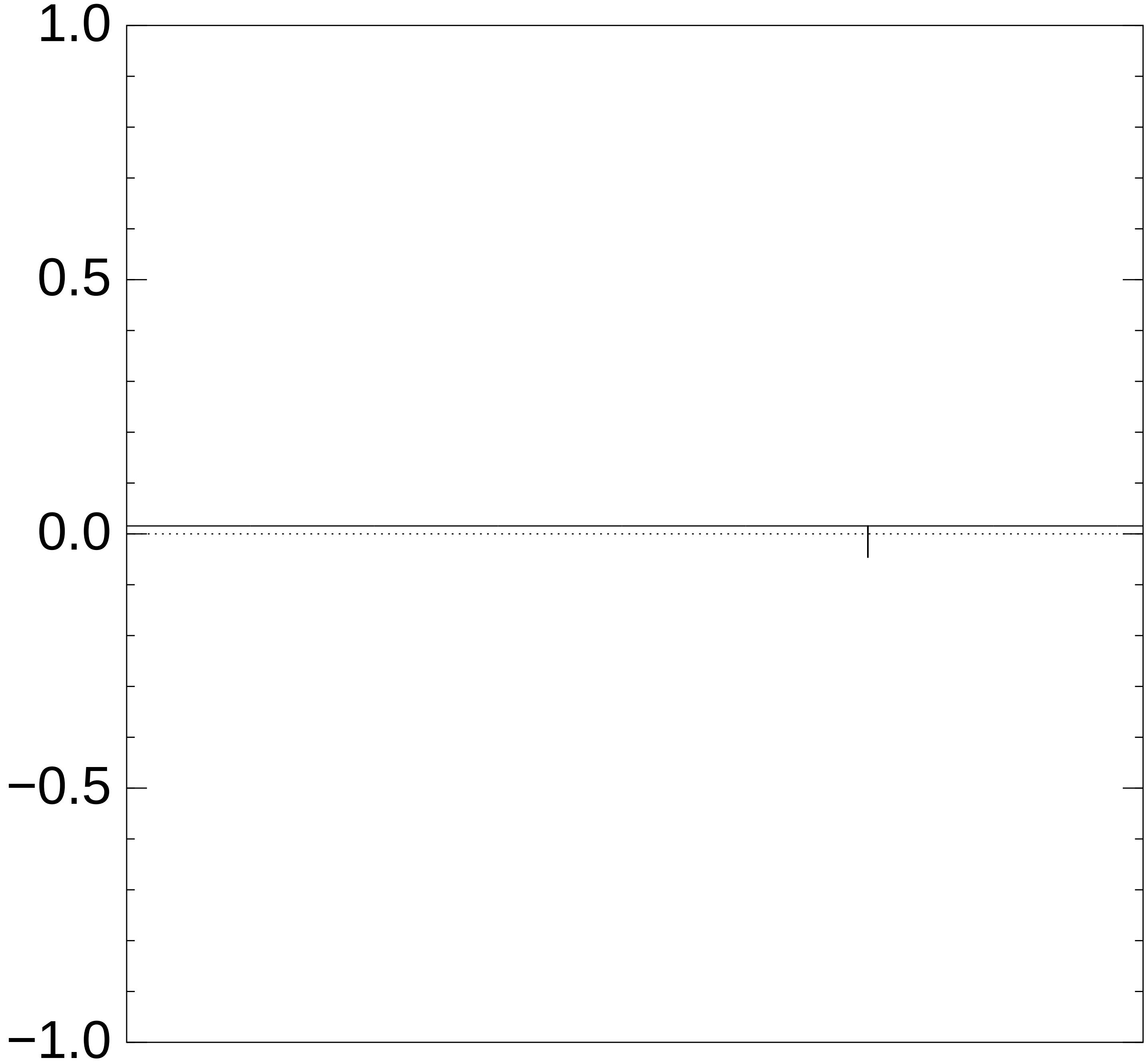
Graph of $q \mapsto a_q$
for an example with $n = 12$
after Step 1:



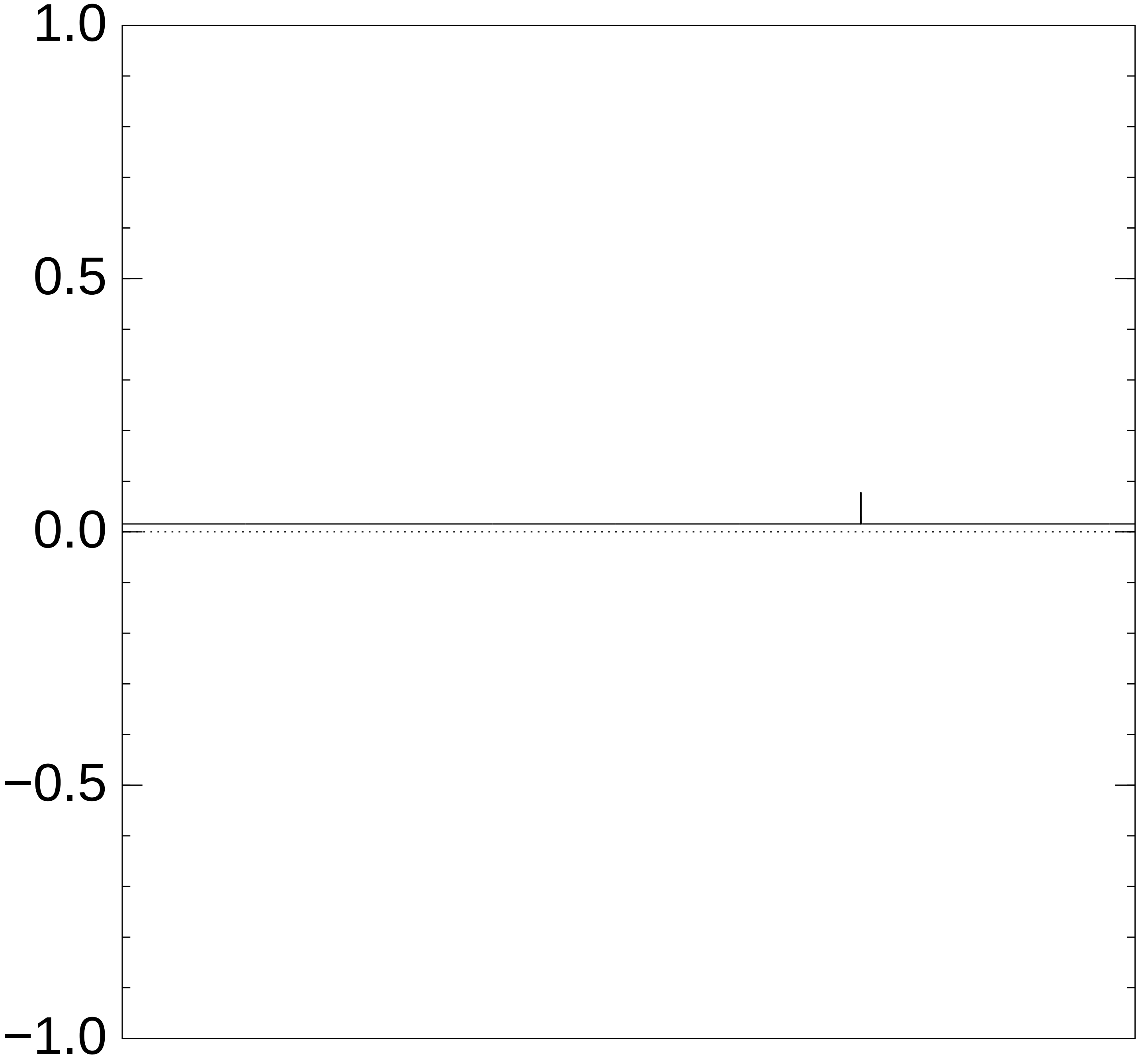
Graph of $q \mapsto a_q$
for an example with $n = 12$
after Step $1 +$ Step 2:



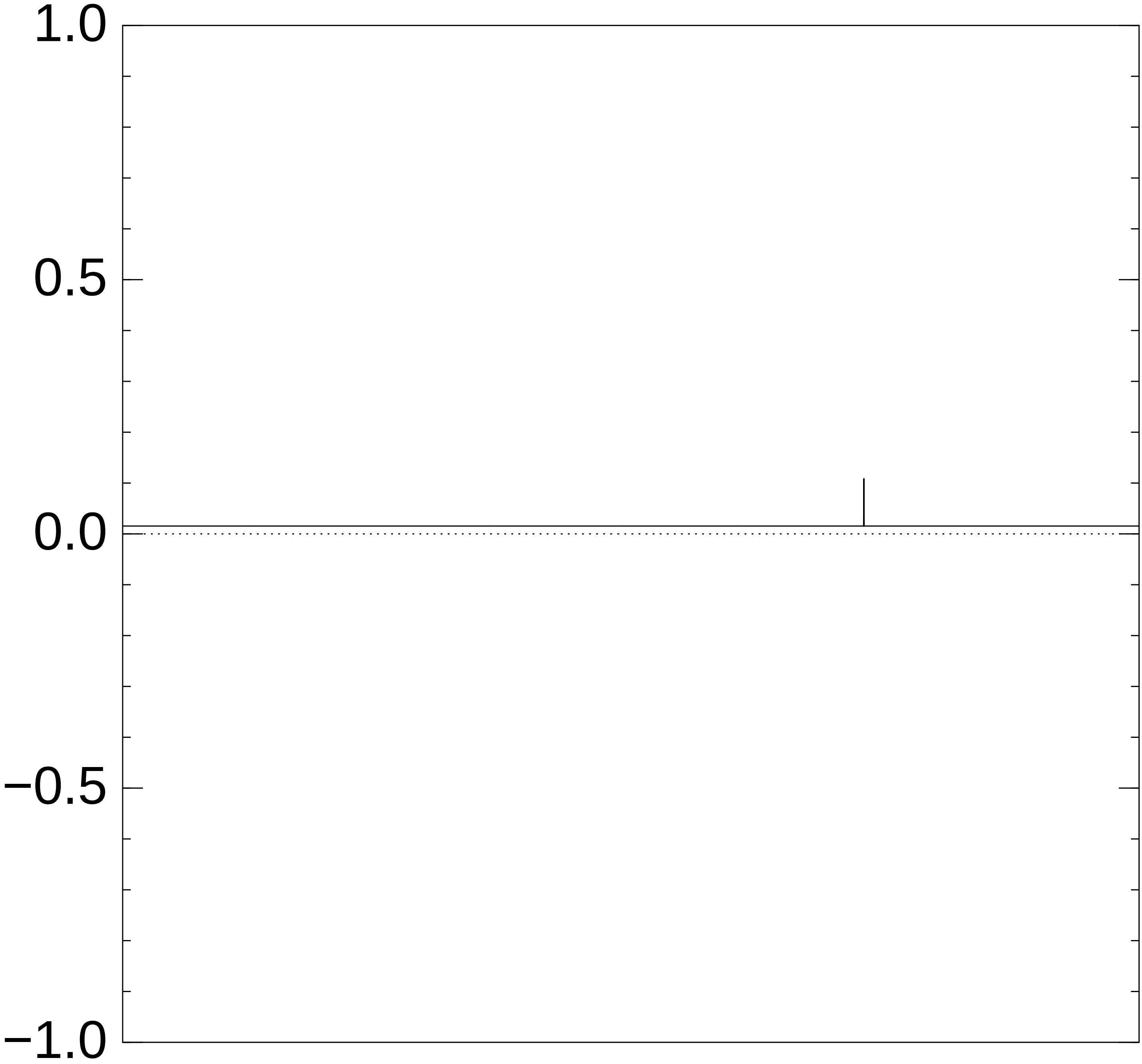
Graph of $q \mapsto a_q$
for an example with $n = 12$
after Step $1 +$ Step $2 +$ Step 1:



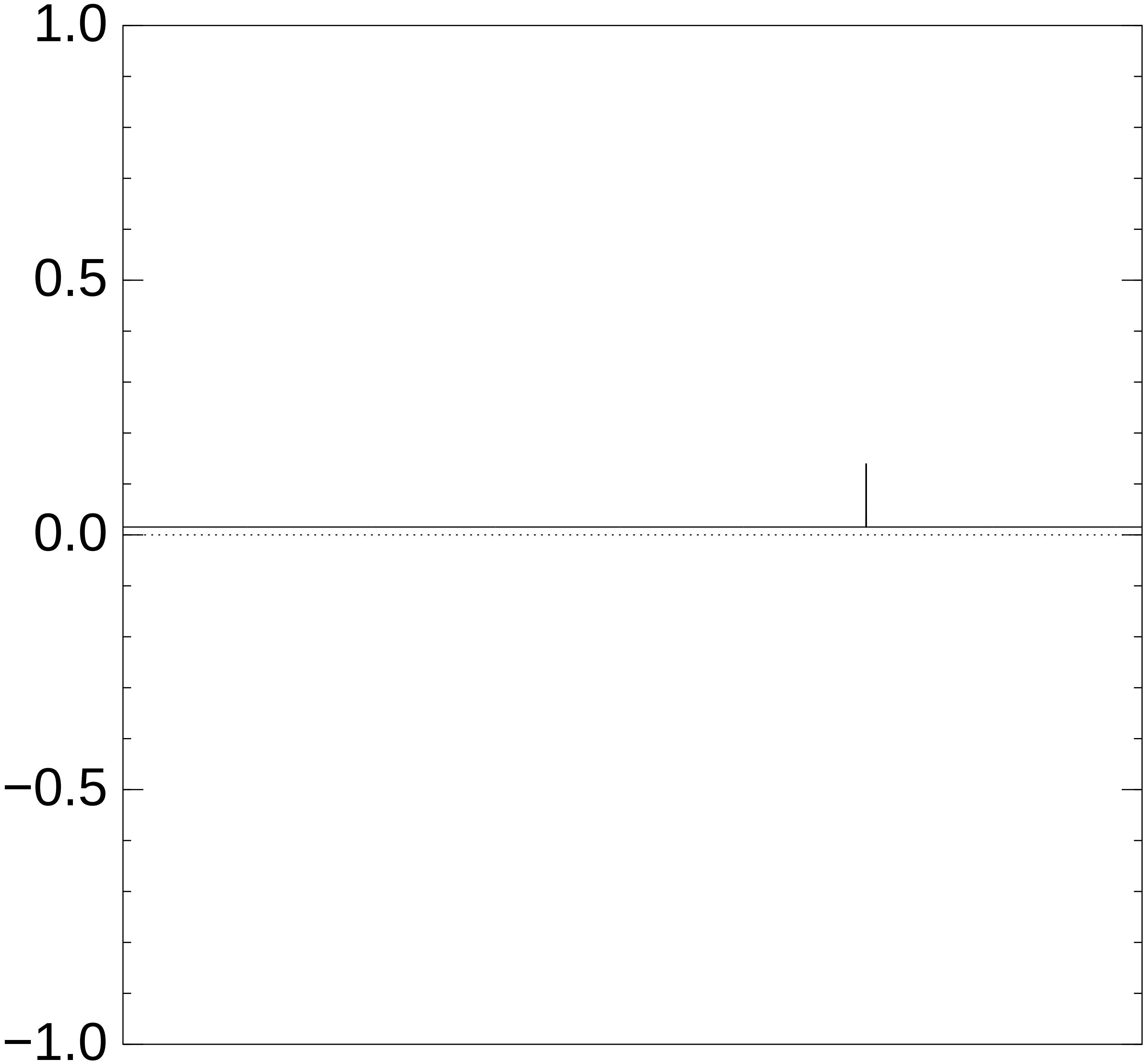
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $2 \times$ (Step $1 +$ Step 2):



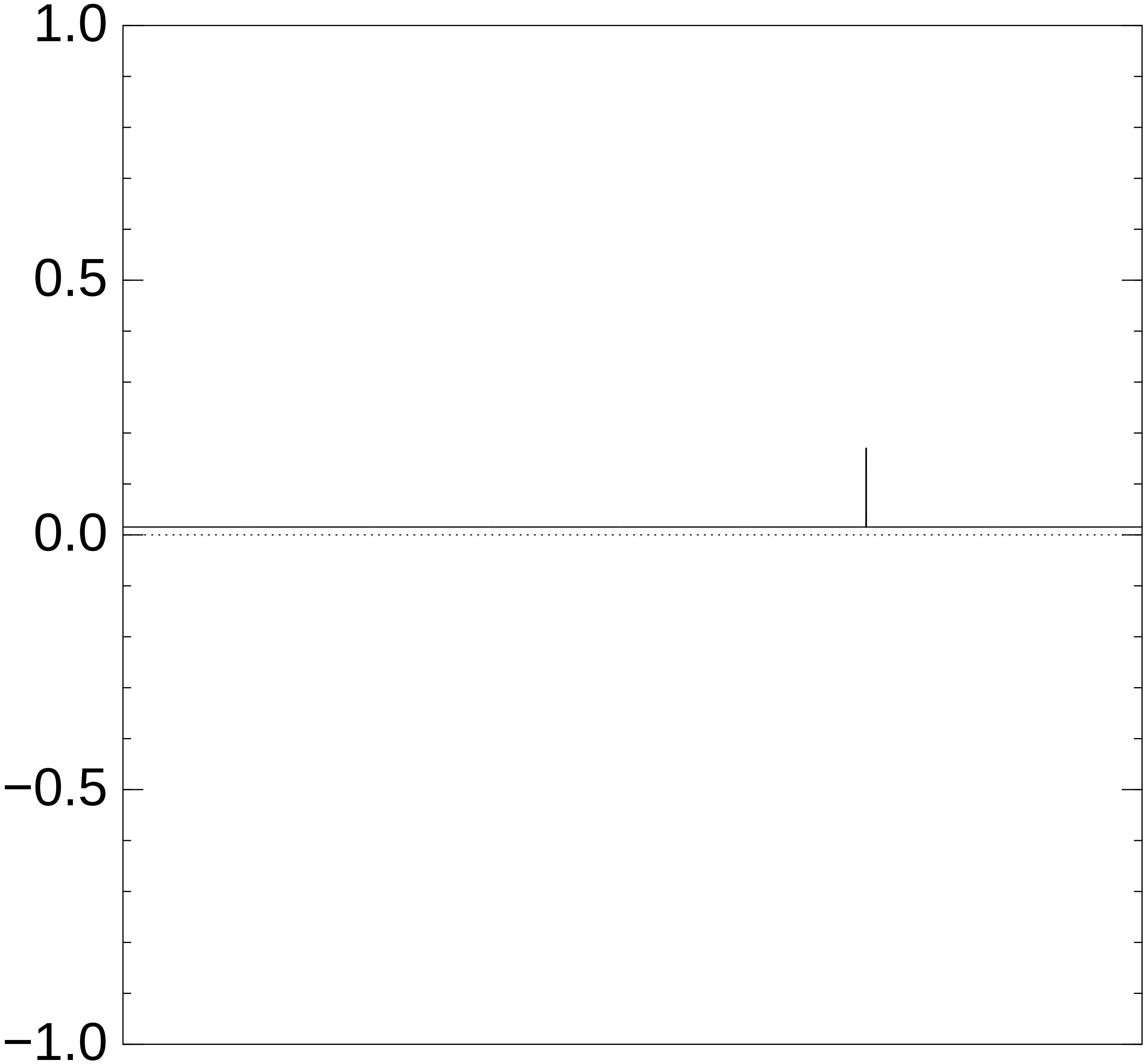
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $3 \times$ (Step $1 +$ Step 2):



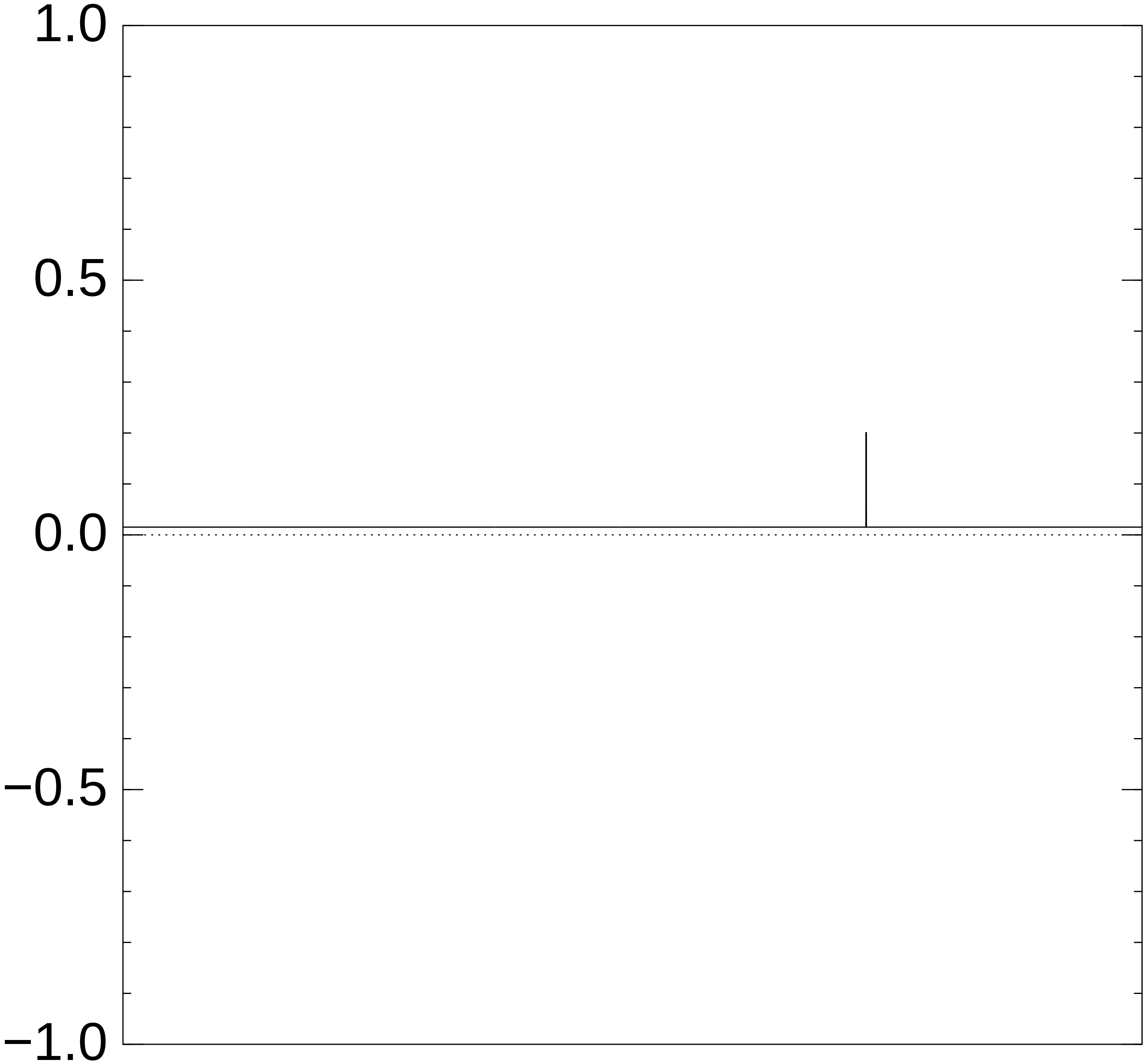
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $4 \times (\text{Step } 1 + \text{Step } 2)$:



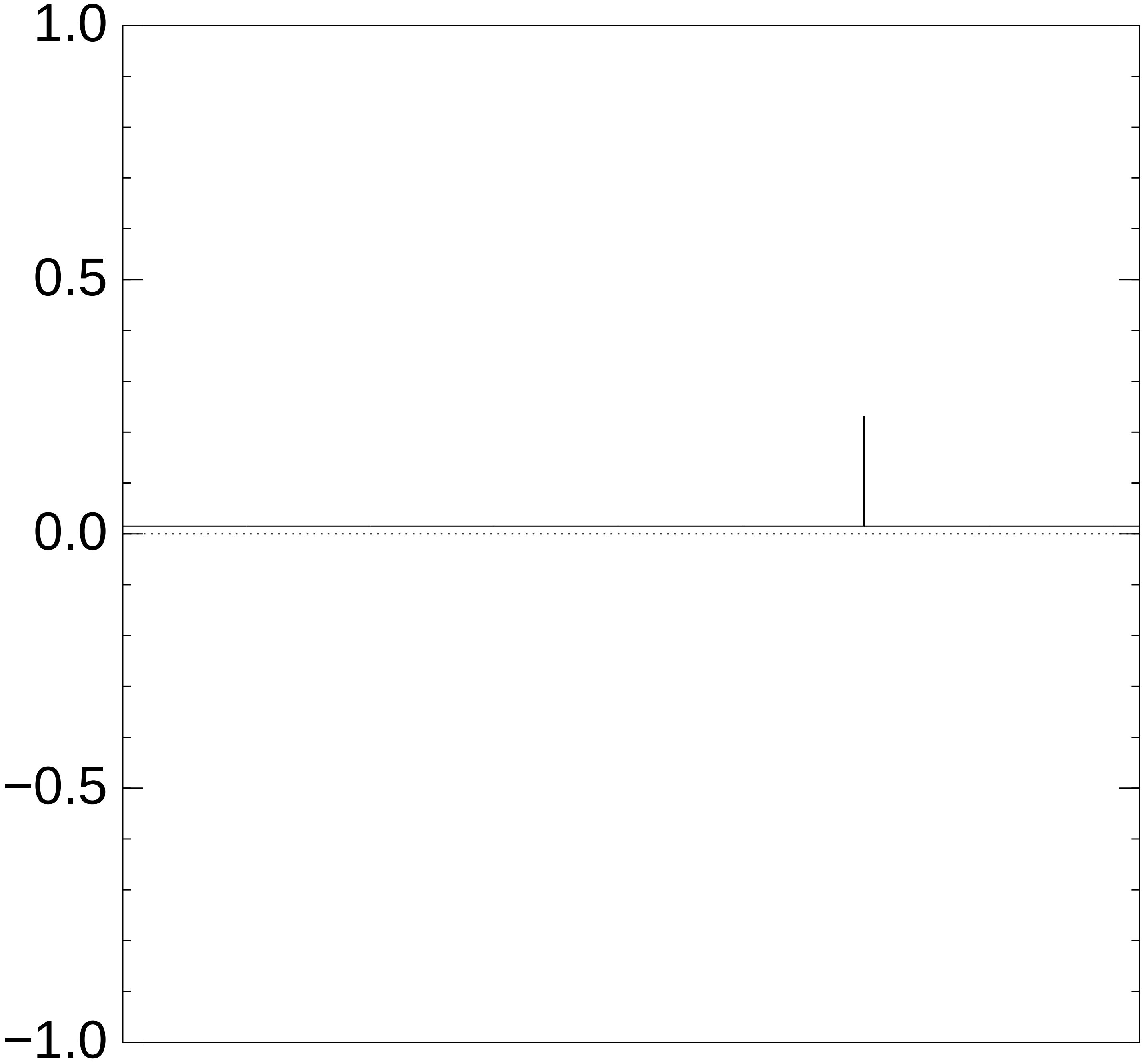
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $5 \times (\text{Step } 1 + \text{Step } 2)$:



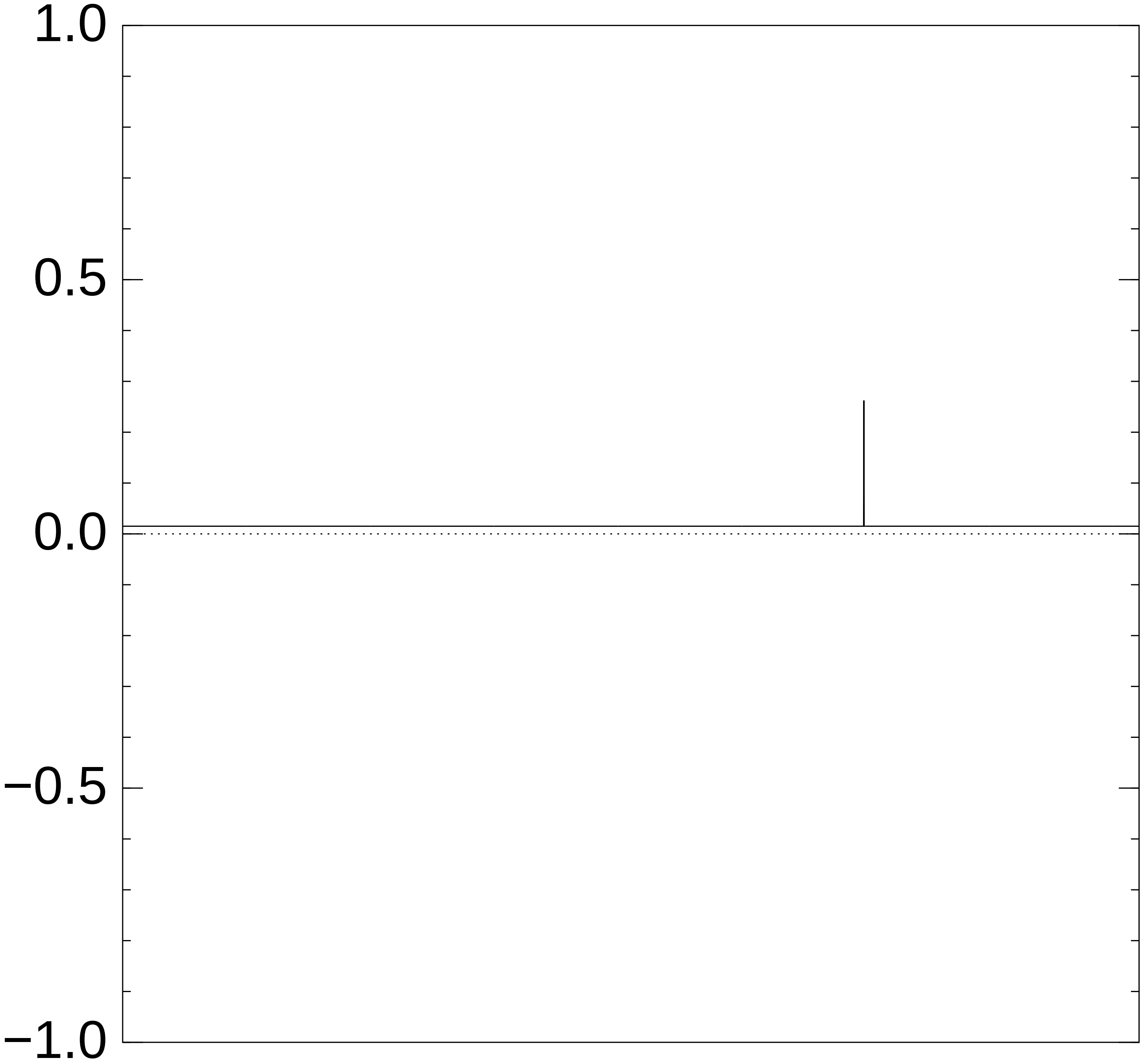
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $6 \times$ (Step $1 +$ Step 2):



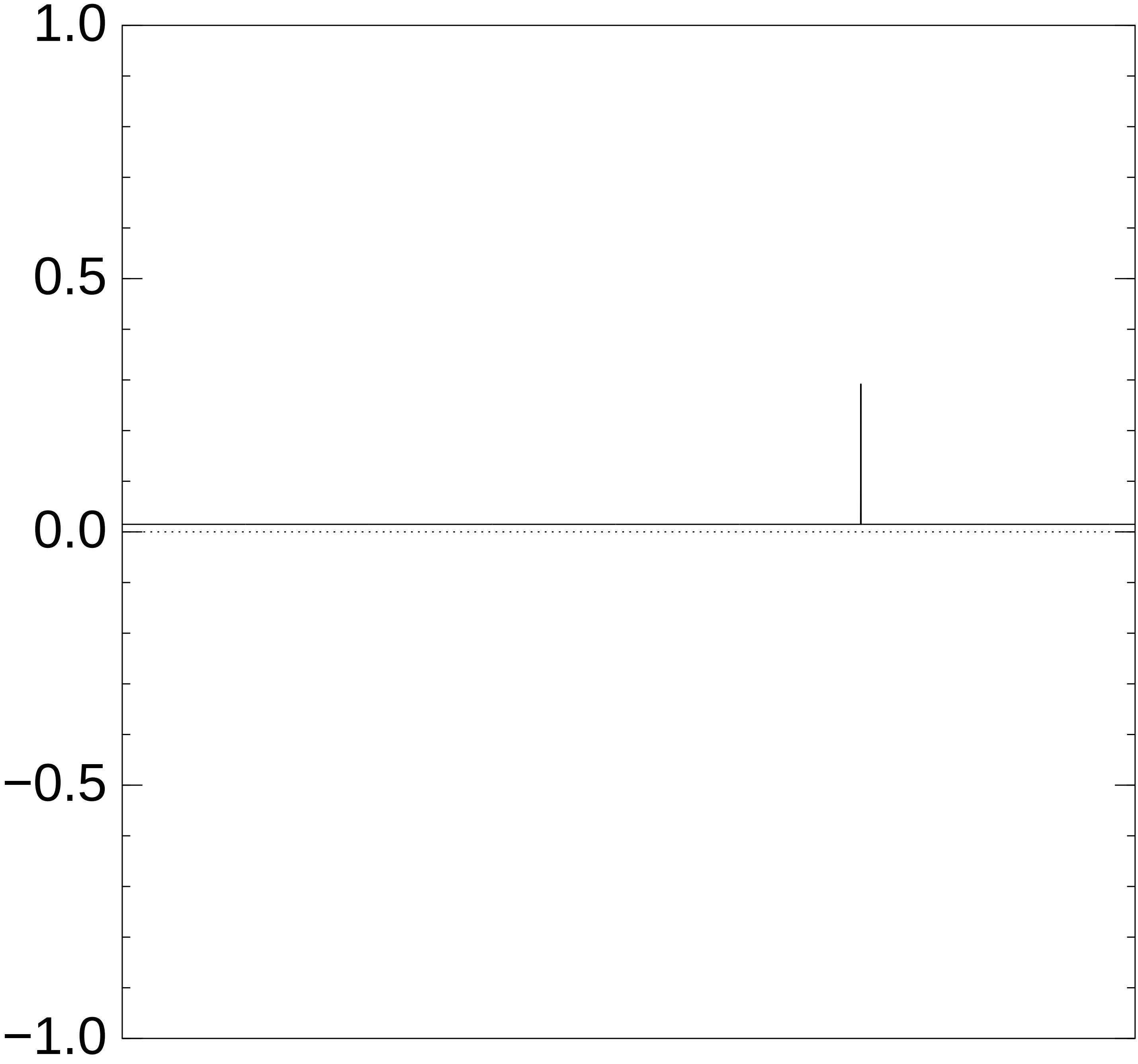
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $7 \times$ (Step $1 +$ Step 2):



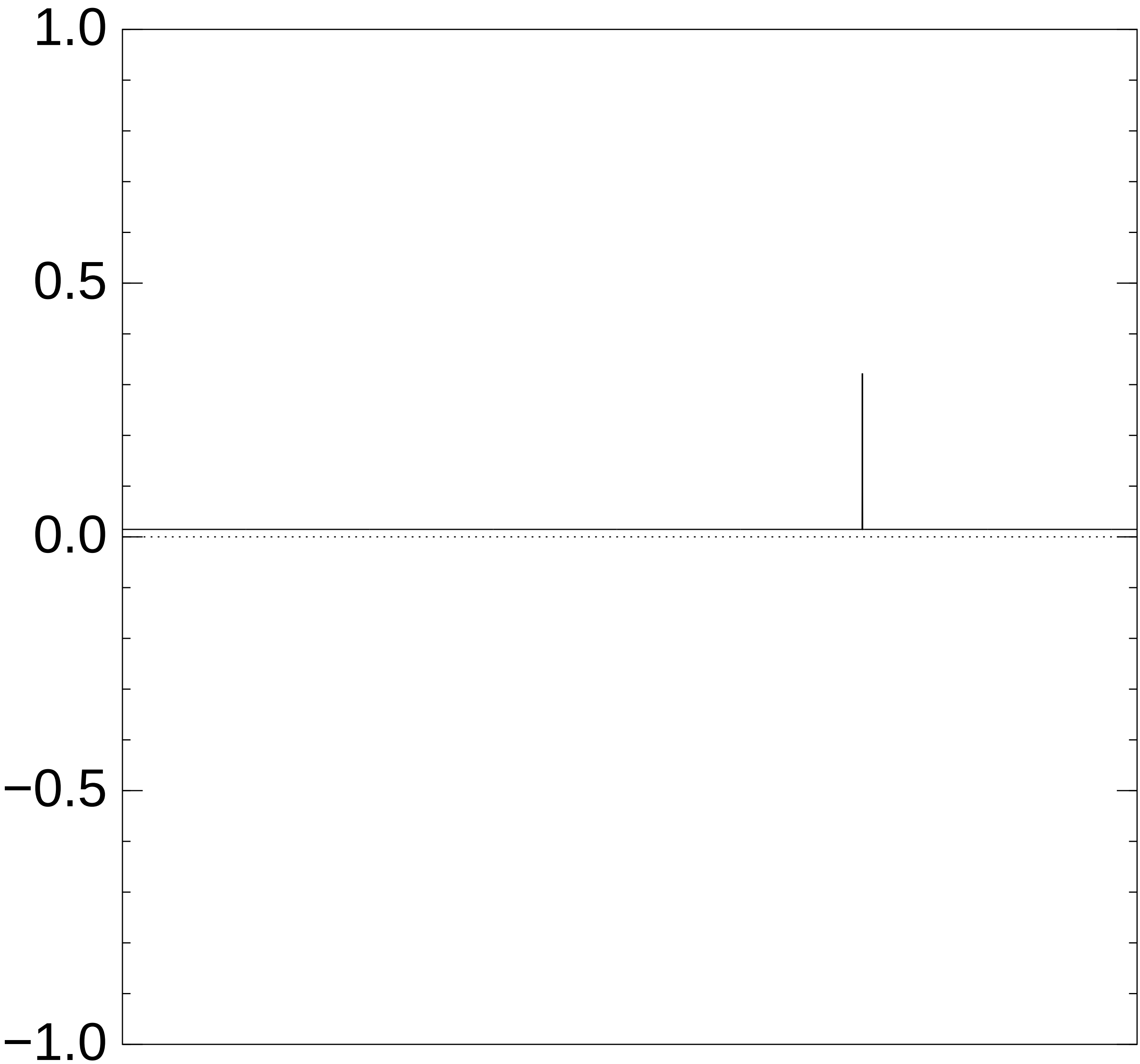
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $8 \times$ (Step $1 +$ Step 2):



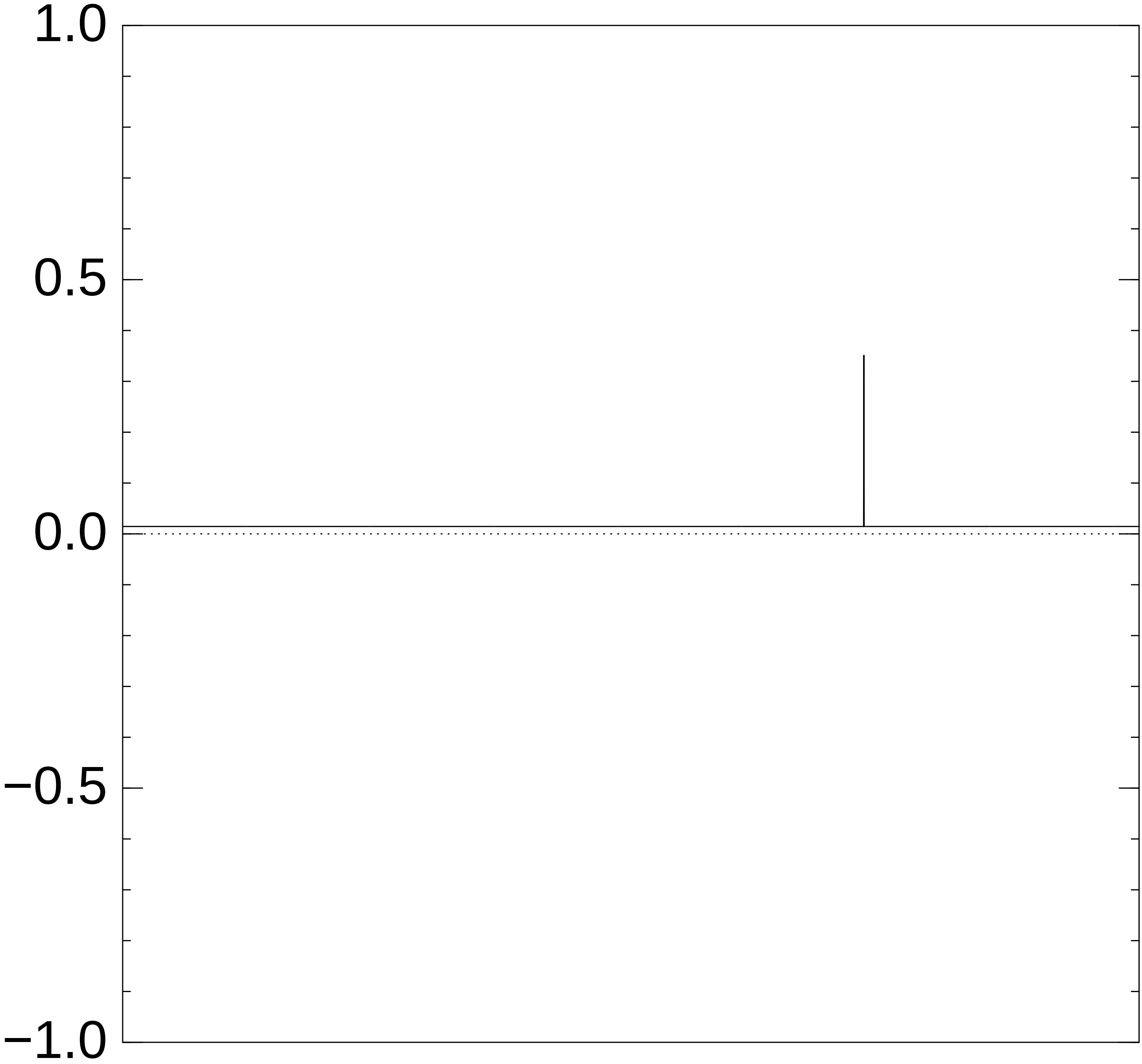
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $9 \times$ (Step $1 +$ Step 2):



Graph of $q \mapsto a_q$
for an example with $n = 12$
after $10 \times$ (Step $1 +$ Step 2):



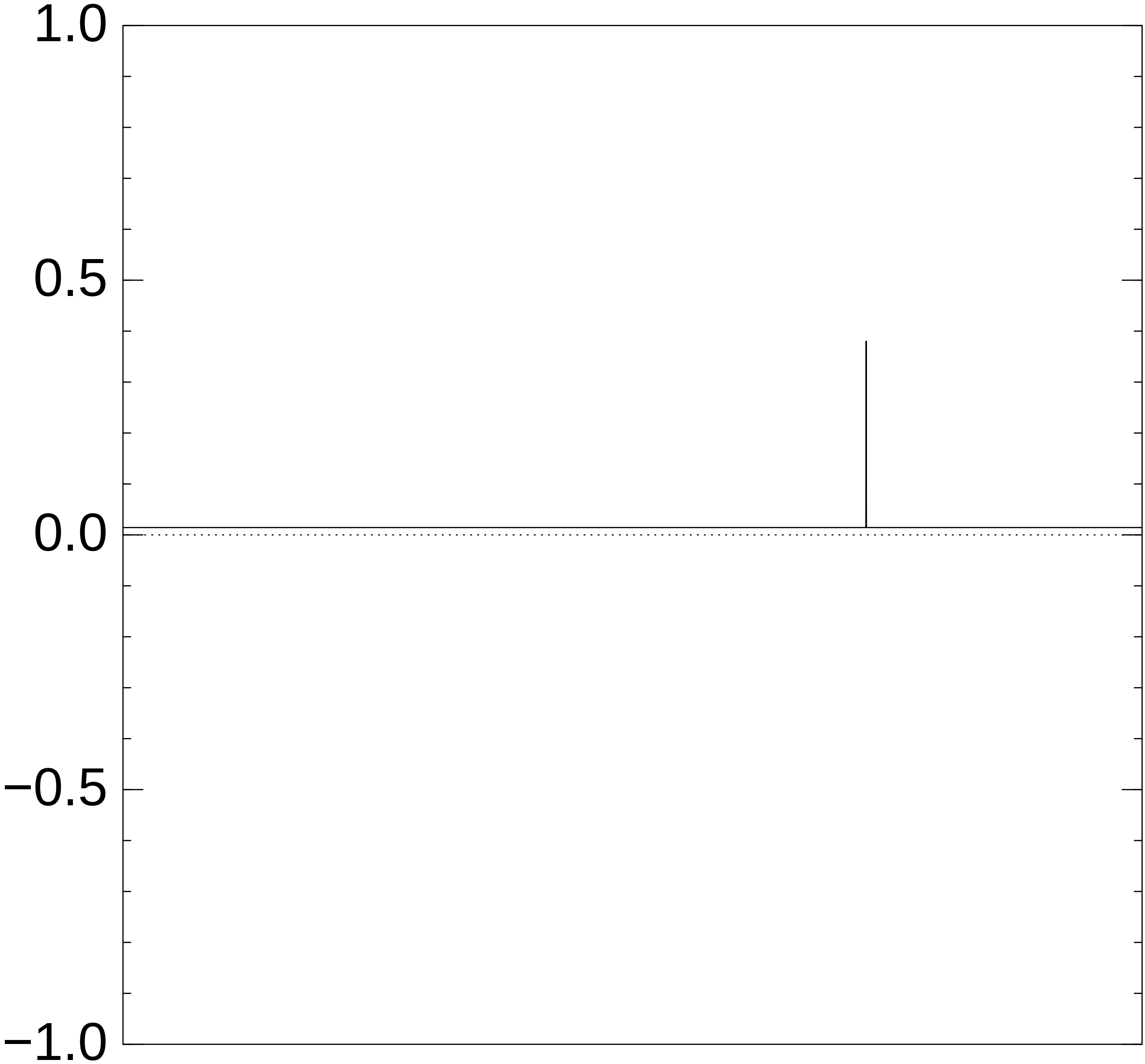
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $11 \times (\text{Step } 1 + \text{Step } 2)$:



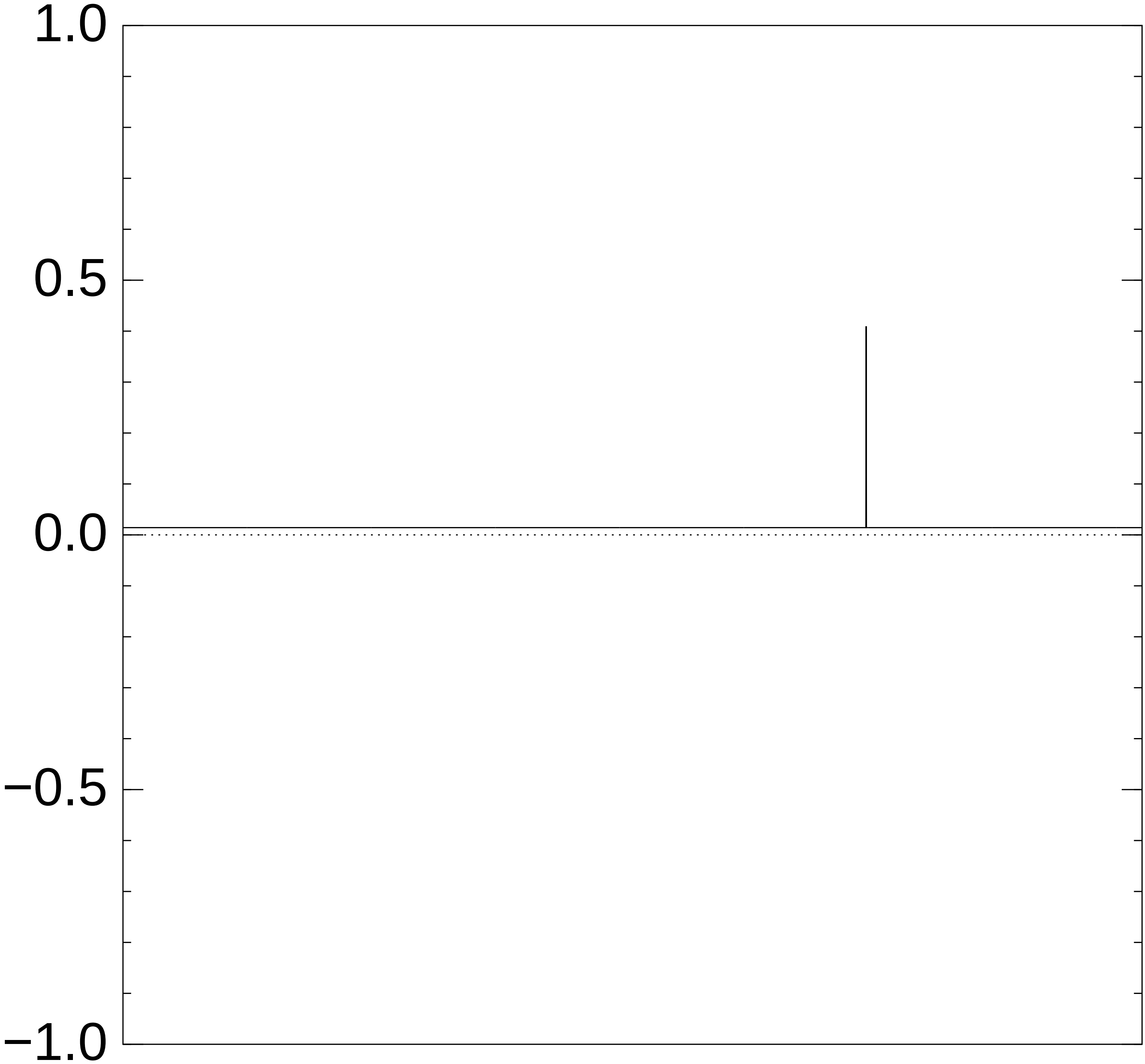
Graph of $q \mapsto a_q$

for an example with $n = 12$

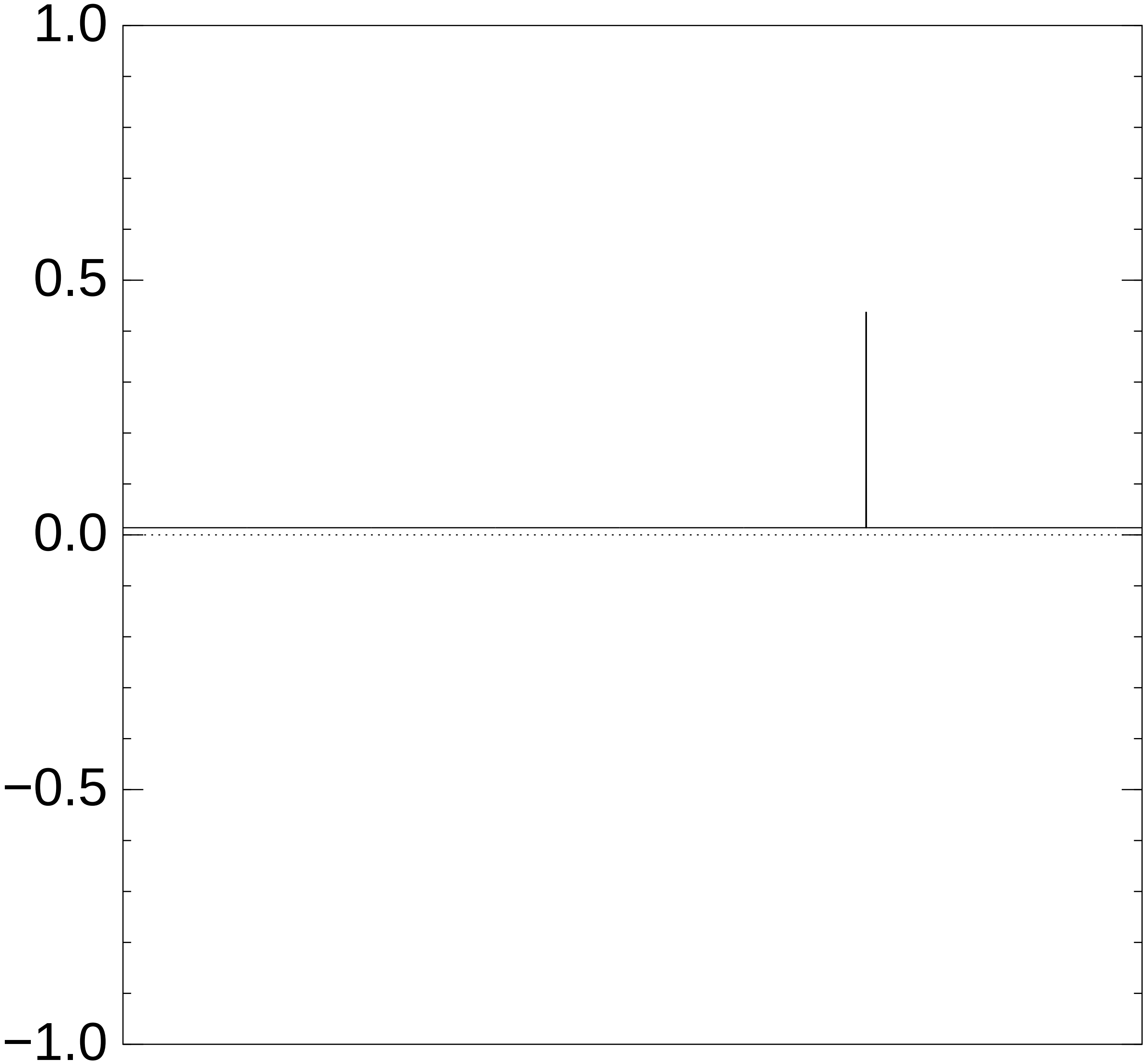
after $12 \times$ (Step $1 +$ Step 2):



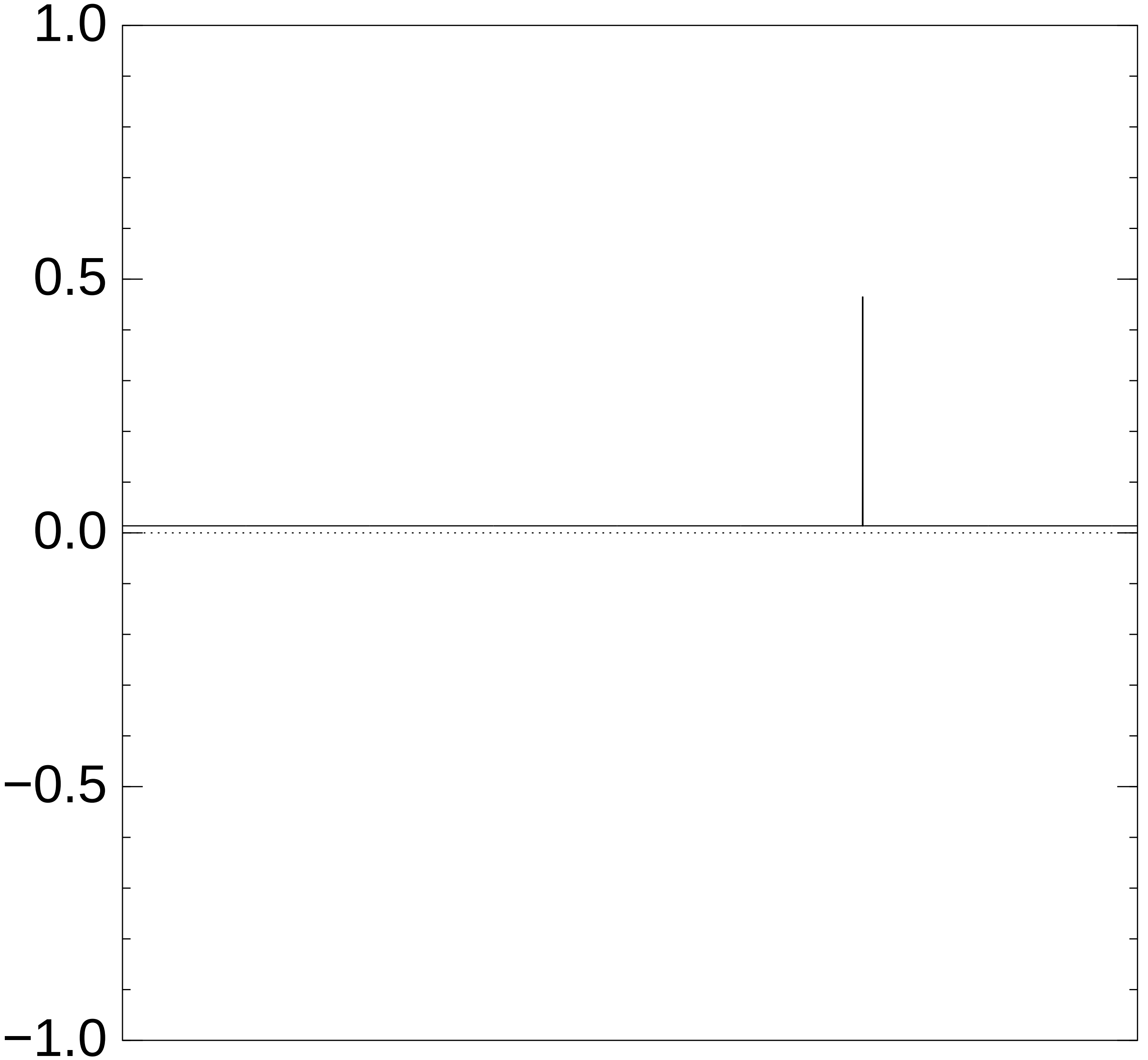
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $13 \times$ (Step $1 +$ Step 2):



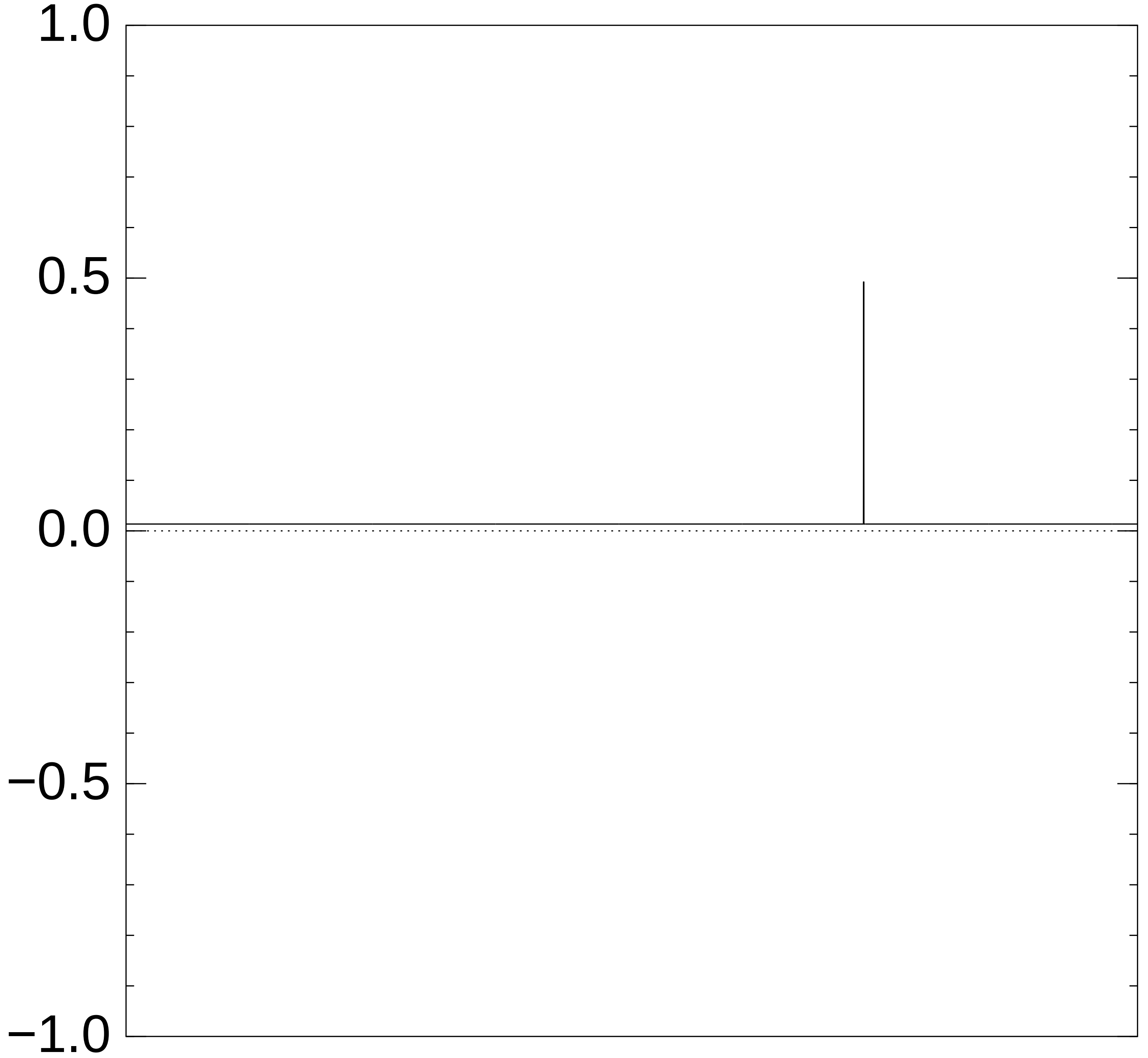
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $14 \times (\text{Step } 1 + \text{Step } 2)$:



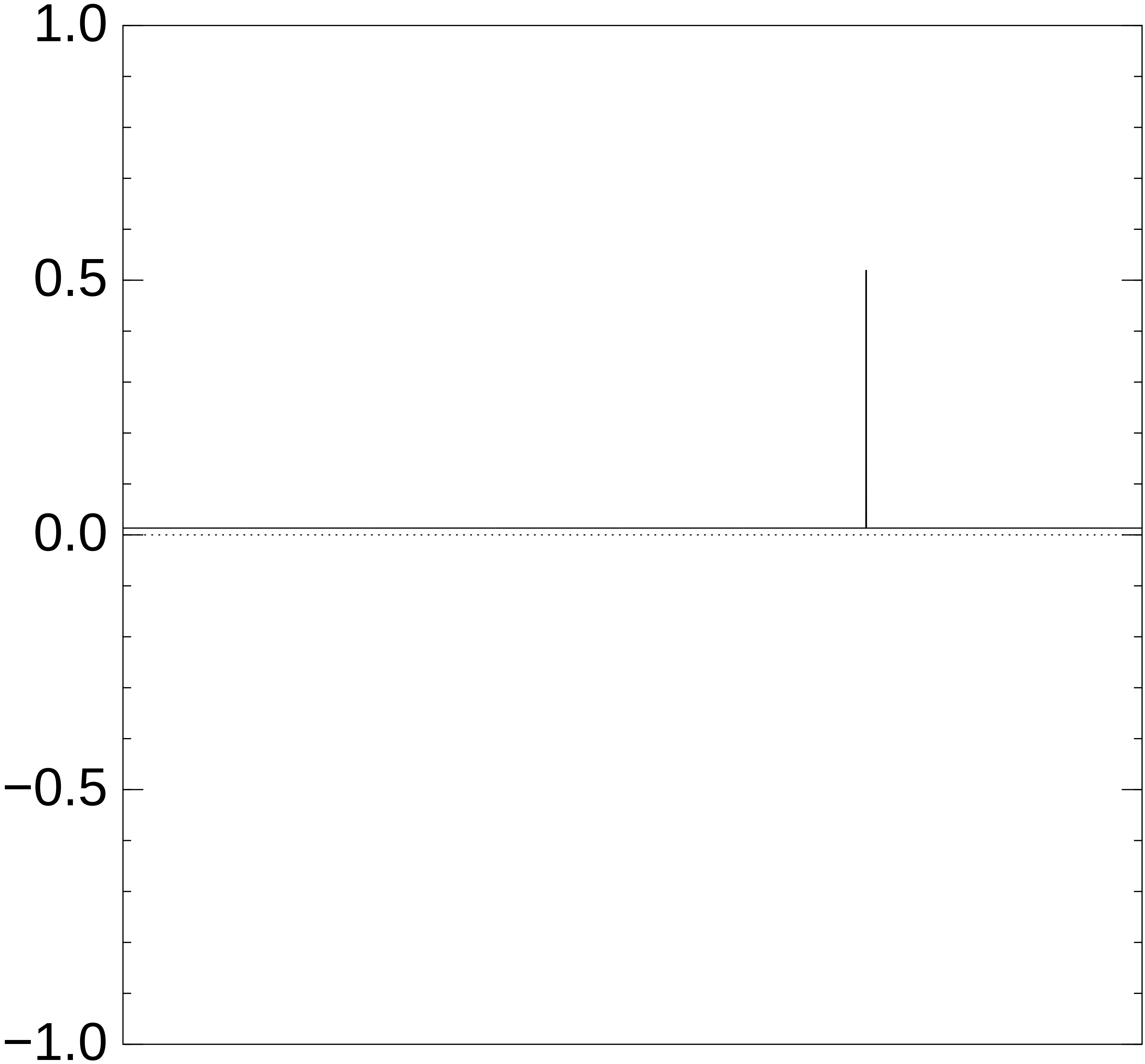
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $15 \times$ (Step $1 +$ Step 2):



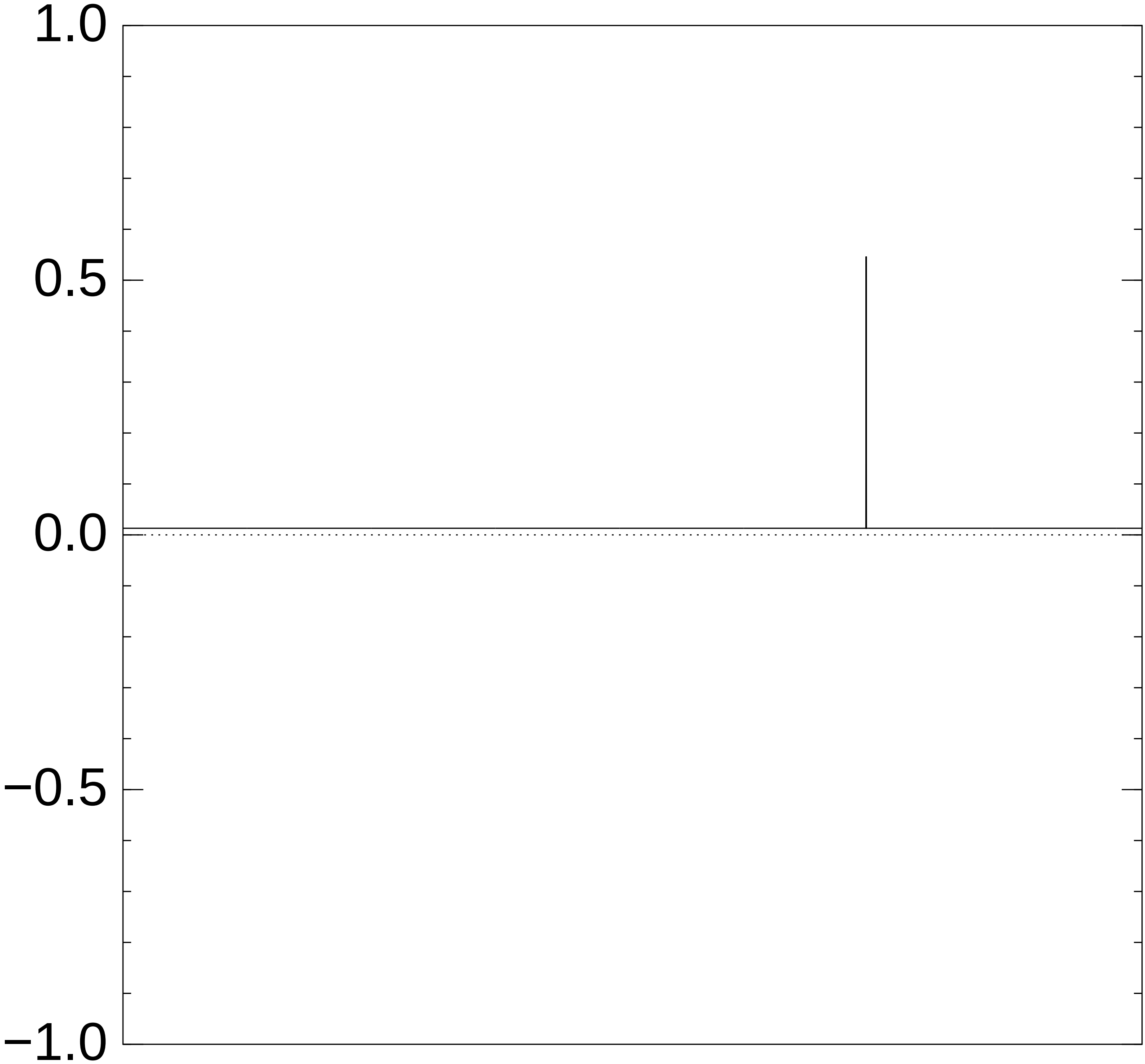
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $16 \times$ (Step $1 +$ Step 2):



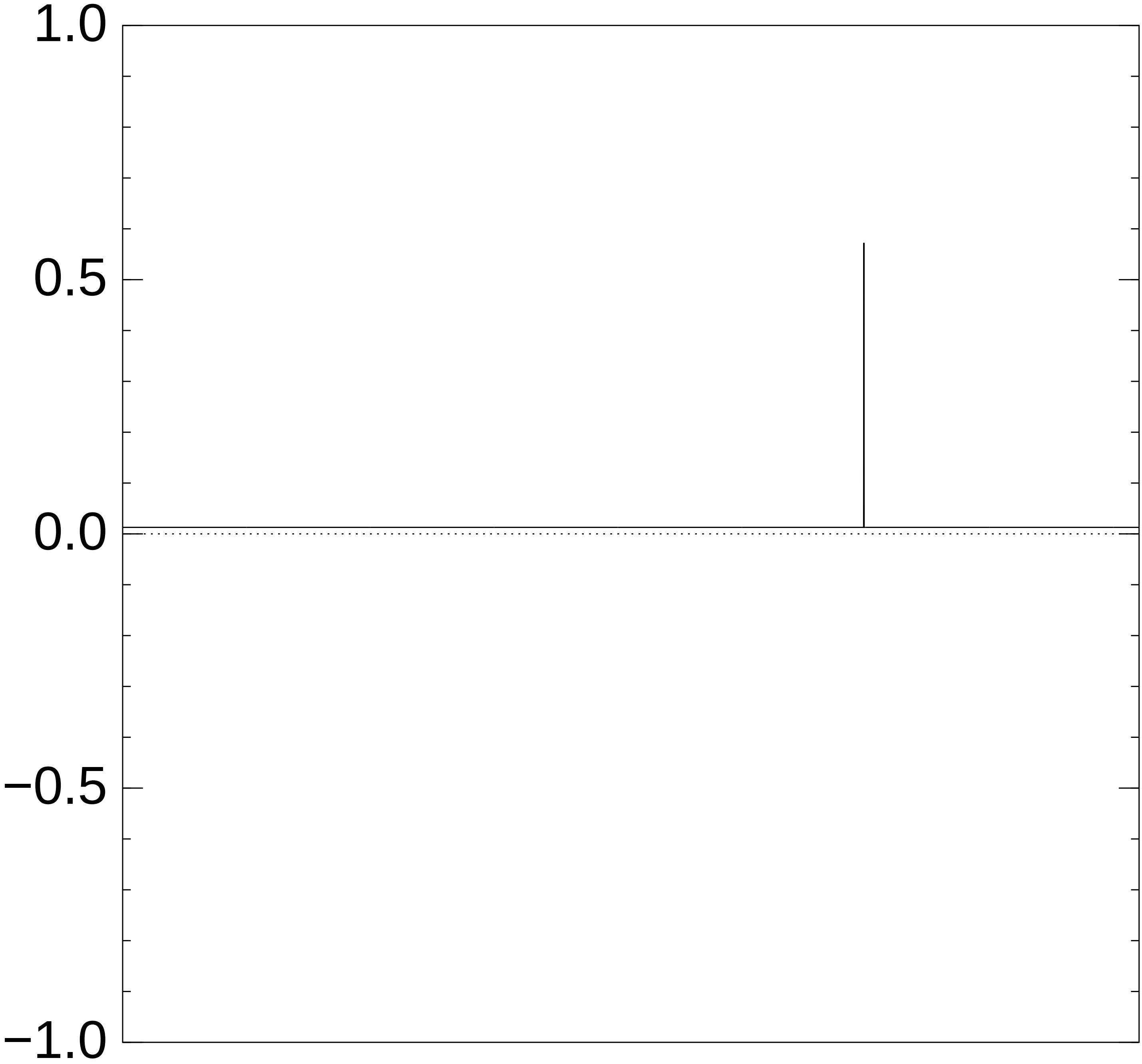
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $17 \times$ (Step $1 +$ Step 2):



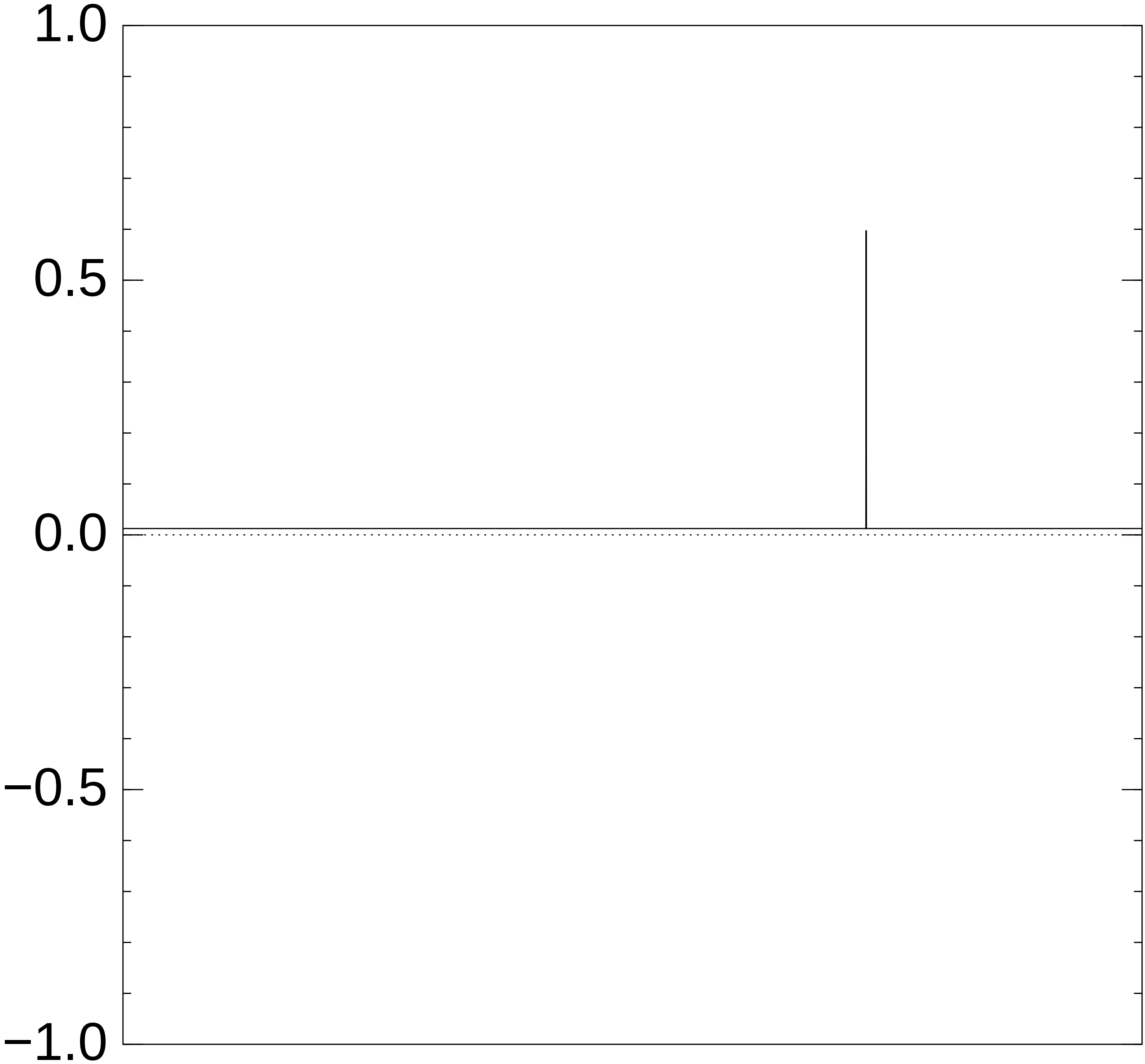
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $18 \times (\text{Step } 1 + \text{Step } 2)$:



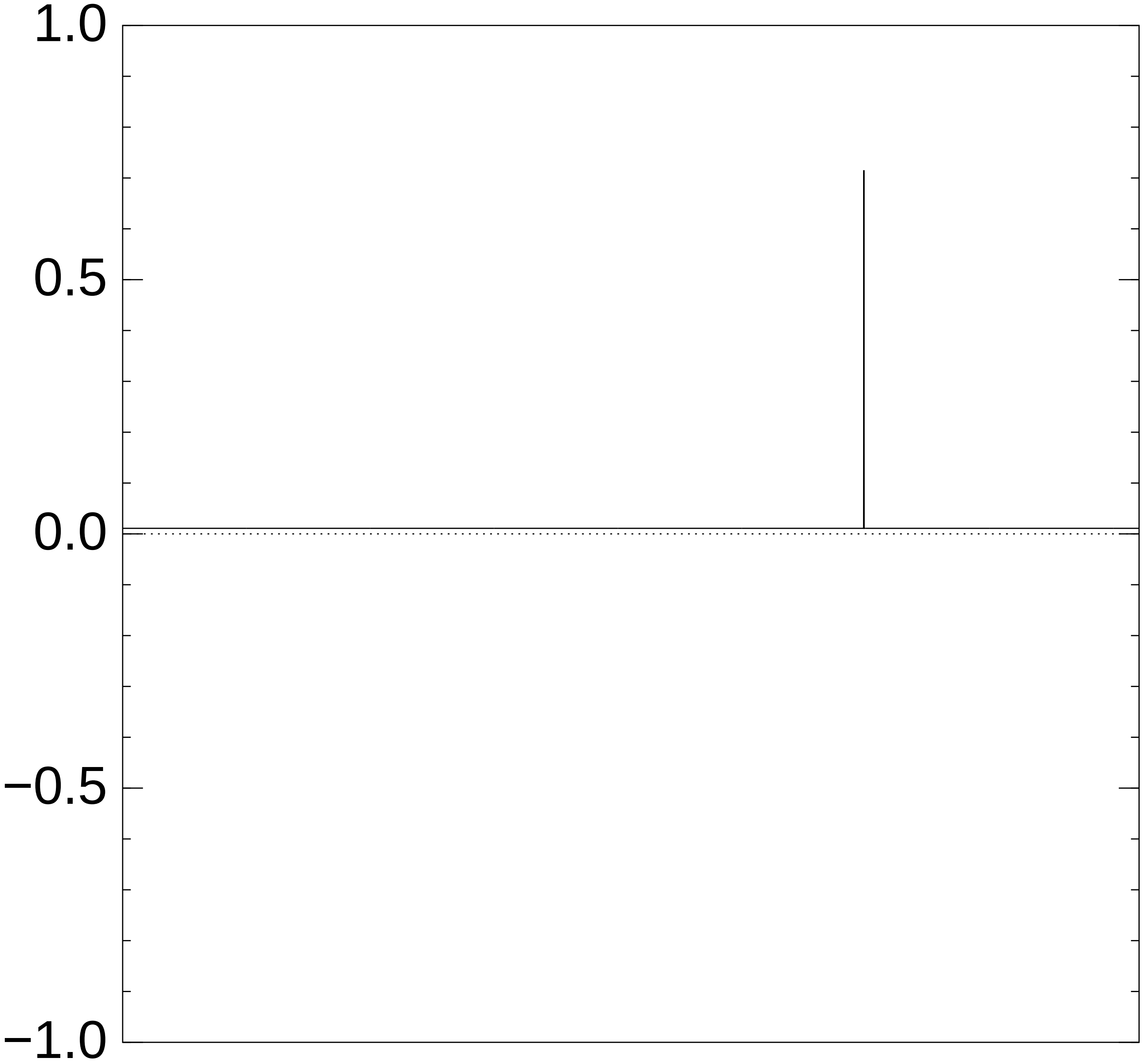
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $19 \times (\text{Step } 1 + \text{Step } 2)$:



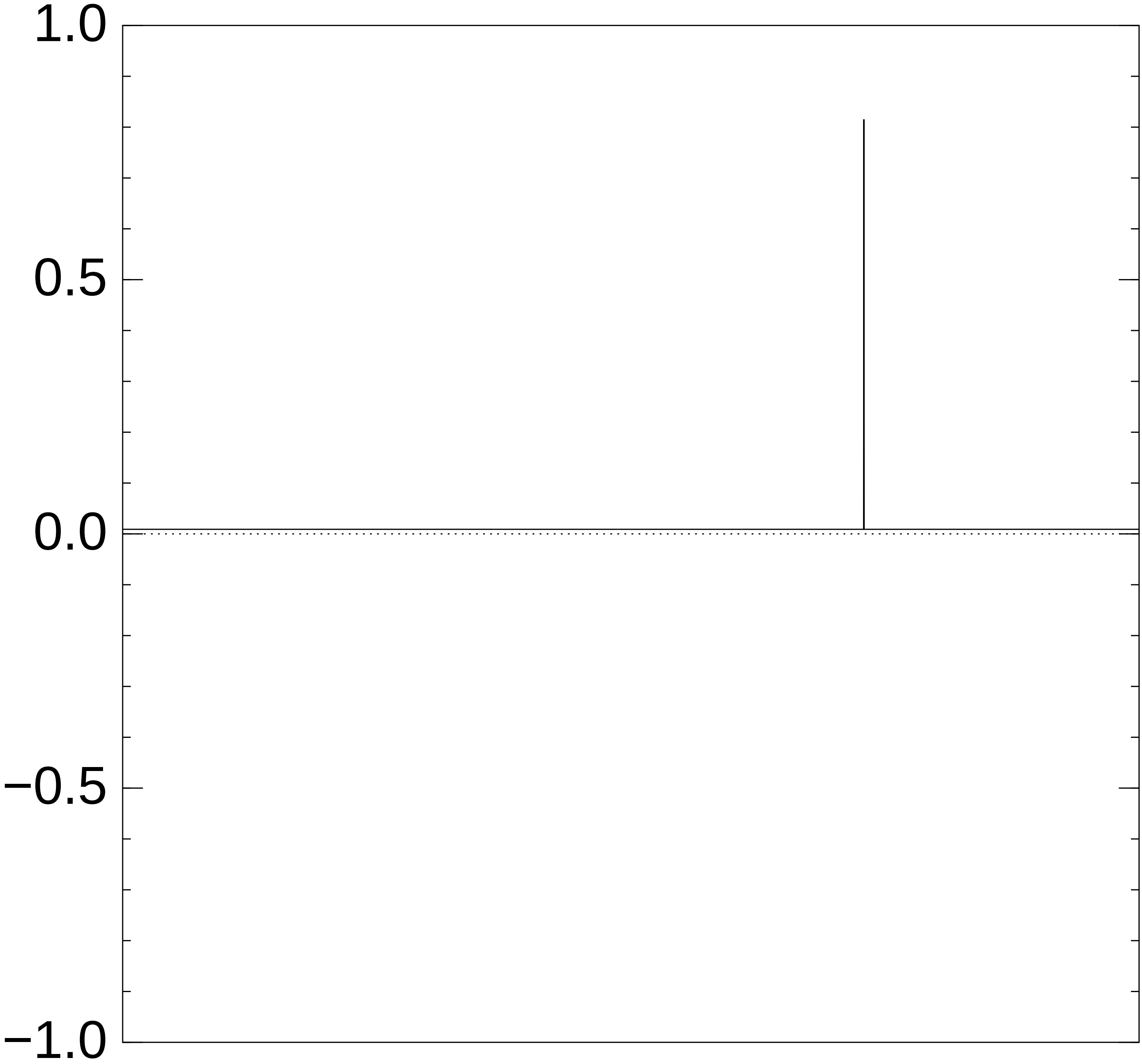
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $20 \times$ (Step $1 +$ Step 2):



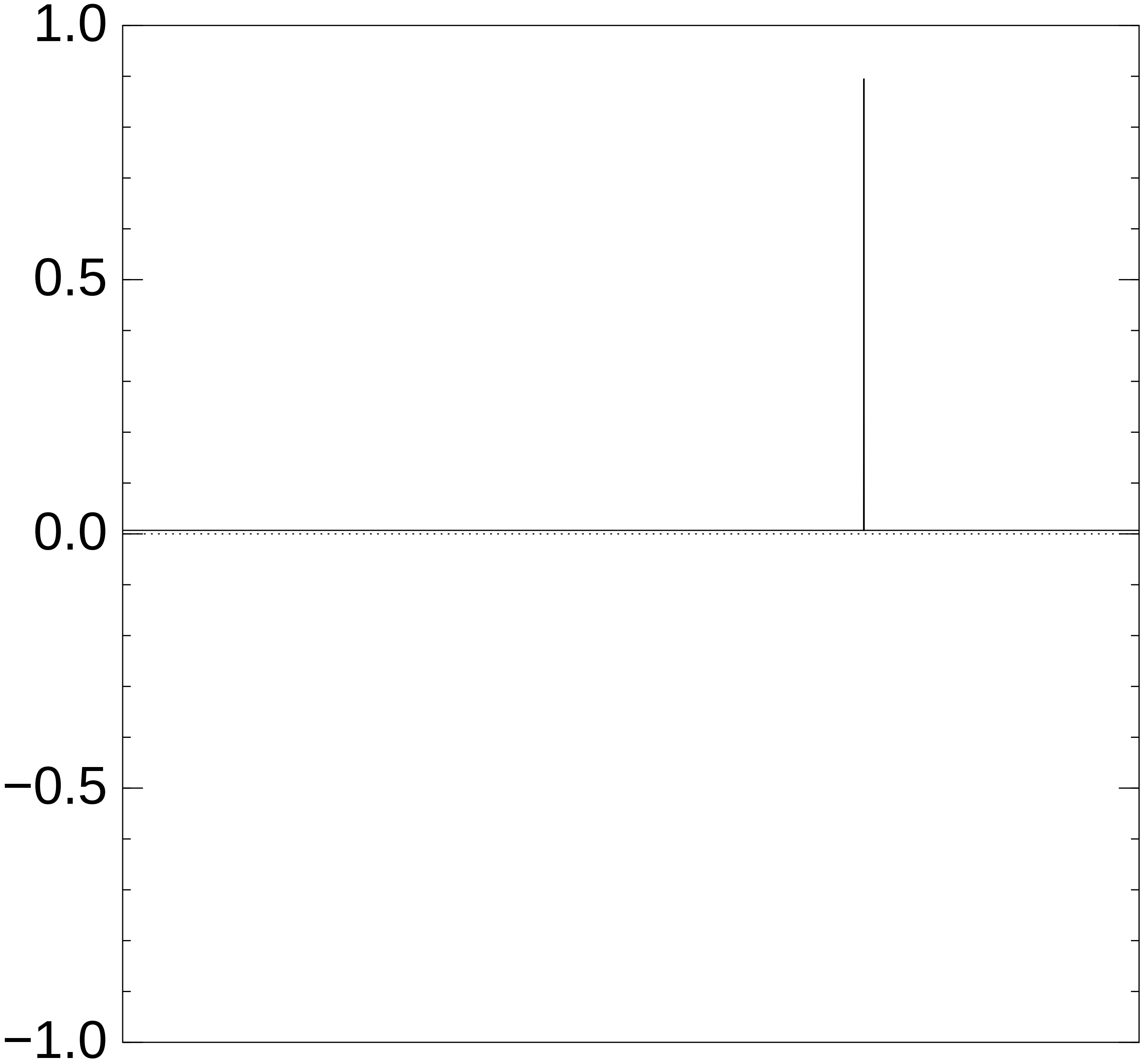
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $25 \times$ (Step $1 +$ Step 2):



Graph of $q \mapsto a_q$
for an example with $n = 12$
after $30 \times (\text{Step } 1 + \text{Step } 2)$:

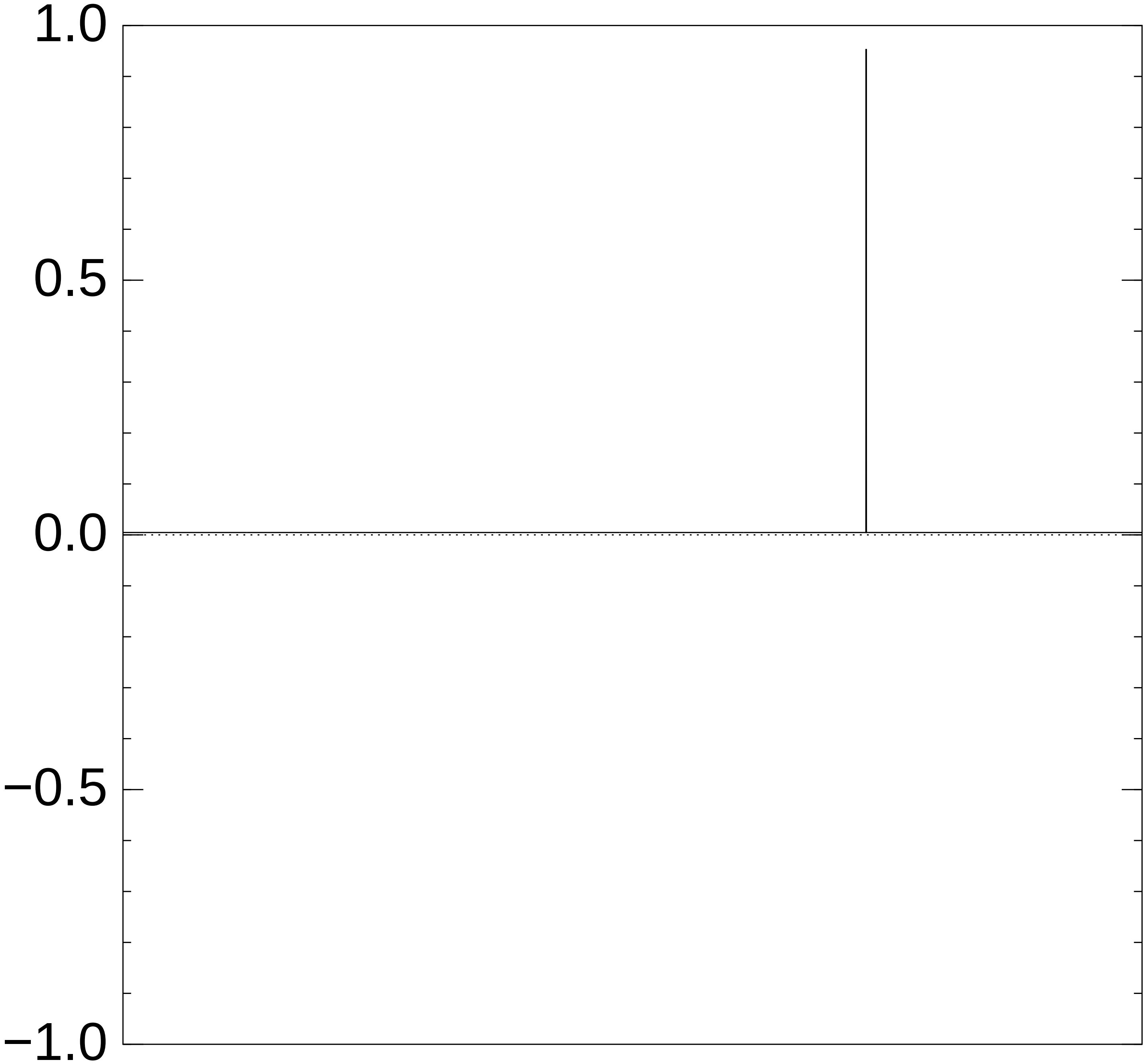


Graph of $q \mapsto a_q$
for an example with $n = 12$
after $35 \times$ (Step $1 +$ Step 2):

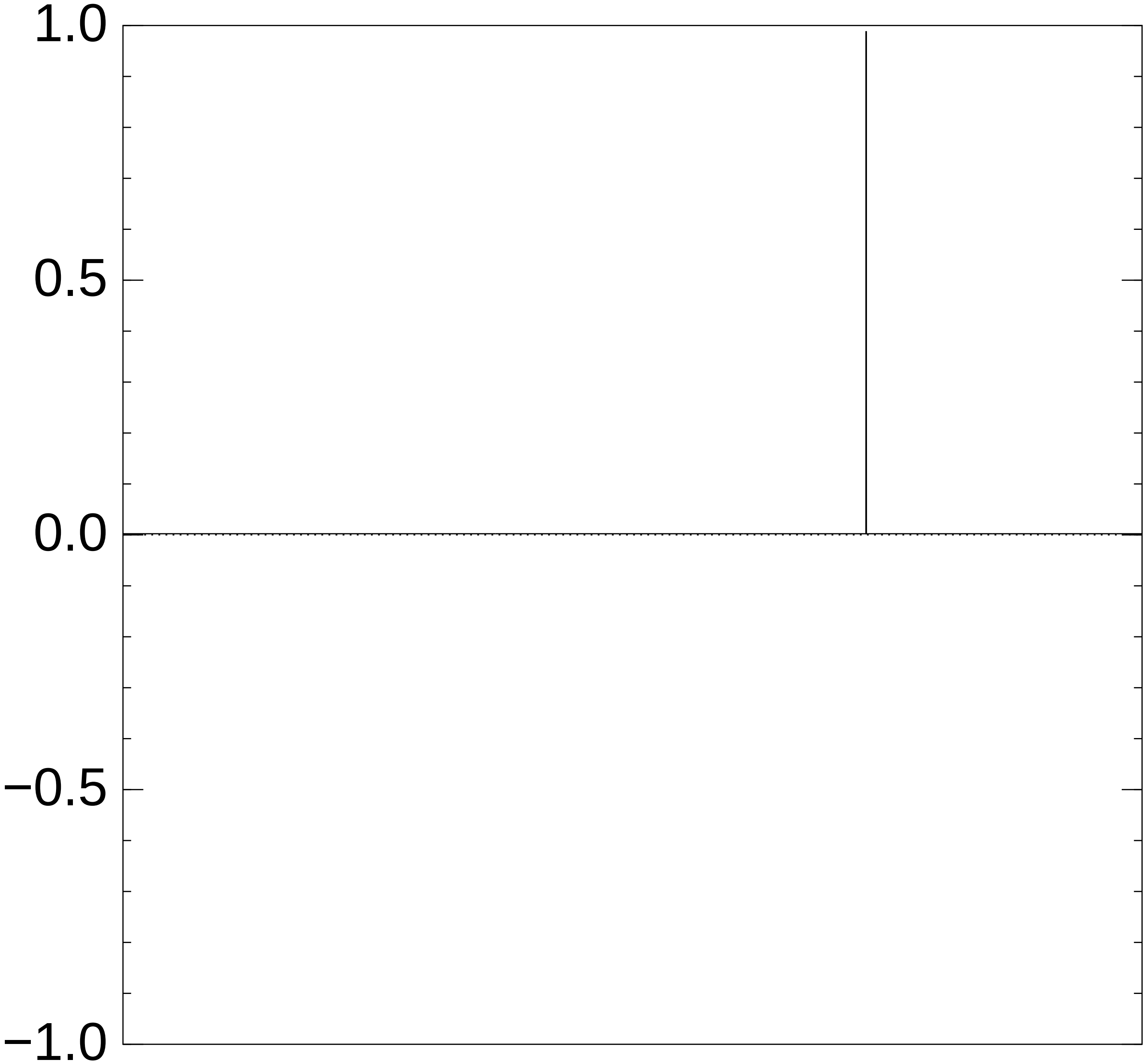


Good moment to stop, measure.

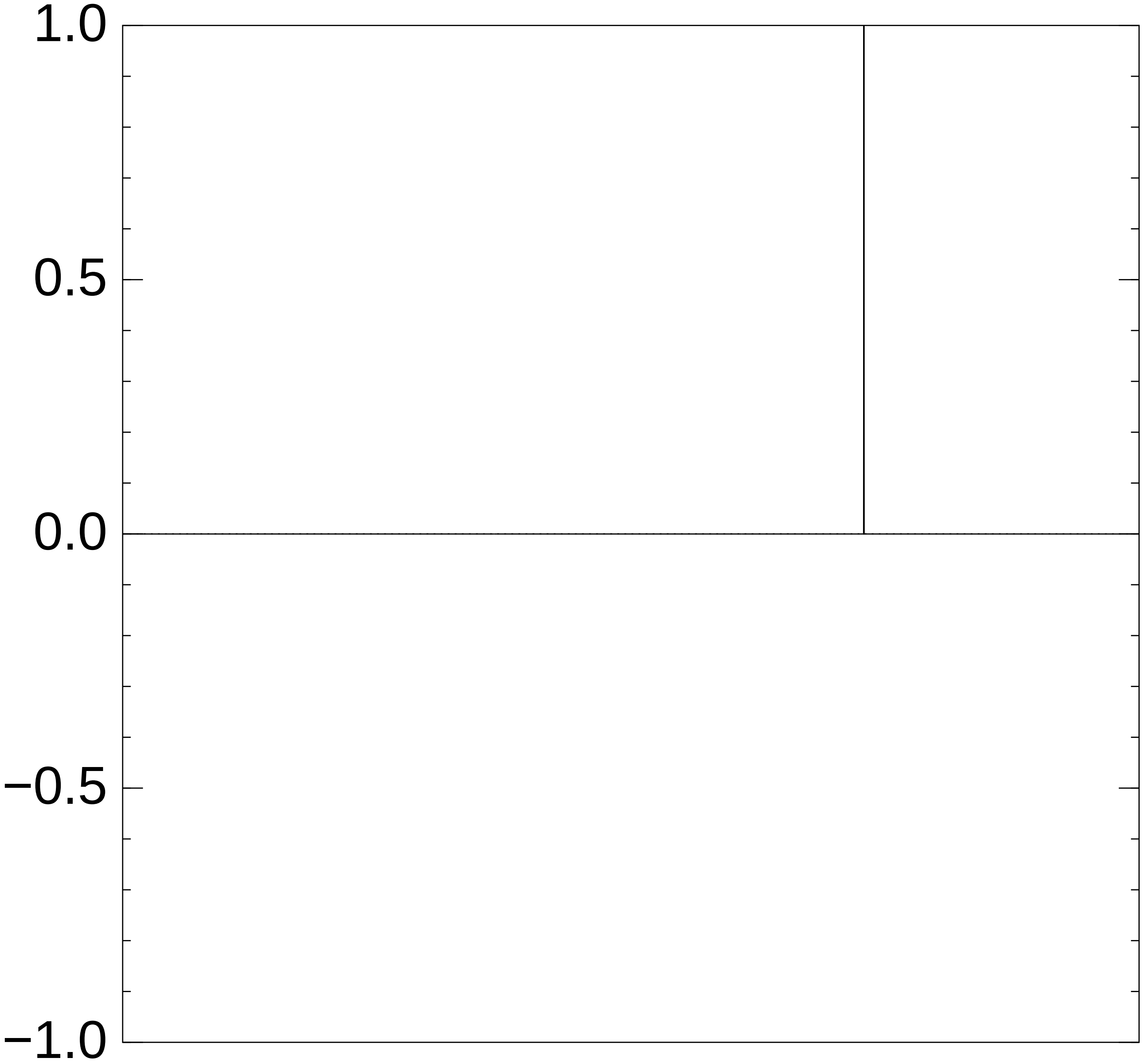
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $40 \times$ (Step $1 +$ Step 2):



Graph of $q \mapsto a_q$
for an example with $n = 12$
after $45 \times (\text{Step } 1 + \text{Step } 2)$:

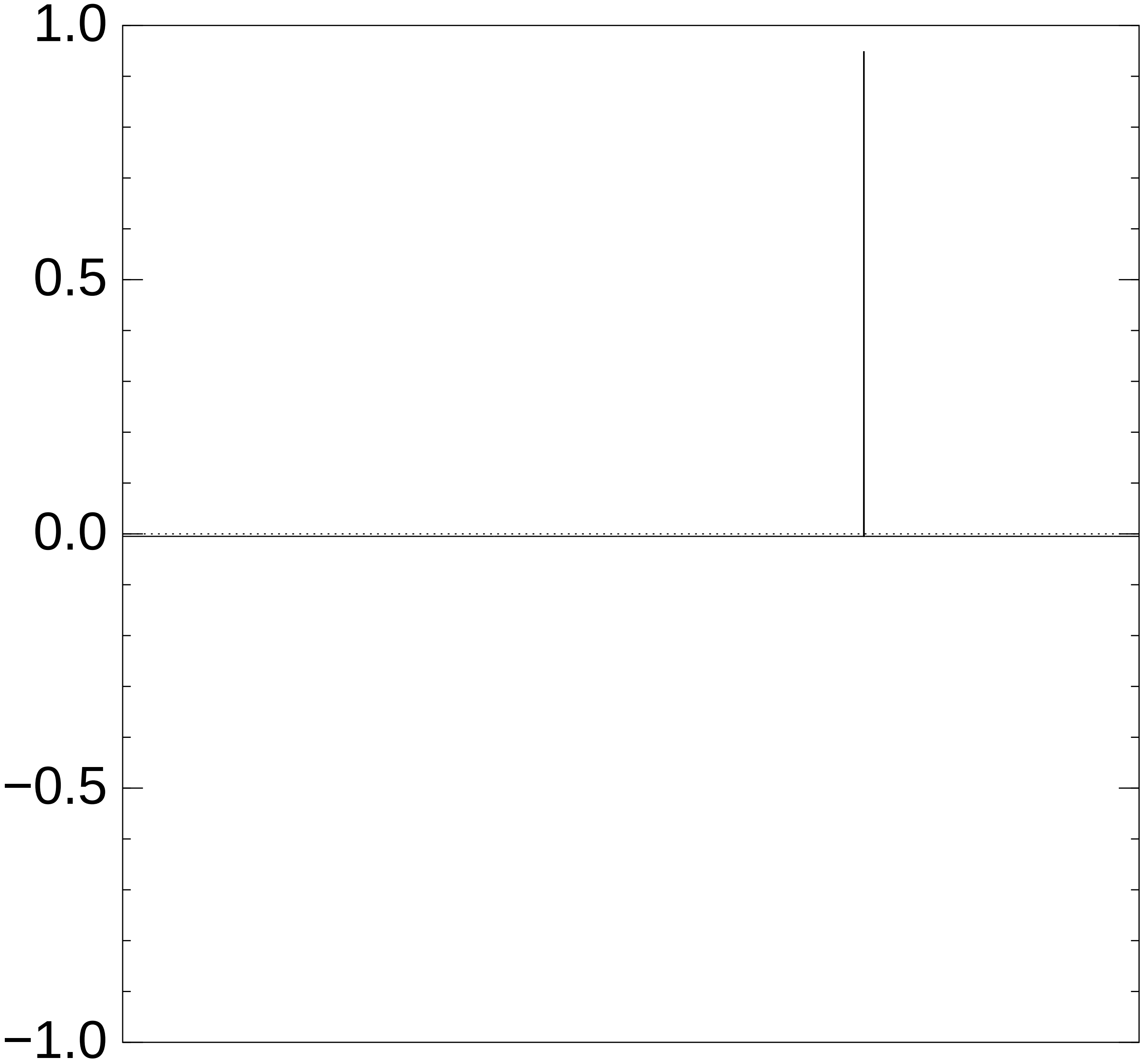


Graph of $q \mapsto a_q$
for an example with $n = 12$
after $50 \times$ (Step 1 + Step 2):

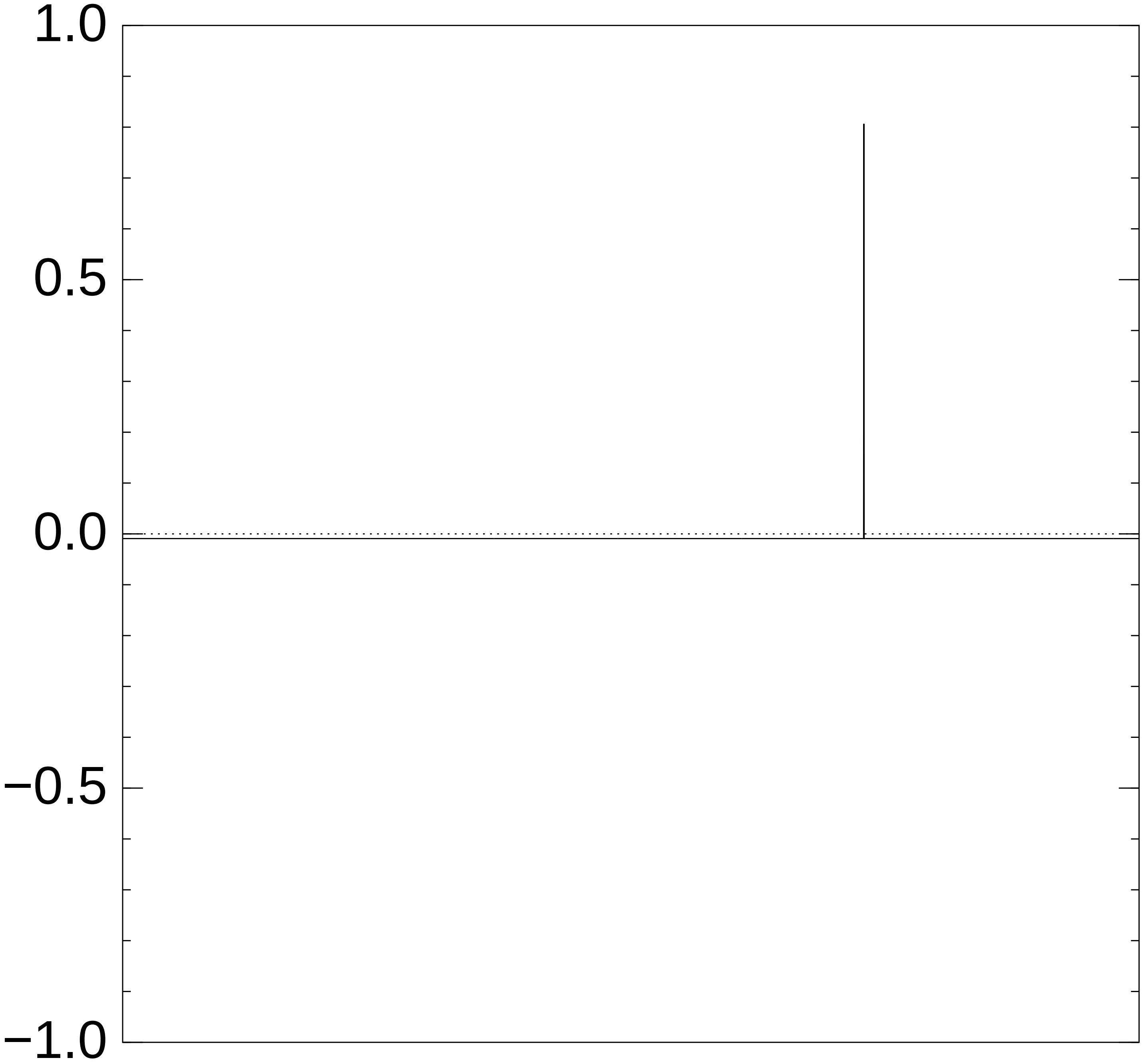


Traditional stopping point.

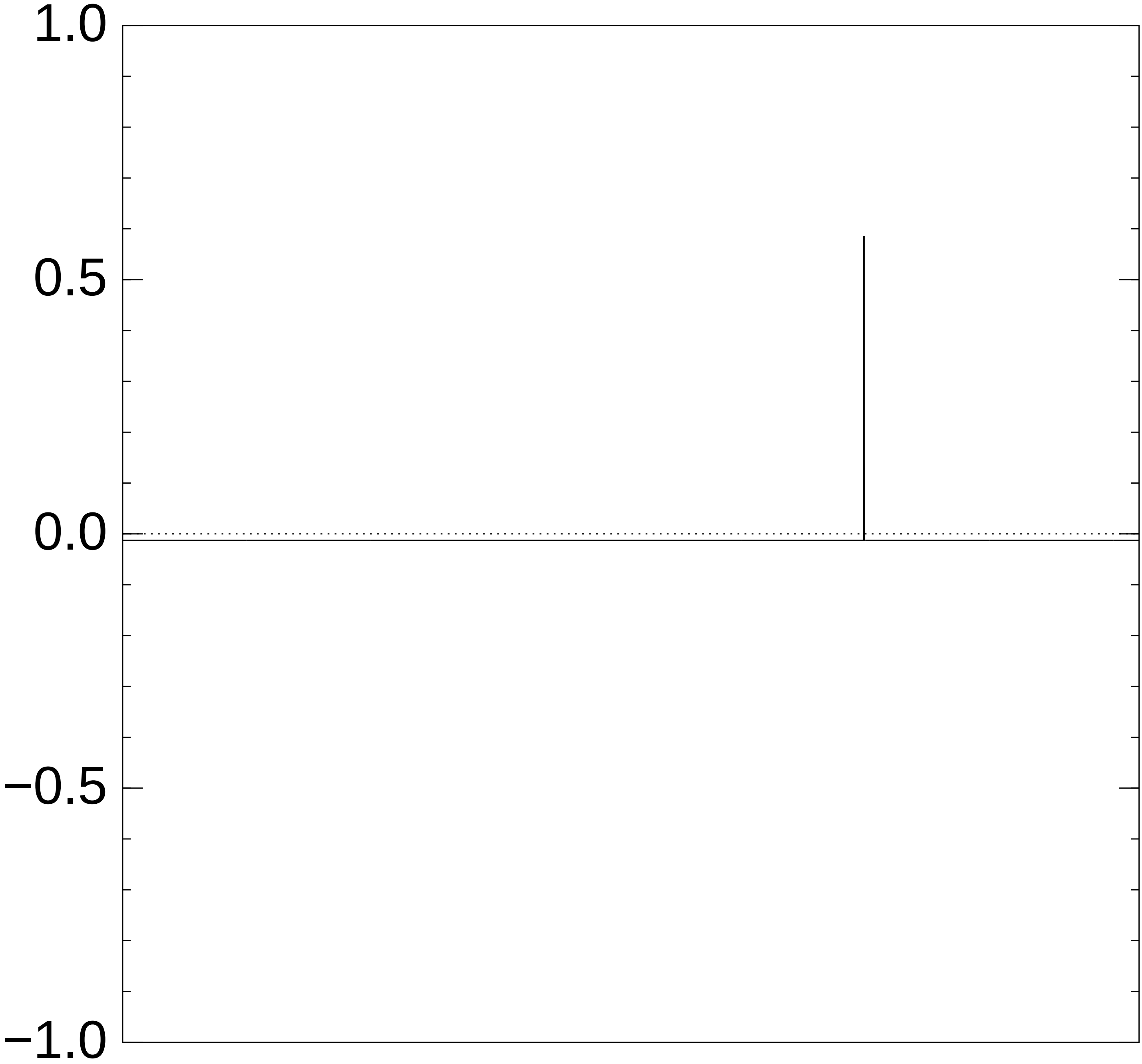
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $60 \times (\text{Step } 1 + \text{Step } 2)$:



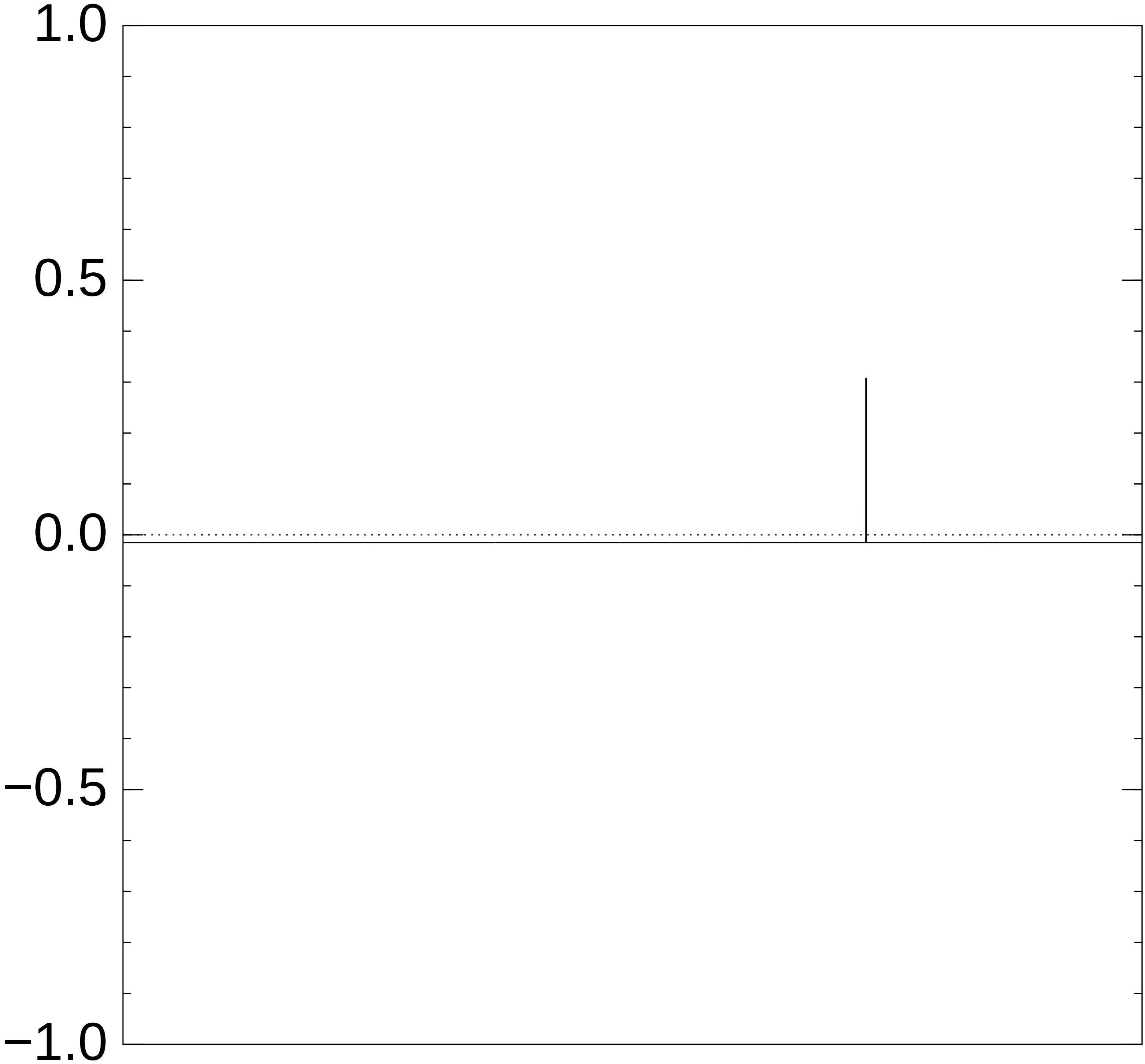
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $70 \times$ (Step $1 +$ Step 2):



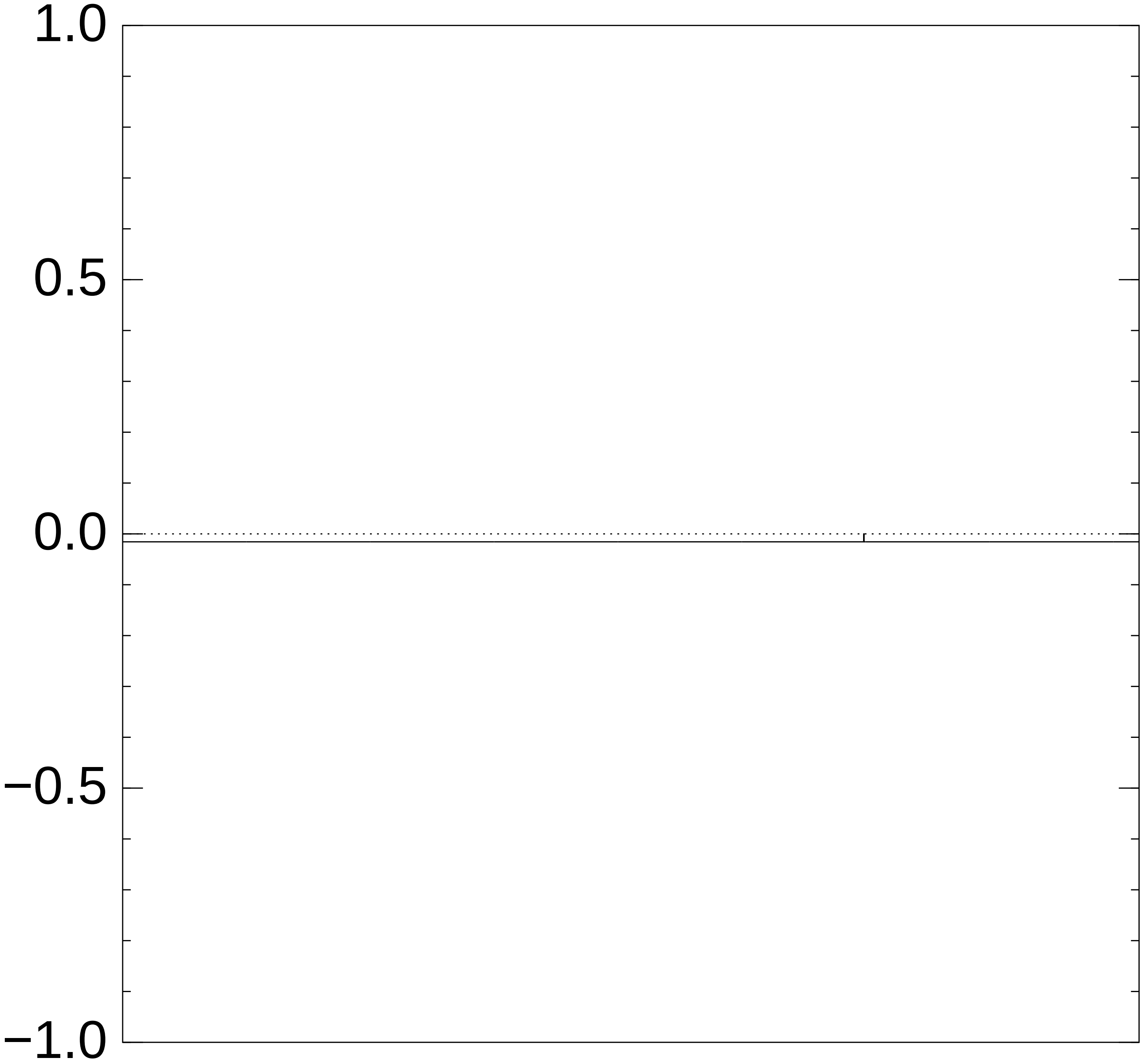
Graph of $q \mapsto a_q$
for an example with $n = 12$
after $80 \times$ (Step $1 +$ Step 2):



Graph of $q \mapsto a_q$
for an example with $n = 12$
after $90 \times$ (Step $1 +$ Step 2):



Graph of $q \mapsto a_q$
for an example with $n = 12$
after $100 \times$ (Step $1 +$ Step 2):



Very bad stopping point.

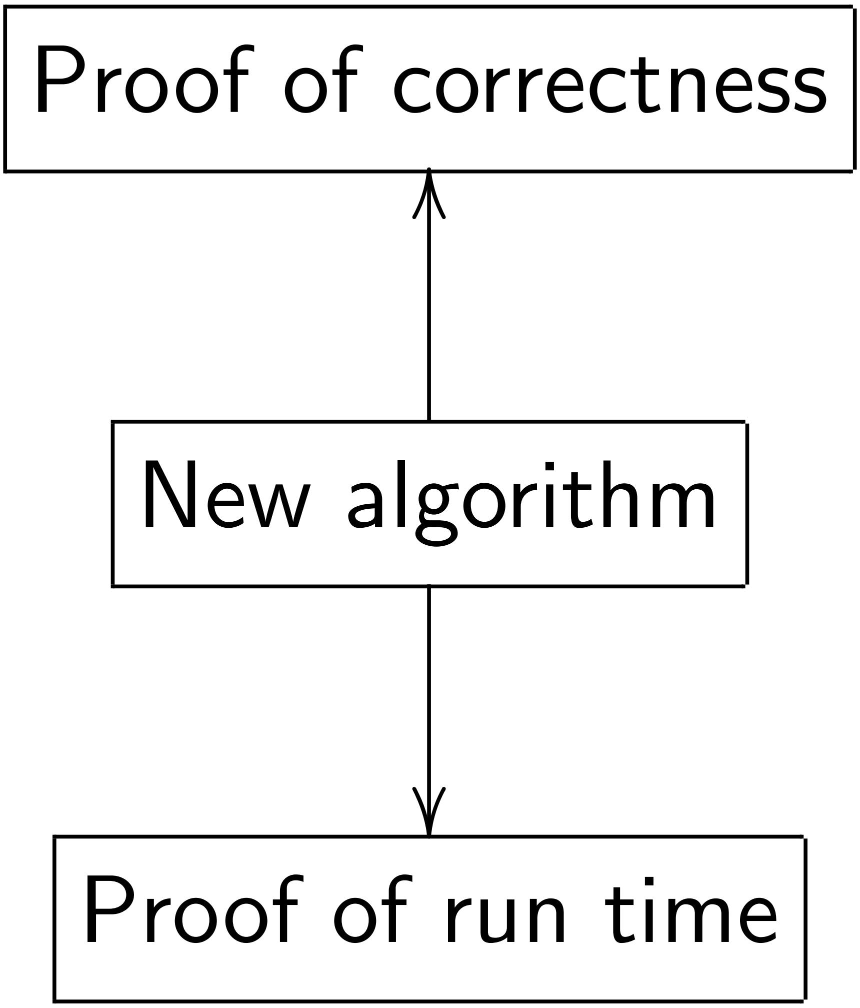
$q \mapsto a_q$ is completely described
by a vector of two numbers
(with fixed multiplicities):
(1) a_q for roots q;
(2) a_q for non-roots q.

Step 1 + Step 2
act linearly on this vector.

Easily compute eigenvalues
and powers of this linear map
to understand evolution
of state of Grover's algorithm.
$\Rightarrow$ Probability is ≈ 1
after $\approx(\pi/4)2^{0.5n}$ iterations.

Notes on provability

Textbook algorithm analysis:



Mislead students into thinking that best algorithm = best *proven* algorithm.

Reality: state-of-the-art cryptanalytic algorithms are almost never proven.

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"Work harder, find proofs!"

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Consensus of the experts:
proofs probably do not *exist*
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So demanding proofs is silly.

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Ignorant response:
"Work harder, find proofs!"

Consensus of the experts: proofs probably do not *exist* for most of these algorithms. So demanding proofs is silly.

Without proofs, how do we analyze correctness+speed? Answer: Real algorithm analysis relies critically on heuristics and **computer experiments**.

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Want to analyze, optimize
quantum algorithms *today*
to figure out safe crypto
against *future* quantum attack.

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2. Faster algorithm-specific simulation? Yes, sometimes.

3. Fast **trapdoor simulation.**
Simulator (like prover) knows more than the algorithm does.
Tung Chou has implemented this, found errors in two publications.