Batch NFS

D. J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Tanja Lange
Technische Universiteit Eindhoven

In this talk $\log L$ means
$(1 + o(1))(\log N)^{1/3}(\log\log N)^{2/3}$.
L is often written
"$L_N(1/3)$" or "$L_N(1/3)^{1+o(1)}$".

Exponents of L in this talk
are limited to $10^{-6}\mathbf{Z}$.

Rigorously proven? Ha ha ha.

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Best result known for *one* key
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Eratosthenes for s

Sieving small integ
using primes $2, 3, 5$

1	
2	2
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4	2 2
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6	2 3
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8	2 2 2
9	3 3
10	2 5
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12	2 2 3
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14	2 7
15	3 5
16	2 2 2 2
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Eratosthenes for smoothness

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
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9	3 3
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etc.

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1	
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3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
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12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

The **Q** s

Sieving
using pri

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8	2 2 2
9	3
10	2
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12	2 2 3
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14	2
15	3
16	2 2 2 2
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18	2 3
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Eratosthenes for smoothness

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

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3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

The **Q** sieve

Sieving i and 611
using primes $2, 3, 5$

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

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622	2
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624	2
625	
626	2
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628	2
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630	2
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Eratosthenes for smoothness

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

The **Q** sieve

Sieving i and $611 + i$ for sm
using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2 2 3 3
613	
614	2
615	3 5
616	2 2 2
617	
618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	
624	2 2 2 2 3
625	5
626	2
627	3
628	2 2
629	
630	2 3 3 5
631	

etc.

Eratosthenes for smoothness

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

etc.

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
13	
14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2 2 3 3
613	
614	2
615	3 5
616	2 2 2 7
617	
618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	7
624	2 2 2 2 3
625	5 5 5 5
626	2
627	3
628	2 2
629	
630	2 3 3 5 7
631	

etc.

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small integers $i > 0$
imes 2, 3, 5, 7:

	5
	7
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	5
	7
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The **Q** sieve

Sieving i and $611 + i$ for small i
using primes 2, 3, 5, 7:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
11	
12	2 2 3
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14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
20	2 2 5

612	2 2 3 3
613	
614	2
615	3 5
616	2 2 2 7
617	
618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	7
624	2 2 2 2 3
625	5 5 5 5
626	2
627	3
628	2 2
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etc.

Have co
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$14 \cdot 625$
$64 \cdot 675$
$75 \cdot 686$

$14 \cdot 64 \cdot$
$= 2^8 3^4 5$

$\gcd\{611$
$= 47.$

$611 = 4$

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gers $i > 0$
$5, 7$:

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1				
2	2			
3		3		
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8	2	2	2	
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12	2	2	3	
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15		3	5	
16	2	2	2	2
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18	2	3	3	
19				
20	2	2	5	

612	2	2	3	3		
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616	2	2	2			7
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618	2		3			
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622	2					
623						7
624	2	2	2	2	3	
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626	2					
627			3			
628	2	2				
629						
630	2		3	3	5	7
631						

etc.

Have complete fac
the congruences i
for some i's.

$$14 \cdot 625 = 2^1 3^0 5^4$$
$$64 \cdot 675 = 2^6 3^3 5^2$$
$$75 \cdot 686 = 2^1 3^1 5^2$$

$$14 \cdot 64 \cdot 75 \cdot 625 \cdot 6$$
$$= 2^8 3^4 5^8 7^4 = (2^4$$

$$\gcd\{611, 14 \cdot 64 \cdot$$
$$= 47.$$

$$611 = 47 \cdot 13.$$

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The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1	
2	2
3	3
4	2 2
5	5
6	2 3
7	7
8	2 2 2
9	3 3
10	2 5
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12	2 2 3
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14	2 7
15	3 5
16	2 2 2 2
17	
18	2 3 3
19	
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613	
614	2
615	3 5
616	2 2 2 7
617	
618	2 3
619	
620	2 2 5
621	3 3 3
622	2
623	7
624	2 2 2 2 3
625	5 5 5 5
626	2
627	3
628	2 2
629	
630	2 3 3 5 7
631	

etc.

Have complete factorization
the congruences $i \equiv 611 + i$
for some i's.

$$14 \cdot 625 = 2^1 3^0 5^4 7^1.$$
$$64 \cdot 675 = 2^6 3^3 5^2 7^0.$$
$$75 \cdot 686 = 2^1 3^1 5^2 7^3.$$

$$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$$
$$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2.$$

$$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5$$
$$= 47.$$

$$611 = 47 \cdot 13.$$

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1				
2	2			
3		3		
4	2	2		
5			5	
6	2	3		
7				7
8	2	2	2	
9		3	3	
10	2		5	
11				
12	2	2	3	
13				
14	2			7
15		3	5	
16	2	2	2	2
17				
18	2	3	3	
19				
20	2	2	5	

612	2	2		3	3			
613								
614	2							
615				3		5		
616	2	2	2					7
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618	2			3				
619								
620	2	2				5		
621				3	3	3		
622	2							
623								7
624	2	2	2	2	3			
625						5	5	5 5
626	2							
627				3				
628	2	2						
629								
630	2			3	3	5		7
631								

etc.

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

$$14 \cdot 625 = 2^1 3^0 5^4 7^1.$$
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$$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$$
$$= 47.$$

$$611 = 47 \cdot 13.$$

sieve

i and $611 + i$ for small i
imes $2, 3, 5, 7$:

5									
	7								
3									
5									
	7								
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612	2	2			3	3			
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615					3			5	
616	2	2	2						7
617									
618	2				3				
619									
620	2	2						5	
621					3	3	3		
622	2								
623									7
624	2	2	2	2	3				
625							5	5	5 5
626	2								
627					3				
628	2	2							
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630	2				3	3		5	7
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$611 = 47 \cdot 13$.

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Optimal
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5, 7:

2		3	3				
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		3					
2				5			
		3	3	3			
							7
2	2	2	3				
				5	5	5	5
		3					
2							
		3	3		5		7

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

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The number-field

Generalize $i \equiv i +$

$\to a \equiv a + bN$ (

$\to a - bm \equiv a - b$

for root $\alpha \in \mathbf{C}$

of nonzero integer

For any m can fin

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Optimal choice of

$(\mu + o(1))(\log N)^2$

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5 5 5

7

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
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$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \pmod{}$
$\to a \equiv a + bN \pmod{N}$
$\to a - bm \equiv a - b\alpha \pmod{}$
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
produces factorization of N.

Optimal choice of $\log m$ is
$(\mu + o(1))(\log N)^{2/3}(\log \log$

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

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$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

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The number-field sieve

Generalize $i \equiv i + N \pmod{N}$
$\to a \equiv a + bN \pmod{N}$
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gruences $i \equiv 611 + i$
e i's.

$= 2^1 3^0 5^4 7^1$.
$= 2^6 3^3 5^2 7^0$.
$= 2^1 3^1 5^2 7^3$.

$75 \cdot 625 \cdot 675 \cdot 686$
$^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$

$7 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \pmod{N}$
$\to a \equiv a + bN \pmod{N}$
$\to a - bm \equiv a - b\alpha \pmod{m - \alpha}$
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Sieve L^1
Find $L^{0.}$
with $a -$
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$\equiv 611 + i$

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The number-field sieve

Generalize $i \equiv i + N \pmod{N}$

$\to a \equiv a + bN \pmod{N}$

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RAM cost analysis

1993 Buhler–Lenst

Smoothness bound

Sieve $L^{1.923000}$ pa

Find $L^{0.961500}$ pair

with $a - bm$ and

Total RAM time L

using multiple num

(Multiple number

don't seem to com

with AT, factory,

of

$5^4 7^2\}$

The number-field sieve

Generalize $i \equiv i + N \pmod{N}$

$\to a \equiv a + bN \pmod{N}$

$\to a - bm \equiv a - b\alpha \pmod{m - \alpha}$

for root $\alpha \in \mathbf{C}$

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For any m can find α

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$(\mu + o(1))(\log N)^{2/3}(\log \log N)^{1/3}$.

RAM cost analysis

1993 Buhler–Lenstra–Pomer

Smoothness bound $L^{0.961500}$

Sieve $L^{1.923000}$ pairs (a, b).

Find $L^{0.961500}$ pairs

with $a - bm$ and $a - b\alpha$ sm

Total RAM time $L^{1.923000}$.

1993 Coppersmith:

Total RAM time $L^{1.901884}$

using multiple number fields

(Multiple number fields

don't seem to combine well

with AT, factory, et al.)

The number-field sieve

Generalize $i \equiv i + N \pmod{N}$

$\to a \equiv a + bN \pmod{N}$

$\to a - bm \equiv a - b\alpha \pmod{m - \alpha}$

for root $\alpha \in \mathbf{C}$

of nonzero integer poly.

For any m can find α

so that factoring $m - \alpha$

produces factorization of N.

Optimal choice of $\log m$ is

$(\mu + o(1))(\log N)^{2/3}(\log \log N)^{1/3}$.

RAM cost analysis

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The factorization factory

1993 Coppersmith:
There *exists* an algorithm
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with same #bits as N
in RAM time $L^{1.638587}$.

Smoothness bound $L^{0.819290}$
Smaller than before,
so need more (a, b).

Algorithm *knows* all (a, b)
such that $a - bm$ is smooth
Note: one m works for all N
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Goal: Optimize AT asymptotics.

1. Generate (a, b) in parallel.
Test $a - bm$ for smoothness.

2. Make many copies of each N,
close to each (a, b) generator.
When smooth $a - bm$ is found,
test each $a - b\alpha_N$ for smoothness.

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Generate (a, b)
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If so, store.
Repeat.
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Is $a - b\alpha_1$ smooth? If so, store. Send (a, b). right. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b). up. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b). right. Repeat.
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Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.

Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.

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Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.

Is $a-b\alpha_1$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_2$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_3$ smooth? If so, store. Send (a,b). right. Repeat.	I If S do
Is $a-b\alpha_5$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a-b\alpha_6$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_7$ smooth? If so, store. Send (a,b). left. Repeat.	I If S le
Is $a-b\alpha_9$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{10}$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{11}$ smooth? If so, store. Send (a,b). right. Repeat.	Is If S do
Is $a-b\alpha_{13}$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a-b\alpha_{14}$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_{15}$ smooth? If so, store. Send (a,b). left. Repeat.	Is If S le

Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.

Is $a - b\alpha_1$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.
Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.	Generate (a,b). Is $a-bm$ smooth? If so, store. Repeat.

Is $a-b\alpha_1$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_2$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_3$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_4$ smooth? If so, store. Send (a,b). down. Repeat.
Is $a-b\alpha_5$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a-b\alpha_6$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_7$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_8$ smooth? If so, store. Send (a,b). left. Repeat.
Is $a-b\alpha_9$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{10}$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{11}$ smooth? If so, store. Send (a,b). right. Repeat.	Is $a-b\alpha_{12}$ smooth? If so, store. Send (a,b). down. Repeat.
Is $a-b\alpha_{13}$ smooth? If so, store. Send (a,b). up. Repeat.	Is $a-b\alpha_{14}$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_{15}$ smooth? If so, store. Send (a,b). left. Repeat.	Is $a-b\alpha_{16}$ smooth? If so, store. Send (a,b). left. Repeat.

$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}
$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}
$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}
$N_1, N_2, N_3,$ $N_5, N_6, N_7,$ N_9, N_{10}, N_{11} N_{13}, N_{14}, N_{15}

Generate (a, b).	Generate (a, b).
Is $a - bm$	Is $a - bm$
smooth?	smooth?
If so, store.	If so, store.
Repeat.	Repeat.
Generate (a, b).	Generate (a, b).
Is $a - bm$	Is $a - bm$
smooth?	smooth?
If so, store.	If so, store.
Repeat.	Repeat.
Generate (a, b).	Generate (a, b).
Is $a - bm$	Is $a - bm$
smooth?	smooth?
If so, store.	If so, store.
Repeat.	Repeat.
Generate (a, b).	Generate (a, b).
Is $a - bm$	Is $a - bm$
smooth?	smooth?
If so, store.	If so, store.
Repeat.	Repeat.

Is $a - b\alpha_1$	Is $a - b\alpha_2$	Is $a - b\alpha_3$	Is $a - b\alpha_4$
smooth?	smooth?	smooth?	smooth?
If so, store.	If so, store.	If so, store.	If so, store.
Send (a, b).	Send (a, b).	Send (a, b).	Send (a, b).
right. Repeat.	right. Repeat.	right. Repeat.	down. Repeat.
Is $a - b\alpha_5$	Is $a - b\alpha_6$	Is $a - b\alpha_7$	Is $a - b\alpha_8$
smooth?	smooth?	smooth?	smooth?
If so, store.	If so, store.	If so, store.	If so, store.
Send (a, b).	Send (a, b).	Send (a, b).	Send (a, b).
up. Repeat.	left. Repeat.	left. Repeat.	left. Repeat.
Is $a - b\alpha_9$	Is $a - b\alpha_{10}$	Is $a - b\alpha_{11}$	Is $a - b\alpha_{12}$
smooth?	smooth?	smooth?	smooth?
If so, store.	If so, store.	If so, store.	If so, store.
Send (a, b).	Send (a, b).	Send (a, b).	Send (a, b).
right. Repeat.	right. Repeat.	right. Repeat.	down. Repeat.
Is $a - b\alpha_{13}$	Is $a - b\alpha_{14}$	Is $a - b\alpha_{15}$	Is $a - b\alpha_{16}$
smooth?	smooth?	smooth?	smooth?
If so, store.	If so, store.	If so, store.	If so, store.
Send (a, b).	Send (a, b).	Send (a, b).	Send (a, b).
up. Repeat.	left. Repeat.	left. Repeat.	left. Repeat.

N_1, N_2, N_3, N_4	N_1, N_2
N_5, N_6, N_7, N_8	N_5, N_6
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10},$
$N_{13}, N_{14}, N_{15}, N_{16}$	N_{13}, N_{14}
N_1, N_2, N_3, N_4	N_1, N_2
N_5, N_6, N_7, N_8	N_5, N_6
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10},$
$N_{13}, N_{14}, N_{15}, N_{16}$	N_{13}, N_{14}
N_1, N_2, N_3, N_4	N_1, N_2
N_5, N_6, N_7, N_8	N_5, N_6
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10},$
$N_{13}, N_{14}, N_{15}, N_{16}$	N_{13}, N_{14}
N_1, N_2, N_3, N_4	N_1, N_2
N_5, N_6, N_7, N_8	N_5, N_6
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10},$
$N_{13}, N_{14}, N_{15}, N_{16}$	N_{13}, N_{14}
N_1, N_2, N_3, N_4	N_1, N_2
N_5, N_6, N_7, N_8	N_5, N_6
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10},$
$N_{13}, N_{14}, N_{15}, N_{16}$	N_{13}, N_{14}

nerate (a, b).
Is $a - bm$
smooth?
f so, store.
Repeat.
nerate (a, b).
Is $a - bm$
smooth?
f so, store.
Repeat.
nerate (a, b).
Is $a - bm$
smooth?
f so, store.
Repeat.
nerate (a, b).
Is $a - bm$
smooth?
f so, store.
Repeat.

Is $a - b\alpha_1$	Is $a - b\alpha_2$	Is $a - b\alpha_3$	Is $a - b\alpha_4$
smooth?	smooth?	smooth?	smooth?
If so, store.	If so, store.	If so, store.	If so, store.
Send (a, b).	Send (a, b).	Send (a, b).	Send (a, b).
right. Repeat.	right. Repeat.	right. Repeat.	down. Repeat.
Is $a - b\alpha_5$	Is $a - b\alpha_6$	Is $a - b\alpha_7$	Is $a - b\alpha_8$
smooth?	smooth?	smooth?	smooth?
If so, store.	If so, store.	If so, store.	If so, store.
Send (a, b).	Send (a, b).	Send (a, b).	Send (a, b).
up. Repeat.	left. Repeat.	left. Repeat.	left. Repeat.
Is $a - b\alpha_9$	Is $a - b\alpha_{10}$	Is $a - b\alpha_{11}$	Is $a - b\alpha_{12}$
smooth?	smooth?	smooth?	smooth?
If so, store.	If so, store.	If so, store.	If so, store.
Send (a, b).	Send (a, b).	Send (a, b).	Send (a, b).
right. Repeat.	right. Repeat.	right. Repeat.	down. Repeat.
Is $a - b\alpha_{13}$	Is $a - b\alpha_{14}$	Is $a - b\alpha_{15}$	Is $a - b\alpha_{16}$
smooth?	smooth?	smooth?	smooth?
If so, store.	If so, store.	If so, store.	If so, store.
Send (a, b).	Send (a, b).	Send (a, b).	Send (a, b).
up. Repeat.	left. Repeat.	left. Repeat.	left. Repeat.

N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9,$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13},$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9,$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13},$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9,$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13},$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9,$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13},$

Is $a - b\alpha_1$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b). up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$

Is $a - b\alpha_2$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_6$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$

Is $a - b\alpha_3$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_7$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b). left. Repeat.
Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b). right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b). down. Repeat.
Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b). left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b). left. Repeat.

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N N_5, N_6, N_7, N $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N N_5, N_6, N_7, N $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N N_5, N_6, N_7, N $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N N_5, N_6, N_7, N $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N N_5, N_6, N_7, N $N_9, N_{10}, N_{11},$ $N_{13}, N_{14}, N_{15},$

ls $a - b\alpha_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
smooth?	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
so, store.	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
Send (a, b).	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
wn. Repeat.	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
ls $a - b\alpha_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
smooth?	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
so, store.	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
Send (a, b).	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
ft. Repeat.	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
s $a - b\alpha_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
smooth?	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
so, store.	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
Send (a, b).	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
wn. Repeat.	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
s $a - b\alpha_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
smooth?	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
so, store.	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
Send (a, b).	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
ft. Repeat.	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$

N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$	N_1, N_2, N_3, N_4 N_5, N_6, N_7, N_8 $N_9, N_{10}, N_{11}, N_{12}$ $N_{13}, N_{14}, N_{15}, N_{16}$

N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$

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(a, b) $(a,$
(a, b) $(a,$
(a, b) $(a,$

, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$

Linear algebra for N_1	L
using congruences	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
Linear algebra for N_5	L
using congruences	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
Linear algebra for N_9	L
using congruences	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
Linear algebra for N_{13}	L
using congruences	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	
(a, b) (a, b) (a, b)	

$, N_2, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_6, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{10}, N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
N_{14}, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
$, N_2, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_6, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{10}, N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
N_{14}, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
$, N_2, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_6, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{10}, N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
N_{14}, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
$, N_2, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_6, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{10}, N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
N_{14}, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
$, N_2, N_3, N_4$	N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
$, N_6, N_7, N_8$	N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
N_{10}, N_{11}, N_{12}	$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
N_{14}, N_{15}, N_{16}	$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$

Linear algebra for N_1	Linear algebra for
using congruences	using congruences
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
Linear algebra for N_5	Linear algebra for
using congruences	using congruences
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
Linear algebra for N_9	Linear algebra for
using congruences	using congruences
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
Linear algebra for N_{13}	Linear algebra for
using congruences	using congruences
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$
(a,b) (a,b) (a,b)	(a,b) (a,b) $(a,b$

N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4	N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8	N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$	$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$	$N_{13}, N_{14}, N_{15}, N_{16}$

Linear algebra for N_1	Linear algebra for N_2	Linear al
using congruences	using congruences	using
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
Linear algebra for N_5	Linear algebra for N_6	Linear al
using congruences	using congruences	using
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
Linear algebra for N_9	Linear algebra for N_{10}	Linear alg
using congruences	using congruences	using
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
Linear algebra for N_{13}	Linear algebra for N_{14}	Linear alg
using congruences	using congruences	using
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b)

N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$
N_1, N_2, N_3, N_4
N_5, N_6, N_7, N_8
$N_9, N_{10}, N_{11}, N_{12}$
$N_{13}, N_{14}, N_{15}, N_{16}$

Linear algebra for N_1 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_2 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_3 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_5 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_6 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_7 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_9 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{10} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{11} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_{13} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{14} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{15} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)

N_4
N_8
N_{12}
N_{16}
N_4
N_8
N_{12}
N_{16}
N_4
N_8
N_{12}
N_{16}
N_4
N_8
N_{12}
N_{16}

Linear algebra for N_1	Linear algebra for N_2	Linear algebra for N_3	Linear algebra fo
using congruences	using congruences	using congruences	using congruen
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
Linear algebra for N_5	Linear algebra for N_6	Linear algebra for N_7	Linear algebra fo
using congruences	using congruences	using congruences	using congruen
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
Linear algebra for N_9	Linear algebra for N_{10}	Linear algebra for N_{11}	Linear algebra fo
using congruences	using congruences	using congruences	using congruen
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
Linear algebra for N_{13}	Linear algebra for N_{14}	Linear algebra for N_{15}	Linear algebra fo
using congruences	using congruences	using congruences	using congruen
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$
(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) (a,b)	(a,b) (a,b) $(a$

Linear algebra for N_1 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_2 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_3 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_4 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_5 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_6 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_7 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_8 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_9 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{10} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{11} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{12} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_{13} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{14} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{15} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{16} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)

Linear algebra for N_1 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_2 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_3 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_4 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_5 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_6 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_7 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_8 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_9 using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{10} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{11} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{12} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)
Linear algebra for N_{13} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{14} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{15} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)	Linear algebra for N_{16} using congruences (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b) (a,b)