Hyper-and-elliptic-curve cryptography

(which is not the same as: hyperelliptic-curve cryptography and elliptic-curve cryptography)

Daniel J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Tanja Lange
Technische Universiteit Eindhoven

"Through our inefficient use of energy (gas guzzling vehicles, badly insulated buildings, poorly optimized crypto, etc) we needlessly throw away almost a third of the energy we use."
—Greenpeace UK

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DH speed records

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2011 Bernstein–D
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2012 Hamburg:
2012 Longa–Sica:
2013 Bos–Costell
Lauter:
2013 Oliveira–Lóp
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DH speed records

Sandy Bridge cycles for high
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2011 Bernstein–Duif–Lange–
Schwabe–Yang: 19
2012 Hamburg: 15
2012 Longa–Sica: 13
2013 Bos–Costello–Hisil–
Lauter: 12
2013 Oliveira–López–Aranha
Rodríguez-Henríquez: 11
2013 Faz-Hernández–Longa–
Sánchez: 9
2014 Bernstein–Chuengsatia
Lange–Schwabe: 9

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DH speed records

Sandy Bridge cycles for high-security constant-time $a, P \mapsto aP$ ("?" if not SUPERCOP-verified):

2011 Bernstein–Duif–Lange–Schwabe–Yang:	194036
2012 Hamburg:	153000?
2012 Longa–Sica:	137000?
2013 Bos–Costello–Hisil–Lauter:	122716
2013 Oliveira–López–Aranha–Rodríguez-Henríquez:	114800?
2013 Faz-Hernández–Longa–Sánchez:	96000?
2014 Bernstein–Chuengsatiansup–Lange–Schwabe:	91320

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<u>DH speed records</u>

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25**M** for $X(P), X(Q), X(Q$
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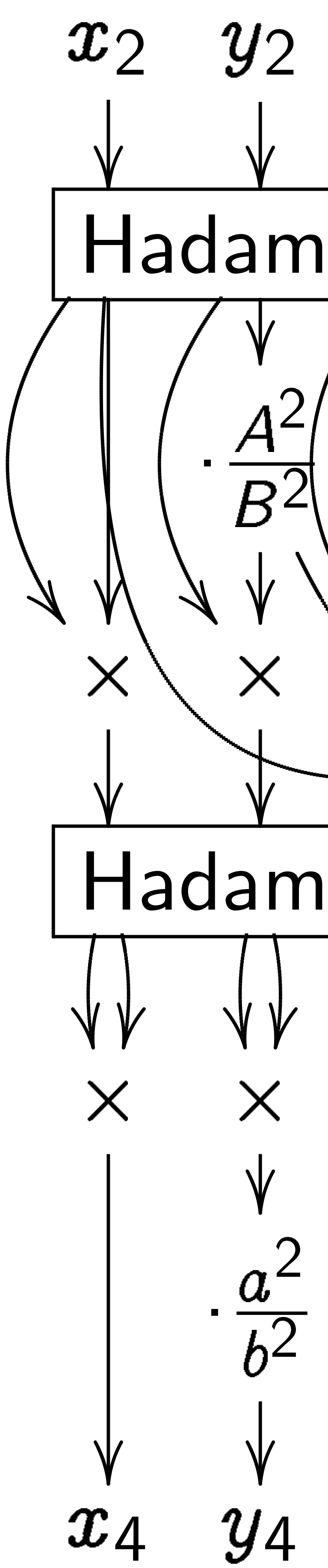

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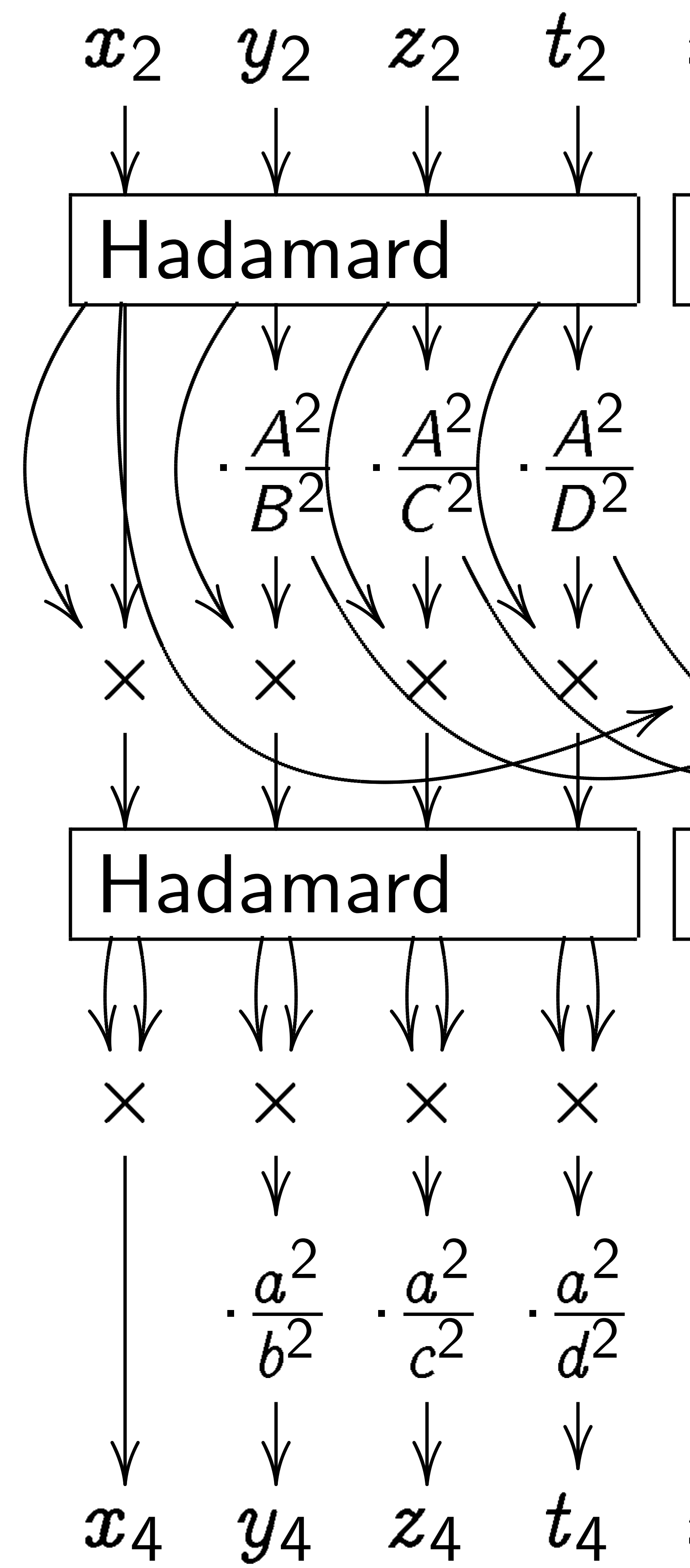

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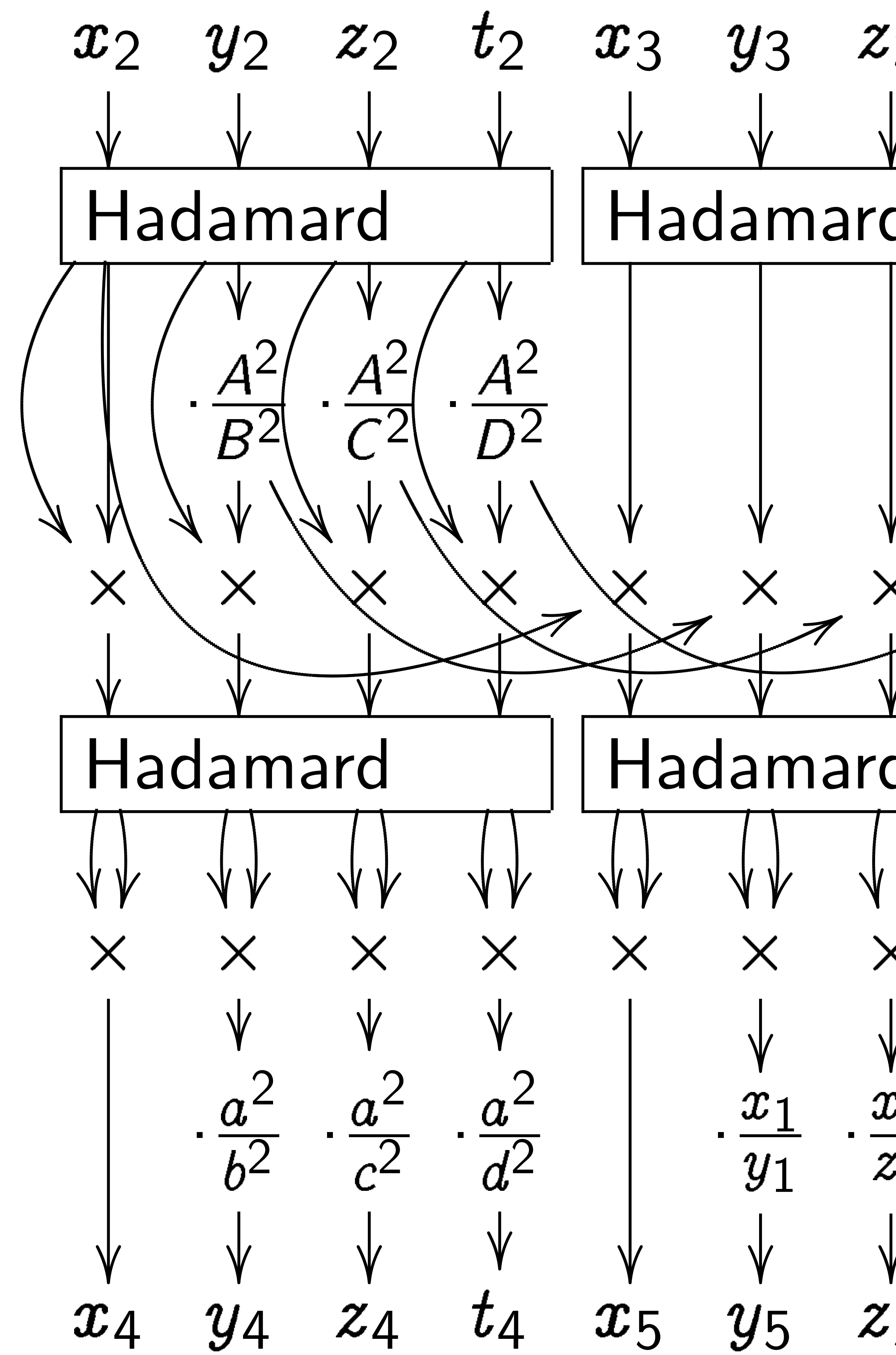

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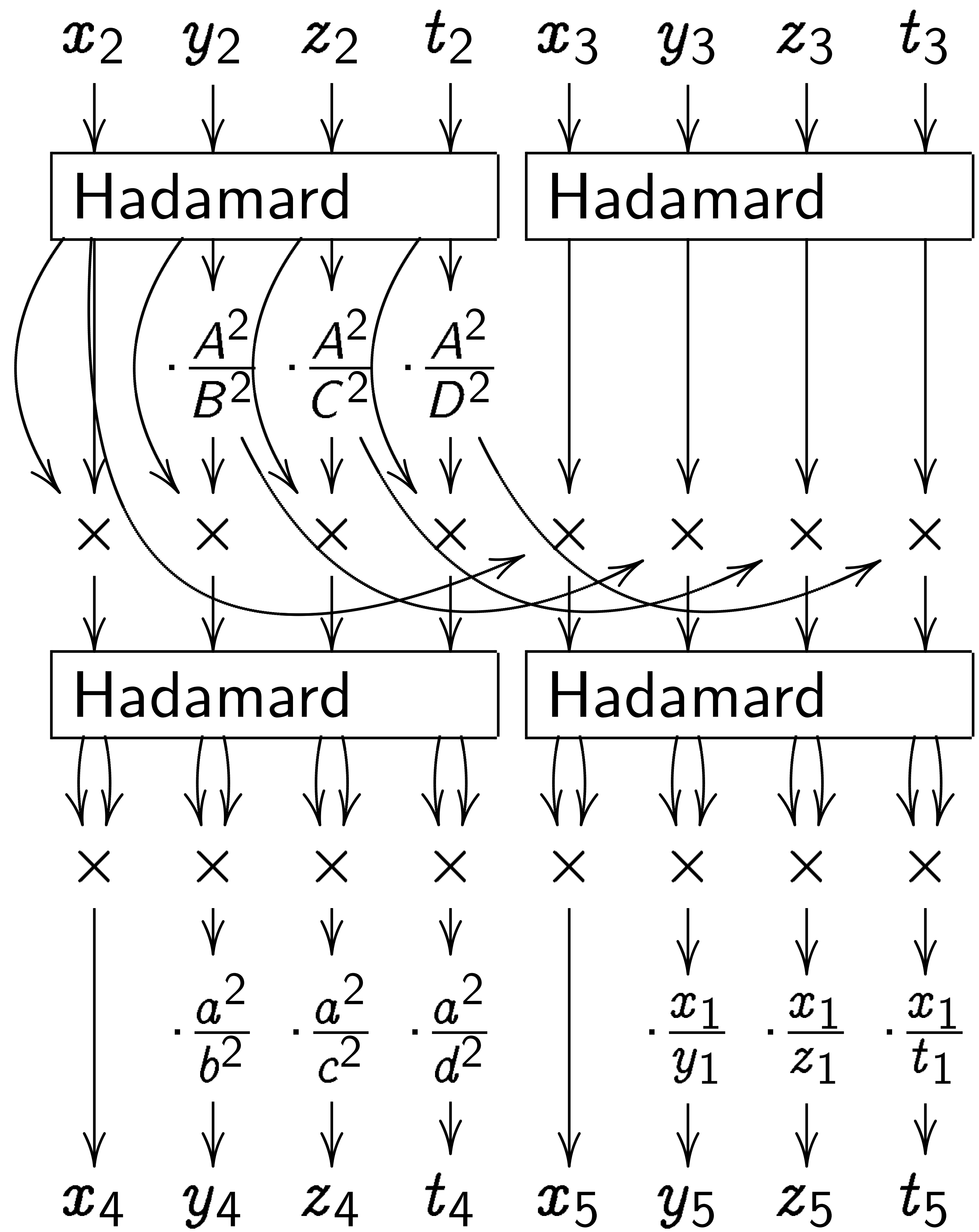

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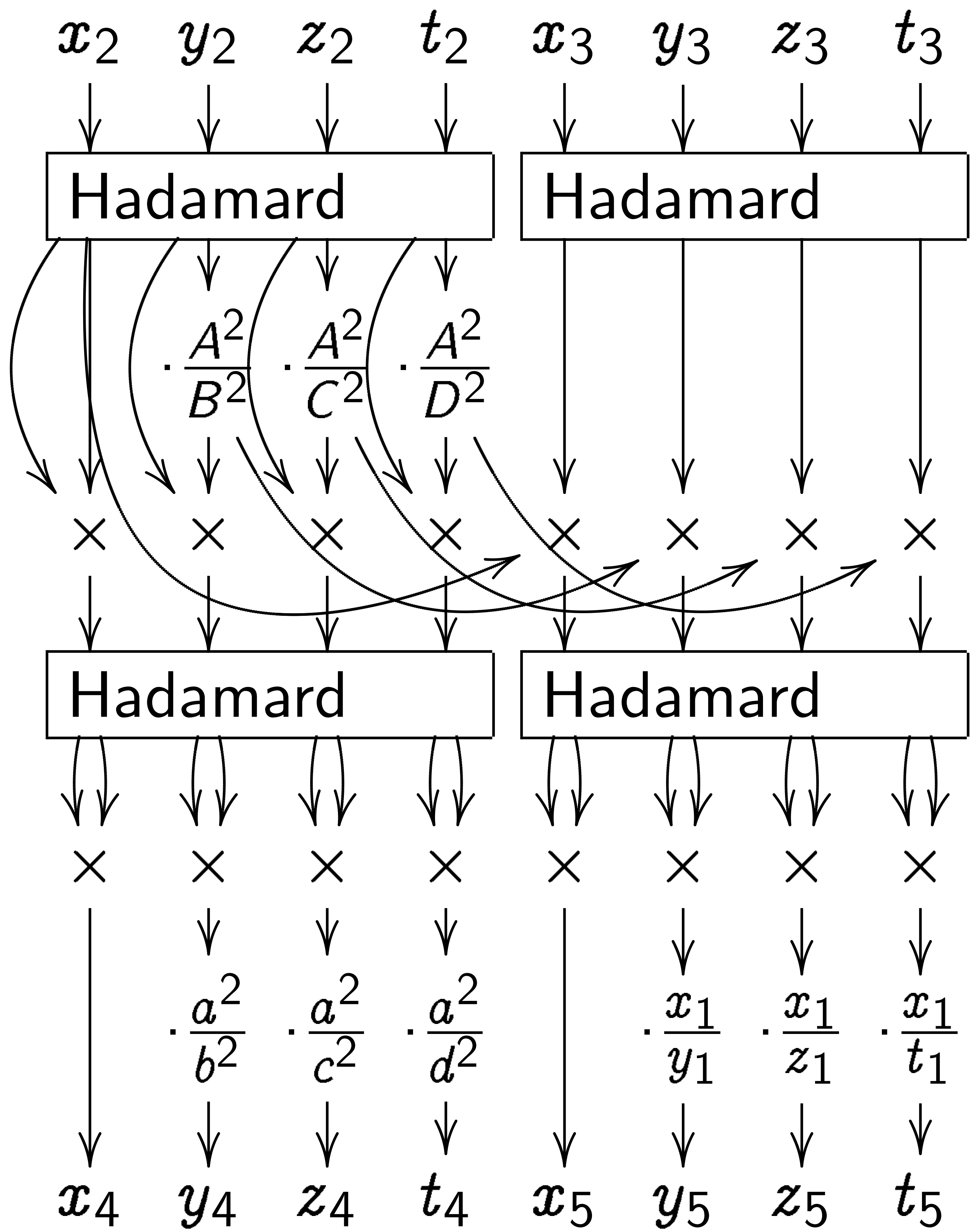

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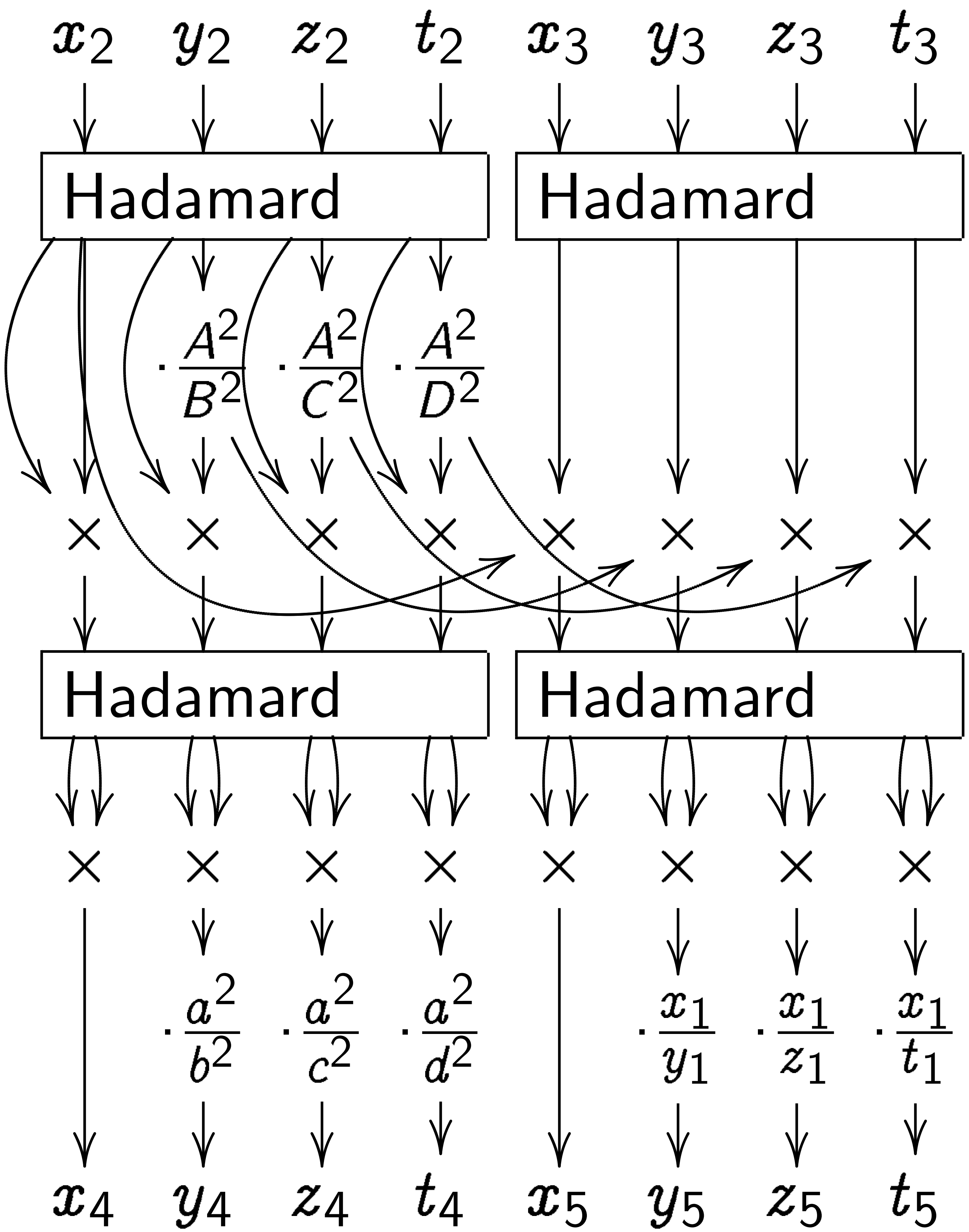

Strategies to build

with known $\#J(\mathbf{F}$

	CM
fast build	yes
any curve	no
many curves	no
secure curves	yes
twist-secure	yes
Kummer	yes
small coeff	no
fastest DH	no
fastest keygen	no
complete add	no

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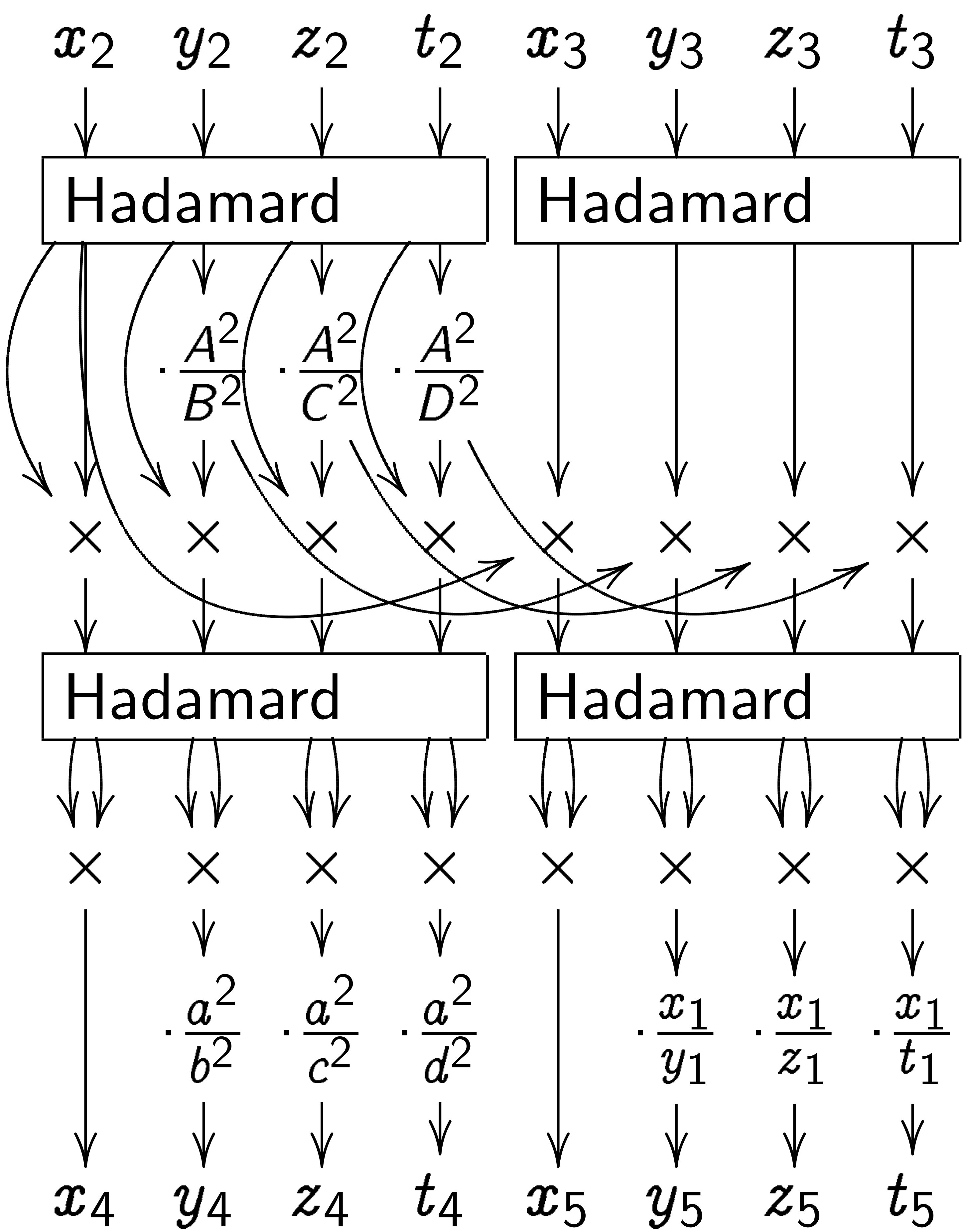
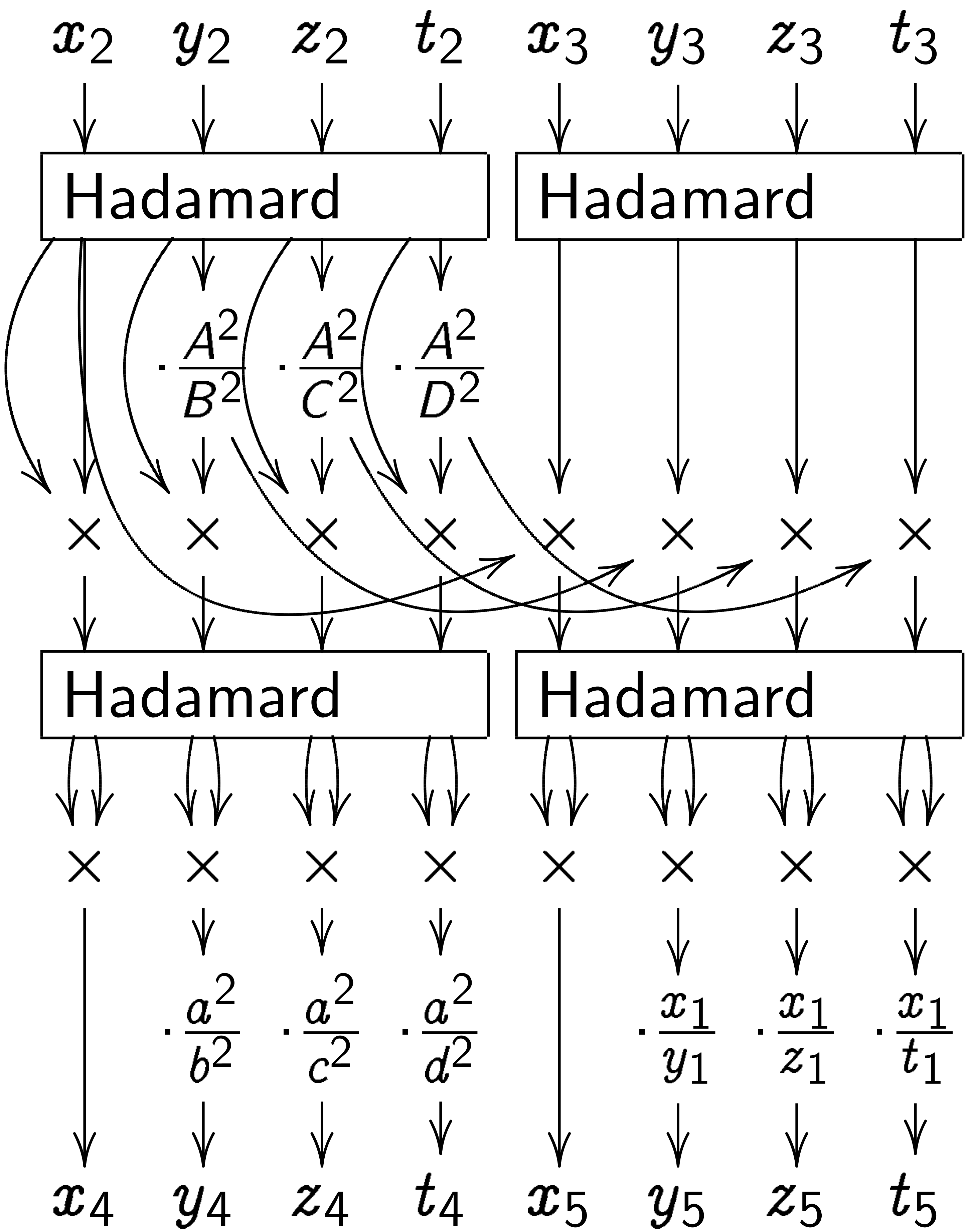


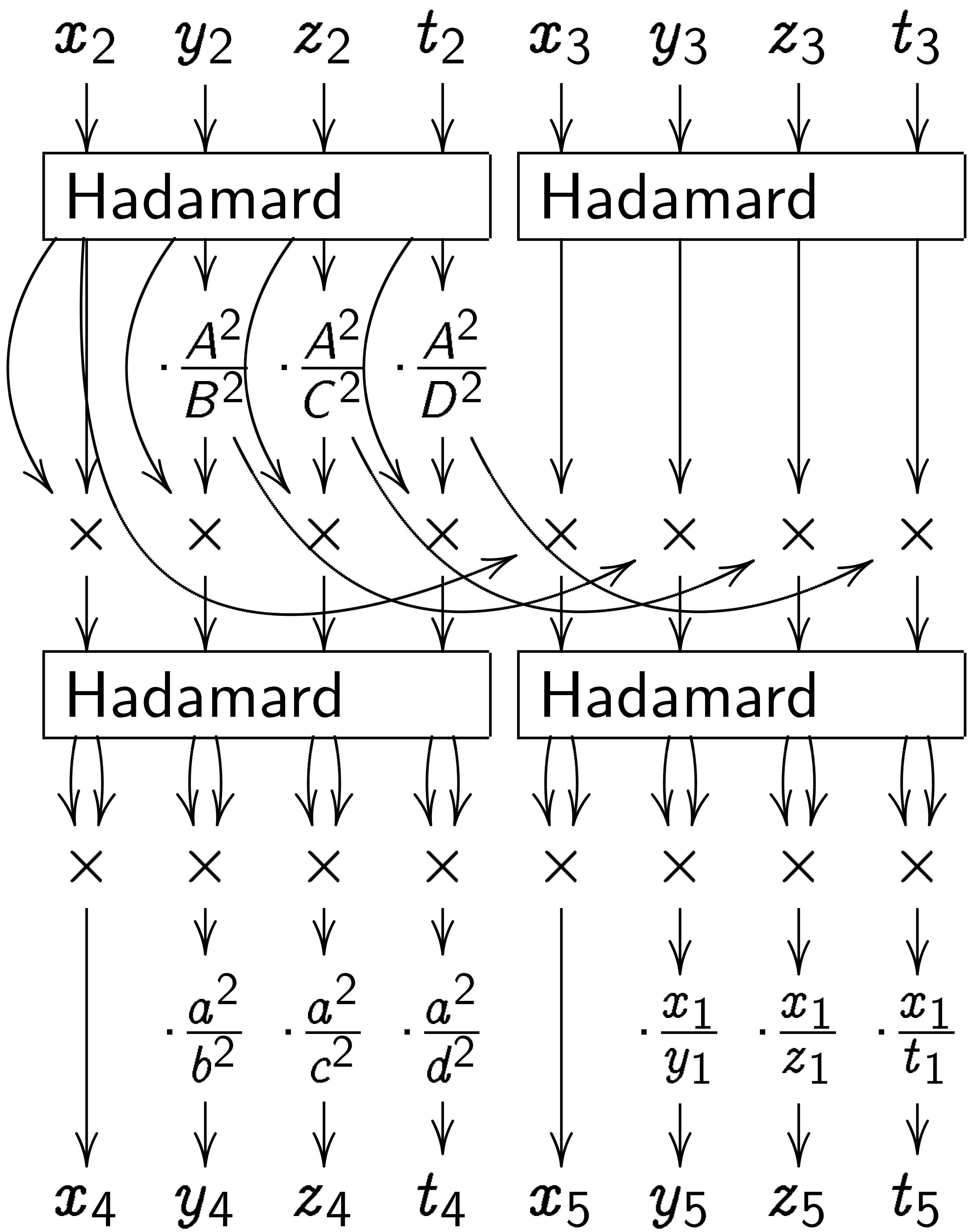
Strategies to build dim-2 J/
with known $\#J(\mathbf{F}_p)$, large p

	CM	Pila	nev
fast build	yes	no	yes
any curve	no	yes	no
many curves	no	yes	yes
secure curves	yes	yes	yes
twist-secure	yes	yes	yes
Kummer	yes	yes	yes
small coeff	no	yes	yes
fastest DH	no	yes	yes
fastest keygen	no	no	yes
complete add	no	no	yes



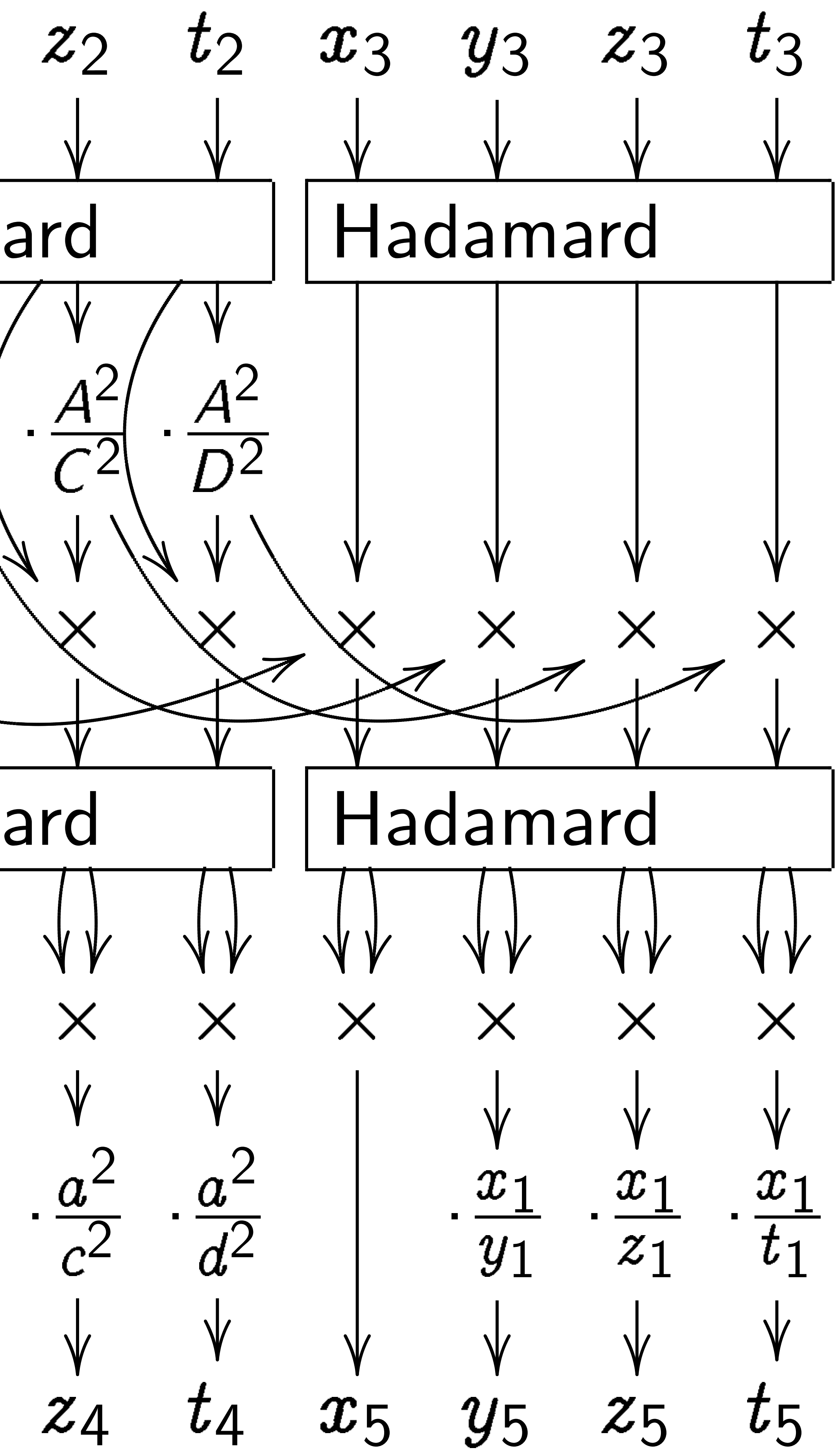
Strategies to build dim-2 $J/\mathbf{F}_p$ with known $\#J(\mathbf{F}_p)$, large p:

	CM	Pila	new
fast build	**yes**	no	**yes**
any curve	no	**yes**	no
many curves	no	**yes**	**yes**
secure curves	**yes**	**yes**	**yes**
twist-secure	**yes**	**yes**	**yes**
Kummer	**yes**	**yes**	**yes**
small coeff	no	**yes**	**yes**
fastest DH	no	**yes**	**yes**
fastest keygen	no	no	**yes**
complete add	no	no	**yes**



Strategies to build dim-2 $J/\mathbf{F}_p$ with known $\#J(\mathbf{F}_p)$, large p:

	CM	Pila	Stn	new
fast build	**yes**	no	**yes**	**yes**
any curve	no	**yes**	no	no
many curves	no	**yes**	**yes**	**yes**
secure curves	**yes**	**yes**	**yes**	**yes**
twist-secure	**yes**	**yes**	**yes**	**yes**
Kummer	**yes**	**yes**	**yes**	**yes**
small coeff	no	**yes**	no	**yes**
fastest DH	no	**yes**	no	**yes**
fastest keygen	no	no	no	**yes**
complete add	no	no	no	**yes**



Strategies to build dim-2 $J/\mathbf{F}_p$ with known $\#J(\mathbf{F}_p)$, large p:

	CM	Pila	Stn	new
fast build	**yes**	no	**yes**	**yes**
any curve	no	**yes**	no	no
many curves	no	**yes**	**yes**	**yes**
secure curves	**yes**	**yes**	**yes**	**yes**
twist-secure	**yes**	**yes**	**yes**	**yes**
Kummer	**yes**	**yes**	**yes**	**yes**
small coeff	no	**yes**	no	**yes**
fastest DH	no	**yes**	no	**yes**
fastest keygen	no	no	no	**yes**
complete add	no	no	no	**yes**

Hyper-a

Typical

$H : y^2 =$

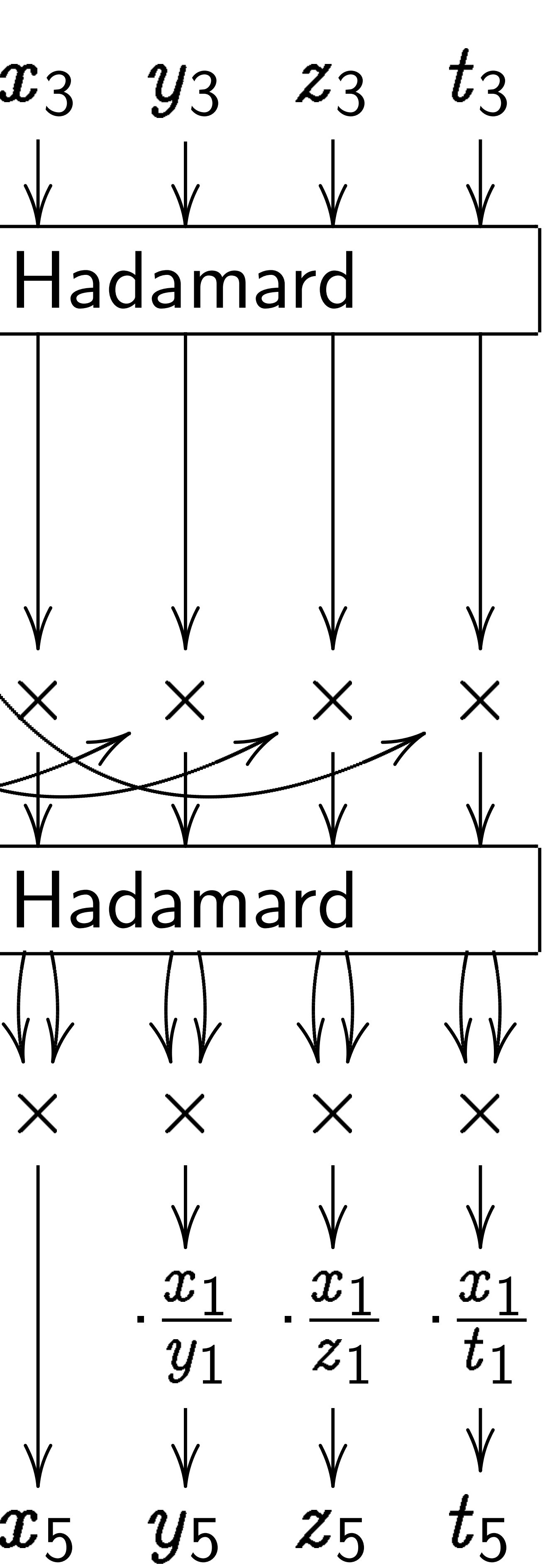
$(z$

over $\mathbf{F}_p$

$J =$ Jac

surface

Small K



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	CM	Pila	Stn	new
fast build	**yes**	no	**yes**	**yes**
any curve	no	**yes**	no	no
many curves	no	**yes**	**yes**	**yes**
secure curves	**yes**	**yes**	**yes**	**yes**
twist-secure	**yes**	**yes**	**yes**	**yes**
Kummer	**yes**	**yes**	**yes**	**yes**
small coeff	no	**yes**	no	**yes**
fastest DH	no	**yes**	no	**yes**
fastest keygen	no	no	no	**yes**
complete add	no	no	no	**yes**

Hyper-and-elliptic-

Typical example:

$H : y^2 = (z-1)($

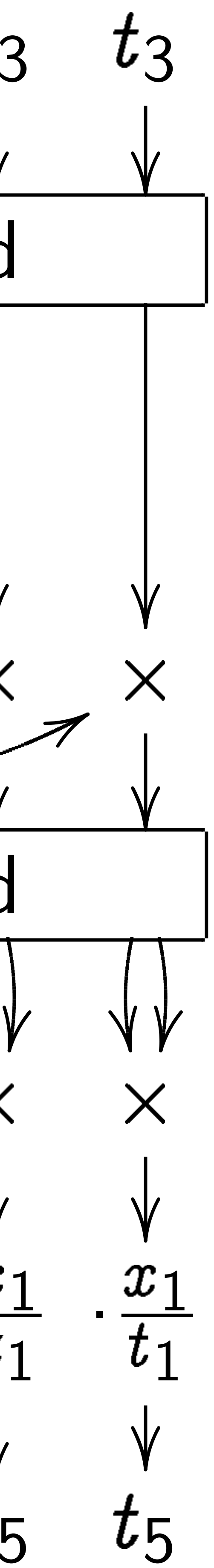
$(z - 1/2)(z +$

over $\mathbf{F}_p$ with $p =$

$J = \mathrm{Jac}\,H$; traditi

surface K; traditio

Small K coeffs (2



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	CM	Pila	Stn	new
fast build	yes	no	yes	yes
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fastest keygen	no	no	no	yes
complete add	no	no	no	yes

Hyper-and-elliptic-curve cryp

Typical example: Define

$H : y^2 = (z-1)(z+1)(z+$
$(z-1/2)(z+3/2)(z-$

over $\mathbf{F}_p$ with $p = 2^{127} - 309$

$J = \mathrm{Jac}\, H$; traditional Kumm

surface K; traditional $X : J$

Small K coeffs $(20 : 1 : 20 :$

Strategies to build dim-2 $J/\mathbf{F}_p$ with known $\#J(\mathbf{F}_p)$, large p:

	CM	Pila	Stn	new
fast build	yes	no	yes	yes
any curve	no	yes	no	no
many curves	no	yes	yes	yes
secure curves	yes	yes	yes	yes
twist-secure	yes	yes	yes	yes
Kummer	yes	yes	yes	yes
small coeff	no	yes	no	yes
fastest DH	no	yes	no	yes
fastest keygen	no	no	no	yes
complete add	no	no	no	yes

Hyper-and-elliptic-curve crypto

Typical example: Define

$$H : y^2 = (z-1)(z+1)(z+2)(z-1/2)(z+3/2)(z-2/3)$$

over $\mathbf{F}_p$ with $p = 2^{127} - 309$; $J = \text{Jac}\, H$; traditional Kummer surface K; traditional $X : J \to K$. Small K coeffs $(20 : 1 : 20 : 40)$.

Strategies to build dim-2 $J/\mathbf{F}_p$ with known $\#J(\mathbf{F}_p)$, large p:

	CM	Pila	Stn	new
fast build	yes	no	yes	yes
any curve	no	yes	no	no
many curves	no	yes	yes	yes
secure curves	yes	yes	yes	yes
twist-secure	yes	yes	yes	yes
Kummer	yes	yes	yes	yes
small coeff	no	yes	no	yes
fastest DH	no	yes	no	yes
fastest keygen	no	no	no	yes
complete add	no	no	no	yes

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$J = \text{Jac}\, H$; traditional Kummer surface K; traditional $X : J \to K$.
Small K coeffs $(20 : 1 : 20 : 40)$.

Warning: There are typos in the Rosenhain/Mumford/Kummer formulas in 2007 Gaudry, 2010 Cosset, 2013 Bos–Costello–Hisil–Lauter. We have simpler, computer-verified formulas.

es to build dim-2 $J/\mathbf{F}_p$

own $\#J(\mathbf{F}_p)$, large p:

	CM	Pila	Stn	new
ld	**yes**	no	**yes**	**yes**
ve	no	**yes**	no	no
urves	no	**yes**	**yes**	**yes**
curves	**yes**	**yes**	**yes**	**yes**
cure	**yes**	**yes**	**yes**	**yes**
r	**yes**	**yes**	**yes**	**yes**
oeff	no	**yes**	no	**yes**
DH	no	**yes**	no	**yes**
keygen	no	no	no	**yes**
e add	no	no	no	**yes**

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$\#J(\mathbf{F}_p)$

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p), large p:

M	Pila	Stn	new
s	no	yes	yes
	yes	no	no
	yes	yes	yes
s	yes	yes	yes
s	yes	yes	yes
s	yes	yes	yes
	yes	no	yes
	yes	no	yes
	no	no	yes
	no	no	yes

Hyper-and-elliptic-curve crypto

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$$H : y^2 = (z-1)(z+1)(z+2)$$
$$(z-1/2)(z+3/2)(z-2/3)$$

over $\mathbf{F}_p$ with $p = 2^{127} - 309$;

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$\#J(\mathbf{F}_p) = 16\ell$

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Security $\approx 2^{125}$ ag

Order of ℓ in $(\mathbf{Z}/p$

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Switch to $p = 2^{127}$

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$\mathbf{F}_p$

):

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s	yes
	yes
	yes
	yes
	yes

Hyper-and-elliptic-curve crypto

Typical example: Define

$H : y^2 = (z-1)(z+1)(z+2)$
$\quad (z-1/2)(z+3/2)(z-2/3)$

over $\mathbf{F}_p$ with $p = 2^{127} - 309$;

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$\#J(\mathbf{F}_p) = 16\ell$

where ℓ is the prime

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Security $\approx 2^{125}$ against rho.

Order of ℓ in $(\mathbf{Z}/p)^*$ is

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122563150387.

Twist security $\approx 2^{75}$.

(Want more twist security?

Switch to $p = 2^{127} - 94825$

cofactors $16 \cdot 3269239$, 4.)

Hyper-and-elliptic-curve crypto

Typical example: Define
$H : y^2 = (z-1)(z+1)(z+2)$
$(z-1/2)(z+3/2)(z-2/3)$
over $\mathbf{F}_p$ with $p = 2^{127} - 309$;
$J = \text{Jac}\, H$; traditional Kummer surface K; traditional $X : J \to K$.
Small K coeffs $(20 : 1 : 20 : 40)$.

Warning: There are typos in the Rosenhain/Mumford/Kummer formulas in 2007 Gaudry, 2010 Cosset, 2013 Bos–Costello–Hisil–Lauter. We have simpler, computer-verified formulas.

$\#J(\mathbf{F}_p) = 16\ell$
where ℓ is the prime
1809251394333065553493296640760748553649194606010814289531455285792829679923.

Security $\approx 2^{125}$ against rho.

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Fast point-countin

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Bob views aG in $W(\mathbf{F}_p)$,
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Scholten: Define $\phi : H \to E$ as

$$(z, y) \mapsto \left(\frac{(1+iz)^2}{(1-iz)^2}, \frac{\omega y}{(1-iz)^3} \right).$$

Composition of $\phi_2 : (P_1, P_2) \mapsto \phi(P_1)+\phi(P_2)$ and standard $E \to W$ is composition of standard $H \times H \to J$ and some $\iota' : J \to W$.

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$$(z, y) \mapsto \left(\frac{(1+iz)^2}{(1-iz)^2}, \frac{\omega y}{(1-iz)^3} \right).$$

Composition of $\phi_2 : (P_1, P_2) \mapsto \phi(P_1)+\phi(P_2)$ and standard $E \to W$ is composition of standard $H \times H \to J$ and some $\iota' : J \to W$.

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Much easier: We applied ϕ_2 to random points in $H(\mathbf{F}_p) \times H(\mathbf{F}_p)$, interpolated coefficients of ι'. Similarly interpolated formulas for ι; verified composition.

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Define $s = -r(\rho_1 + \rho_2 + \rho_3)$. Then $rx^3 + sx^2 + \overline{s}x + \overline{r} = r(x - \rho_1)(x - \rho_2)(x - \rho_3)$.

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$r = 8648575 - 15$
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Example: Choose $\Delta = -1$;

$\rho_1 = (i)^2$, $\rho_2 = ((3+4i)/5$

$\rho_3 = ((5+12i)/13)^2$; $r = 3$

$s = 159 + 56i$, $\beta = i$.

One Rosenhain choice is $\lambda = 10$, $\mu = 5/8$, $\nu = 25$.

Then $\dfrac{\lambda\mu}{\nu} = \dfrac{1}{2^2}$

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Larger example:

$r = 8648575 - 15615600i$,

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and $\dfrac{\mu(\mu-1)(\lambda-\nu)}{\nu(\nu-1)(\lambda-\mu)} = \dfrac{1}{40^2}$.

Larger example:
$r = 8648575 - 15615600i$,
$s = -40209279 - 33245520i$;
coeffs (6137 : 833 : 2275 : 2275).

$\overline{\Delta}$) with $\beta \notin \mathbf{Q}$

, $\rho_2, \rho_3\}$.

n curve

$\overline{\beta}^2\beta^4 + \overline{r}\beta^6)y^2 =$

$\overline{\beta}z)^4(1-\beta z)^2+$

$z)^4 + \overline{r}(1-\beta z)^6$

over $\mathbf{Q}$.

responding

ters λ, μ, ν

$-1)(\lambda-\nu)$

$-1)(\lambda-\mu)$

,

er $\mathbf{Q}$.

: see paper.)

Example: Choose $\Delta = -1$;

$\rho_1 = (i)^2$, $\rho_2 = ((3+4i)/5)^2$,

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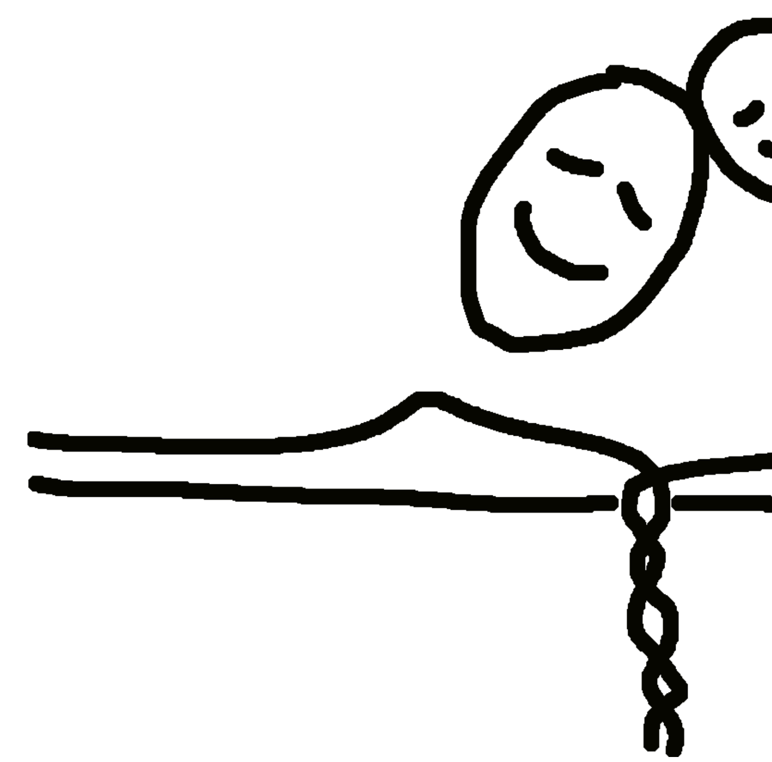

$\notin$ **Q**

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$\beta z)^2 +$

$-\beta z)^6$

g

ν

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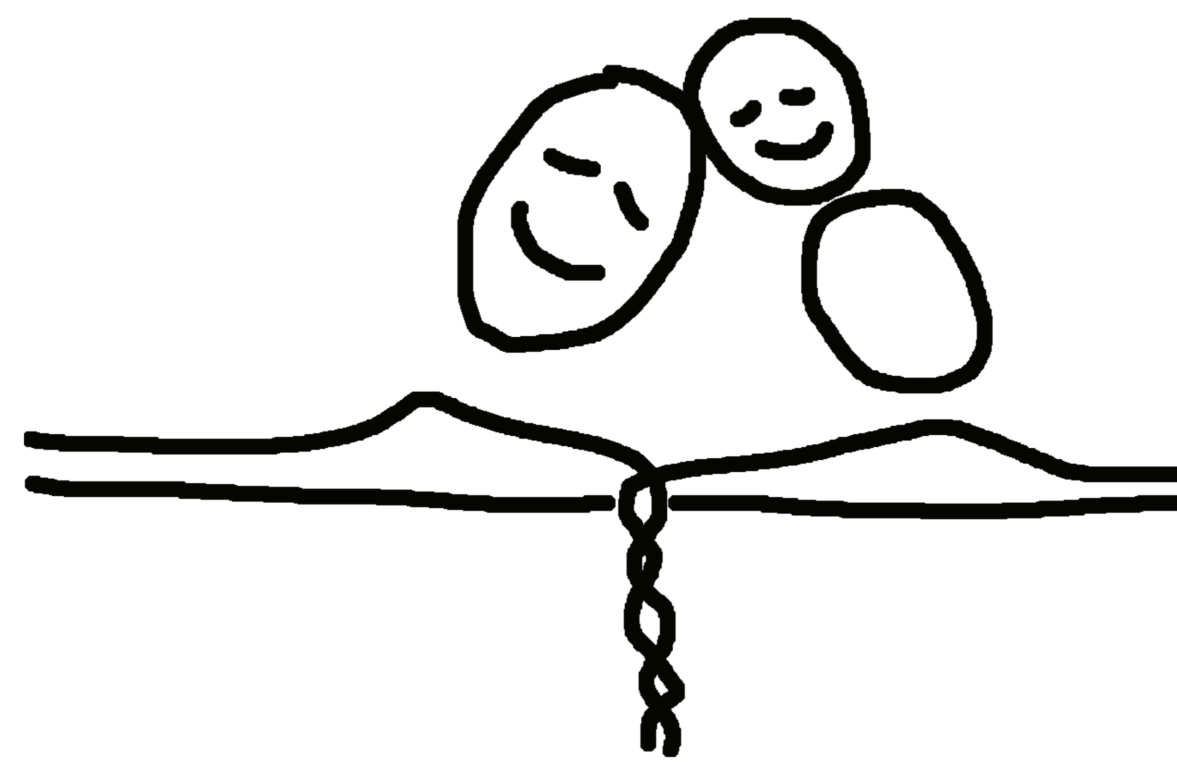

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