Complexity news:
discrete logarithms in
multiplicative groups of
small-characteristic finite fields—
the algorithm of Barbulescu,
Gaudry, Joux, Thomé

D. J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Advertisement, maybe related:
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2013.10.07–11, Tahiti.
Submit talks this month!

Also somewhat related:

I'm starting to analyze
cost of NFS + CVP
for class groups, unit groups,
short generators of ideals, etc.;
exploiting subfields
(find short *norms* first),
small Galois groups, etc.
Anyone else working on this?

Cryptanalytic applications:
attack NTRU, Ring-LWE, FHE.
I think NTRU should switch to
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Algorithm input:
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Algorithm output:
$\log_g h_1, \log_g h_2, \ldots$
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"$\log_g$" means the
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$$= (cx + d)(ax + b$$

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$\prod_{\alpha \in \mathbf{F}_p} (x - \alpha) \equiv x^2 - x + \beta$.

"$\equiv$" for $\mathbf{F}_{p^2}[x]$: equal mod $x^p - x^2 - \beta$; forces $=$ in $\mathbf{F}_q$.

Hope that $x^2 - x + \beta$ splits in $\mathbf{F}_{p^2}[x]$, say as $f_1 \cdot f_2$.
Not an unreasonable hope: $\approx$50% of quadratics split.

Then $\log_g f_1 + \log_g f_2 = \sum_{\alpha \in \mathbf{F}_p} \log_g (x - \alpha)$.

This is a "relation" among discrete logs of monic linear polys.

More relations for $D = 1$

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Larger discrete logs

What if $D < \deg h \leq 2D$?

Use same equation:

$$(ch + d) \prod_{\alpha \in \mathbf{F}_p} (ah + b - \alpha(ch + d))$$

$$\equiv (ch + d)(a^p h(x^2 + \beta) + b^p)$$

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Occasionally right side is product of small polys.

We now know those discrete logs.

Left side is product on new factor base: $\{h + \gamma : \gamma \in \mathbf{F}_{p^2}\}$.

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Total cost $p^{\Theta(\log n/\log\log n)}$

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