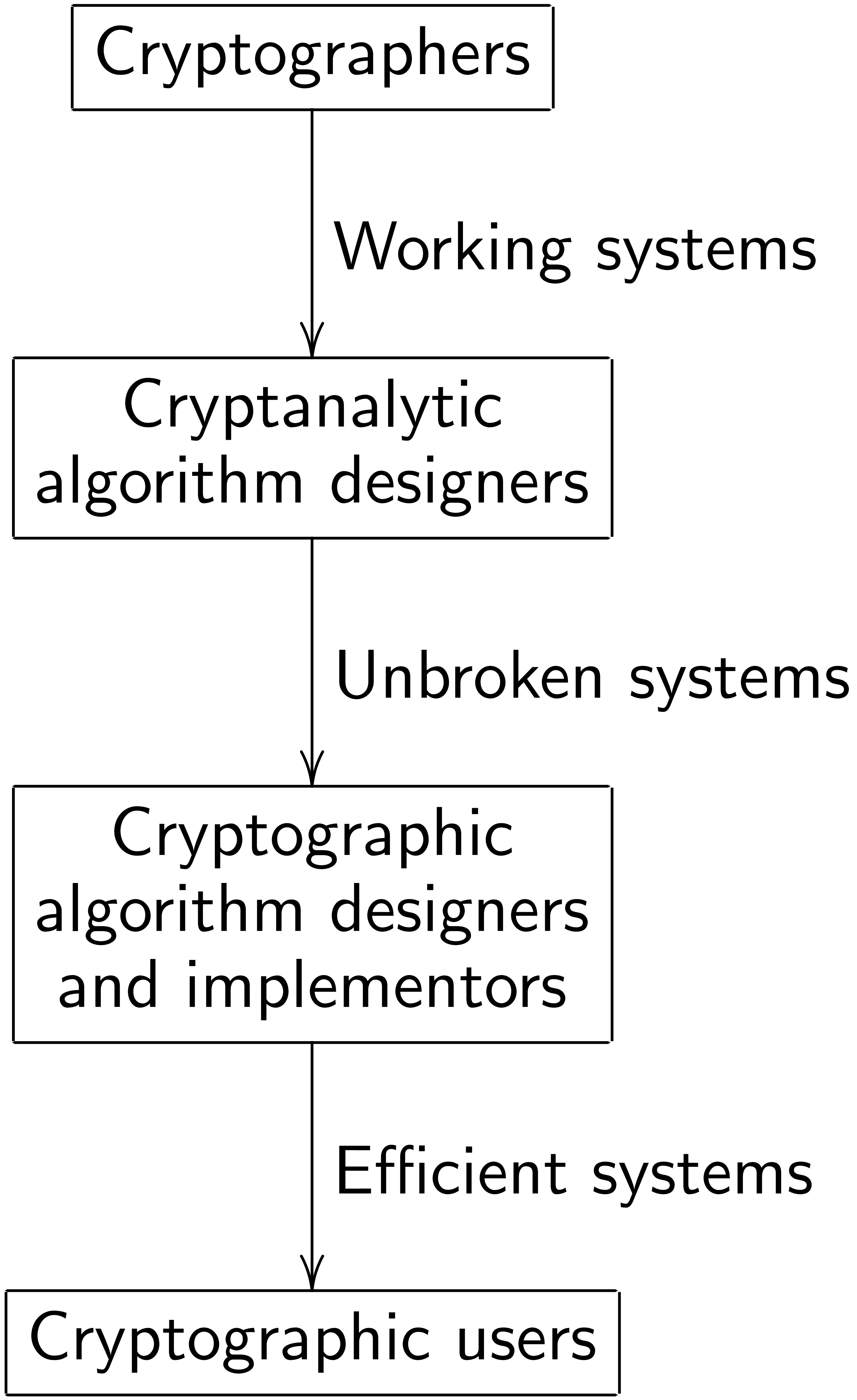
High-speed cryptography, part 3: more cryptosystems

Daniel J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven



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Working systems

Cryptanalytic algorithm designers

Unbroken systems

Cryptographic algorithm designers and implementors

Efficient systems

Cryptographic users

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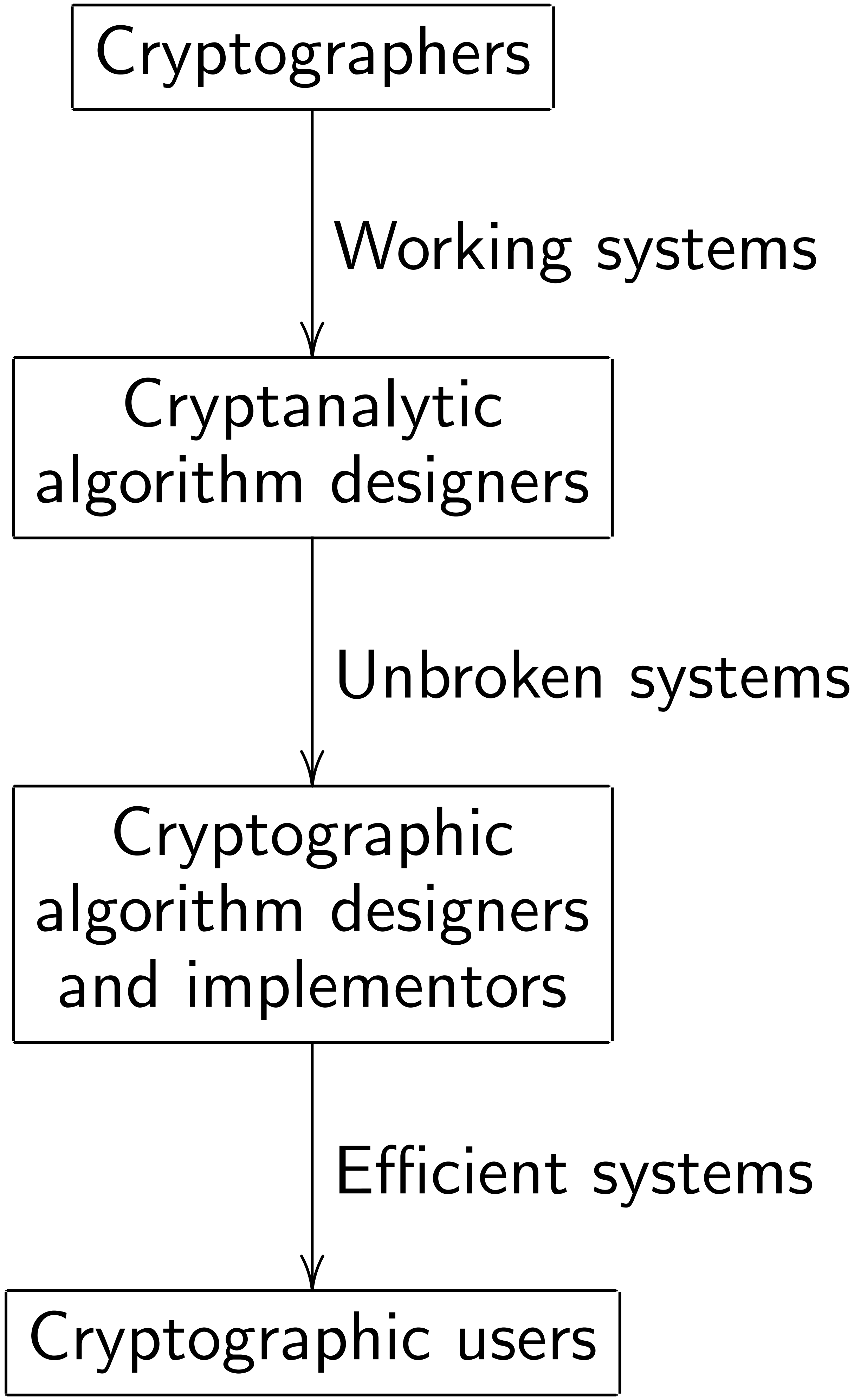

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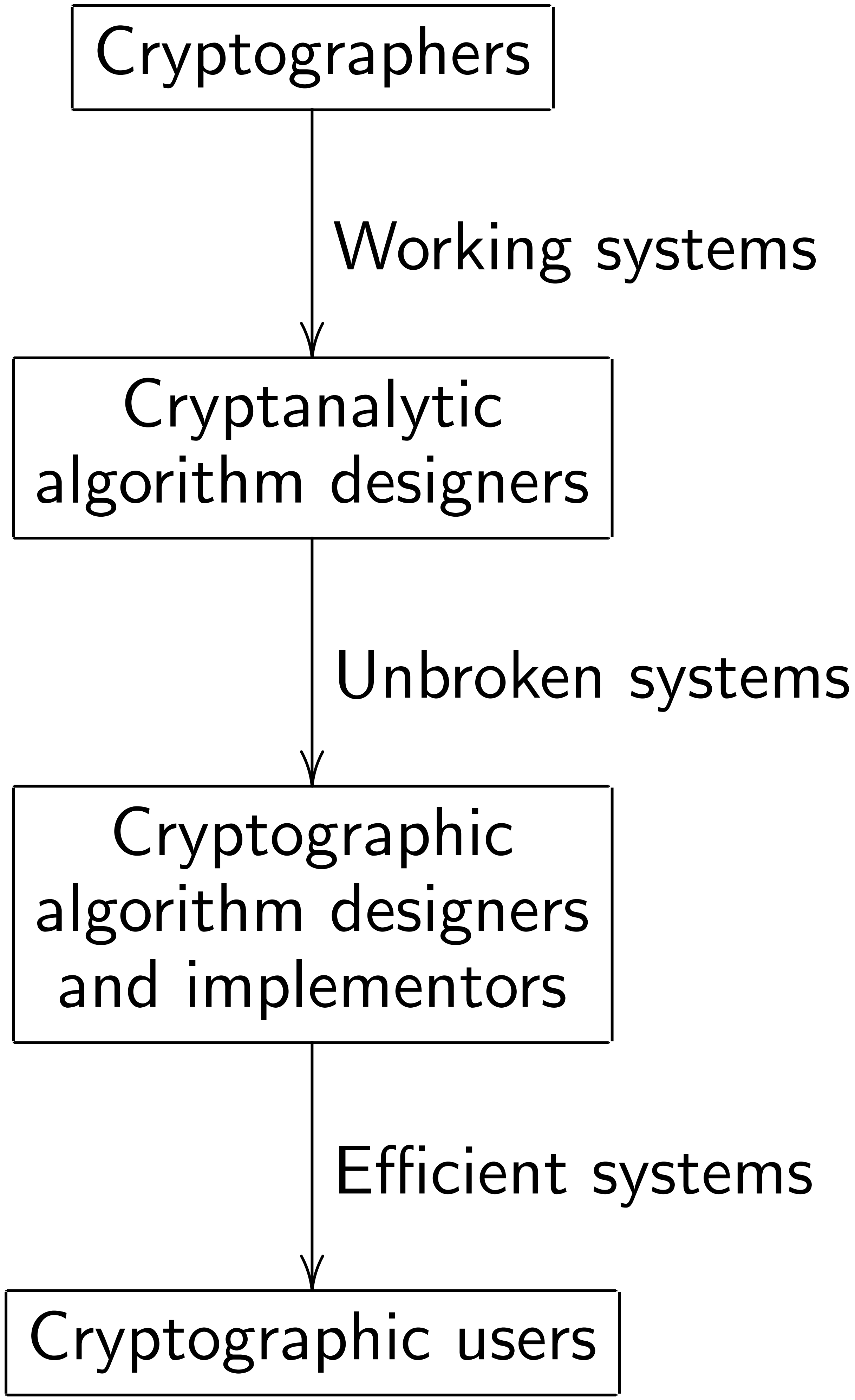

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Many answers:
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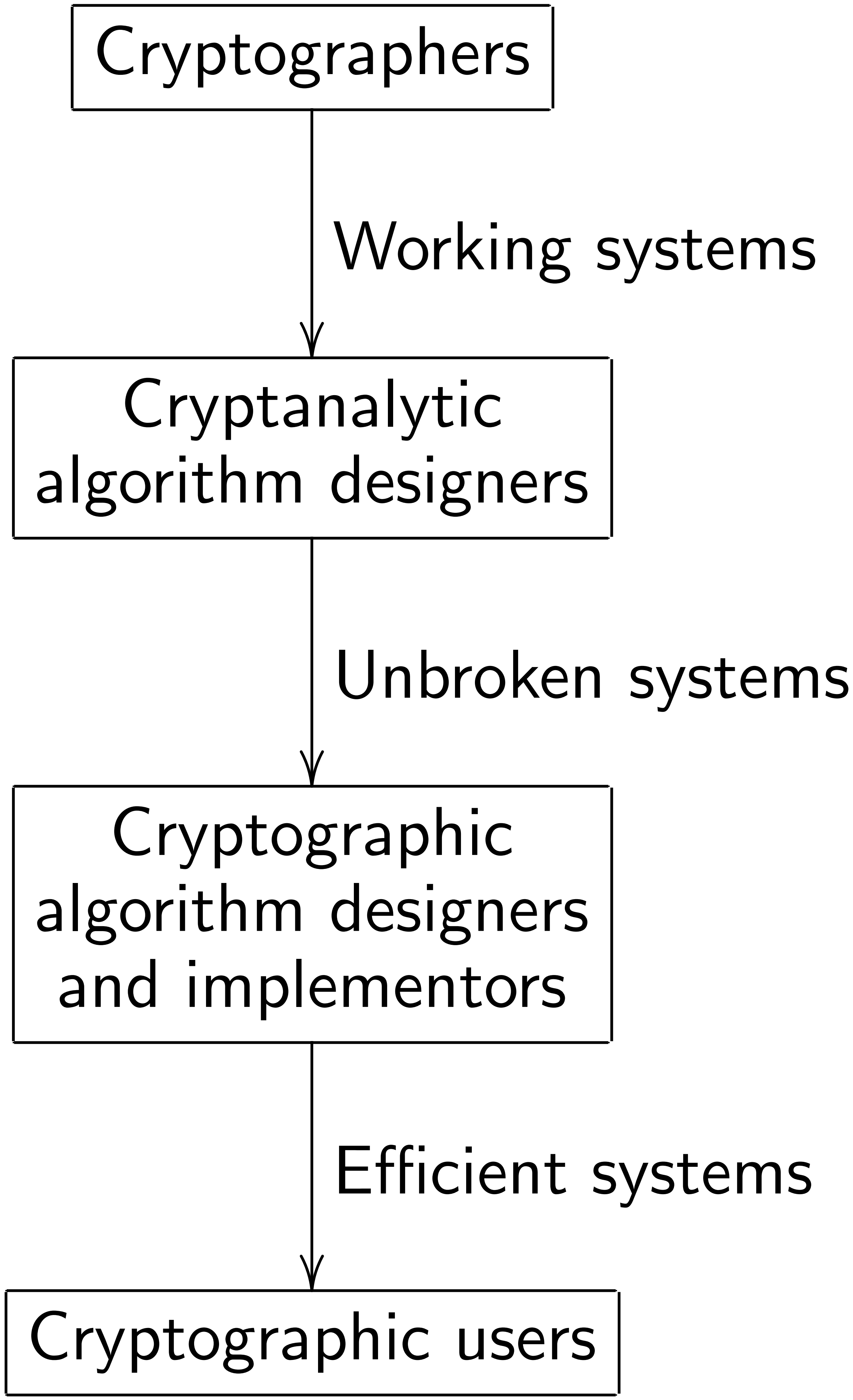

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1. Working systems

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RSA, McEliece encryption,
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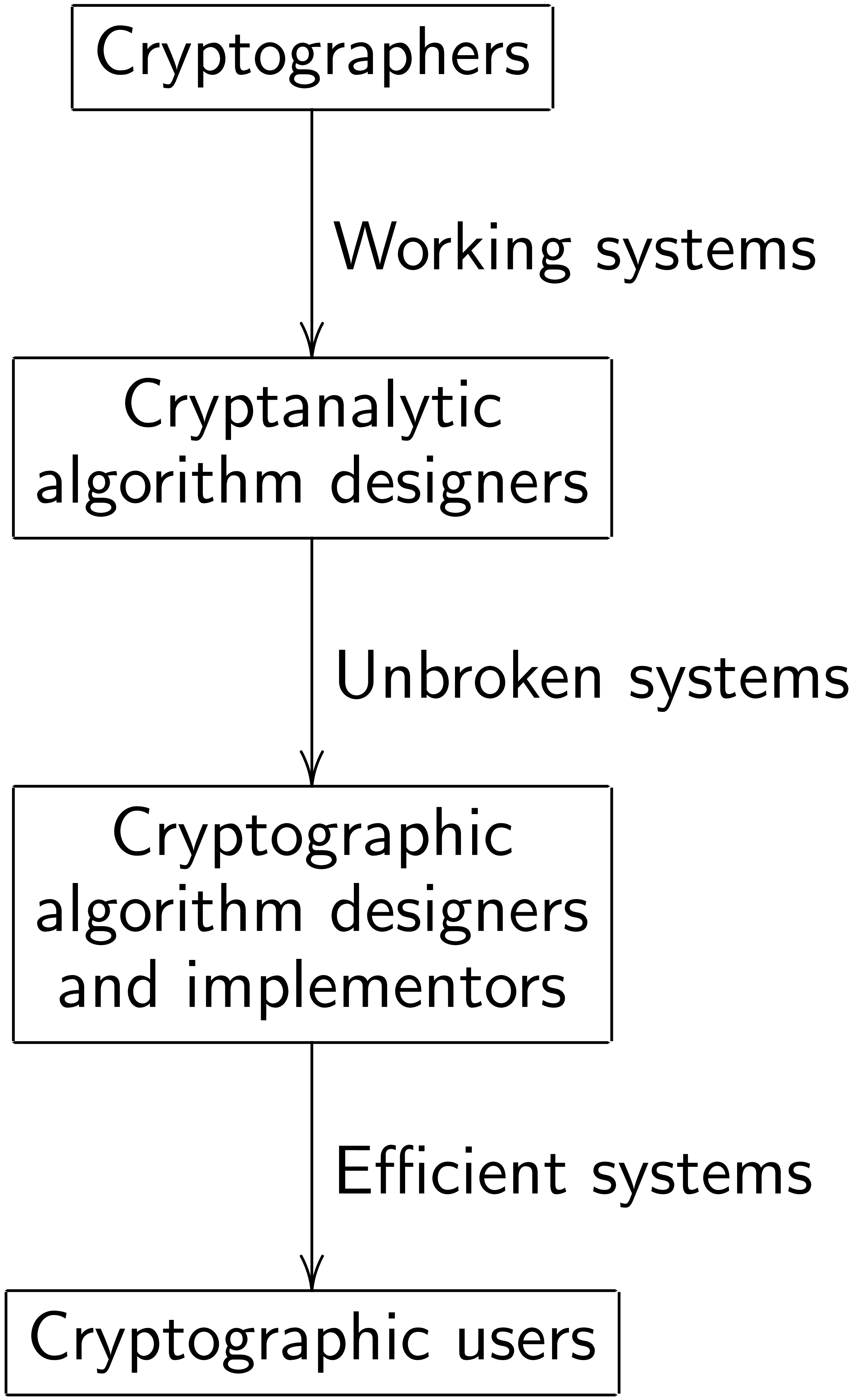

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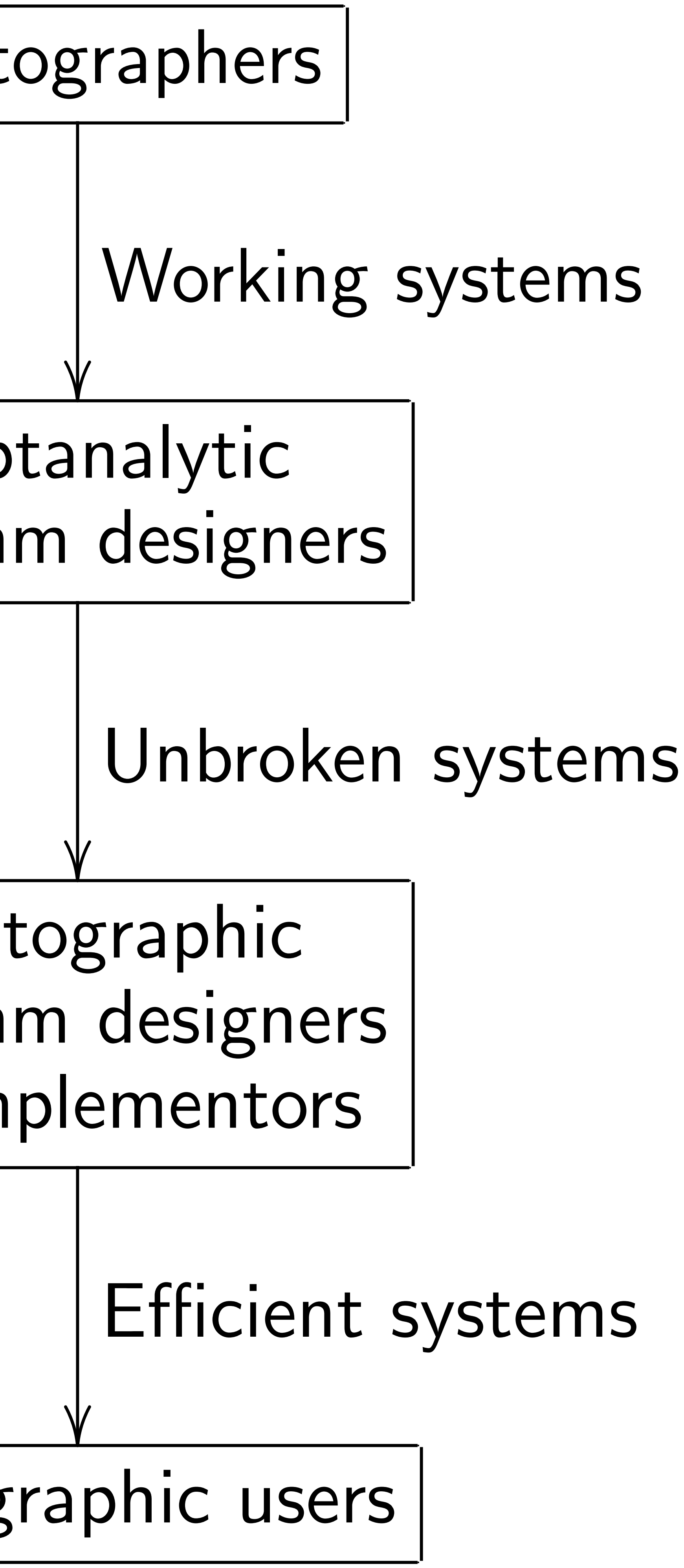

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Note 1: $\lg = \log_2$.

Note 2: $o(1)$ says
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Triple DES (for $b \le 112$),
AES-256 (for $b \le 256$),
RSA with $b^{3+o(1)}$-bit modul
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Cryptographic systems surviving *pre-quantum* cryptanalysis:

Triple DES (for $b \le 112$), AES-256 (for $b \le 256$), RSA with $b^{3+o(1)}$-bit modulus, McEliece with code length $b^{1+o(1)}$, Merkle signatures with "strong" $b^{1+o(1)}$-bit hash, BW with "strong" $b^{2+o(1)}$-bit discriminant, ECDSA with "strong" $b^{1+o(1)}$-bit curve, HFE$^{\text{v}-}$ with $b^{1+o(1)}$ polynomials, NTRU with $b^{1+o(1)}$ bits, et al.

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Typical algorithmic tools for
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Spectacular example:
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1994 Shor factors pq into p, q
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To push this beyond 2^b,
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$(b^2 \lg b)$.

e is asymptotically faster.

Even faster decryption.

bonus: Post-quantum.

Algorithmic advances can change the competition. Examples:

1. Speed up ECC: can reduce $\lg\lg b$ using 2007 Fürer; maybe someday eliminate $\lg\lg b$?

2. Faster attacks on McEliece:
2010 Bernstein–Lange–Peters,
2011 May–Meurer–Thomae,
2012 Becker–Joux–May–Meurer.
… but still $\Theta(b^2 \lg b)$.

3. We're optimizing "subfield AG" variant of McEliece.
Conjecture: Fastest attacks cost $2^{(\alpha+o(1))n}$; encryption $\Theta(b^2)$.

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Specifies linear $\mathbf{F}_2^n$

Typically $t \lg n \approx$
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Goppa codes

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$t \in \{2, 3, \ldots, \lfloor(q$
$n \in \{t \lg q + 1, t \lg$
e.g. $q = 1024$, $t =$
or $q = 4096$, $t = 1$

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$n \in \{t \lg q + 1, t \lg q + 2, \ldots$
e.g. $q = 1024$, $t = 50$, $n =$
or $q = 4096$, $t = 150$, $n = 3$

Receiver builds a matrix H
as the parity-check matrix
for the classical (genus-0)
irreducible length-n degree-t
binary Goppa code defined b
a monic degree-t irreducible
polynomial $g \in \mathbf{F}_q[x]$ and
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e.g. $q = 1024$, $t = 50$, $n = 1024$.
or $q = 4096$, $t = 150$, $n = 3600$.

Receiver builds a matrix H as the parity-check matrix for the classical (genus-0) irreducible length-n degree-t binary Goppa code defined by a monic degree-t irreducible polynomial $g \in \mathbf{F}_q[x]$ and distinct $a_1, a_2, \ldots, a_n \in \mathbf{F}_q$.

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. . . which means:

$$\left(\begin{array}{ll} \dfrac{1}{g(a_1)} & \cdots \\ \dfrac{a_1}{g(a_1)} & \cdots \\ \vdots & \ddots \\ \dfrac{a_1^{t-1}}{g(a_1)} & \cdots \end{array} \right.$$

View each element
as a column in $\mathbf{F}_2^{\lg}$
Then $H : \mathbf{F}_2^n \to \mathbf{F}$

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Then $H : \mathbf{F}_2^n \to \mathbf{F}_2^{t \lg q}$.

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from $\mathbf{F}_2^n$ to $\mathbf{F}_q[x]/g$.

H is the matrix for this map where $\mathbf{F}_2^n$ has standard basi
and $\mathbf{F}_q[x]/g$ has basis
$\lfloor g/x \rfloor$, $\lfloor g/x^2 \rfloor$, …, $\lfloor g/x^t \rfloor$.

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$$\frac{g - g(a_i)}{x - a_i} = \sum_{j \ge 0} a_i^j \left\lfloor g/x^{j+1}\right.$$

Receiver generates key K
as row reduction of H,
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More useful view: Consider the map $m \mapsto \sum_i m_i/(x - a_i)$ from $\mathbf{F}_2^n$ to $\mathbf{F}_q[x]/g$.

H is the matrix for this map where $\mathbf{F}_2^n$ has standard basis and $\mathbf{F}_q[x]/g$ has basis $\lfloor g/x \rfloor$, $\lfloor g/x^2 \rfloor$, …, $\lfloor g/x^t \rfloor$.

One-line proof: In $\mathbf{F}_q[x]$ have

$$\frac{g - g(a_i)}{x - a_i} = \sum_{j \geq 0} a_i^j \left\lfloor g/x^{j+1} \right\rfloor .$$

Receiver generates key K as row reduction of H, revealing only Ker$H$.

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where $f, g \in (\mathbf{Z}/q$
all coeffs in $\{-1, 0$
$\#\{i : f_i{=}{-1}\} =$
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Given ciphertext c
receiver computes
$(1+3f)c = (1+$
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Receiver built $h = 3g/(1 + 3$
where $f, g \in (\mathbf{Z}/q)[x]/(x^p -$
all coeffs in $\{-1, 0, 1\}$,
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Given ciphertext $c = m + rh$,
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$(1 + 3f)c = (1 + 3f)m + 3rg$
in $(\mathbf{Z}/q)[x]/(x^p - 1)$,
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Basic attack tool:
Lift pairs (u, uh)
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Take $p \in \Theta(b)$ for
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Time $b(\lg b)^{2+o(1)}$
to multiply in
$(\mathbf{Z}/q)[x]/(x^p - 1)$

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Take $p \in \Theta(b)$ for security 2
against all known attacks.

$\Theta(b \lg b)$ bits in key.

Time $b(\lg b)^{2+o(1)}$
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1985 Lange–Ruppert:
Explicit complete system
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Reduce formulas to 53 monomials
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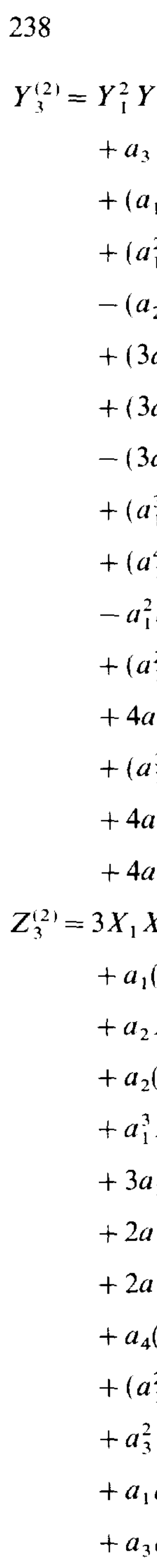

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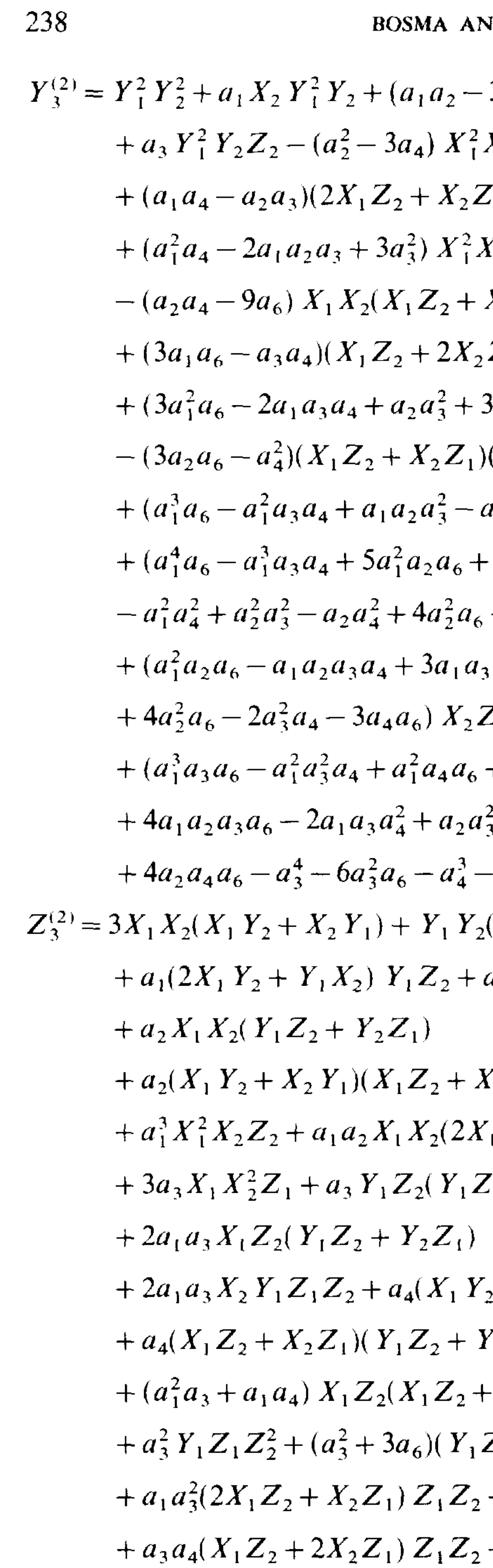

238 BOSMA AN

$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4)\, X_1^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2)\, X_1^2 \\
& - (a_2 a_4 - 9a_6)\, X_1 X_2 (X_1 Z_2 + \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3 \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(\\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6)\, X_2 Z \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (\\
& + a_1 (2X_1 Y_2 + Y_1 X_2)\, Y_1 Z_2 + \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y \\
& + (a_1^2 a_3 + a_1 a_4)\, X_1 Z_2 (X_1 Z_2 + \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1)\, Z_1 Z_2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1)\, Z_1 Z_2
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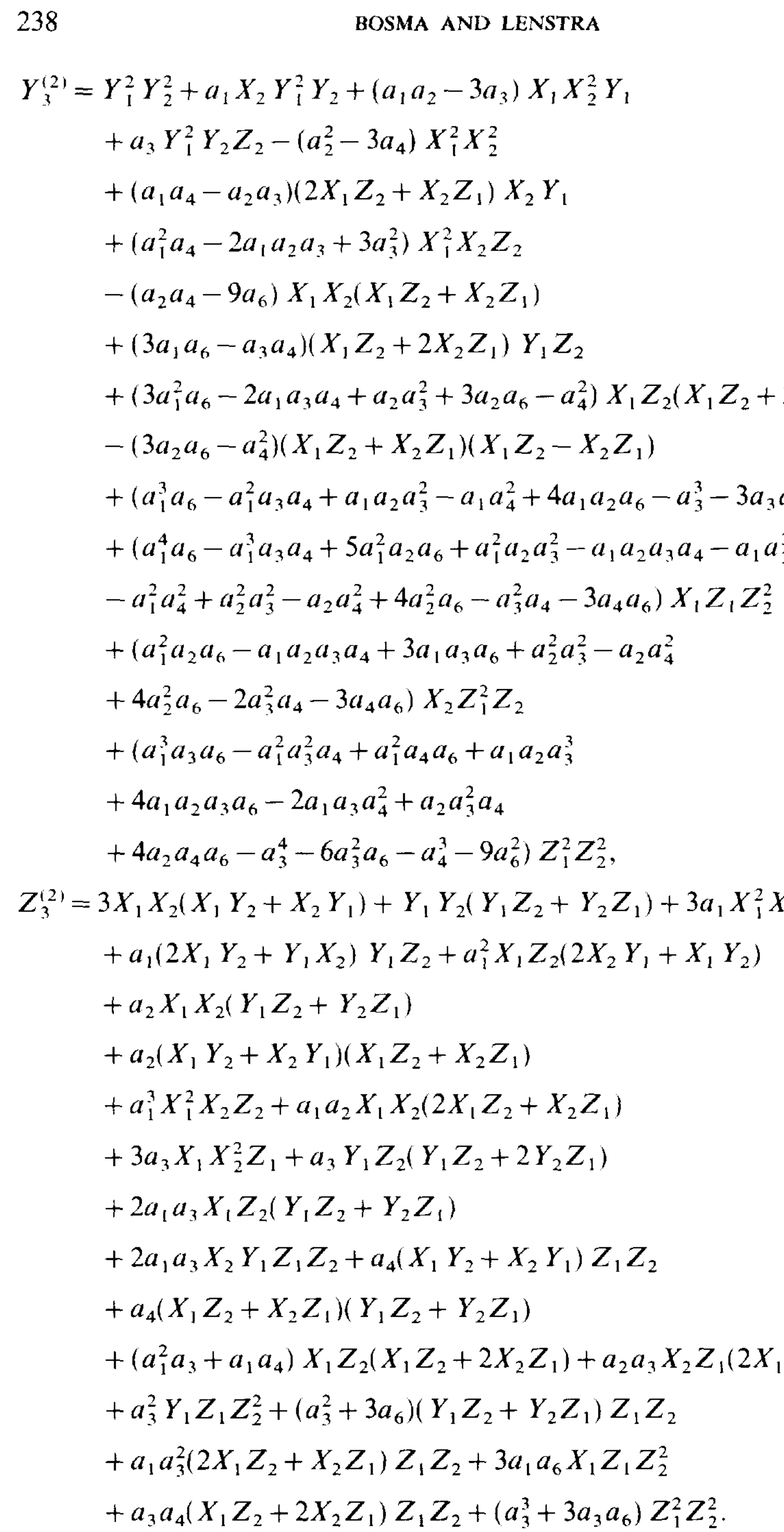

238 BOSMA AND LENSTRA

$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2Y_2^2 + a_1X_2Y_1^2Y_2 + (a_1a_2 - 3a_3)\,X_1X_2^2Y_1 \\
& + a_3Y_1^2Y_2Z_2 - (a_2^2 - 3a_4)\,X_1^2X_2^2 \\
& + (a_1a_4 - a_2a_3)(2X_1Z_2 + X_2Z_1)\,X_2Y_1 \\
& + (a_1^2a_4 - 2a_1a_2a_3 + 3a_3^2)\,X_1^2X_2Z_2 \\
& - (a_2a_4 - 9a_6)\,X_1X_2(X_1Z_2 + X_2Z_1) \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 + 2X_2Z_1)\,Y_1Z_2 \\
& + (3a_1^2a_6 - 2a_1a_3a_4 + a_2a_3^2 + 3a_2a_6 - a_4^2)\,X_1Z_2(X_1Z_2 + \\
& - (3a_2a_6 - a_4^2)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 - a_1a_4^2 + 4a_1a_2a_6 - a_3^3 - 3a_3 \\
& + (a_1^4a_6 - a_1^3a_3a_4 + 5a_1^2a_2a_6 + a_1^2a_2a_3^2 - a_1a_2a_3a_4 - a_1a \\
& - a_1^2a_4^2 + a_2^2a_3^2 - a_2a_4^2 + 4a_2^2a_6 - a_3^2a_4 - 3a_4a_6)\,X_1Z_1Z_2^2 \\
& + (a_1^2a_2a_6 - a_1a_2a_3a_4 + 3a_1a_3a_6 + a_2^2a_3^2 - a_2a_4^2 \\
& + 4a_2^2a_6 - 2a_3^2a_4 - 3a_4a_6)\,X_2Z_1^2Z_2 \\
& + (a_1^3a_3a_6 - a_1^2a_3^2a_4 + a_1^2a_4a_6 + a_1a_2a_3^3 \\
& + 4a_1a_2a_3a_6 - 2a_1a_3a_4^2 + a_2a_3^2a_4 \\
& + 4a_2a_4a_6 - a_3^4 - 6a_3^2a_6 - a_4^3 - 9a_6^2)\,Z_1^2Z_2^2, \\
Z_3^{(2)} = {} & 3X_1X_2(X_1Y_2 + X_2Y_1) + Y_1Y_2(Y_1Z_2 + Y_2Z_1) + 3a_1X_1^2X \\
& + a_1(2X_1Y_2 + Y_1X_2)\,Y_1Z_2 + a_1^2X_1Z_2(2X_2Y_1 + X_1Y_2) \\
& + a_2X_1X_2(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& + a_1^3X_1^2X_2Z_2 + a_1a_2X_1X_2(2X_1Z_2 + X_2Z_1) \\
& + 3a_3X_1X_2^2Z_1 + a_3Y_1Z_2(Y_1Z_2 + 2Y_2Z_1) \\
& + 2a_1a_3X_1Z_2(Y_1Z_2 + Y_2Z_1) \\
& + 2a_1a_3X_2Y_1Z_1Z_2 + a_4(X_1Y_2 + X_2Y_1)\,Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + (a_1^2a_3 + a_1a_4)\,X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_2a_3X_2Z_1(2X_1 \\
& + a_3^2Y_1Z_1Z_2^2 + (a_3^2 + 3a_6)(Y_1Z_2 + Y_2Z_1)\,Z_1Z_2 \\
& + a_1a_3^2(2X_1Z_2 + X_2Z_1)\,Z_1Z_2 + 3a_1a_6X_1Z_1Z_2^2 \\
& + a_3a_4(X_1Z_2 + 2X_2Z_1)\,Z_1Z_2 + (a_3^3 + 3a_3a_6)\,Z_1^2Z_2^2.
\end{aligned}
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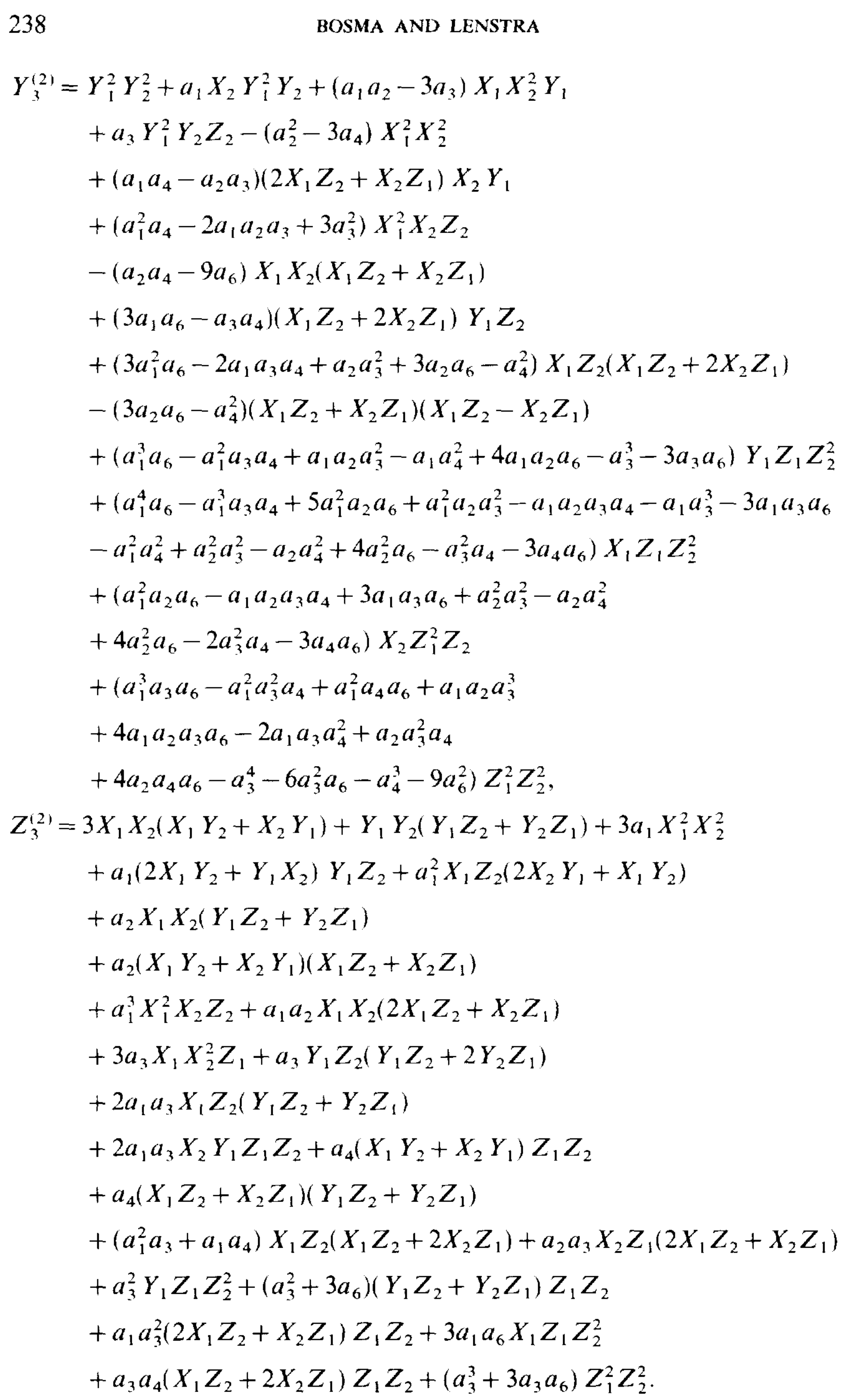

238 BOSMA AND LENSTRA

$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3)\, X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4)\, X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1)\, X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2)\, X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6)\, X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1)\, Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6)\, Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6)\, X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6)\, X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2)\, Z_1^2 Z_2^2, \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1 (2X_1 Y_2 + Y_1 X_2)\, Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1)\, Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1)\, Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1)\, Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1)\, Z_1 Z_2 + (a_3^3 + 3a_3 a_6)\, Z_1^2 Z_2^2.
\end{aligned}
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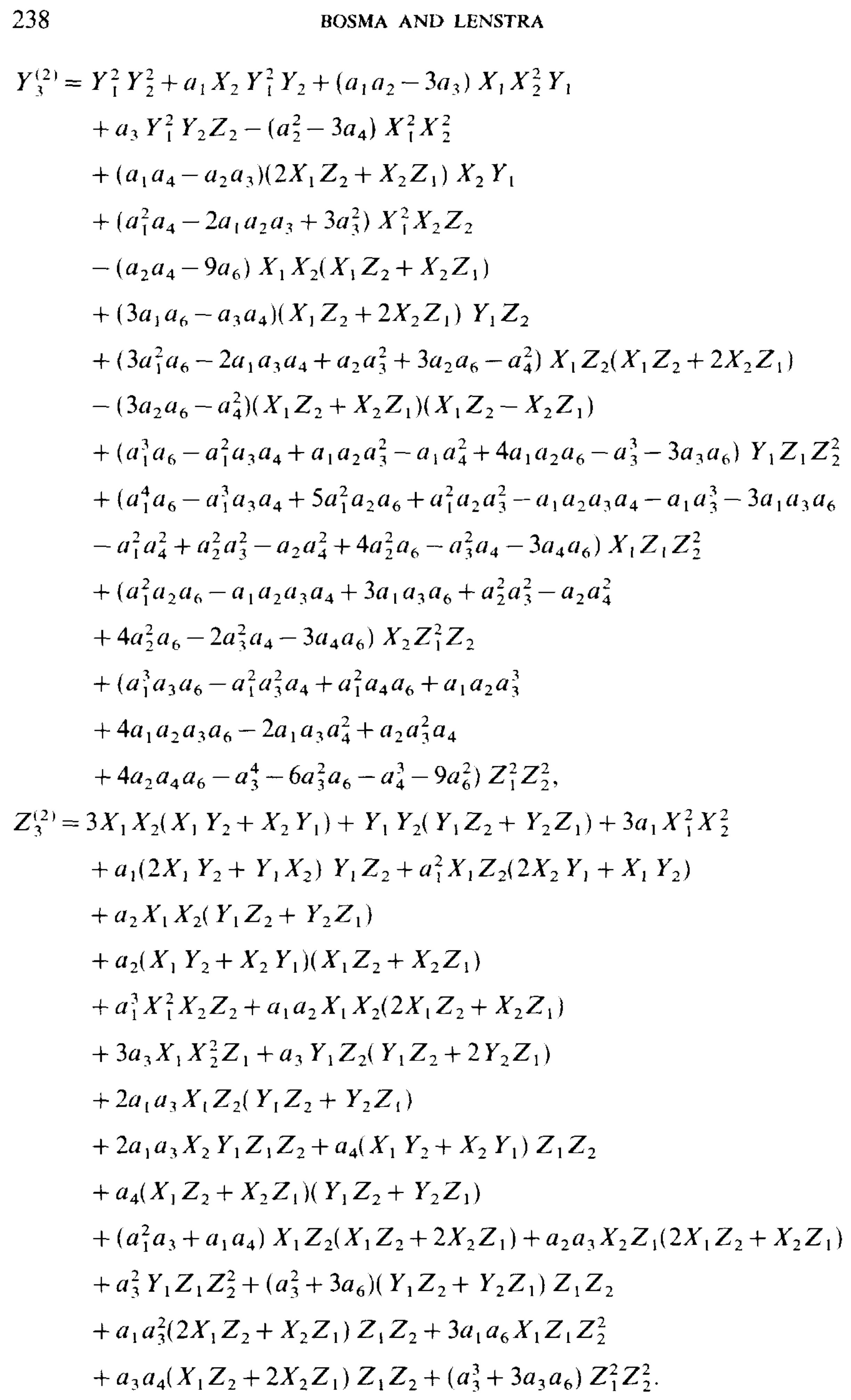

238 BOSMA AND LENSTRA

$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2Y_2^2 + a_1X_2Y_1^2Y_2 + (a_1a_2 - 3a_3)\,X_1X_2^2Y_1 \\
& + a_3Y_1^2Y_2Z_2 - (a_2^2 - 3a_4)\,X_1^2X_2^2 \\
& + (a_1a_4 - a_2a_3)(2X_1Z_2 + X_2Z_1)\,X_2Y_1 \\
& + (a_1^2a_4 - 2a_1a_2a_3 + 3a_3^2)\,X_1^2X_2Z_2 \\
& - (a_2a_4 - 9a_6)\,X_1X_2(X_1Z_2 + X_2Z_1) \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 + 2X_2Z_1)\,Y_1Z_2 \\
& + (3a_1^2a_6 - 2a_1a_3a_4 + a_2a_3^2 + 3a_2a_6 - a_4^2)\,X_1Z_2(X_1Z_2 + 2X_2Z_1) \\
& - (3a_2a_6 - a_4^2)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 - a_1a_4^2 + 4a_1a_2a_6 - a_3^3 - 3a_3a_6)\,Y_1Z_1Z_2^2 \\
& + (a_1^4a_6 - a_1^3a_3a_4 + 5a_1^2a_2a_6 + a_1^2a_2a_3^2 - a_1a_2a_3a_4 - a_1a_3^3 - 3a_1a_3a_6 \\
& - a_1^2a_4^2 + a_2^2a_3^2 - a_2a_4^2 + 4a_2^2a_6 - a_3^2a_4 - 3a_4a_6)\,X_1Z_1Z_2^2 \\
& + (a_1^2a_2a_6 - a_1a_2a_3a_4 + 3a_1a_3a_6 + a_2^2a_3^2 - a_2a_4^2 \\
& + 4a_2^2a_6 - 2a_3^2a_4 - 3a_4a_6)\,X_2Z_1^2Z_2 \\
& + (a_1^3a_3a_6 - a_1^2a_3^2a_4 + a_1^2a_4a_6 + a_1a_2a_3^3 \\
& + 4a_1a_2a_3a_6 - 2a_1a_3a_4^2 + a_2a_3^2a_4 \\
& + 4a_2a_4a_6 - a_3^4 - 6a_3^2a_6 - a_4^3 - 9a_6^2)\,Z_1^2Z_2^2, \\
Z_3^{(2)} = {} & 3X_1X_2(X_1Y_2 + X_2Y_1) + Y_1Y_2(Y_1Z_2 + Y_2Z_1) + 3a_1X_1^2X_2^2 \\
& + a_1(2X_1Y_2 + Y_1X_2)\,Y_1Z_2 + a_1^2X_1Z_2(2X_2Y_1 + X_1Y_2) \\
& + a_2X_1X_2(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& + a_1^3X_1^2X_2Z_2 + a_1a_2X_1X_2(2X_1Z_2 + X_2Z_1) \\
& + 3a_3X_1X_2^2Z_1 + a_3Y_1Z_2(Y_1Z_2 + 2Y_2Z_1) \\
& + 2a_1a_3X_1Z_2(Y_1Z_2 + Y_2Z_1) \\
& + 2a_1a_3X_2Y_1Z_1Z_2 + a_4(X_1Y_2 + X_2Y_1)\,Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + (a_1^2a_3 + a_1a_4)\,X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_2a_3X_2Z_1(2X_1Z_2 + X_2Z_1) \\
& + a_3^2Y_1Z_1Z_2^2 + (a_3^2 + 3a_6)(Y_1Z_2 + Y_2Z_1)\,Z_1Z_2 \\
& + a_1a_3^2(2X_1Z_2 + X_2Z_1)\,Z_1Z_2 + 3a_1a_6X_1Z_1Z_2^2 \\
& + a_3a_4(X_1Z_2 + 2X_2Z_1)\,Z_1Z_2 + (a_3^3 + 3a_3a_6)\,Z_1^2Z_2^2.
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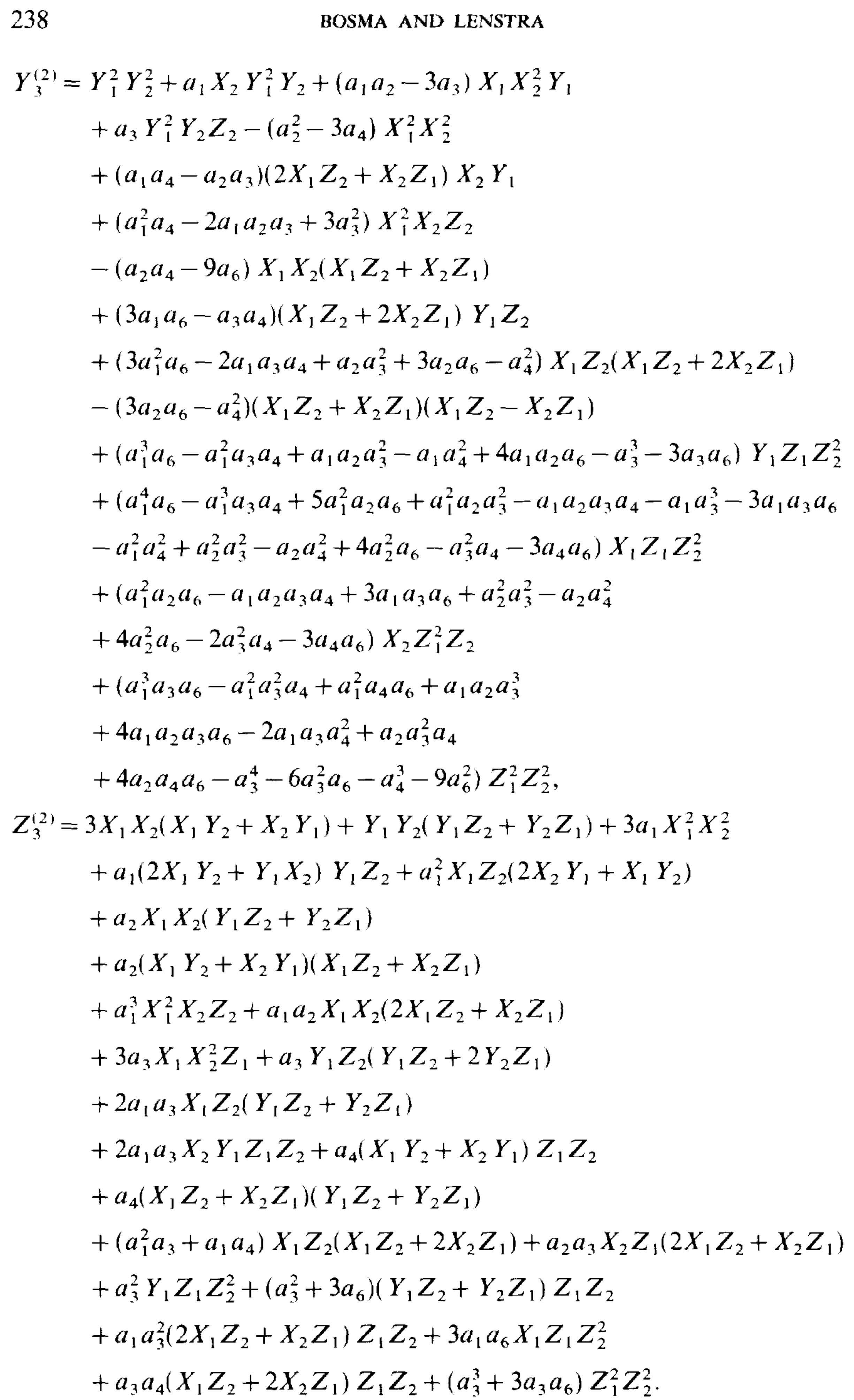

238 BOSMA AND LENSTRA

$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3) X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4) X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1) X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2) X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6) X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1) Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2) X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6) Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6) X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6) X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2) Z_1^2 Z_2^2, \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1 (2X_1 Y_2 + Y_1 X_2) Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1) Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4) X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1) Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1) Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1) Z_1 Z_2 + (a_3^3 + 3a_3 a_6) Z_1^2 Z_2^2.
\end{aligned}$$

1995 Bosma–Lens

Explicit complete s

of 2 addition laws

for long Weierstra

$X_3, Y_3, Z_3, X'_3, Y'_3,$

$\in \mathbf{Z}[a_1, a_2, a_3, a_4,$

X_1, Y_1, Z_1, X_2

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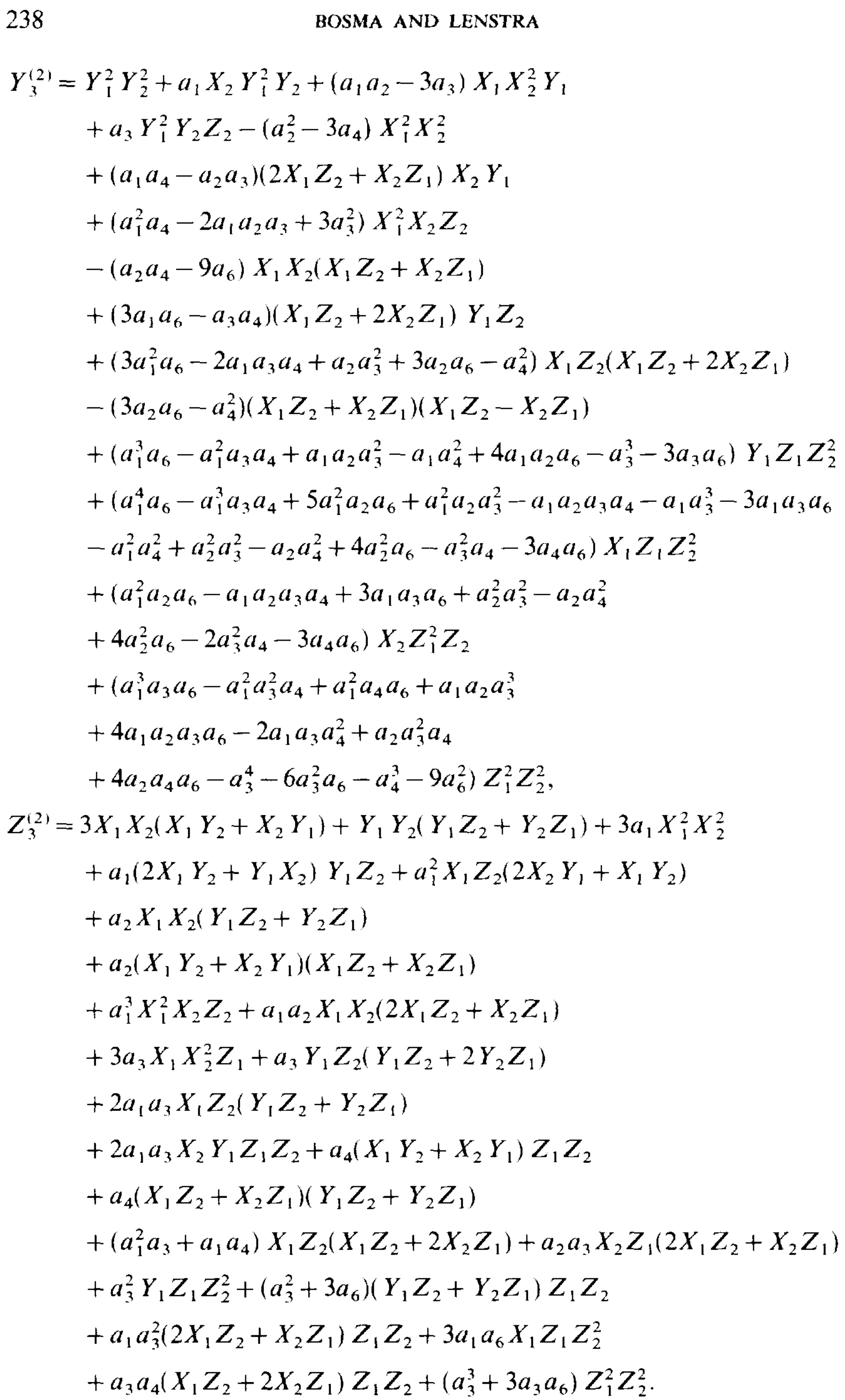

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$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3)\, X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4)\, X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1)\, X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2)\, X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6)\, X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1)\, Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6)\, Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6)\, X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6)\, X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2)\, Z_1^2 Z_2^2, \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1 (2X_1 Y_2 + Y_1 X_2)\, Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1)\, Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1)\, Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1)\, Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1)\, Z_1 Z_2 + (a_3^3 + 3a_3 a_6)\, Z_1^2 Z_2^2.
\end{aligned}$$

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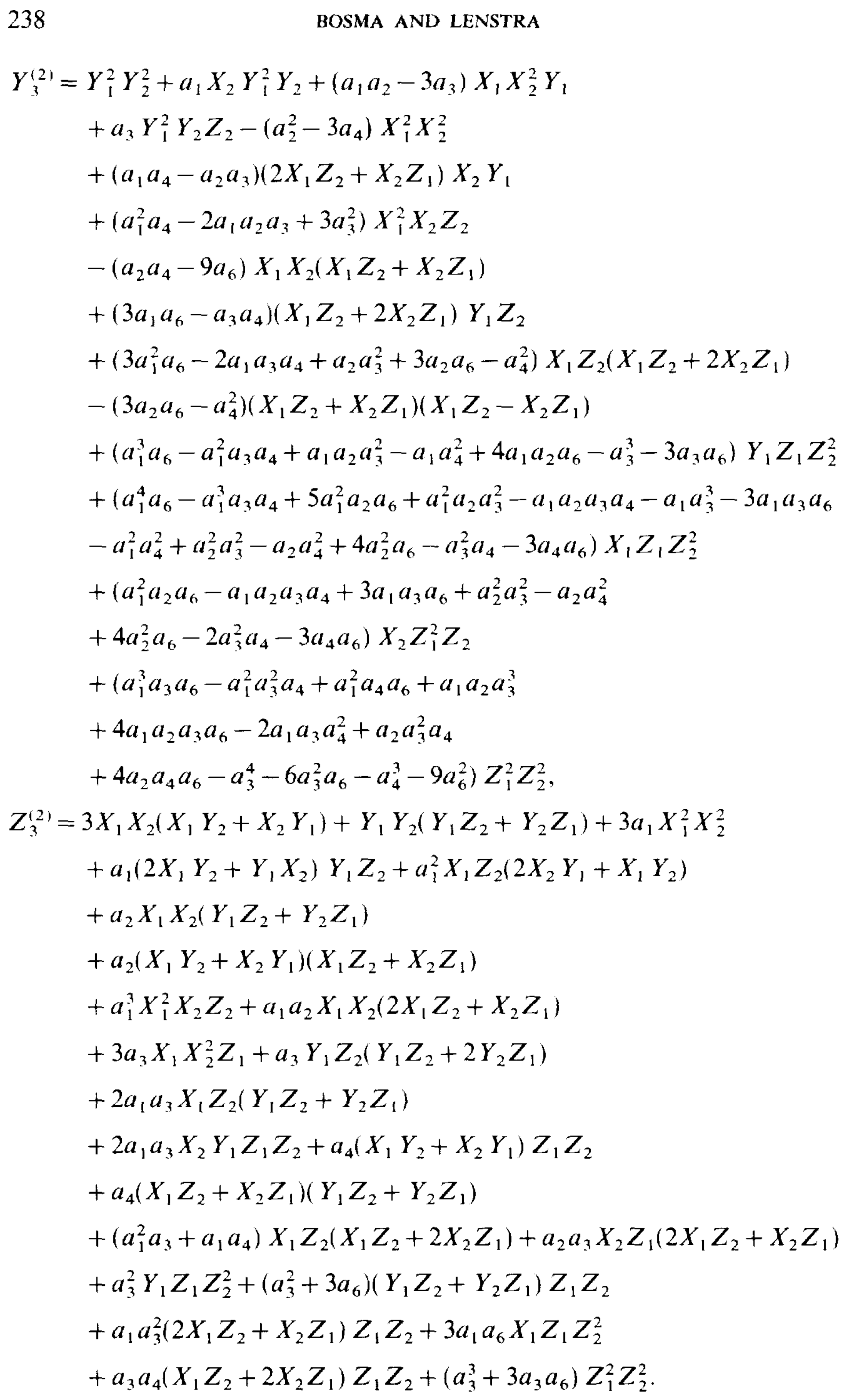
$X_1, Y_1, Z_1, X_2, Y_2, Z_2]$.



238 BOSMA AND LENSTRA

$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3)\, X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4)\, X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1)\, X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2)\, X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6)\, X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1)\, Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6)\, Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6)\, X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6)\, X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2)\, Z_1^2 Z_2^2, \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1 (2X_1 Y_2 + Y_1 X_2)\, Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1)\, Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1)\, Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1)\, Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1)\, Z_1 Z_2 + (a_3^3 + 3a_3 a_6)\, Z_1^2 Z_2^2.
\end{aligned}$$

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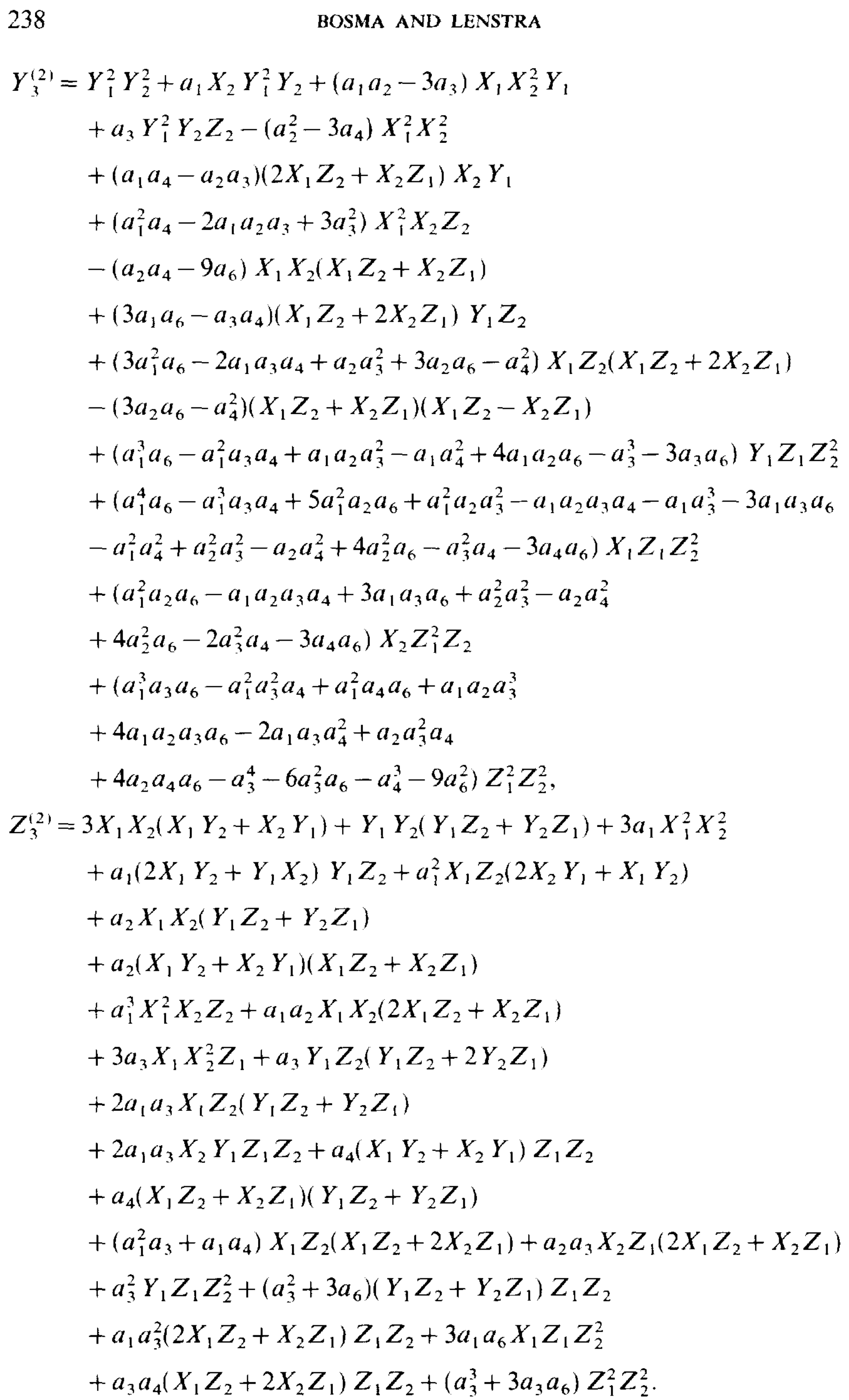
$\in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6,$
$X_1, Y_1, Z_1, X_2, Y_2, Z_2].$



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$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3)\, X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4)\, X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1)\, X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2)\, X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6)\, X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1)\, Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6)\, Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6)\, X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6)\, X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2)\, Z_1^2 Z_2^2, \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1 (2X_1 Y_2 + Y_1 X_2)\, Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
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& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1)\, Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1)\, Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1)\, Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1)\, Z_1 Z_2 + (a_3^3 + 3a_3 a_6)\, Z_1^2 Z_2^2.
\end{aligned}
$$

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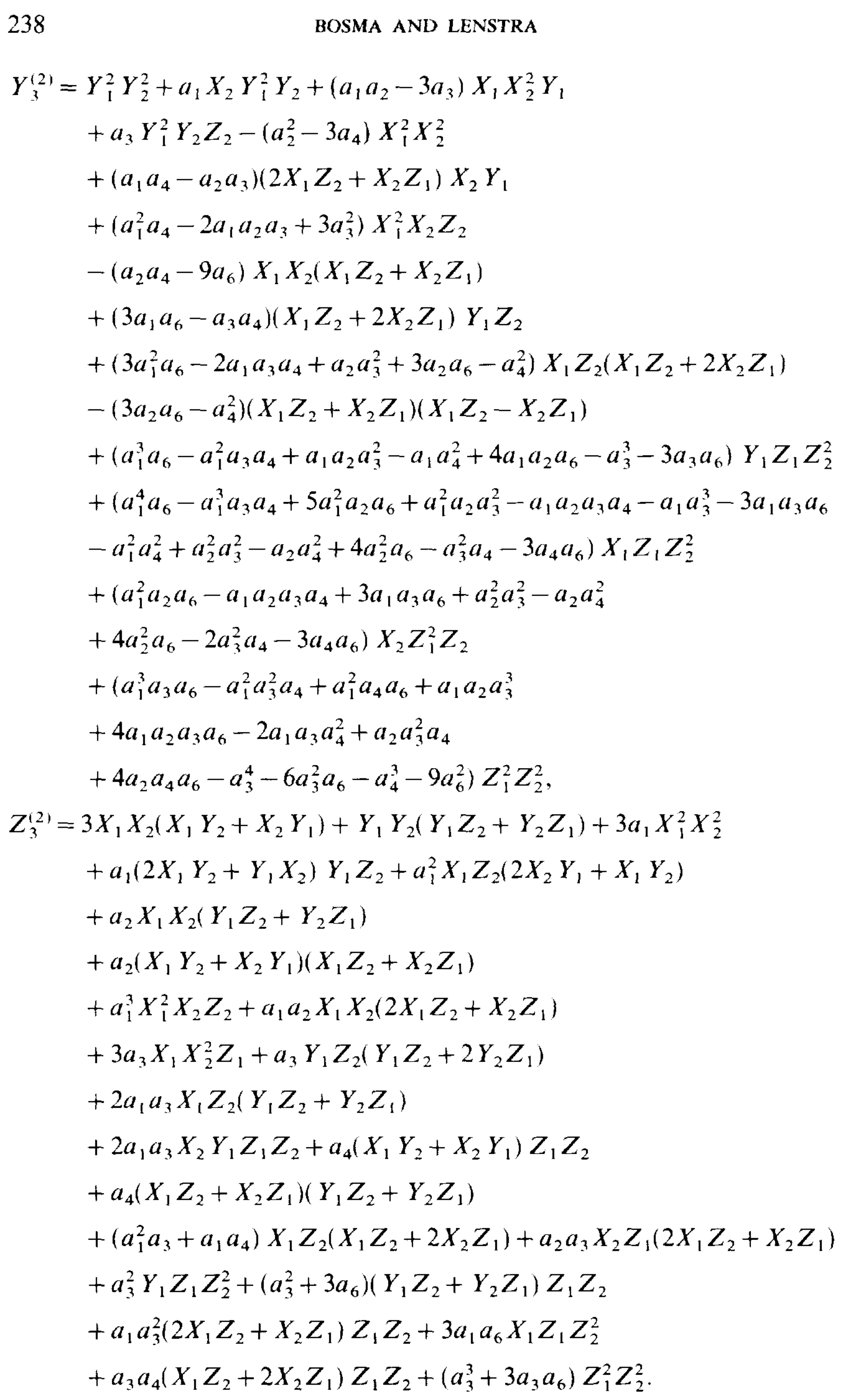
$X_1, Y_1, Z_1, X_2, Y_2, Z_2]$.

My previous slide in this talk:
Bosma–Lenstra Y'_3, Z'_3.



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$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3)\, X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4)\, X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1)\, X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2)\, X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6)\, X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1)\, Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6)\, Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6)\, X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6)\, X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2)\, Z_1^2 Z_2^2, \\
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& + a_1 (2X_1 Y_2 + Y_1 X_2)\, Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1)\, Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1)\, Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1)\, Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1)\, Z_1 Z_2 + (a_3^3 + 3a_3 a_6)\, Z_1^2 Z_2^2.
\end{aligned}
$$

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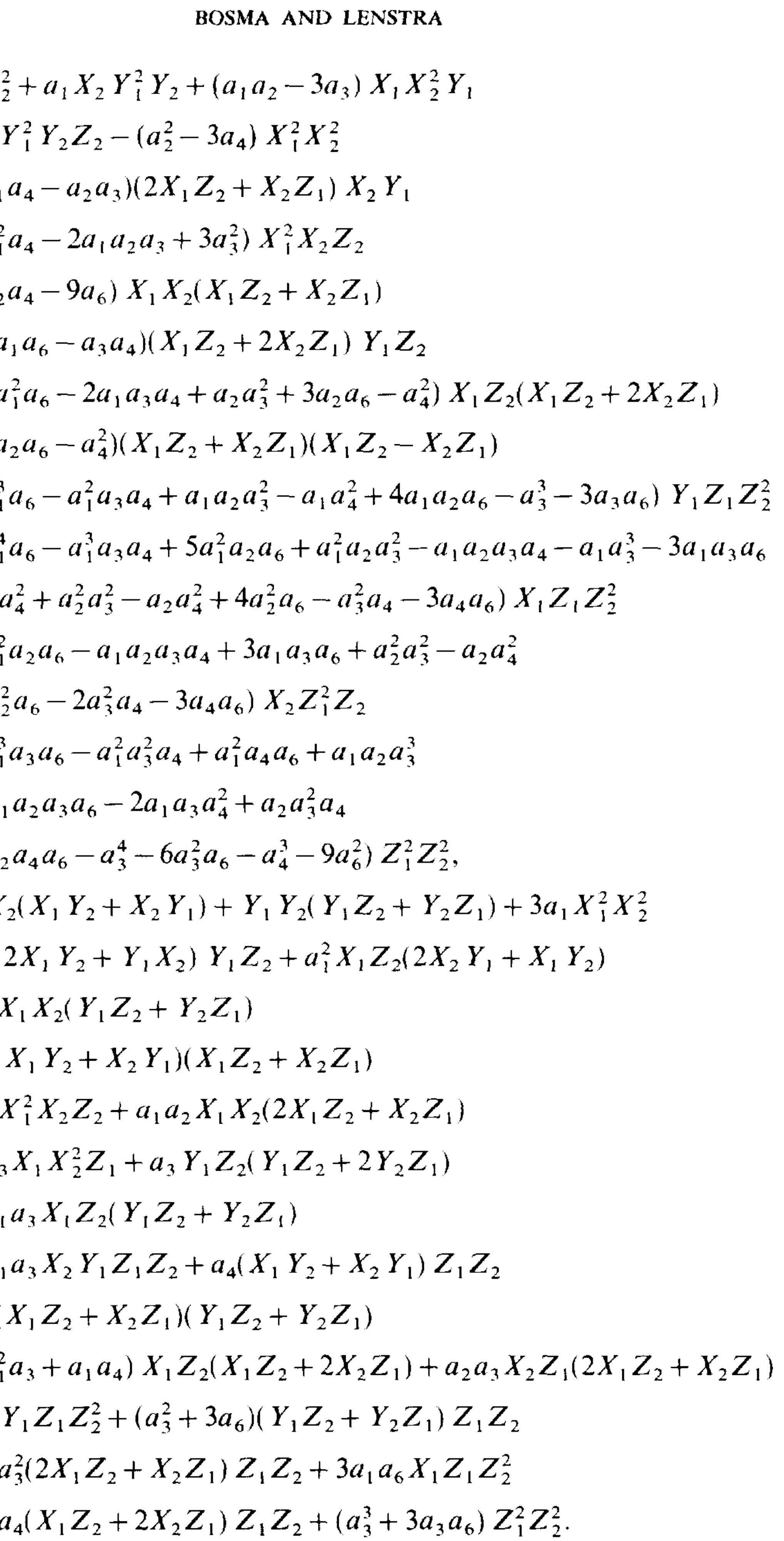
$X_1, Y_1, Z_1, X_2, Y_2, Z_2].$

My previous slide in this talk:
Bosma–Lenstra Y_3', Z_3'.
Actually, slide shows
$\text{Publish}(Y_3'), \text{Publish}(Z_3')$,
where Publish introduces typos.



BOSMA AND LENSTRA

${}^2_2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3) X_1 X_2^2 Y_1$

$Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4) X_1^2 X_2^2$

${}_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1) X_2 Y_1$

${}^2_1 a_4 - 2a_1 a_2 a_3 + 3a_3^2) X_1^2 X_2 Z_2$

${}_2 a_4 - 9a_6) X_1 X_2(X_1 Z_2 + X_2 Z_1)$

$a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1) Y_1 Z_2$

$a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2) X_1 Z_2(X_1 Z_2 + 2X_2 Z_1)$

$a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1)$

${}^3_1 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6) Y_1 Z_1 Z_2^2$

${}^4_1 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6$

$a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6) X_1 Z_1 Z_2^2$

${}^2_1 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2$

${}^2_2 a_6 - 2a_3^2 a_4 - 3a_4 a_6) X_2 Z_1^2 Z_2$

${}^3_1 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3$

${}_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4$

${}_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2) Z_1^2 Z_2^2,$

${}_2(X_1 Y_2 + X_2 Y_1) + Y_1 Y_2(Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2$

$2X_1 Y_2 + Y_1 X_2) Y_1 Z_2 + a_1^2 X_1 Z_2(2X_2 Y_1 + X_1 Y_2)$

$X_1 X_2(Y_1 Z_2 + Y_2 Z_1)$

$X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1)$

$X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2(2X_1 Z_2 + X_2 Z_1)$

${}_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2(Y_1 Z_2 + 2Y_2 Z_1)$

${}_1 a_3 X_1 Z_2(Y_1 Z_2 + Y_2 Z_1)$

${}_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4(X_1 Y_2 + X_2 Y_1) Z_1 Z_2$

$(X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1)$

${}^2_1 a_3 + a_1 a_4) X_1 Z_2(X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1(2X_1 Z_2 + X_2 Z_1)$

$Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1) Z_1 Z_2$

$a_3^2(2X_1 Z_2 + X_2 Z_1) Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2$

$a_4(X_1 Z_2 + 2X_2 Z_1) Z_1 Z_2 + (a_3^3 + 3a_3 a_6) Z_1^2 Z_2^2.$

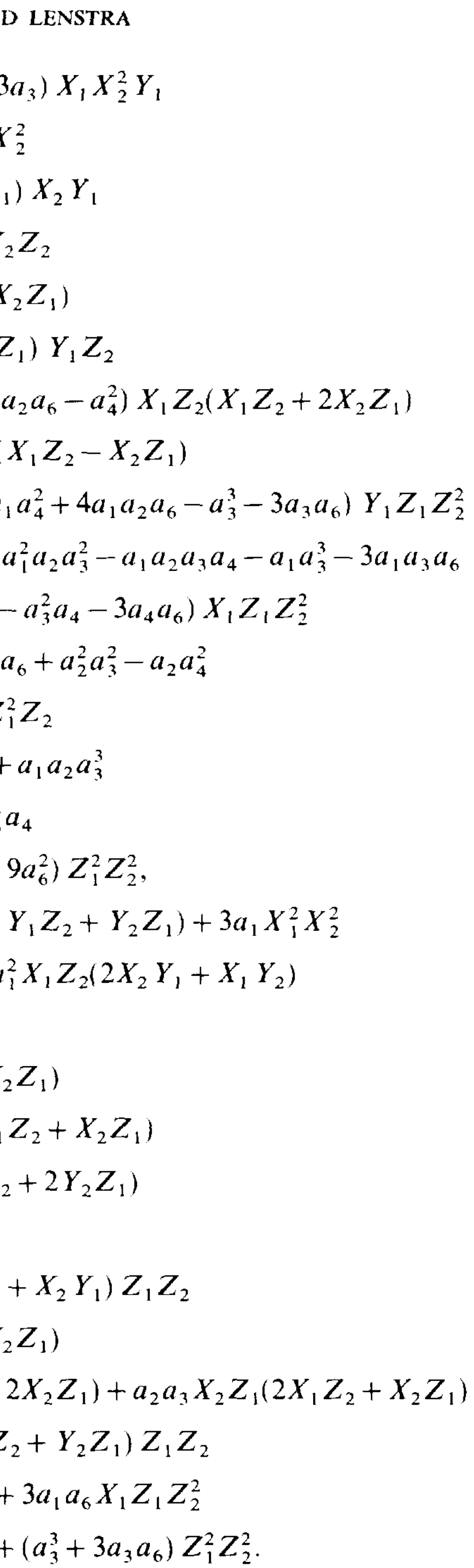
1995 Bosma–Lenstra:
Explicit complete system
of 2 addition laws
for long Weierstrass curves:
$X_3, Y_3, Z_3, X_3', Y_3', Z_3'$
$\in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6,$
$X_1, Y_1, Z_1, X_2, Y_2, Z_2].$

My previous slide in this talk:
Bosma–Lenstra Y_3', Z_3'.
Actually, slide shows
$\text{Publish}(Y_3'), \text{Publish}(Z_3')$,
where Publish introduces typos.

What th

For all fi
all $\mathbf{P}^2$ W
$E/k : Y$
$X^3 + a_2$
all $P_1 =$
all $P_2 =$

$(X_3 : Y_3$
is $P_1 +$
$(X_3' : Y_3'$
is $P_1 +$
at most



D LENSTRA

$3a_3)\,X_1X_2^2Y_1$

X_2^2

$_1)\,X_2Y_1$

$_2Z_2$

$X_2Z_1)$

$Z_1)\,Y_1Z_2$

$a_2a_6-a_4^2)\,X_1Z_2(X_1Z_2+2X_2Z_1)$

$(X_1Z_2-X_2Z_1)$

$_1a_4^2+4a_1a_2a_6-a_3^3-3a_3a_6)\,Y_1Z_1Z_2^2$

$a_1^2a_2a_3^2-a_1a_2a_3a_4-a_1a_3^3-3a_1a_3a_6$

$-a_3^2a_4-3a_4a_6)\,X_1Z_1Z_2^2$

$a_6+a_2^2a_3^2-a_2a_4^2$

$Z_1^2Z_2$

$+a_1a_2a_3^3$

a_4

$9a_6^2)\,Z_1^2Z_2^2,$

$Y_1Z_2+Y_2Z_1)+3a_1X_1^2X_2^2$

$X_1Z_2(2X_2Y_1+X_1Y_2)$

$_2Z_1)$

$Z_2+X_2Z_1)$

$_2+2Y_2Z_1)$

$+X_2Y_1)\,Z_1Z_2$

$_2Z_1)$

$2X_2Z_1)+a_2a_3X_2Z_1(2X_1Z_2+X_2Z_1)$

$Z_2+Y_2Z_1)\,Z_1Z_2$

$+3a_1a_6X_1Z_1Z_2^2$

$+(a_3^3+3a_3a_6)\,Z_1^2Z_2^2.$

1995 Bosma–Lenstra:
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$X_3, Y_3, Z_3, X_3', Y_3', Z_3'$
$\in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6,$
$X_1, Y_1, Z_1, X_2, Y_2, Z_2].$

My previous slide in this talk:
Bosma–Lenstra Y_3', Z_3'.
Actually, slide shows
$\text{Publish}(Y_3'), \text{Publish}(Z_3')$,
where Publish introduces typos.

What this means:

For all fields k,
all $\mathbf{P}^2$ Weierstrass
$E/k : Y^2Z + a_1X$
$X^3 + a_2X^2Z + a_4$
all $P_1 = (X_1 : Y_1 :$
all $P_2 = (X_2 : Y_2 :$

$(X_3 : Y_3 : Z_3)$
is $P_1 + P_2$ or $(0 :$
$(X_3' : Y_3' : Z_3')$
is $P_1 + P_2$ or $(0 :$
at most one of the

$2X_2Z_1)$

$a_6)\ Y_1Z_1Z_2^2$

${}_3^3 - 3a_1a_3a_6$

${}_2^2$

$Z_2 + X_2Z_1)$

1995 Bosma–Lenstra:
Explicit complete system
of 2 addition laws
for long Weierstrass curves:

$$X_3, Y_3, Z_3, X_3', Y_3', Z_3' \in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6, X_1, Y_1, Z_1, X_2, Y_2, Z_2].$$

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What this means:

For all fields k,
all $\mathbf{P}^2$ Weierstrass curves
$E/k : Y^2Z + a_1XYZ + a_3Y$
$X^3 + a_2X^2Z + a_4XZ^2 + a$
all $P_1 = (X_1 : Y_1 : Z_1) \in E($
all $P_2 = (X_2 : Y_2 : Z_2) \in E($

$(X_3 : Y_3 : Z_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
$(X_3' : Y_3' : Z_3')$
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$\in \mathbf{Z}[a_1, a_2, a_3, a_4, a_6,$
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What this means:

For all fields k,
all $\mathbf{P}^2$ Weierstrass curves
$E/k : Y^2Z + a_1XYZ + a_3YZ^2 =$
$X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$,
all $P_1 = (X_1 : Y_1 : Z_1) \in E(k)$,
all $P_2 = (X_2 : Y_2 : Z_2) \in E(k)$:

$(X_3 : Y_3 : Z_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
$(X_3' : Y_3' : Z_3')$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
at most one of these is $(0 : 0 : 0)$.

osma–Lenstra:

complete system
ition laws
Weierstrass curves:
Z_3, X'_3, Y'_3, Z'_3
$a_2, a_3, a_4, a_6,$
$Y_1, Z_1, X_2, Y_2, Z_2]$.

ious slide in this talk:
Lenstra Y'_3, Z'_3.
, slide shows
$Y'_3), \mathrm{Publish}(Z'_3),$
ublish introduces typos.

What this means:

For all fields k,
all $\mathbf{P}^2$ Weierstrass curves
$E/k : Y^2Z + a_1XYZ + a_3YZ^2 =$
$X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$
all $P_1 = (X_1 : Y_1 : Z_1) \in E(k)$,
all $P_2 = (X_2 : Y_2 : Z_2) \in E(k)$:

$(X_3 : Y_3 : Z_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
$(X'_3 : Y'_3 : Z'_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
at most one of these is $(0 : 0 : 0)$.

2009 Be

For all fi
all $\mathbf{P}^1 \times$
$X^2T^2 +$
all P_1, P
$P_1 = (($
$P_2 = (($

$(X_3 : Z_3$
$(X'_3 : Z'_3$
$(Y_3 : T_3)$
$(Y'_3 : T'_3)$
at most

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system

ss curves:

Z_3'

a_6,

$_2, Y_2, Z_2]$.

in this talk:

$', Z_3'$.

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oduces typos.

What this means:

For all fields k,

all $\mathbf{P}^2$ Weierstrass curves

$E/k : Y^2Z + a_1XYZ + a_3YZ^2 =$

$X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$,

all $P_1 = (X_1 : Y_1 : Z_1) \in E(k)$,

all $P_2 = (X_2 : Y_2 : Z_2) \in E(k)$:

$(X_3 : Y_3 : Z_3)$

is $P_1 + P_2$ or $(0 : 0 : 0)$;

$(X_3' : Y_3' : Z_3')$

is $P_1 + P_2$ or $(0 : 0 : 0)$;

at most one of these is $(0 : 0 : 0)$.

2009 Bernstein–T.

For all fields k wit

all $\mathbf{P}^1 \times \mathbf{P}^1$ Edwar

$X^2T^2 + Y^2Z^2 =$

all $P_1, P_2 \in E(k)$,

$P_1 = ((X_1 : Z_1), ($

$P_2 = ((X_2 : Z_2), ($

$(X_3 : Z_3)$ is $x(P_1$

$(X_3' : Z_3')$ is $x(P_1$

$(Y_3 : T_3)$ is $y(P_1 +$

$(Y_3' : T_3')$ is $y(P_1 +$

at most one of the

k:

pos.

What this means:

For all fields k,
all $\mathbf{P}^2$ Weierstrass curves
$E/k : Y^2Z + a_1XYZ + a_3YZ^2 =$
$X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$,
all $P_1 = (X_1 : Y_1 : Z_1) \in E(k)$,
all $P_2 = (X_2 : Y_2 : Z_2) \in E(k)$:

$(X_3 : Y_3 : Z_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
$(X'_3 : Y'_3 : Z'_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
at most one of these is $(0 : 0 : 0)$.

2009 Bernstein–T. Lange:

For all fields k with $2 \neq 0$,
all $\mathbf{P}^1 \times \mathbf{P}^1$ Edwards curves
$X^2T^2 + Y^2Z^2 = Z^2T^2 + d$
all $P_1, P_2 \in E(k)$,
$P_1 = ((X_1 : Z_1), (Y_1 : T_1))$,
$P_2 = ((X_2 : Z_2), (Y_2 : T_2))$:

$(X_3 : Z_3)$ is $x(P_1 + P_2)$ or (
$(X'_3 : Z'_3)$ is $x(P_1 + P_2)$ or (
$(Y_3 : T_3)$ is $y(P_1 + P_2)$ or (0
$(Y'_3 : T'_3)$ is $y(P_1 + P_2)$ or (0
at most one of these is (0 :

What this means:

For all fields k,
all $\mathbf{P}^2$ Weierstrass curves
$E/k : Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$,
all $P_1 = (X_1 : Y_1 : Z_1) \in E(k)$,
all $P_2 = (X_2 : Y_2 : Z_2) \in E(k)$:

$(X_3 : Y_3 : Z_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
$(X'_3 : Y'_3 : Z'_3)$
is $P_1 + P_2$ or $(0 : 0 : 0)$;
at most one of these is $(0 : 0 : 0)$.

2009 Bernstein–T. Lange:

For all fields k with $2 \neq 0$,
all $\mathbf{P}^1 \times \mathbf{P}^1$ Edwards curves E/k :
$X^2T^2 + Y^2Z^2 = Z^2T^2 + dX^2Y^2$,
all $P_1, P_2 \in E(k)$,
$P_1 = ((X_1 : Z_1), (Y_1 : T_1))$,
$P_2 = ((X_2 : Z_2), (Y_2 : T_2))$:

$(X_3 : Z_3)$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(X'_3 : Z'_3)$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3 : T_3)$ is $y(P_1 + P_2)$ or $(0 : 0)$;
$(Y'_3 : T'_3)$ is $y(P_1 + P_2)$ or $(0 : 0)$;
at most one of these is $(0 : 0)$.

is means:

ields k,
eierstrass curves
$^2Z + a_1XYZ + a_3YZ^2 =$
$X^2Z + a_4XZ^2 + a_6Z^3$,
$(X_1 : Y_1 : Z_1) \in E(k)$,
$(X_2 : Y_2 : Z_2) \in E(k)$:

$: Z_3)$
P_2 or $(0 : 0 : 0)$;
$: Z'_3)$
P_2 or $(0 : 0 : 0)$;
one of these is $(0 : 0 : 0)$.

2009 Bernstein–T. Lange:

For all fields k with $2 \neq 0$,
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$X^2T^2 + Y^2Z^2 = Z^2T^2 + dX^2Y^2$,
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$(X_3 : Z_3)$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(X'_3 : Z'_3)$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3 : T_3)$ is $y(P_1 + P_2)$ or $(0 : 0)$;
$(Y'_3 : T'_3)$ is $y(P_1 + P_2)$ or $(0 : 0)$;
at most one of these is $(0 : 0)$.

$X_3 = X$
$Z_3 = Z_1$
$Y_3 = Y_1$
$T_3 = Z_1$

$X'_3 = X$
$Z'_3 = X_1$
$Y'_3 = X_1$
$T'_3 = X_1$

Much, m

Lange–R

Also mu

curves

$YZ + a_3YZ^2 =$

$_4XZ^2 + a_6Z^3$,

$Z_1) \in E(k)$,

$Z_2) \in E(k)$:

$0 : 0)$;

$0 : 0)$;

ese is $(0 : 0 : 0)$.

2009 Bernstein–T. Lange:

For all fields k with $2 \neq 0$,
all $\mathbf{P}^1 \times \mathbf{P}^1$ Edwards curves E/k:
$X^2T^2 + Y^2Z^2 = Z^2T^2 + dX^2Y^2$,
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$P_2 = ((X_2 : Z_2), (Y_2 : T_2))$:

$(X_3 : Z_3)$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(X_3' : Z_3')$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3 : T_3)$ is $y(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3' : T_3')$ is $y(P_1 + P_2)$ or $(0 : 0)$;
at most one of these is $(0 : 0)$.

$X_3 = X_1Y_2Z_2T_1 +$
$Z_3 = Z_1Z_2T_1T_2 +$
$Y_3 = Y_1Y_2Z_1Z_2 -$
$T_3 = Z_1Z_2T_1T_2 -$

$X_3' = X_1Y_1Z_2T_2 +$
$Z_3' = X_1X_2T_1T_2 +$
$Y_3' = X_1Y_1Z_2T_2 -$
$T_3' = X_1Y_2Z_2T_1 -$

Much, much, muc

Lange–Ruppert, B

Also much easier t

$Z^2 =$

${}_6Z^3$,

$k)$,

$k)$:

$0 : 0)$.

2009 Bernstein–T. Lange:

For all fields k with $2 \neq 0$,
all $\mathbf{P}^1 \times \mathbf{P}^1$ Edwards curves E/k :
$X^2T^2 + Y^2Z^2 = Z^2T^2 + dX^2Y^2$,
all $P_1, P_2 \in E(k)$,
$P_1 = ((X_1 : Z_1), (Y_1 : T_1))$,
$P_2 = ((X_2 : Z_2), (Y_2 : T_2))$:

$(X_3 : Z_3)$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(X_3' : Z_3')$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3 : T_3)$ is $y(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3' : T_3')$ is $y(P_1 + P_2)$ or $(0 : 0)$;
at most one of these is $(0 : 0)$.

$X_3 = X_1Y_2Z_2T_1 + X_2Y_1Z_1T$
$Z_3 = Z_1Z_2T_1T_2 + dX_1X_2Y_1$
$Y_3 = Y_1Y_2Z_1Z_2 - X_1X_2T_1T$
$T_3 = Z_1Z_2T_1T_2 - dX_1X_2Y_1$

$X_3' = X_1Y_1Z_2T_2 + X_2Y_2Z_1T$
$Z_3' = X_1X_2T_1T_2 + Y_1Y_2Z_1Z$
$Y_3' = X_1Y_1Z_2T_2 - X_2Y_2Z_1T$
$T_3' = X_1Y_2Z_2T_1 - X_2Y_1Z_1T$

Much, much, much simpler
Lange–Ruppert, Bosma–Len
Also much easier to prove.

2009 Bernstein–T. Lange:

For all fields k with $2 \neq 0$,
all $\mathbf{P}^1 \times \mathbf{P}^1$ Edwards curves E/k :
$X^2T^2 + Y^2Z^2 = Z^2T^2 + dX^2Y^2$,
all $P_1, P_2 \in E(k)$,
$P_1 = ((X_1 : Z_1), (Y_1 : T_1))$,
$P_2 = ((X_2 : Z_2), (Y_2 : T_2))$:

$(X_3 : Z_3)$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(X_3' : Z_3')$ is $x(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3 : T_3)$ is $y(P_1 + P_2)$ or $(0 : 0)$;
$(Y_3' : T_3')$ is $y(P_1 + P_2)$ or $(0 : 0)$;
at most one of these is $(0 : 0)$.

$$X_3 = X_1Y_2Z_2T_1 + X_2Y_1Z_1T_2,$$
$$Z_3 = Z_1Z_2T_1T_2 + dX_1X_2Y_1Y_2,$$
$$Y_3 = Y_1Y_2Z_1Z_2 - X_1X_2T_1T_2,$$
$$T_3 = Z_1Z_2T_1T_2 - dX_1X_2Y_1Y_2,$$

$$X_3' = X_1Y_1Z_2T_2 + X_2Y_2Z_1T_1,$$
$$Z_3' = X_1X_2T_1T_2 + Y_1Y_2Z_1Z_2,$$
$$Y_3' = X_1Y_1Z_2T_2 - X_2Y_2Z_1T_1,$$
$$T_3' = X_1Y_2Z_2T_1 - X_2Y_1Z_1T_2.$$

Much, much, much simpler than
Lange–Ruppert, Bosma–Lenstra.
Also much easier to prove.

rnstein–T. Lange:

ields k with $2 \neq 0$,

$\mathbf{P}^1$ Edwards curves E/k :

$Y^2Z^2 = Z^2T^2 + dX^2Y^2$,

$_2 \in E(k)$,

$X_1 : Z_1), (Y_1 : T_1))$,

$X_2 : Z_2), (Y_2 : T_2))$:

) is $x(P_1 + P_2)$ or $(0 : 0)$;

) is $x(P_1 + P_2)$ or $(0 : 0)$;

) is $y(P_1 + P_2)$ or $(0 : 0)$;

) is $y(P_1 + P_2)$ or $(0 : 0)$;

one of these is $(0 : 0)$.

$$
\begin{aligned}
X_3 &= X_1Y_2Z_2T_1 + X_2Y_1Z_1T_2,\\
Z_3 &= Z_1Z_2T_1T_2 + dX_1X_2Y_1Y_2,\\
Y_3 &= Y_1Y_2Z_1Z_2 - X_1X_2T_1T_2,\\
T_3 &= Z_1Z_2T_1T_2 - dX_1X_2Y_1Y_2,
\end{aligned}
$$

$$
\begin{aligned}
X_3' &= X_1Y_1Z_2T_2 + X_2Y_2Z_1T_1,\\
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\end{aligned}
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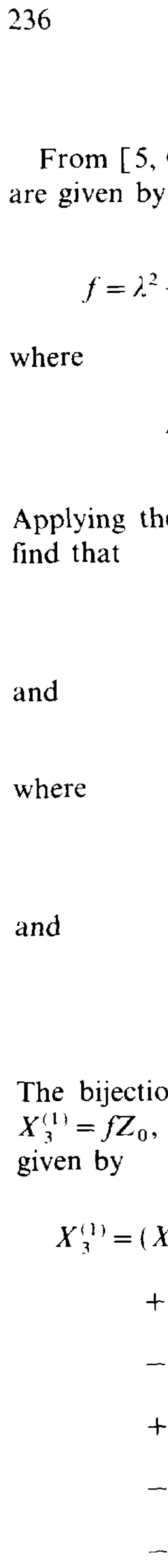

236

From [5,

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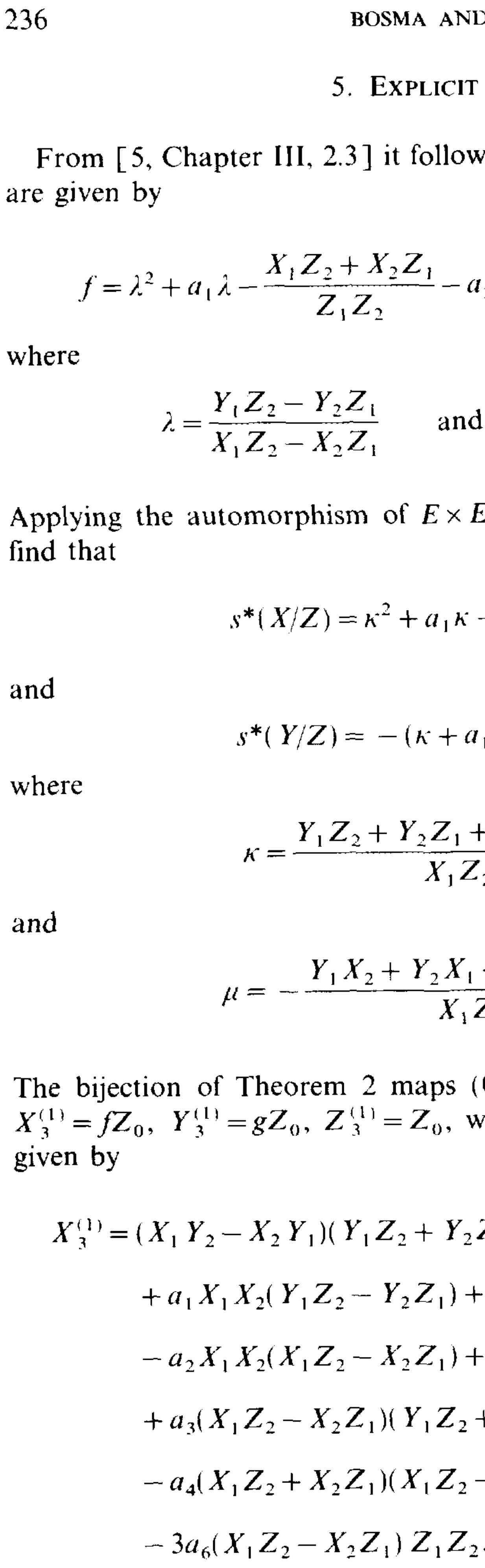

236 BOSMA AND

5. EXPLICIT

From [5, Chapter III, 2.3] it follow
are given by

$$f = \lambda^2 + a_1\lambda - \frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a$$

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Applying the automorphism of $E \times E$
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$$X_3^{(1)} = (X_1Y_2 - X_2Y_1)(Y_1Z_2 + Y_2$$
$$+ a_1X_1X_2(Y_1Z_2 - Y_2Z_1) +$$
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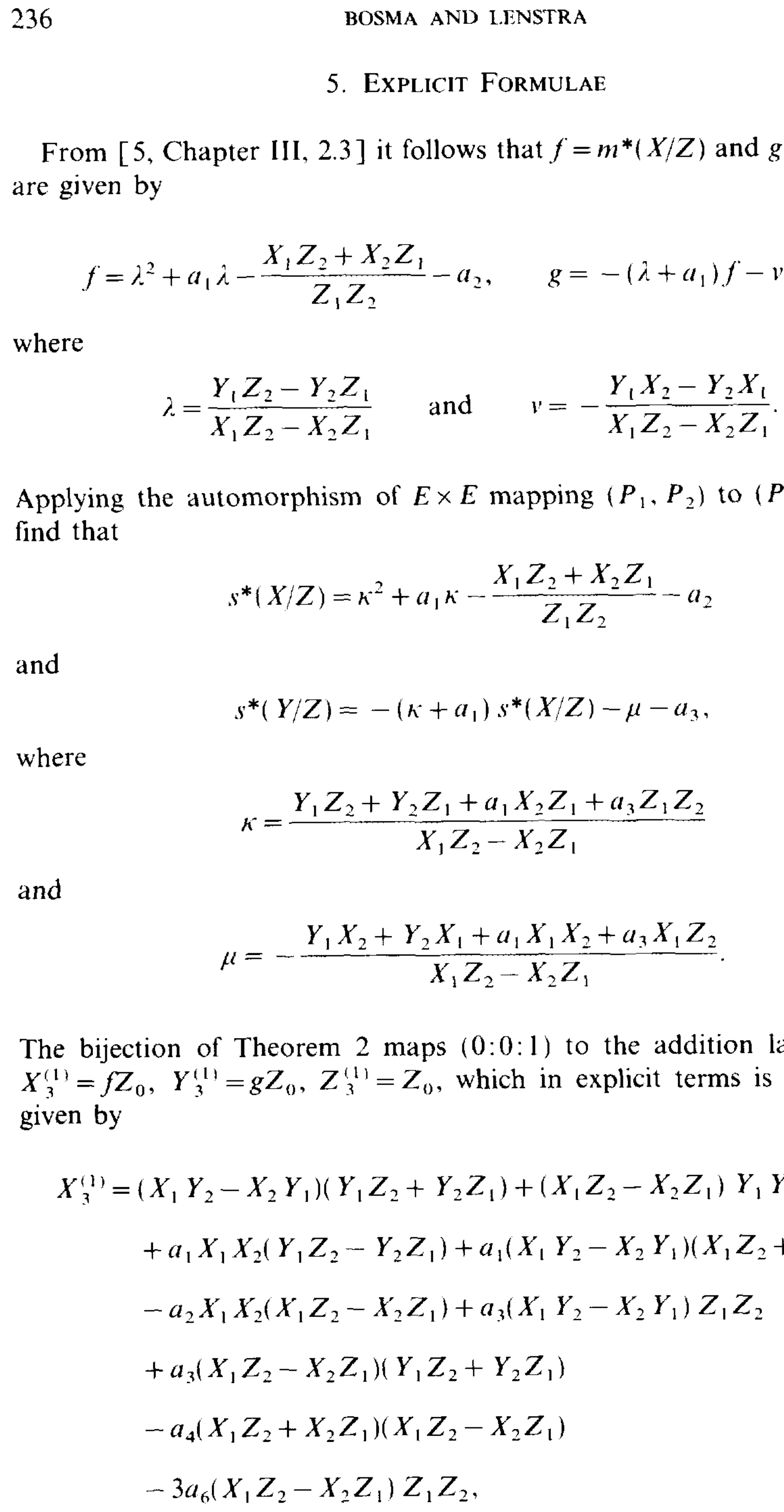

236 BOSMA AND LENSTRA

5. EXPLICIT FORMULAE

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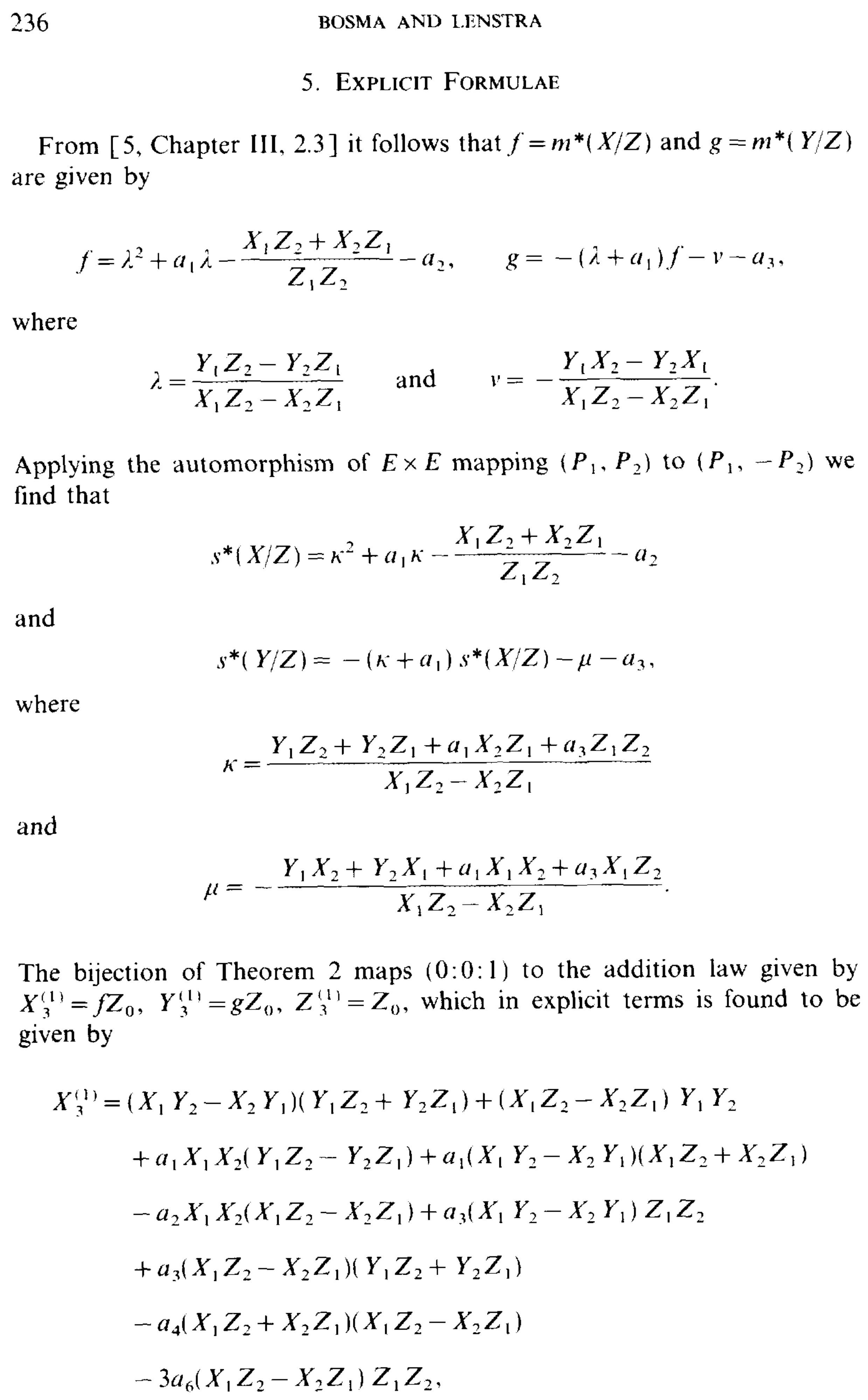

236 BOSMA AND LENSTRA

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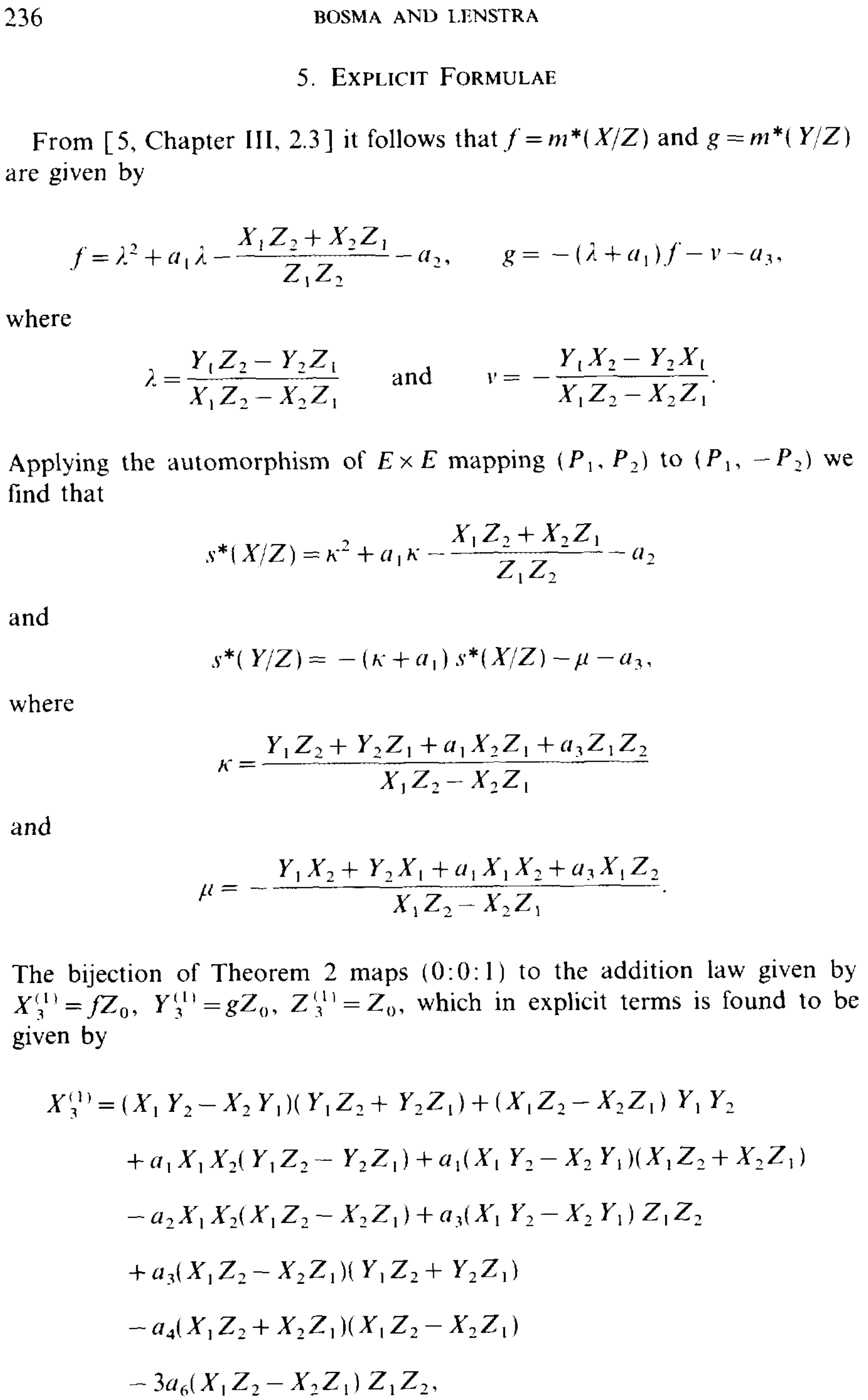

5. EXPLICIT FORMULAE

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Applying the automorphism of $E \times E$ mapping (P_1, P_2) to $(P_1, -P_2)$ we find that

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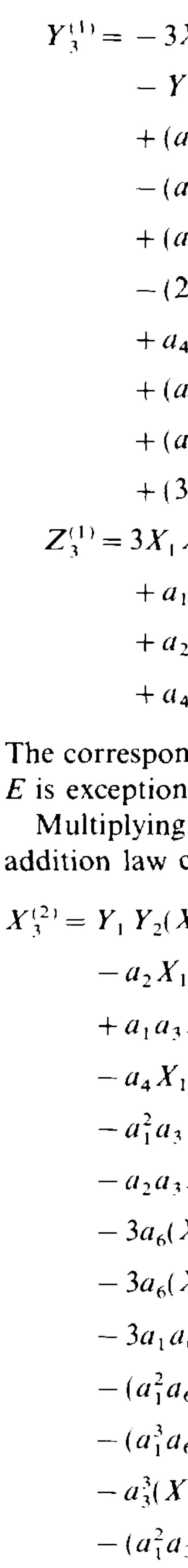
\end{aligned}$$



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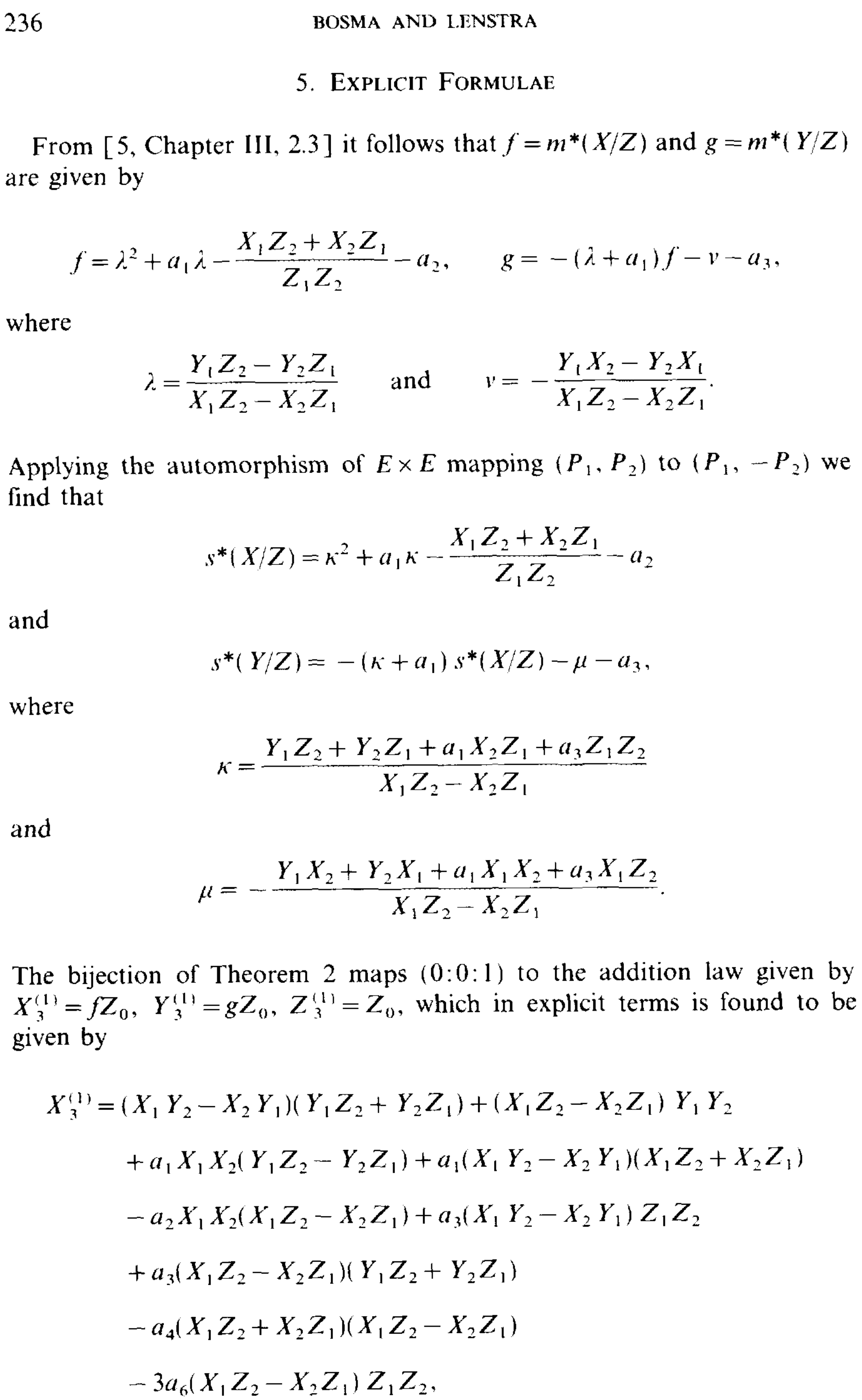

5. Explicit Formulae

From [5, Chapter III, 2.3] it follows that $f = m^*(X/Z)$ and $g = m^*(Y/Z)$ are given by

$$f = \lambda^2 + a_1\lambda - \frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a_2, \qquad g = -(\lambda + a_1)f - \nu - a_3,$$

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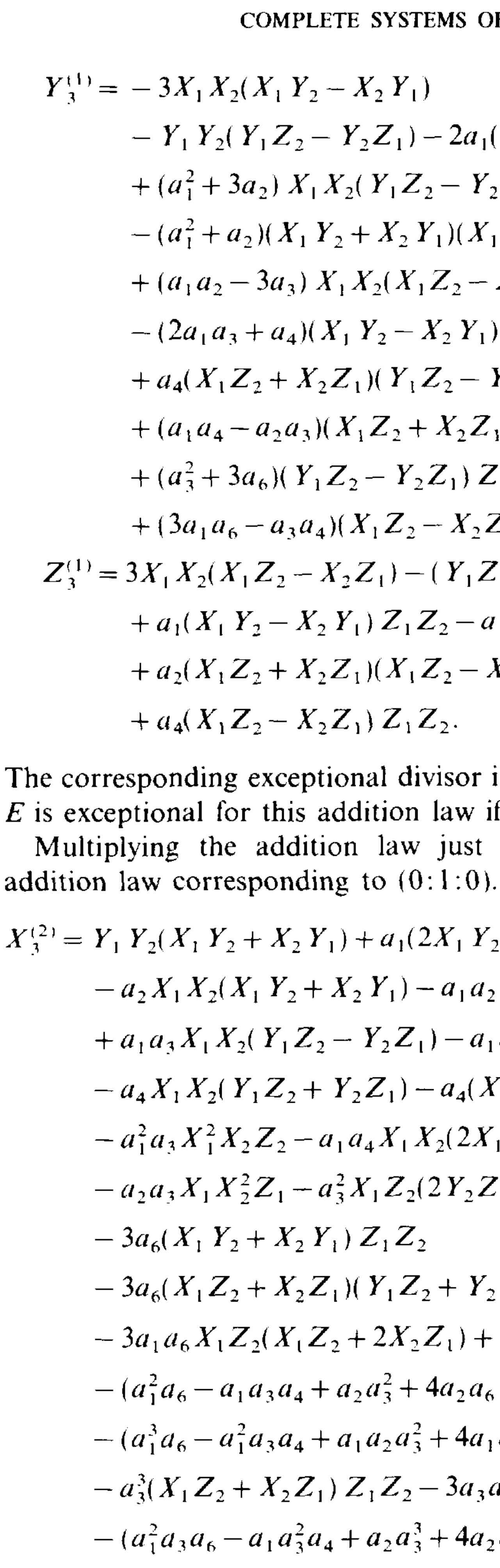
The bijection of Theorem 2 maps $(0:0:1)$ to the addition law given by $X_3^{(1)} = fZ_0$, $Y_3^{(1)} = gZ_0$, $Z_3^{(1)} = Z_0$, which in explicit terms is found to be given by

$$\begin{aligned} X_3^{(1)} = {} & (X_1Y_2 - X_2Y_1)(Y_1Z_2 + Y_2Z_1) + (X_1Z_2 - X_2Z_1)\,Y_1Y_2 \\ & + a_1X_1X_2(Y_1Z_2 - Y_2Z_1) + a_1(X_1Y_2 - X_2Y_1)(X_1Z_2 + X_2Z_1) \\ & - a_2X_1X_2(X_1Z_2 - X_2Z_1) + a_3(X_1Y_2 - X_2Y_1)\,Z_1Z_2 \\ & + a_3(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\ & - a_4(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\ & - 3a_6(X_1Z_2 - X_2Z_1)\,Z_1Z_2, \end{aligned}$$



$$\begin{aligned} Y_3^{(1)} = {} & -3X_1X_2(X_1Y_2 - X_2Y_1) \\ & - Y_1Y_2(Y_1Z_2 - Y_2Z_1) - 2a_1(\\ & + (a_1^2 + 3a_2)\,X_1X_2(Y_1Z_2 - Y_2 \\ & - (a_1^2 + a_2)(X_1Y_2 + X_2Y_1)(X_1 \\ & + (a_1a_2 - 3a_3)\,X_1X_2(X_1Z_2 - \\ & - (2a_1a_3 + a_4)(X_1Y_2 - X_2Y_1) \\ & + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 - \\ & + (a_1a_4 - a_2a_3)(X_1Z_2 + X_2Z_1 \\ & + (a_3^2 + 3a_6)(Y_1Z_2 - Y_2Z_1)\,Z \\ & + (3a_1a_6 - a_3a_4)(X_1Z_2 - X_2Z \end{aligned}$$

$$\begin{aligned} Z_3^{(1)} = {} & 3X_1X_2(X_1Z_2 - X_2Z_1) - (Y_1Z \\ & + a_1(X_1Y_2 - X_2Y_1)\,Z_1Z_2 - a \\ & + a_2(X_1Z_2 + X_2Z_1)(X_1Z_2 - \\ & + a_4(X_1Z_2 - X_2Z_1)\,Z_1Z_2. \end{aligned}$$

The corresponding exceptional divisor i
E is exceptional for this addition law if
Multiplying the addition law just
addition law corresponding to $(0:1:0)$.

$$\begin{aligned} X_3^{(2)} = {} & Y_1Y_2(X_1Y_2 + X_2Y_1) + a_1(2X_1Y_2 \\ & - a_2X_1X_2(X_1Y_2 + X_2Y_1) - a_1a_2 \\ & + a_1a_3X_1X_2(Y_1Z_2 - Y_2Z_1) - a_1 \\ & - a_4X_1X_2(Y_1Z_2 + Y_2Z_1) - a_4(X \\ & - a_1^2a_3X_1^2X_2Z_2 - a_1a_4X_1X_2(2X_1 \\ & - a_2a_3X_1X_2^2Z_1 - a_3^2X_1Z_2(2Y_2Z \\ & - 3a_6(X_1Y_2 + X_2Y_1)\,Z_1Z_2 \\ & - 3a_6(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2 \\ & - 3a_1a_6X_1Z_2(X_1Z_2 + 2X_2Z_1) + \\ & - (a_1^2a_6 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 \\ & - (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 + 4a_1 \\ & - a_3^3(X_1Z_2 + X_2Z_1)\,Z_1Z_2 - 3a_3a \\ & - (a_1^2a_3a_6 - a_1a_3^2a_4 + a_2a_3^3 + 4a_2 \end{aligned}$$

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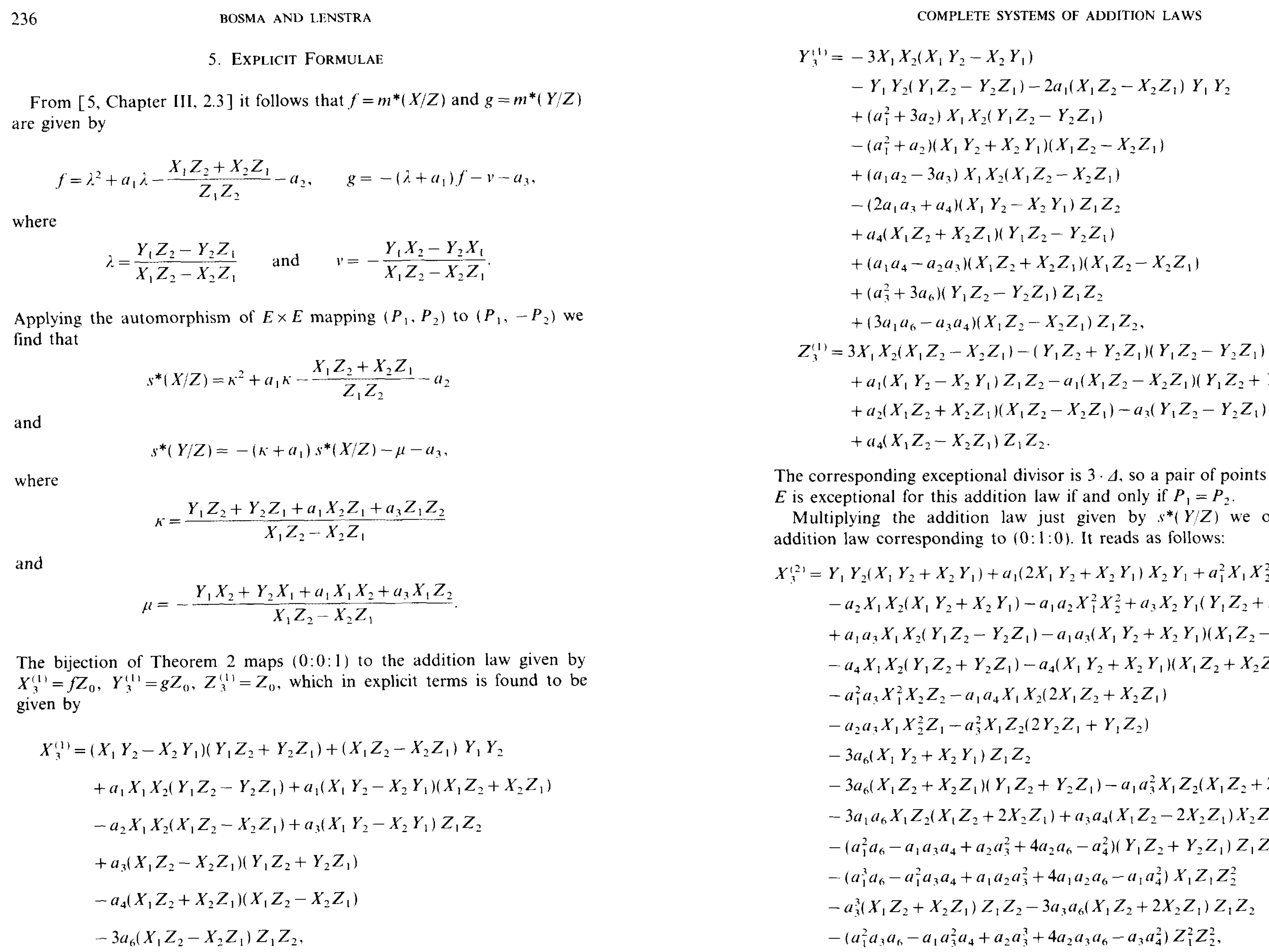

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$$\mu = -\frac{Y_1X_2 + Y_2X_1 + a_1X_1X_2 + a_3X_1Z_2}{X_1Z_2 - X_2Z_1}.$$

The bijection of Theorem 2 maps $(0:0:1)$ to the addition law given by $X_3^{(1)} = fZ_0$, $Y_3^{(1)} = gZ_0$, $Z_3^{(1)} = Z_0$, which in explicit terms is found to be given by

$$\begin{aligned} X_3^{(1)} = {} & (X_1Y_2 - X_2Y_1)(Y_1Z_2 + Y_2Z_1) + (X_1Z_2 - X_2Z_1)\, Y_1Y_2 \\ & + a_1X_1X_2(Y_1Z_2 - Y_2Z_1) + a_1(X_1Y_2 - X_2Y_1)(X_1Z_2 + X_2Z_1) \\ & - a_2X_1X_2(X_1Z_2 - X_2Z_1) + a_3(X_1Y_2 - X_2Y_1)\, Z_1Z_2 \\ & + a_3(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\ & - a_4(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\ & - 3a_6(X_1Z_2 - X_2Z_1)\, Z_1Z_2, \end{aligned}$$

$$\begin{aligned} Y_3^{(1)} = {} & -3X_1X_2(X_1Y_2 - X_2Y_1) \\ & - Y_1Y_2(Y_1Z_2 - Y_2Z_1) - 2a_1(X_1Z_2 - X_2Z_1)\, Y_1Y_2 \\ & + (a_1^2 + 3a_2)\, X_1X_2(Y_1Z_2 - Y_2Z_1) \\ & - (a_1^2 + a_2)(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\ & + (a_1a_2 - 3a_3)\, X_1X_2(X_1Z_2 - X_2Z_1) \\ & - (2a_1a_3 + a_4)(X_1Y_2 - X_2Y_1)\, Z_1Z_2 \\ & + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 - Y_2Z_1) \\ & + (a_1a_4 - a_2a_3)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\ & + (a_3^2 + 3a_6)(Y_1Z_2 - Y_2Z_1)\, Z_1Z_2 \\ & + (3a_1a_6 - a_3a_4)(X_1Z_2 - X_2Z_1)\, Z_1Z_2, \\ Z_3^{(1)} = {} & 3X_1X_2(X_1Z_2 - X_2Z_1) - (Y_1Z_2 + Y_2Z_1)(Y_1Z_2 - Y_2Z_1) \\ & + a_1(X_1Y_2 - X_2Y_1)\, Z_1Z_2 - a_1(X_1Z_2 - X_2Z_1)(Y_1Z_2 + \\ & + a_2(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) - a_3(Y_1Z_2 - Y_2Z_1) \\ & + a_4(X_1Z_2 - X_2Z_1)\, Z_1Z_2. \end{aligned}$$

The corresponding exceptional divisor is $3 \cdot \varDelta$, so a pair of points
E is exceptional for this addition law if and only if $P_1 = P_2$.

Multiplying the addition law just given by $s^*(Y/Z)$ we o
addition law corresponding to $(0:1:0)$. It reads as follows:

$$\begin{aligned} X_3^{(2)} = {} & Y_1Y_2(X_1Y_2 + X_2Y_1) + a_1(2X_1Y_2 + X_2Y_1)\, X_2Y_1 + a_1^2X_1X \\ & - a_2X_1X_2(X_1Y_2 + X_2Y_1) - a_1a_2X_1^2X_2^2 + a_3X_2Y_1(Y_1Z_2 + \\ & + a_1a_3X_1X_2(Y_1Z_2 - Y_2Z_1) - a_1a_3(X_1Y_2 + X_2Y_1)(X_1Z_2 - \\ & - a_4X_1X_2(Y_1Z_2 + Y_2Z_1) - a_4(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z \\ & - a_1^2a_3X_1^2X_2Z_2 - a_1a_4X_1X_2(2X_1Z_2 + X_2Z_1) \\ & - a_2a_3X_1X_2^2Z_1 - a_3^2X_1Z_2(2Y_2Z_1 + Y_1Z_2) \\ & - 3a_6(X_1Y_2 + X_2Y_1)\, Z_1Z_2 \\ & - 3a_6(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) - a_1a_3^2X_1Z_2(X_1Z_2 + \\ & - 3a_1a_6X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_3a_4(X_1Z_2 - 2X_2Z_1)\, X_2Z \\ & - (a_1^2a_6 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 - a_4^2)(Y_1Z_2 + Y_2Z_1)\, Z_1Z \\ & - (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 + 4a_1a_2a_6 - a_1a_4^2)\, X_1Z_1Z_2^2 \\ & - a_3^3(X_1Z_2 + X_2Z_1)\, Z_1Z_2 - 3a_3a_6(X_1Z_2 + 2X_2Z_1)\, Z_1Z_2 \\ & - (a_1^2a_3a_6 - a_1a_3^2a_4 + a_2a_3^3 + 4a_2a_3a_6 - a_3a_4^2)\, Z_1^2Z_2^2, \end{aligned}$$

5. EXPLICIT FORMULAE

From [5, Chapter III, 2.3] it follows that $f = m^*(X/Z)$ and $g = m^*(Y/Z)$ are given by

$$f = \lambda^2 + a_1\lambda - \frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a_2, \qquad g = -(\lambda + a_1)f - \nu - a_3,$$

where

$$\lambda = \frac{Y_1Z_2 - Y_2Z_1}{X_1Z_2 - X_2Z_1} \quad \text{and} \quad \nu = -\frac{Y_1X_2 - Y_2X_1}{X_1Z_2 - X_2Z_1}.$$

Applying the automorphism of $E \times E$ mapping (P_1, P_2) to $(P_1, -P_2)$ we find that

$$s^*(X/Z) = \kappa^2 + a_1\kappa - \frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a_2$$

and

$$s^*(Y/Z) = -(\kappa + a_1)\, s^*(X/Z) - \mu - a_3,$$

where

$$\kappa = \frac{Y_1Z_2 + Y_2Z_1 + a_1X_2Z_1 + a_3Z_1Z_2}{X_1Z_2 - X_2Z_1}$$

and

$$\mu = -\frac{Y_1X_2 + Y_2X_1 + a_1X_1X_2 + a_3X_1Z_2}{X_1Z_2 - X_2Z_1}.$$

The bijection of Theorem 2 maps $(0:0:1)$ to the addition law given by $X_3^{(1)} = fZ_0$, $Y_3^{(1)} = gZ_0$, $Z_3^{(1)} = Z_0$, which in explicit terms is found to be given by

$$\begin{aligned}
X_3^{(1)} = {} & (X_1Y_2 - X_2Y_1)(Y_1Z_2 + Y_2Z_1) + (X_1Z_2 - X_2Z_1)\, Y_1Y_2 \\
& + a_1X_1X_2(Y_1Z_2 - Y_2Z_1) + a_1(X_1Y_2 - X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& - a_2X_1X_2(X_1Z_2 - X_2Z_1) + a_3(X_1Y_2 - X_2Y_1)\, Z_1Z_2 \\
& + a_3(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& - a_4(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& - 3a_6(X_1Z_2 - X_2Z_1)\, Z_1Z_2,
\end{aligned}$$

$$\begin{aligned}
Y_3^{(1)} = {} & -3X_1X_2(X_1Y_2 - X_2Y_1) \\
& - Y_1Y_2(Y_1Z_2 - Y_2Z_1) - 2a_1(X_1Z_2 - X_2Z_1)\, Y_1Y_2 \\
& + (a_1^2 + 3a_2)\, X_1X_2(Y_1Z_2 - Y_2Z_1) \\
& - (a_1^2 + a_2)(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1a_2 - 3a_3)\, X_1X_2(X_1Z_2 - X_2Z_1) \\
& - (2a_1a_3 + a_4)(X_1Y_2 - X_2Y_1)\, Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 - Y_2Z_1) \\
& + (a_1a_4 - a_2a_3)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_3^2 + 3a_6)(Y_1Z_2 - Y_2Z_1)\, Z_1Z_2 \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 - X_2Z_1)\, Z_1Z_2, \\
Z_3^{(1)} = {} & 3X_1X_2(X_1Z_2 - X_2Z_1) - (Y_1Z_2 + Y_2Z_1)(Y_1Z_2 - Y_2Z_1) \\
& + a_1(X_1Y_2 - X_2Y_1)\, Z_1Z_2 - a_1(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) - a_3(Y_1Z_2 - Y_2Z_1)\, Z_1Z_2 \\
& + a_4(X_1Z_2 - X_2Z_1)\, Z_1Z_2.
\end{aligned}$$

The corresponding exceptional divisor is $3 \cdot \Delta$, so a pair of points P_1, P_2 on E is exceptional for this addition law if and only if $P_1 = P_2$.

Multiplying the addition law just given by $s^*(Y/Z)$ we obtain the addition law corresponding to $(0:1:0)$. It reads as follows:

$$\begin{aligned}
X_3^{(2)} = {} & Y_1Y_2(X_1Y_2 + X_2Y_1) + a_1(2X_1Y_2 + X_2Y_1)\, X_2Y_1 + a_1^2X_1X_2^2Y_1 \\
& - a_2X_1X_2(X_1Y_2 + X_2Y_1) - a_1a_2X_1^2X_2^2 + a_3X_2Y_1(Y_1Z_2 + 2Y_2Z_1) \\
& + a_1a_3X_1X_2(Y_1Z_2 - Y_2Z_1) - a_1a_3(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\
& - a_4X_1X_2(Y_1Z_2 + Y_2Z_1) - a_4(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& - a_1^2a_3X_1^2X_2Z_2 - a_1a_4X_1X_2(2X_1Z_2 + X_2Z_1) \\
& - a_2a_3X_1X_2^2Z_1 - a_3^2X_1Z_2(2Y_2Z_1 + Y_1Z_2) \\
& - 3a_6(X_1Y_2 + X_2Y_1)\, Z_1Z_2 \\
& - 3a_6(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) - a_1a_3^2X_1Z_2(X_1Z_2 + 2X_2Z_1) \\
& - 3a_1a_6X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_3a_4(X_1Z_2 - 2X_2Z_1)\, X_2Z_1 \\
& - (a_1^2a_6 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 - a_4^2)(Y_1Z_2 + Y_2Z_1)\, Z_1Z_2 \\
& - (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 + 4a_1a_2a_6 - a_1a_4^2)\, X_1Z_1Z_2^2 \\
& - a_3^3(X_1Z_2 + X_2Z_1)\, Z_1Z_2 - 3a_3a_6(X_1Z_2 + 2X_2Z_1)\, Z_1Z_2 \\
& - (a_1^2a_3a_6 - a_1a_3^2a_4 + a_2a_3^3 + 4a_2a_3a_6 - a_3a_4^2)\, Z_1^2Z_2^2,
\end{aligned}$$

5. Explicit Formulae

Chapter III, 2.3] it follows that $f = m^*(X/Z)$ and $g = m^*(Y/Z)$

$$+ a_1\lambda - \frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a_2, \qquad g = -(\lambda + a_1)f - \nu - a_3,$$

$$\lambda = \frac{Y_1Z_2 - Y_2Z_1}{X_1Z_2 - X_2Z_1} \qquad \text{and} \qquad \nu = -\frac{Y_1X_2 - Y_2X_1}{X_1Z_2 - X_2Z_1}.$$

e automorphism of $E \times E$ mapping (P_1, P_2) to $(P_1, -P_2)$ we

$$s^*(X/Z) = \kappa^2 + a_1\kappa - \frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a_2$$

$$s^*(Y/Z) = -(\kappa + a_1)\, s^*(X/Z) - \mu - a_3,$$

$$\kappa = \frac{Y_1Z_2 + Y_2Z_1 + a_1X_2Z_1 + a_3Z_1Z_2}{X_1Z_2 - X_2Z_1}$$

$$\mu = -\frac{Y_1X_2 + Y_2X_1 + a_1X_1X_2 + a_3X_1Z_2}{X_1Z_2 - X_2Z_1}.$$

n of Theorem 2 maps (0:0:1) to the addition law given by $Y_3^{(1)} = gZ_0$, $Z_3^{(1)} = Z_0$, which in explicit terms is found to be

$$
\begin{aligned}
&{}_1Y_2 - X_2Y_1)(Y_1Z_2 + Y_2Z_1) + (X_1Z_2 - X_2Z_1)\,Y_1Y_2\\
&a_1X_1X_2(Y_1Z_2 - Y_2Z_1) + a_1(X_1Y_2 - X_2Y_1)(X_1Z_2 + X_2Z_1)\\
&a_2X_1X_2(X_1Z_2 - X_2Z_1) + a_3(X_1Y_2 - X_2Y_1)\,Z_1Z_2\\
&a_3(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1)\\
&a_4(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1)\\
&3a_6(X_1Z_2 - X_2Z_1)\,Z_1Z_2,
\end{aligned}
$$

$$
\begin{aligned}
Y_3^{(1)} = {}& -3X_1X_2(X_1Y_2 - X_2Y_1)\\
&- Y_1Y_2(Y_1Z_2 - Y_2Z_1) - 2a_1(X_1Z_2 - X_2Z_1)\,Y_1Y_2\\
&+ (a_1^2 + 3a_2)\,X_1X_2(Y_1Z_2 - Y_2Z_1)\\
&- (a_1^2 + a_2)(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1)\\
&+ (a_1a_2 - 3a_3)\,X_1X_2(X_1Z_2 - X_2Z_1)\\
&- (2a_1a_3 + a_4)(X_1Y_2 - X_2Y_1)\,Z_1Z_2\\
&+ a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 - Y_2Z_1)\\
&+ (a_1a_4 - a_2a_3)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1)\\
&+ (a_3^2 + 3a_6)(Y_1Z_2 - Y_2Z_1)\,Z_1Z_2\\
&+ (3a_1a_6 - a_3a_4)(X_1Z_2 - X_2Z_1)\,Z_1Z_2,\\
Z_3^{(1)} = {}& 3X_1X_2(X_1Z_2 - X_2Z_1) - (Y_1Z_2 + Y_2Z_1)(Y_1Z_2 - Y_2Z_1)\\
&+ a_1(X_1Y_2 - X_2Y_1)\,Z_1Z_2 - a_1(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1)\\
&+ a_2(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) - a_3(Y_1Z_2 - Y_2Z_1)\,Z_1Z_2\\
&+ a_4(X_1Z_2 - X_2Z_1)\,Z_1Z_2.
\end{aligned}
$$

The corresponding exceptional divisor is $3 \cdot \Delta$, so a pair of points P_1, P_2 on E is exceptional for this addition law if and only if $P_1 = P_2$.

Multiplying the addition law just given by $s^*(Y/Z)$ we obtain the addition law corresponding to (0:1:0). It reads as follows:

$$
\begin{aligned}
X_3^{(2)} = {}& Y_1Y_2(X_1Y_2 + X_2Y_1) + a_1(2X_1Y_2 + X_2Y_1)\,X_2Y_1 + a_1^2X_1X_2^2Y_1\\
&- a_2X_1X_2(X_1Y_2 + X_2Y_1) - a_1a_2X_1^2X_2^2 + a_3X_2Y_1(Y_1Z_2 + 2Y_2Z_1)\\
&+ a_1a_3X_1X_2(Y_1Z_2 - Y_2Z_1) - a_1a_3(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1)\\
&- a_4X_1X_2(Y_1Z_2 + Y_2Z_1) - a_4(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1)\\
&- a_1^2a_3X_1^2X_2Z_2 - a_1a_4X_1X_2(2X_1Z_2 + X_2Z_1)\\
&- a_2a_3X_1X_2^2Z_1 - a_3^2X_1Z_2(2Y_2Z_1 + Y_1Z_2)\\
&- 3a_6(X_1Y_2 + X_2Y_1)\,Z_1Z_2\\
&- 3a_6(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) - a_1a_3^2X_1Z_2(X_1Z_2 + 2X_2Z_1)\\
&- 3a_1a_6X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_3a_4(X_1Z_2 - 2X_2Z_1)\,X_2Z_1\\
&- (a_1^2a_6 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 - a_4^2)(Y_1Z_2 + Y_2Z_1)\,Z_1Z_2\\
&- (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 + 4a_1a_2a_6 - a_1a_4^2)\,X_1Z_1Z_2^2\\
&- a_3^3(X_1Z_2 + X_2Z_1)\,Z_1Z_2 - 3a_3a_6(X_1Z_2 + 2X_2Z_1)\,Z_1Z_2\\
&- (a_1^2a_3a_6 - a_1a_3^2a_4 + a_2a_3^3 + 4a_2a_3a_6 - a_3a_4^2)\,Z_1^2Z_2^2,
\end{aligned}
$$

$$
\begin{aligned}
Y_3^{(2)} = {}& Y_1^2Y\\
&+ a_3\\
&+ (a_1\\
&+ (a\\
&- (a\\
&+ (3\\
&+ (3\\
&- (3\\
&+ (a\\
&+ (a\\
&- a_1^2\\
&+ (a\\
&+ 4a\\
&+ (a\\
&+ 4a\\
&+ 4a\\
Z_3^{(2)} = {}& 3X_1X\\
&+ a_1(\\
&+ a_2\\
&+ a_2(\\
&+ a_1^3\\
&+ 3a\\
&+ 2a\\
&+ 2a\\
&+ a_4(\\
&+ (a\\
&+ a_3^2\\
&+ a_1\\
&+ a_3
\end{aligned}
$$

FORMULAE

s that $f = m^*(X/Z)$ and $g = m^*(Y/Z)$

$${}_2, \qquad g = -(\lambda + a_1) f - \nu - a_3,$$

$$\nu = -\frac{Y_1X_2 - Y_2X_1}{X_1Z_2 - X_2Z_1}.$$

mapping (P_1, P_2) to $(P_1, -P_2)$ we

$$\frac{X_1Z_2 + X_2Z_1}{Z_1Z_2} - a_2$$

$${}_1)\, s^*(X/Z) - \mu - a_3,$$

$$\frac{a_1X_2Z_1 + a_3Z_1Z_2}{{}_2 - X_2Z_1}$$

$$\frac{+ a_1X_1X_2 + a_3X_1Z_2}{Z_2 - X_2Z_1}.$$

0:0:1) to the addition law given by hich in explicit terms is found to be

$$\begin{aligned}
&Z_1) + (X_1Z_2 - X_2Z_1)\, Y_1Y_2 \\
&a_1(X_1Y_2 - X_2Y_1)(X_1Z_2 + X_2Z_1) \\
&a_3(X_1Y_2 - X_2Y_1)\, Z_1Z_2 \\
&- Y_2Z_1) \\
&- X_2Z_1)
\end{aligned}$$

,

$$\begin{aligned}
Y_3^{(1)} = {} & -3X_1X_2(X_1Y_2 - X_2Y_1) \\
& - Y_1Y_2(Y_1Z_2 - Y_2Z_1) - 2a_1(X_1Z_2 - X_2Z_1)\, Y_1Y_2 \\
& + (a_1^2 + 3a_2)\, X_1X_2(Y_1Z_2 - Y_2Z_1) \\
& - (a_1^2 + a_2)(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1a_2 - 3a_3)\, X_1X_2(X_1Z_2 - X_2Z_1) \\
& - (2a_1a_3 + a_4)(X_1Y_2 - X_2Y_1)\, Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 - Y_2Z_1) \\
& + (a_1a_4 - a_2a_3)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_3^2 + 3a_6)(Y_1Z_2 - Y_2Z_1)\, Z_1Z_2 \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 - X_2Z_1)\, Z_1Z_2, \\
Z_3^{(1)} = {} & 3X_1X_2(X_1Z_2 - X_2Z_1) - (Y_1Z_2 + Y_2Z_1)(Y_1Z_2 - Y_2Z_1) \\
& + a_1(X_1Y_2 - X_2Y_1)\, Z_1Z_2 - a_1(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) - a_3(Y_1Z_2 - Y_2Z_1)\, Z_1Z_2 \\
& + a_4(X_1Z_2 - X_2Z_1)\, Z_1Z_2.
\end{aligned}$$

The corresponding exceptional divisor is $3 \cdot \Delta$, so a pair of points P_1, P_2 on E is exceptional for this addition law if and only if $P_1 = P_2$.

Multiplying the addition law just given by $s^*(Y/Z)$ we obtain the addition law corresponding to $(0:1:0)$. It reads as follows:

$$\begin{aligned}
X_3^{(2)} = {} & Y_1Y_2(X_1Y_2 + X_2Y_1) + a_1(2X_1Y_2 + X_2Y_1)\, X_2Y_1 + a_1^2X_1X_2^2Y_1 \\
& - a_2X_1X_2(X_1Y_2 + X_2Y_1) - a_1a_2X_1^2X_2^2 + a_3X_2Y_1(Y_1Z_2 + 2Y_2Z_1) \\
& + a_1a_3X_1X_2(Y_1Z_2 - Y_2Z_1) - a_1a_3(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\
& - a_4X_1X_2(Y_1Z_2 + Y_2Z_1) - a_4(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& - a_1^2a_3X_1^2X_2Z_2 - a_1a_4X_1X_2(2X_1Z_2 + X_2Z_1) \\
& - a_2a_3X_1X_2^2Z_1 - a_3^2X_1Z_2(2Y_2Z_1 + Y_1Z_2) \\
& - 3a_6(X_1Y_2 + X_2Y_1)\, Z_1Z_2 \\
& - 3a_6(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) - a_1a_3^2X_1Z_2(X_1Z_2 + 2X_2Z_1) \\
& - 3a_1a_6X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_3a_4(X_1Z_2 - 2X_2Z_1)\, X_2Z_1 \\
& - (a_1^2a_6 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 - a_4^2)(Y_1Z_2 + Y_2Z_1)\, Z_1Z_2 \\
& - (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 + 4a_1a_2a_6 - a_1a_4^2)\, X_1Z_1Z_2^2 \\
& - a_3^3(X_1Z_2 + X_2Z_1)\, Z_1Z_2 - 3a_3a_6(X_1Z_2 + 2X_2Z_1)\, Z_1Z_2 \\
& - (a_1^2a_3a_6 - a_1a_3^2a_4 + a_2a_3^3 + 4a_2a_3a_6 - a_3a_4^2)\, Z_1^2Z_2^2,
\end{aligned}$$

$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2Y_2^2 + a_1X_2Y_1^2Y_2 + (a_1a_2 - \\
& + a_3Y_1^2Y_2Z_2 - (a_2^2 - 3a_4)\, X_1^2 \\
& + (a_1a_4 - a_2a_3)(2X_1Z_2 + X_2Z \\
& + (a_1^2a_4 - 2a_1a_2a_3 + 3a_3^2)\, X_1^2X \\
& - (a_2a_4 - 9a_6)\, X_1X_2(X_1Z_2 + \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 + 2X_2 \\
& + (3a_1^2a_6 - 2a_1a_3a_4 + a_2a_3^2 + 3 \\
& - (3a_2a_6 - a_4^2)(X_1Z_2 + X_2Z_1) \\
& + (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 - a \\
& + (a_1^4a_6 - a_1^3a_3a_4 + 5a_1^2a_2a_6 + \\
& - a_1^2a_4^2 + a_2^2a_3^2 - a_2a_4^2 + 4a_2^2a_6 \\
& + (a_1^2a_2a_6 - a_1a_2a_3a_4 + 3a_1a_3 \\
& + 4a_2^2a_6 - 2a_3^2a_4 - 3a_4a_6)\, X_2Z \\
& + (a_1^3a_3a_6 - a_1^2a_3^2a_4 + a_1^2a_4a_6 \\
& + 4a_1a_2a_3a_6 - 2a_1a_3a_4^2 + a_2a_3^2 \\
& + 4a_2a_4a_6 - a_3^4 - 6a_3^2a_6 - a_4^3 - \\
Z_3^{(2)} = {} & 3X_1X_2(X_1Y_2 + X_2Y_1) + Y_1Y_2(\\
& + a_1(2X_1Y_2 + Y_1X_2)\, Y_1Z_2 + a \\
& + a_2X_1X_2(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Y_2 + X_2Y_1)(X_1Z_2 + X \\
& + a_1^3X_1^2X_2Z_2 + a_1a_2X_1X_2(2X \\
& + 3a_3X_1X_2^2Z_1 + a_3Y_1Z_2(Y_1Z \\
& + 2a_1a_3X_1Z_2(Y_1Z_2 + Y_2Z_1) \\
& + 2a_1a_3X_2Y_1Z_1Z_2 + a_4(X_1Y_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y \\
& + (a_1^2a_3 + a_1a_4)\, X_1Z_2(X_1Z_2 + \\
& + a_3^2Y_1Z_1Z_2^2 + (a_3^2 + 3a_6)(Y_1Z \\
& + a_1a_3^2(2X_1Z_2 + X_2Z_1)\, Z_1Z_2 \\
& + a_3a_4(X_1Z_2 + 2X_2Z_1)\, Z_1Z_2
\end{aligned}$$

$= m^*(Y/Z)$

$- a_3,$

$_1, -P_2)$ we

aw given by
found to be

$_2$

$- X_2Z_1)$

$$\begin{aligned}
Y_3^{(1)} = {} & -3X_1X_2(X_1Y_2 - X_2Y_1) \\
& - Y_1Y_2(Y_1Z_2 - Y_2Z_1) - 2a_1(X_1Z_2 - X_2Z_1)\,Y_1Y_2 \\
& + (a_1^2 + 3a_2)\,X_1X_2(Y_1Z_2 - Y_2Z_1) \\
& - (a_1^2 + a_2)(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1a_2 - 3a_3)\,X_1X_2(X_1Z_2 - X_2Z_1) \\
& - (2a_1a_3 + a_4)(X_1Y_2 - X_2Y_1)\,Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 - Y_2Z_1) \\
& + (a_1a_4 - a_2a_3)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_3^2 + 3a_6)(Y_1Z_2 - Y_2Z_1)\,Z_1Z_2 \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 - X_2Z_1)\,Z_1Z_2, \\
Z_3^{(1)} = {} & 3X_1X_2(X_1Z_2 - X_2Z_1) - (Y_1Z_2 + Y_2Z_1)(Y_1Z_2 - Y_2Z_1) \\
& + a_1(X_1Y_2 - X_2Y_1)\,Z_1Z_2 - a_1(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) - a_3(Y_1Z_2 - Y_2Z_1)\,Z_1Z_2 \\
& + a_4(X_1Z_2 - X_2Z_1)\,Z_1Z_2.
\end{aligned}$$

The corresponding exceptional divisor is $3 \cdot \Delta$, so a pair of points P_1, P_2 on E is exceptional for this addition law if and only if $P_1 = P_2$.

Multiplying the addition law just given by $s^*(Y/Z)$ we obtain the addition law corresponding to $(0:1:0)$. It reads as follows:

$$\begin{aligned}
X_3^{(2)} = {} & Y_1Y_2(X_1Y_2 + X_2Y_1) + a_1(2X_1Y_2 + X_2Y_1)\,X_2Y_1 + a_1^2X_1X_2^2Y_1 \\
& - a_2X_1X_2(X_1Y_2 + X_2Y_1) - a_1a_2X_1^2X_2^2 + a_3X_2Y_1(Y_1Z_2 + 2Y_2Z_1) \\
& + a_1a_3X_1X_2(Y_1Z_2 - Y_2Z_1) - a_1a_3(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1) \\
& - a_4X_1X_2(Y_1Z_2 + Y_2Z_1) - a_4(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& - a_1^2a_3X_1^2X_2Z_2 - a_1a_4X_1X_2(2X_1Z_2 + X_2Z_1) \\
& - a_2a_3X_1X_2^2Z_1 - a_3^2X_1Z_2(2Y_2Z_1 + Y_1Z_2) \\
& - 3a_6(X_1Y_2 + X_2Y_1)\,Z_1Z_2 \\
& - 3a_6(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) - a_1a_3^2X_1Z_2(X_1Z_2 + 2X_2Z_1) \\
& - 3a_1a_6X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_3a_4(X_1Z_2 - 2X_2Z_1)\,X_2Z_1 \\
& - (a_1^2a_6 - a_1a_3a_4 + a_2a_3^2 + 4a_2a_6 - a_4^2)(Y_1Z_2 + Y_2Z_1)\,Z_1Z_2 \\
& - (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 + 4a_1a_2a_6 - a_1a_4^2)\,X_1Z_1Z_2^2 \\
& - a_3^3(X_1Z_2 + X_2Z_1)\,Z_1Z_2 - 3a_3a_6(X_1Z_2 + 2X_2Z_1)\,Z_1Z_2 \\
& - (a_1^2a_3a_6 - a_1a_3^2a_4 + a_2a_3^3 + 4a_2a_3a_6 - a_3a_4^2)\,Z_1^2Z_2^2,
\end{aligned}$$

$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2Y_2^2 + a_1X_2Y_1^2Y_2 + (a_1a_2 - 3a_3)\,X_1X_2^2Y_1 \\
& + a_3Y_1^2Y_2Z_2 - (a_2^2 - 3a_4)\,X_1^2X_2^2 \\
& + (a_1a_4 - a_2a_3)(2X_1Z_2 + X_2Z_1)\,X_2Y_1 \\
& + (a_1^2a_4 - 2a_1a_2a_3 + 3a_3^2)\,X_1^2X_2Z_2 \\
& - (a_2a_4 - 9a_6)\,X_1X_2(X_1Z_2 + X_2Z_1) \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 + 2X_2Z_1)\,Y_1Z_2 \\
& + (3a_1^2a_6 - 2a_1a_3a_4 + a_2a_3^2 + 3a_2a_6 - a_4^2)\,X_1Z_2(X_1Z_2 + \\
& - (3a_2a_6 - a_4^2)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 - a_1a_4^2 + 4a_1a_2a_6 - a_3^3 - 3a_3 \\
& + (a_1^4a_6 - a_1^3a_3a_4 + 5a_1^2a_2a_6 + a_1^2a_2a_3^2 - a_1a_2a_3a_4 - a_1a \\
& - a_1^2a_4^2 + a_2^2a_3^2 - a_2a_4^2 + 4a_2^2a_6 - a_3^2a_4 - 3a_4a_6)\,X_1Z_1Z_2^2 \\
& + (a_1^2a_2a_6 - a_1a_2a_3a_4 + 3a_1a_3a_6 + a_2^2a_3^2 - a_2a_4^2 \\
& + 4a_2^2a_6 - 2a_3^2a_4 - 3a_4a_6)\,X_2Z_1^2Z_2 \\
& + (a_1^3a_3a_6 - a_1^2a_3^2a_4 + a_1^2a_4a_6 + a_1a_2a_3^3 \\
& + 4a_1a_2a_3a_6 - 2a_1a_3a_4^2 + a_2a_3^2a_4 \\
& + 4a_2a_4a_6 - a_3^4 - 6a_3^2a_6 - a_4^3 - 9a_6^2)\,Z_1^2Z_2^2, \\
Z_3^{(2)} = {} & 3X_1X_2(X_1Y_2 + X_2Y_1) + Y_1Y_2(Y_1Z_2 + Y_2Z_1) + 3a_1X_1^2X \\
& + a_1(2X_1Y_2 + Y_1X_2)\,Y_1Z_2 + a_1^2X_1Z_2(2X_2Y_1 + X_1Y_2) \\
& + a_2X_1X_2(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& + a_1^3X_1^2X_2Z_2 + a_1a_2X_1X_2(2X_1Z_2 + X_2Z_1) \\
& + 3a_3X_1X_2^2Z_1 + a_3Y_1Z_2(Y_1Z_2 + 2Y_2Z_1) \\
& + 2a_1a_3X_1Z_2(Y_1Z_2 + Y_2Z_1) \\
& + 2a_1a_3X_2Y_1Z_1Z_2 + a_4(X_1Y_2 + X_2Y_1)\,Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + (a_1^2a_3 + a_1a_4)\,X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_2a_3X_2Z_1(2X_1 \\
& + a_3^2Y_1Z_1Z_2^2 + (a_3^2 + 3a_6)(Y_1Z_2 + Y_2Z_1)\,Z_1Z_2 \\
& + a_1a_3^2(2X_1Z_2 + X_2Z_1)\,Z_1Z_2 + 3a_1a_6X_1Z_1Z_2^2 \\
& + a_3a_4(X_1Z_2 + 2X_2Z_1)\,Z_1Z_2 + (a_3^3 + 3a_3a_6)\,Z_1^2Z_2^2.
\end{aligned}$$

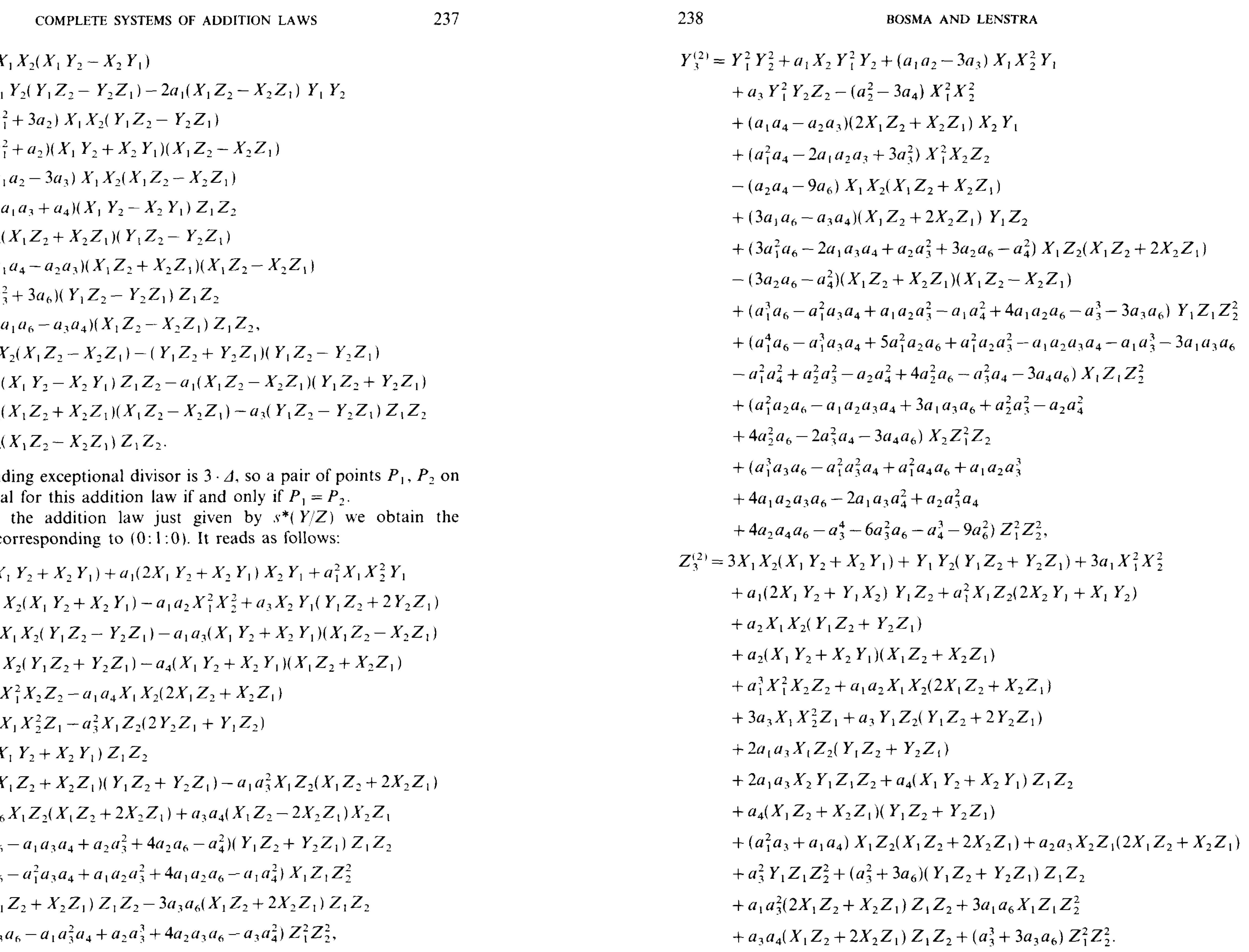
$$\begin{aligned}
Y_3^{(1)} = {} & -3X_1X_2(X_1Y_2-X_2Y_1) \\
& -Y_1Y_2(Y_1Z_2-Y_2Z_1)-2a_1(X_1Z_2-X_2Z_1)\,Y_1Y_2 \\
& +(a_1^2+3a_2)\,X_1X_2(Y_1Z_2-Y_2Z_1) \\
& -(a_1^2+a_2)(X_1Y_2+X_2Y_1)(X_1Z_2-X_2Z_1) \\
& +(a_1a_2-3a_3)\,X_1X_2(X_1Z_2-X_2Z_1) \\
& -(2a_1a_3+a_4)(X_1Y_2-X_2Y_1)\,Z_1Z_2 \\
& +a_4(X_1Z_2+X_2Z_1)(Y_1Z_2-Y_2Z_1) \\
& +(a_1a_4-a_2a_3)(X_1Z_2+X_2Z_1)(X_1Z_2-X_2Z_1) \\
& +(a_3^2+3a_6)(Y_1Z_2-Y_2Z_1)\,Z_1Z_2 \\
& +(3a_1a_6-a_3a_4)(X_1Z_2-X_2Z_1)\,Z_1Z_2, \\
Z_3^{(1)} = {} & 3X_1X_2(X_1Z_2-X_2Z_1)-(Y_1Z_2+Y_2Z_1)(Y_1Z_2-Y_2Z_1) \\
& +a_1(X_1Y_2-X_2Y_1)\,Z_1Z_2-a_1(X_1Z_2-X_2Z_1)(Y_1Z_2+Y_2Z_1) \\
& +a_2(X_1Z_2+X_2Z_1)(X_1Z_2-X_2Z_1)-a_3(Y_1Z_2-Y_2Z_1)\,Z_1Z_2 \\
& +a_4(X_1Z_2-X_2Z_1)\,Z_1Z_2.
\end{aligned}$$

The corresponding exceptional divisor is $3\cdot\Delta$, so a pair of points P_1, P_2 on E is exceptional for this addition law if and only if $P_1=P_2$.

Multiplying the addition law just given by $s^*(Y/Z)$ we obtain the addition law corresponding to $(0:1:0)$. It reads as follows:

$$\begin{aligned}
X_3^{(2)} = {} & Y_1Y_2(X_1Y_2+X_2Y_1)+a_1(2X_1Y_2+X_2Y_1)\,X_2Y_1+a_1^2X_1X_2^2Y_1 \\
& -a_2X_1X_2(X_1Y_2+X_2Y_1)-a_1a_2X_1^2X_2^2+a_3X_2Y_1(Y_1Z_2+2Y_2Z_1) \\
& +a_1a_3X_1X_2(Y_1Z_2-Y_2Z_1)-a_1a_3(X_1Y_2+X_2Y_1)(X_1Z_2-X_2Z_1) \\
& -a_4X_1X_2(Y_1Z_2+Y_2Z_1)-a_4(X_1Y_2+X_2Y_1)(X_1Z_2+X_2Z_1) \\
& -a_1^2a_3X_1^2X_2Z_2-a_1a_4X_1X_2(2X_1Z_2+X_2Z_1) \\
& -a_2a_3X_1X_2^2Z_1-a_3^2X_1Z_2(2Y_2Z_1+Y_1Z_2) \\
& -3a_6(X_1Y_2+X_2Y_1)\,Z_1Z_2 \\
& -3a_6(X_1Z_2+X_2Z_1)(Y_1Z_2+Y_2Z_1)-a_1a_3^2X_1Z_2(X_1Z_2+2X_2Z_1) \\
& -3a_1a_6X_1Z_2(X_1Z_2+2X_2Z_1)+a_3a_4(X_1Z_2-2X_2Z_1)\,X_2Z_1 \\
& -(a_1^2a_6-a_1a_3a_4+a_2a_3^2+4a_2a_6-a_4^2)(Y_1Z_2+Y_2Z_1)\,Z_1Z_2 \\
& -(a_1^3a_6-a_1^2a_3a_4+a_1a_2a_3^2+4a_1a_2a_6-a_1a_4^2)\,X_1Z_1Z_2^2 \\
& -a_3^3(X_1Z_2+X_2Z_1)\,Z_1Z_2-3a_3a_6(X_1Z_2+2X_2Z_1)\,Z_1Z_2 \\
& -(a_1^2a_3a_6-a_1a_3^2a_4+a_2a_3^3+4a_2a_3a_6-a_3a_4^2)\,Z_1^2Z_2^2,
\end{aligned}$$

$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2Y_2^2+a_1X_2Y_1^2Y_2+(a_1a_2-3a_3)\,X_1X_2^2Y_1 \\
& +a_3Y_1^2Y_2Z_2-(a_2^2-3a_4)\,X_1^2X_2^2 \\
& +(a_1a_4-a_2a_3)(2X_1Z_2+X_2Z_1)\,X_2Y_1 \\
& +(a_1^2a_4-2a_1a_2a_3+3a_3^2)\,X_1^2X_2Z_2 \\
& -(a_2a_4-9a_6)\,X_1X_2(X_1Z_2+X_2Z_1) \\
& +(3a_1a_6-a_3a_4)(X_1Z_2+2X_2Z_1)\,Y_1Z_2 \\
& +(3a_1^2a_6-2a_1a_3a_4+a_2a_3^2+3a_2a_6-a_4^2)\,X_1Z_2(X_1Z_2+2X_2Z_1) \\
& -(3a_2a_6-a_4^2)(X_1Z_2+X_2Z_1)(X_1Z_2-X_2Z_1) \\
& +(a_1^3a_6-a_1^2a_3a_4+a_1a_2a_3^2-a_1a_4^2+4a_1a_2a_6-a_3^3-3a_3a_6)\,Y_1Z_1Z_2^2 \\
& +(a_1^4a_6-a_1^3a_3a_4+5a_1^2a_2a_6+a_1^2a_2a_3^2-a_1a_2a_3a_4-a_1a_3^3-3a_1a_3a_6 \\
& -a_1^2a_4^2+a_2^2a_3^2-a_2a_4^2+4a_2^2a_6-a_3^2a_4-3a_4a_6)\,X_1Z_1Z_2^2 \\
& +(a_1^2a_2a_6-a_1a_2a_3a_4+3a_1a_3a_6+a_2^2a_3^2-a_2a_4^2 \\
& +4a_2^2a_6-2a_3^2a_4-3a_4a_6)\,X_2Z_1^2Z_2 \\
& +(a_1^3a_3a_6-a_1^2a_3^2a_4+a_1^2a_4a_6+a_1a_2a_3^3 \\
& +4a_1a_2a_3a_6-2a_1a_3a_4^2+a_2a_3^2a_4 \\
& +4a_2a_4a_6-a_3^4-6a_3^2a_6-a_4^3-9a_6^2)\,Z_1^2Z_2^2, \\
Z_3^{(2)} = {} & 3X_1X_2(X_1Y_2+X_2Y_1)+Y_1Y_2(Y_1Z_2+Y_2Z_1)+3a_1X_1^2X_2^2 \\
& +a_1(2X_1Y_2+Y_1X_2)\,Y_1Z_2+a_1^2X_1Z_2(2X_2Y_1+X_1Y_2) \\
& +a_2X_1X_2(Y_1Z_2+Y_2Z_1) \\
& +a_2(X_1Y_2+X_2Y_1)(X_1Z_2+X_2Z_1) \\
& +a_1^3X_1^2X_2Z_2+a_1a_2X_1X_2(2X_1Z_2+X_2Z_1) \\
& +3a_3X_1X_2^2Z_1+a_3Y_1Z_2(Y_1Z_2+2Y_2Z_1) \\
& +2a_1a_3X_1Z_2(Y_1Z_2+Y_2Z_1) \\
& +2a_1a_3X_2Y_1Z_1Z_2+a_4(X_1Y_2+X_2Y_1)\,Z_1Z_2 \\
& +a_4(X_1Z_2+X_2Z_1)(Y_1Z_2+Y_2Z_1) \\
& +(a_1^2a_3+a_1a_4)\,X_1Z_2(X_1Z_2+2X_2Z_1)+a_2a_3X_2Z_1(2X_1Z_2+X_2Z_1) \\
& +a_3^2Y_1Z_1Z_2^2+(a_3^2+3a_6)(Y_1Z_2+Y_2Z_1)\,Z_1Z_2 \\
& +a_1a_3^2(2X_1Z_2+X_2Z_1)\,Z_1Z_2+3a_1a_6X_1Z_1Z_2^2 \\
& +a_3a_4(X_1Z_2+2X_2Z_1)\,Z_1Z_2+(a_3^3+3a_3a_6)\,Z_1^2Z_2^2.
\end{aligned}$$



$$X_1 X_2(X_1 Y_2 - X_2 Y_1)$$
$$_1 Y_2(Y_1 Z_2 - Y_2 Z_1) - 2a_1(X_1 Z_2 - X_2 Z_1) Y_1 Y_2$$
$$^2_1 + 3a_2) X_1 X_2(Y_1 Z_2 - Y_2 Z_1)$$
$$^2_1 + a_2)(X_1 Y_2 + X_2 Y_1)(X_1 Z_2 - X_2 Z_1)$$
$$_1 a_2 - 3a_3) X_1 X_2(X_1 Z_2 - X_2 Z_1)$$
$$a_1 a_3 + a_4)(X_1 Y_2 - X_2 Y_1) Z_1 Z_2$$
$$(X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 - Y_2 Z_1)$$
$$_1 a_4 - a_2 a_3)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1)$$
$$^2_3 + 3a_6)(Y_1 Z_2 - Y_2 Z_1) Z_1 Z_2$$
$$a_1 a_6 - a_3 a_4)(X_1 Z_2 - X_2 Z_1) Z_1 Z_2,$$
$$X_2(X_1 Z_2 - X_2 Z_1) - (Y_1 Z_2 + Y_2 Z_1)(Y_1 Z_2 - Y_2 Z_1)$$
$$(X_1 Y_2 - X_2 Y_1) Z_1 Z_2 - a_1(X_1 Z_2 - X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1)$$
$$(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) - a_3(Y_1 Z_2 - Y_2 Z_1) Z_1 Z_2$$
$$(X_1 Z_2 - X_2 Z_1) Z_1 Z_2.$$

ding exceptional divisor is $3 \cdot \Delta$, so a pair of points P_1, P_2 on
al for this addition law if and only if $P_1 = P_2$.
the addition law just given by $s^*(Y/Z)$ we obtain the
corresponding to $(0:1:0)$. It reads as follows:

$$_1 Y_2 + X_2 Y_1) + a_1(2X_1 Y_2 + X_2 Y_1) X_2 Y_1 + a_1^2 X_1 X_2^2 Y_1$$
$$X_2(X_1 Y_2 + X_2 Y_1) - a_1 a_2 X_1^2 X_2^2 + a_3 X_2 Y_1(Y_1 Z_2 + 2Y_2 Z_1)$$
$$X_1 X_2(Y_1 Z_2 - Y_2 Z_1) - a_1 a_3(X_1 Y_2 + X_2 Y_1)(X_1 Z_2 - X_2 Z_1)$$
$$X_2(Y_1 Z_2 + Y_2 Z_1) - a_4(X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1)$$
$$X_1^2 X_2 Z_2 - a_1 a_4 X_1 X_2(2X_1 Z_2 + X_2 Z_1)$$
$$X_1 X_2^2 Z_1 - a_3^2 X_1 Z_2(2Y_2 Z_1 + Y_1 Z_2)$$
$$X_1 Y_2 + X_2 Y_1) Z_1 Z_2$$
$$X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) - a_1 a_3^2 X_1 Z_2(X_1 Z_2 + 2X_2 Z_1)$$
$$_6 X_1 Z_2(X_1 Z_2 + 2X_2 Z_1) + a_3 a_4(X_1 Z_2 - 2X_2 Z_1) X_2 Z_1$$
$$_6 - a_1 a_3 a_4 + a_2 a_3^2 + 4a_2 a_6 - a_4^2)(Y_1 Z_2 + Y_2 Z_1) Z_1 Z_2$$
$$_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 + 4a_1 a_2 a_6 - a_1 a_4^2) X_1 Z_1 Z_2^2$$
$$_1 Z_2 + X_2 Z_1) Z_1 Z_2 - 3a_3 a_6(X_1 Z_2 + 2X_2 Z_1) Z_1 Z_2$$
$$_3 a_6 - a_1 a_3^2 a_4 + a_2 a_3^3 + 4a_2 a_3 a_6 - a_3 a_4^2) Z_1^2 Z_2^2,$$

$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3) X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4) X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1) X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2) X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6) X_1 X_2(X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1) Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2) X_1 Z_2(X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6) Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6) X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6) X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2) Z_1^2 Z_2^2,
\end{aligned}$$

$$\begin{aligned}
Z_3^{(2)} = {} & 3X_1 X_2(X_1 Y_2 + X_2 Y_1) + Y_1 Y_2(Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1(2X_1 Y_2 + Y_1 X_2) Y_1 Z_2 + a_1^2 X_1 Z_2(2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2(Y_1 Z_2 + Y_2 Z_1) \\
& + a_2(X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2(2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2(Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2(Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4(X_1 Y_2 + X_2 Y_1) Z_1 Z_2 \\
& + a_4(X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4) X_1 Z_2(X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1(2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1) Z_1 Z_2 \\
& + a_1 a_3^2(2X_1 Z_2 + X_2 Z_1) Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4(X_1 Z_2 + 2X_2 Z_1) Z_1 Z_2 + (a_3^3 + 3a_3 a_6) Z_1^2 Z_2^2.
\end{aligned}$$

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Define **F**

X, Y, Z

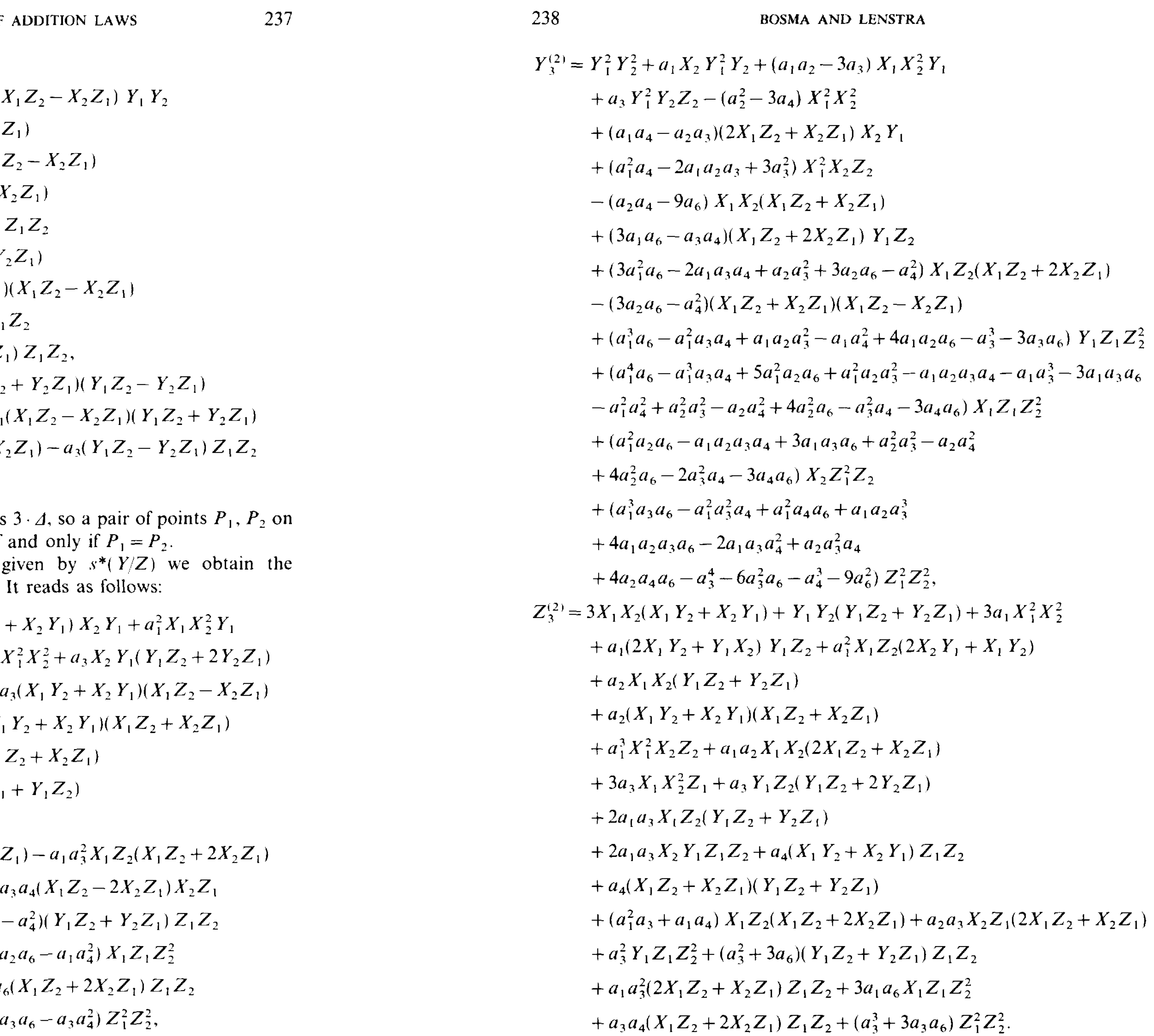
where $($

$\{(\lambda X, \lambda$

Define E

$\{(X : Y$

$Y^2 Z =$



F ADDITION LAWS 237

$X_1Z_2 - X_2Z_1)\ Y_1Y_2$

$Z_1)$

$Z_2 - X_2Z_1)$

$X_2Z_1)$

Z_1Z_2

$Y_2Z_1)$

$)(X_1Z_2 - X_2Z_1)$

${}_1Z_2$

$Z_1)\ Z_1Z_2,$

${}_2 + Y_2Z_1)(Y_1Z_2 - Y_2Z_1)$

${}_1(X_1Z_2 - X_2Z_1)(Y_1Z_2 + Y_2Z_1)$

$Y_2Z_1) - a_3(Y_1Z_2 - Y_2Z_1)\ Z_1Z_2$

s $3 \cdot \Delta$, so a pair of points P_1, P_2 on
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It reads as follows:

$+ X_2Y_1)\ X_2Y_1 + a_1^2X_1X_2^2Y_1$

$X_1^2X_2^2 + a_3X_2Y_1(Y_1Z_2 + 2Y_2Z_1)$

$a_3(X_1Y_2 + X_2Y_1)(X_1Z_2 - X_2Z_1)$

${}_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1)$

$Z_2 + X_2Z_1)$

${}_1 + Y_1Z_2)$

$Z_1) - a_1a_3^2X_1Z_2(X_1Z_2 + 2X_2Z_1)$

$a_3a_4(X_1Z_2 - 2X_2Z_1)X_2Z_1$

$- a_4^2)(Y_1Z_2 + Y_2Z_1)\ Z_1Z_2$

$a_2a_6 - a_1a_4^2)\ X_1Z_1Z_2^2$

${}_6(X_1Z_2 + 2X_2Z_1)\ Z_1Z_2$

$a_3a_6 - a_3a_4^2)\ Z_1^2Z_2^2,$

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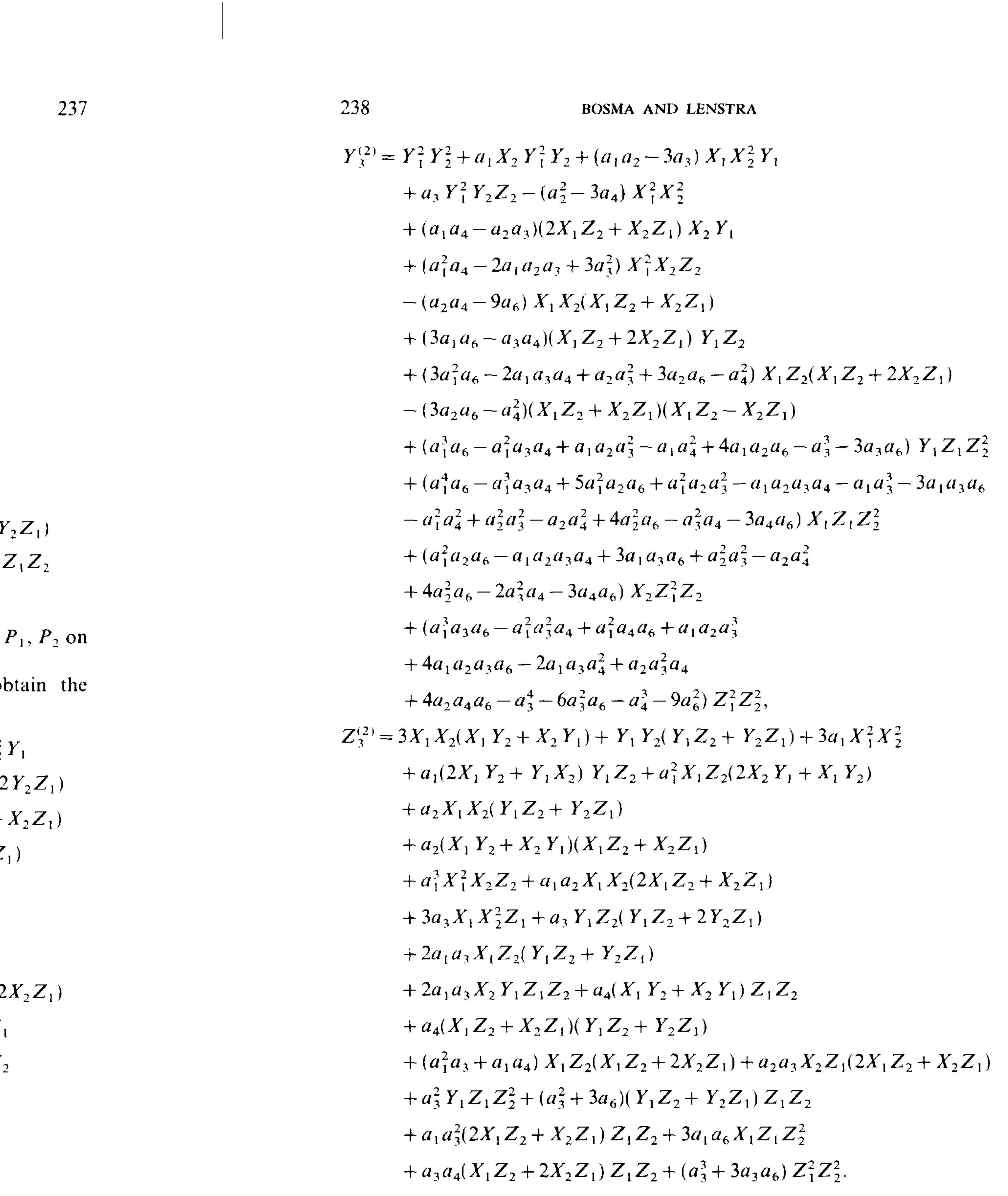
$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2Y_2^2 + a_1X_2Y_1^2Y_2 + (a_1a_2 - 3a_3)\ X_1X_2^2Y_1 \\
& + a_3Y_1^2Y_2Z_2 - (a_2^2 - 3a_4)\ X_1^2X_2^2 \\
& + (a_1a_4 - a_2a_3)(2X_1Z_2 + X_2Z_1)\ X_2Y_1 \\
& + (a_1^2a_4 - 2a_1a_2a_3 + 3a_3^2)\ X_1^2X_2Z_2 \\
& - (a_2a_4 - 9a_6)\ X_1X_2(X_1Z_2 + X_2Z_1) \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 + 2X_2Z_1)\ Y_1Z_2 \\
& + (3a_1^2a_6 - 2a_1a_3a_4 + a_2a_3^2 + 3a_2a_6 - a_4^2)\ X_1Z_2(X_1Z_2 + 2X_2Z_1) \\
& - (3a_2a_6 - a_4^2)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 - a_1a_4^2 + 4a_1a_2a_6 - a_3^3 - 3a_3a_6)\ Y_1Z_1Z_2^2 \\
& + (a_1^4a_6 - a_1^3a_3a_4 + 5a_1^2a_2a_6 + a_1^2a_2a_3^2 - a_1a_2a_3a_4 - a_1a_3^3 - 3a_1a_3a_6 \\
& - a_1^2a_4^2 + a_2^2a_3^2 - a_2a_4^2 + 4a_2^2a_6 - a_3^2a_4 - 3a_4a_6)\ X_1Z_1Z_2^2 \\
& + (a_1^2a_2a_6 - a_1a_2a_3a_4 + 3a_1a_3a_6 + a_2^2a_3^2 - a_2a_4^2 \\
& + 4a_2^2a_6 - 2a_3^2a_4 - 3a_4a_6)\ X_2Z_1^2Z_2 \\
& + (a_1^3a_3a_6 - a_1^2a_3^2a_4 + a_1^2a_4a_6 + a_1a_2a_3^3 \\
& + 4a_1a_2a_3a_6 - 2a_1a_3a_4^2 + a_2a_3^2a_4 \\
& + 4a_2a_4a_6 - a_3^4 - 6a_3^2a_6 - a_4^3 - 9a_6^2)\ Z_1^2Z_2^2, \\
Z_3^{(2)} = {} & 3X_1X_2(X_1Y_2 + X_2Y_1) + Y_1Y_2(Y_1Z_2 + Y_2Z_1) + 3a_1X_1^2X_2^2 \\
& + a_1(2X_1Y_2 + Y_1X_2)\ Y_1Z_2 + a_1^2X_1Z_2(2X_2Y_1 + X_1Y_2) \\
& + a_2X_1X_2(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& + a_1^3X_1^2X_2Z_2 + a_1a_2X_1X_2(2X_1Z_2 + X_2Z_1) \\
& + 3a_3X_1X_2^2Z_1 + a_3Y_1Z_2(Y_1Z_2 + 2Y_2Z_1) \\
& + 2a_1a_3X_1Z_2(Y_1Z_2 + Y_2Z_1) \\
& + 2a_1a_3X_2Y_1Z_1Z_2 + a_4(X_1Y_2 + X_2Y_1)\ Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + (a_1^2a_3 + a_1a_4)\ X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_2a_3X_2Z_1(2X_1Z_2 + X_2Z_1) \\
& + a_3^2Y_1Z_1Z_2^2 + (a_3^2 + 3a_6)(Y_1Z_2 + Y_2Z_1)\ Z_1Z_2 \\
& + a_1a_3^2(2X_1Z_2 + X_2Z_1)\ Z_1Z_2 + 3a_1a_6X_1Z_1Z_2^2 \\
& + a_3a_4(X_1Z_2 + 2X_2Z_1)\ Z_1Z_2 + (a_3^3 + 3a_3a_6)\ Z_1^2Z_2^2.
\end{aligned}
$$

1987 Lenstra: Use
complete system c
to computationally
$E(R)$ for more ge
rings with trivial c

Define $\mathbf{P}^2(R) = \{$
$X, Y, Z \in R$; XR
where $(X : Y : Z)$
$\{(\lambda X, \lambda Y, \lambda Z) : \lambda$

Define $E(R) =$
$\{(X : Y : Z) \in \mathbf{P}^2$
$Y^2Z = X^3 + a_4X$



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$Y_2Z_1)$

Z_1Z_2

P_1, P_2 on

btain the

Y_1

$2Y_2Z_1)$

$X_2Z_1)$

$Z_1)$

$2X_2Z_1)$

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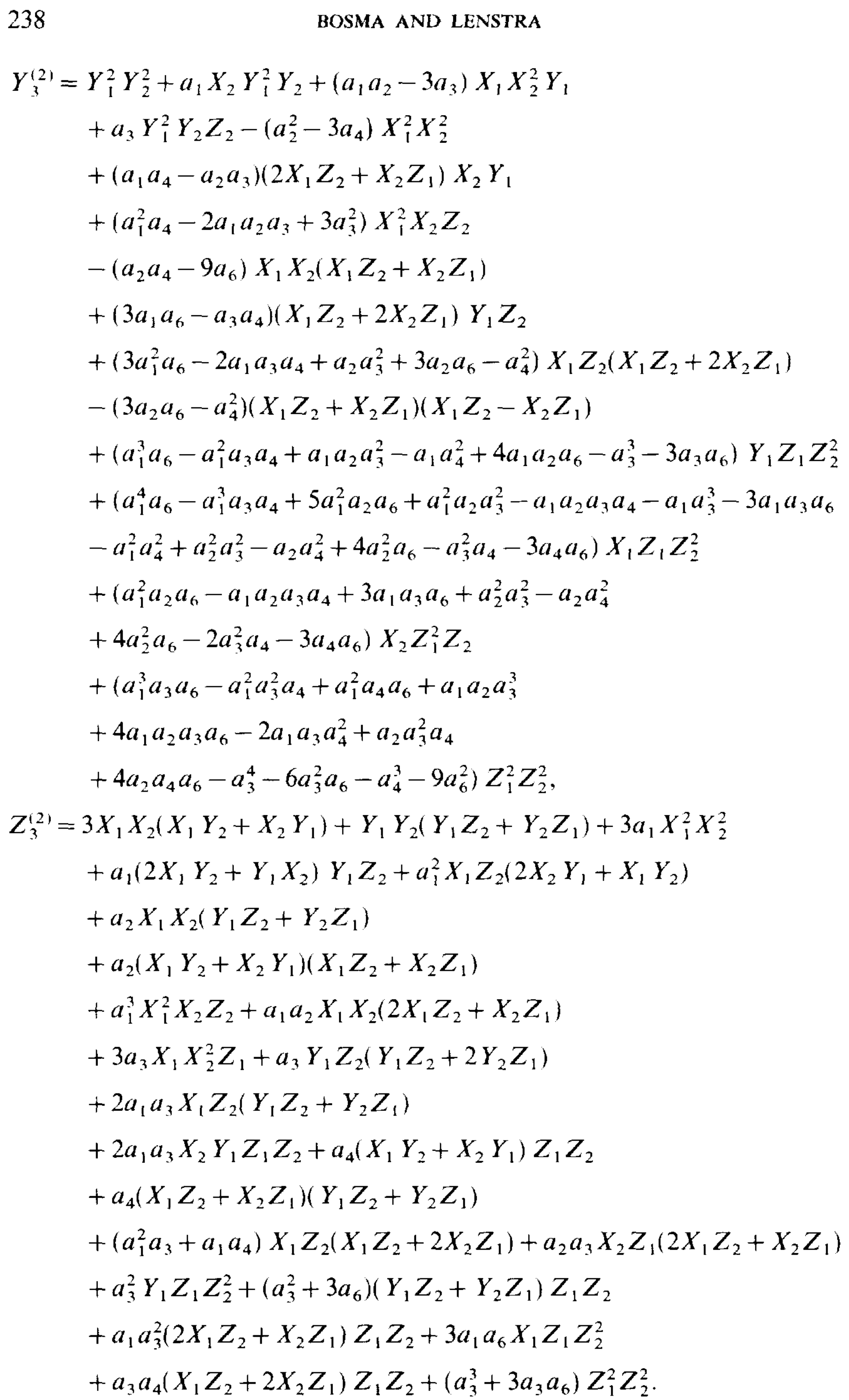
$$
\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1a_2 - 3a_3)\, X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4)\, X_1^2 X_2^2 \\
& + (a_1a_4 - a_2a_3)(2X_1Z_2 + X_2Z_1)\, X_2 Y_1 \\
& + (a_1^2a_4 - 2a_1a_2a_3 + 3a_3^2)\, X_1^2 X_2 Z_2 \\
& - (a_2a_4 - 9a_6)\, X_1X_2(X_1Z_2 + X_2Z_1) \\
& + (3a_1a_6 - a_3a_4)(X_1Z_2 + 2X_2Z_1)\, Y_1Z_2 \\
& + (3a_1^2a_6 - 2a_1a_3a_4 + a_2a_3^2 + 3a_2a_6 - a_4^2)\, X_1Z_2(X_1Z_2 + 2X_2Z_1) \\
& - (3a_2a_6 - a_4^2)(X_1Z_2 + X_2Z_1)(X_1Z_2 - X_2Z_1) \\
& + (a_1^3a_6 - a_1^2a_3a_4 + a_1a_2a_3^2 - a_1a_4^2 + 4a_1a_2a_6 - a_3^3 - 3a_3a_6)\, Y_1Z_1Z_2^2 \\
& + (a_1^4a_6 - a_1^3a_3a_4 + 5a_1^2a_2a_6 + a_1^2a_2a_3^2 - a_1a_2a_3a_4 - a_1a_3^3 - 3a_1a_3a_6 \\
& - a_1^2a_4^2 + a_2^2a_3^2 - a_2a_4^2 + 4a_2^2a_6 - a_3^2a_4 - 3a_4a_6)\, X_1Z_1Z_2^2 \\
& + (a_1^2a_2a_6 - a_1a_2a_3a_4 + 3a_1a_3a_6 + a_2^2a_3^2 - a_2a_4^2 \\
& + 4a_2^2a_6 - 2a_3^2a_4 - 3a_4a_6)\, X_2Z_1^2Z_2 \\
& + (a_1^3a_3a_6 - a_1^2a_3^2a_4 + a_1^2a_4a_6 + a_1a_2a_3^3 \\
& + 4a_1a_2a_3a_6 - 2a_1a_3a_4^2 + a_2a_3^2a_4 \\
& + 4a_2a_4a_6 - a_3^4 - 6a_3^2a_6 - a_4^3 - 9a_6^2)\, Z_1^2Z_2^2, \\
Z_3^{(2)} = {} & 3X_1X_2(X_1Y_2 + X_2Y_1) + Y_1Y_2(Y_1Z_2 + Y_2Z_1) + 3a_1X_1^2X_2^2 \\
& + a_1(2X_1Y_2 + Y_1X_2)\, Y_1Z_2 + a_1^2X_1Z_2(2X_2Y_1 + X_1Y_2) \\
& + a_2X_1X_2(Y_1Z_2 + Y_2Z_1) \\
& + a_2(X_1Y_2 + X_2Y_1)(X_1Z_2 + X_2Z_1) \\
& + a_1^3X_1^2X_2Z_2 + a_1a_2X_1X_2(2X_1Z_2 + X_2Z_1) \\
& + 3a_3X_1X_2^2Z_1 + a_3Y_1Z_2(Y_1Z_2 + 2Y_2Z_1) \\
& + 2a_1a_3X_1Z_2(Y_1Z_2 + Y_2Z_1) \\
& + 2a_1a_3X_2Y_1Z_1Z_2 + a_4(X_1Y_2 + X_2Y_1)\, Z_1Z_2 \\
& + a_4(X_1Z_2 + X_2Z_1)(Y_1Z_2 + Y_2Z_1) \\
& + (a_1^2a_3 + a_1a_4)\, X_1Z_2(X_1Z_2 + 2X_2Z_1) + a_2a_3X_2Z_1(2X_1Z_2 + X_2Z_1) \\
& + a_3^2Y_1Z_1Z_2^2 + (a_3^2 + 3a_6)(Y_1Z_2 + Y_2Z_1)\, Z_1Z_2 \\
& + a_1a_3^2(2X_1Z_2 + X_2Z_1)\, Z_1Z_2 + 3a_1a_6X_1Z_1Z_2^2 \\
& + a_3a_4(X_1Z_2 + 2X_2Z_1)\, Z_1Z_2 + (a_3^3 + 3a_3a_6)\, Z_1^2Z_2^2.
\end{aligned}
$$

1987 Lenstra: Use Lange–R
complete system of addition
to computationally define gr
$E(R)$ for more general rings
rings with trivial class group

Define $\mathbf{P}^2(R) = \{(X : Y : Z$
$X, Y, Z \in R;\ XR{+}YR{+}ZR$
where $(X : Y : Z)$ is the mo
$\{(\lambda X, \lambda Y, \lambda Z) : \lambda \in R\}$.

Define $E(R) =$
$\{(X : Y : Z) \in \mathbf{P}^2(R) :$
$Y^2Z = X^3 + a_4XZ^2 + a_6Z$



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$$\begin{aligned}
Y_3^{(2)} = {} & Y_1^2 Y_2^2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3)\, X_1 X_2^2 Y_1 \\
& + a_3 Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4)\, X_1^2 X_2^2 \\
& + (a_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1)\, X_2 Y_1 \\
& + (a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2)\, X_1^2 X_2 Z_2 \\
& - (a_2 a_4 - 9a_6)\, X_1 X_2 (X_1 Z_2 + X_2 Z_1) \\
& + (3a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1)\, Y_1 Z_2 \\
& + (3a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) \\
& - (3a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1) \\
& + (a_1^3 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6)\, Y_1 Z_1 Z_2^2 \\
& + (a_1^4 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6 \\
& - a_1^2 a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6)\, X_1 Z_1 Z_2^2 \\
& + (a_1^2 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2 \\
& + 4a_2^2 a_6 - 2a_3^2 a_4 - 3a_4 a_6)\, X_2 Z_1^2 Z_2 \\
& + (a_1^3 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3 \\
& + 4a_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4 \\
& + 4a_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2)\, Z_1^2 Z_2^2, \\
Z_3^{(2)} = {} & 3X_1 X_2 (X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2 \\
& + a_1 (2X_1 Y_2 + Y_1 X_2)\, Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2) \\
& + a_2 X_1 X_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + a_2 (X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1) \\
& + a_1^3 X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1) \\
& + 3a_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1) \\
& + 2a_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1) \\
& + 2a_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1)\, Z_1 Z_2 \\
& + a_4 (X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1) \\
& + (a_1^2 a_3 + a_1 a_4)\, X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1) \\
& + a_3^2 Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1)\, Z_1 Z_2 \\
& + a_1 a_3^2 (2X_1 Z_2 + X_2 Z_1)\, Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2 \\
& + a_3 a_4 (X_1 Z_2 + 2X_2 Z_1)\, Z_1 Z_2 + (a_3^3 + 3a_3 a_6)\, Z_1^2 Z_2^2.
\end{aligned}$$

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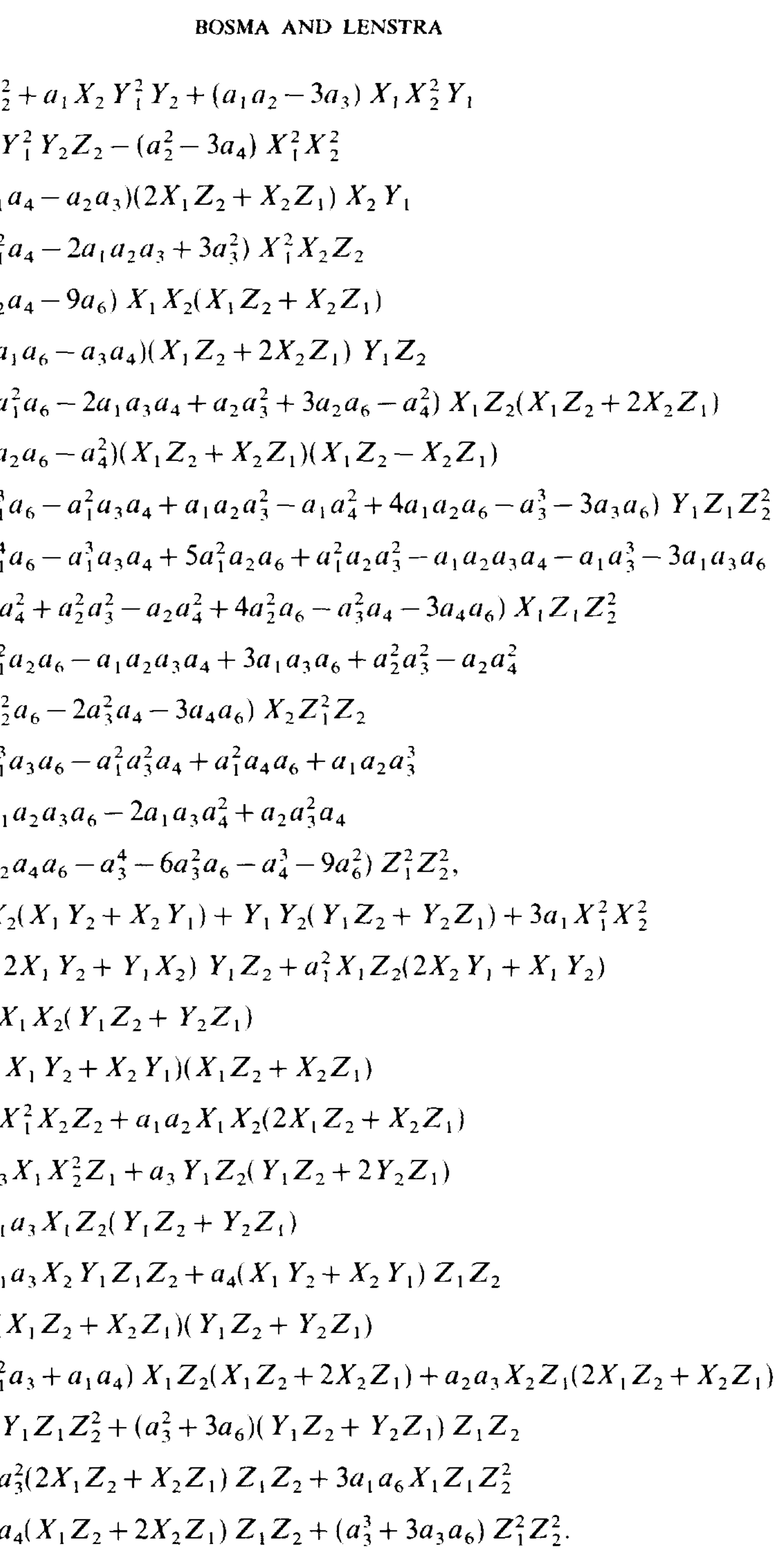

BOSMA AND LENSTRA

$${}^2_2 + a_1 X_2 Y_1^2 Y_2 + (a_1 a_2 - 3a_3) X_1 X_2^2 Y_1$$
$$Y_1^2 Y_2 Z_2 - (a_2^2 - 3a_4) X_1^2 X_2^2$$
$${}_1 a_4 - a_2 a_3)(2X_1 Z_2 + X_2 Z_1) X_2 Y_1$$
$${}^2_1 a_4 - 2a_1 a_2 a_3 + 3a_3^2) X_1^2 X_2 Z_2$$
$${}_2 a_4 - 9a_6) X_1 X_2 (X_1 Z_2 + X_2 Z_1)$$
$$a_1 a_6 - a_3 a_4)(X_1 Z_2 + 2X_2 Z_1) Y_1 Z_2$$
$$a_1^2 a_6 - 2a_1 a_3 a_4 + a_2 a_3^2 + 3a_2 a_6 - a_4^2) X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1)$$
$$a_2 a_6 - a_4^2)(X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1)$$
$${}^3_1 a_6 - a_1^2 a_3 a_4 + a_1 a_2 a_3^2 - a_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6) Y_1 Z_1 Z_2^2$$
$${}^4_1 a_6 - a_1^3 a_3 a_4 + 5a_1^2 a_2 a_6 + a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6$$
$$a_4^2 + a_2^2 a_3^2 - a_2 a_4^2 + 4a_2^2 a_6 - a_3^2 a_4 - 3a_4 a_6) X_1 Z_1 Z_2^2$$
$${}^2_1 a_2 a_6 - a_1 a_2 a_3 a_4 + 3a_1 a_3 a_6 + a_2^2 a_3^2 - a_2 a_4^2$$
$${}^2_2 a_6 - 2a_3^2 a_4 - 3a_4 a_6) X_2 Z_1^2 Z_2$$
$${}^3_1 a_3 a_6 - a_1^2 a_3^2 a_4 + a_1^2 a_4 a_6 + a_1 a_2 a_3^3$$
$${}_1 a_2 a_3 a_6 - 2a_1 a_3 a_4^2 + a_2 a_3^2 a_4$$
$${}_2 a_4 a_6 - a_3^4 - 6a_3^2 a_6 - a_4^3 - 9a_6^2) Z_1^2 Z_2^2,$$
$${}_2(X_1 Y_2 + X_2 Y_1) + Y_1 Y_2 (Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2$$
$$2X_1 Y_2 + Y_1 X_2) Y_1 Z_2 + a_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2)$$
$$X_1 X_2 (Y_1 Z_2 + Y_2 Z_1)$$
$$X_1 Y_2 + X_2 Y_1)(X_1 Z_2 + X_2 Z_1)$$
$$X_1^2 X_2 Z_2 + a_1 a_2 X_1 X_2 (2X_1 Z_2 + X_2 Z_1)$$
$${}_3 X_1 X_2^2 Z_1 + a_3 Y_1 Z_2 (Y_1 Z_2 + 2Y_2 Z_1)$$
$${}_1 a_3 X_1 Z_2 (Y_1 Z_2 + Y_2 Z_1)$$
$${}_1 a_3 X_2 Y_1 Z_1 Z_2 + a_4 (X_1 Y_2 + X_2 Y_1) Z_1 Z_2$$
$$(X_1 Z_2 + X_2 Z_1)(Y_1 Z_2 + Y_2 Z_1)$$
$${}^2_1 a_3 + a_1 a_4) X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1)$$
$$Y_1 Z_1 Z_2^2 + (a_3^2 + 3a_6)(Y_1 Z_2 + Y_2 Z_1) Z_1 Z_2$$
$$a_3^2 (2X_1 Z_2 + X_2 Z_1) Z_1 Z_2 + 3a_1 a_6 X_1 Z_1 Z_2^2$$
$$a_4 (X_1 Z_2 + 2X_2 Z_1) Z_1 Z_2 + (a_3^3 + 3a_3 a_6) Z_1^2 Z_2^2.$$

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To defin

$(X_1 : Y_1$

Consider

Lange–R

$(X'_3 : Y'_3$

Add thes

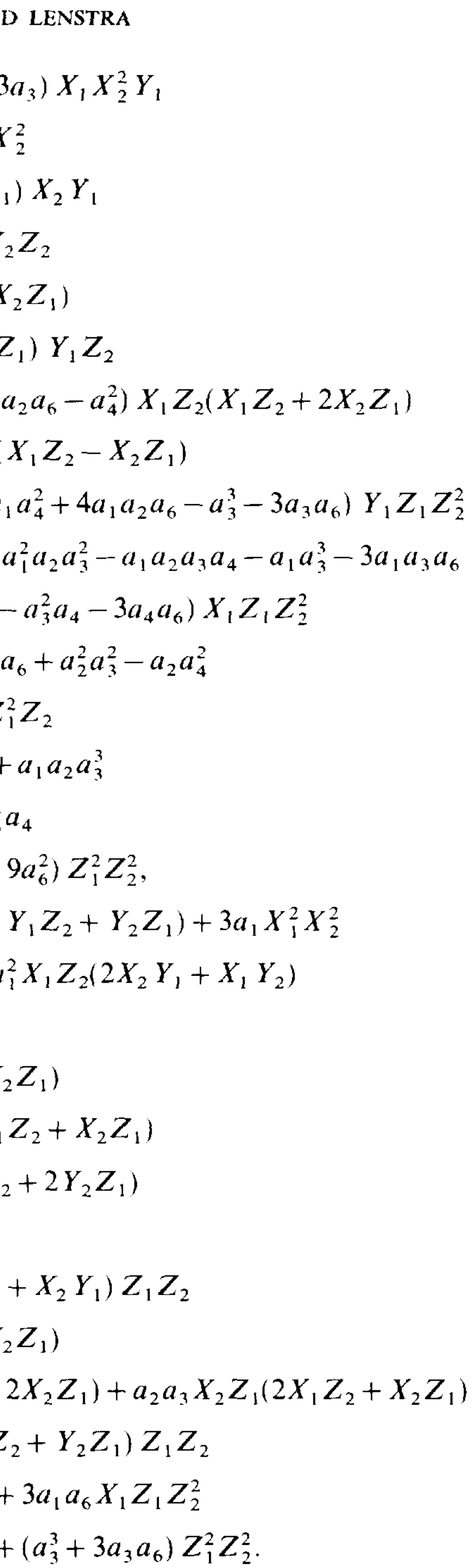
$\{\quad (\lambda X_3$

$+ (\lambda' X$

$+ (\lambda'' X$

Express

using tri



$3a_3)\ X_1 X_2^2 Y_1$

Y_2^2

$_1)\ X_2 Y_1$

$_2 Z_2$

$X_2 Z_1)$

$Z_1)\ Y_1 Z_2$

$a_2 a_6 - a_4^2)\ X_1 Z_2 (X_1 Z_2 + 2X_2 Z_1)$

$(X_1 Z_2 - X_2 Z_1)$

$_1 a_4^2 + 4a_1 a_2 a_6 - a_3^3 - 3a_3 a_6)\ Y_1 Z_1 Z_2^2$

$a_1^2 a_2 a_3^2 - a_1 a_2 a_3 a_4 - a_1 a_3^3 - 3a_1 a_3 a_6$

$- a_3^2 a_4 - 3a_4 a_6)\ X_1 Z_1 Z_2^2$

$a_6 + a_2^2 a_3^2 - a_2 a_4^2$

$_1^2 Z_2$

$+ a_1 a_2 a_3^3$

a_4

$9a_6^2)\ Z_1^2 Z_2^2,$

$Y_1 Z_2 + Y_2 Z_1) + 3a_1 X_1^2 X_2^2$

$_1^2 X_1 Z_2 (2X_2 Y_1 + X_1 Y_2)$

$_2 Z_1)$

$Z_2 + X_2 Z_1)$

$_2 + 2Y_2 Z_1)$

$+ X_2 Y_1)\ Z_1 Z_2$

$_2 Z_1)$

$2X_2 Z_1) + a_2 a_3 X_2 Z_1 (2X_1 Z_2 + X_2 Z_1)$

$Z_2 + Y_2 Z_1)\ Z_1 Z_2$

$+ 3a_1 a_6 X_1 Z_1 Z_2^2$

$+ (a_3^3 + 3a_3 a_6)\ Z_1^2 Z_2^2.$

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To define (and co

$(X_1 : Y_1 : Z_1) + ($

Consider (and com

Lange–Ruppert $(X$

$(X'_3 : Y'_3 : Z'_3)$, $(X$

Add these R-modu

$\{ \quad (\lambda X_3, \lambda Y_3, \lambda Z_3$

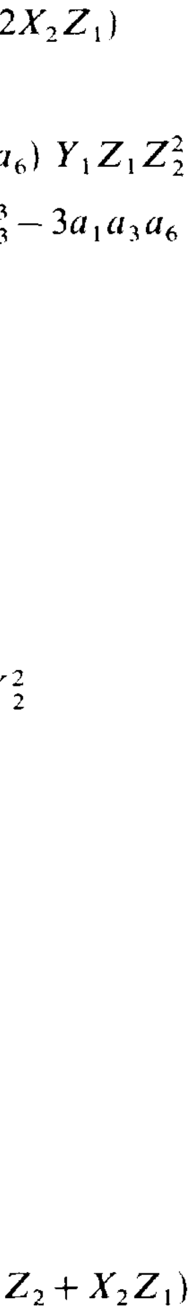
$+ (\lambda' X'_3, \lambda' Y'_3, \lambda'$

$+ (\lambda'' X''_3, \lambda'' Y''_3,$

λ, λ

Express as $(X : Y$

using trivial class



$2X_2Z_1)$

$a_6)\ Y_1Z_1Z_2^2$

$^3_3 - 3a_1a_3a_6$

$^2_2$

$Z_2 + X_2Z_1)$

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Define $E(R) =$ $\{(X : Y : Z) \in \mathbf{P}^2(R) :$ $Y^2Z = X^3 + a_4XZ^2 + a_6Z^3\}$.

To define (and compute) su

$(X_1 : Y_1 : Z_1) + (X_2 : Y_2 : Z$

Consider (and compute)

Lange–Ruppert $(X_3 : Y_3 : Z$

$(X_3' : Y_3' : Z_3')$, $(X_3'' : Y_3'' : Z$

Add these R-modules:

$\{\ \ (\lambda X_3, \lambda Y_3, \lambda Z_3)$

$+ (\lambda' X_3', \lambda' Y_3', \lambda' Z_3')$

$+ (\lambda'' X_3'', \lambda'' Y_3'', \lambda'' Z_3'') :$

$\lambda, \lambda', \lambda'' \in R$

Express as $(X : Y : Z)$,

using trivial class group of R

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Define $E(R) = \{(X : Y : Z) \in \mathbf{P}^2(R) : Y^2Z = X^3 + a_4XZ^2 + a_6Z^3\}$.

To define (and compute) sum $(X_1 : Y_1 : Z_1) + (X_2 : Y_2 : Z_2)$:

Consider (and compute) Lange–Ruppert $(X_3 : Y_3 : Z_3)$, $(X_3' : Y_3' : Z_3')$, $(X_3'' : Y_3'' : Z_3'')$.

Add these R-modules:

$$\begin{aligned}\{\quad &(\lambda X_3, \lambda Y_3, \lambda Z_3)\\ &+ (\lambda' X_3', \lambda' Y_3', \lambda' Z_3')\\ &+ (\lambda'' X_3'', \lambda'' Y_3'', \lambda'' Z_3'') :\\ &\qquad\qquad \lambda, \lambda', \lambda'' \in R\}.\end{aligned}$$

Express as $(X : Y : Z)$, using trivial class group of R.