

Addition laws on elliptic curves

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Joint work with:

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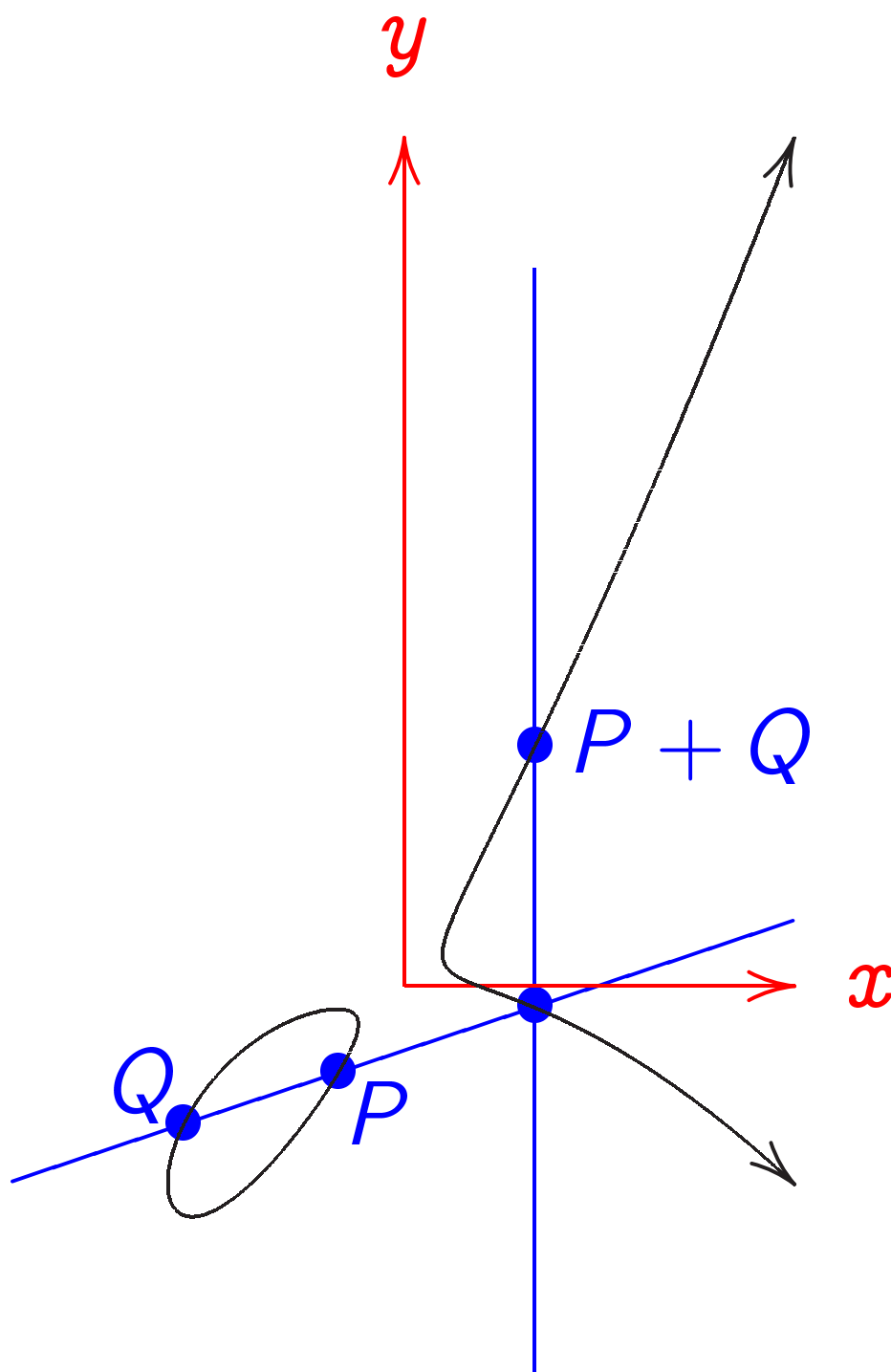
Technische Universiteit Eindhoven

2007.01.10, 09:00 (yikes!),
Leiden University, part of
“Mathematics: Algorithms and
Proofs” week at Lorentz Center:
Harold Edwards speaks on
“Addition on elliptic curves.”



Edwards

What we think when we hear
“addition on elliptic curves”:



Addition on $y^2 - 5xy = x^3 - 7$.

$$\lambda = (y_2 - y_1)/(x_2 - x_1),$$

$$x_3 = \lambda^2 - 5\lambda - x_1 - x_2,$$

$$y_3 = 5x_3 - (y_1 + \lambda(x_3 - x_1))$$

$$\Rightarrow (x_1, y_1) + (x_2, y_2) = (x_3, y_3).$$

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$$\Rightarrow (x_1, y_1) + (x_2, y_2) = (x_3, y_3).$$

Oops, this requires $x_1 \neq x_2$.

$$\lambda = (5y_1 + 3x_1^2)/(2y_1 - 5x_1),$$

$$x_3 = \lambda^2 - 5\lambda - 2x_1,$$

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$$\Rightarrow (x_1, y_1) + (x_1, y_1) = (x_3, y_3).$$

Oops, this requires $2y_1 \neq 5x_1$.

$$(x_1, y_1) + (x_1, 5x_1 - y_1) = \infty.$$

$$(x_1, y_1) + \infty = (x_1, y_1).$$

$$\infty + (x_1, y_1) = (x_1, y_1).$$

$$\infty + \infty = \infty.$$

Despite 09:00,
despite Dutch trains,
we attend the talk.

Edwards says:

Euler–Gauss addition law

on $x^2 + y^2 = 1 - x^2y^2$ is

$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 - x_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 + x_1x_2y_1y_2}.$$



Euler



Gauss

Edwards, continued:

Every elliptic curve over $\overline{\mathbf{Q}}$
is birationally equivalent to
$$x^2 + y^2 = a^2(1 + x^2y^2)$$

for some $a \in \overline{\mathbf{Q}} - \{0, \pm 1, \pm i\}$.

(Euler–Gauss curve \equiv the
“lemniscatic elliptic curve.”)

Edwards, continued:

Every elliptic curve over $\overline{\mathbf{Q}}$
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 $x^2 + y^2 = a^2(1 + x^2y^2)$
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(Euler–Gauss curve \equiv the
“lemniscatic elliptic curve.”)

$x^2 + y^2 = a^2(1 + x^2y^2)$ has
neutral element $(0, a)$, addition
 $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{a(1 + x_1x_2y_1y_2)},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{a(1 - x_1x_2y_1y_2)}.$$

Addition law is “unified”:

$(x_1, y_1) + (x_1, y_1) = (x_3, y_3)$ with

$$x_3 = \frac{x_1 y_1 + y_1 x_1}{a(1 + x_1 x_1 y_1 y_1)},$$

$$y_3 = \frac{y_1 y_1 - x_1 x_1}{a(1 - x_1 x_1 y_1 y_1)}.$$

Have seen unification before.

e.g., 1986 Chudnovsky²:

17M unified addition formulas

for $(S : C : D : Z)$ on Jacobi's

$$S^2 + C^2 = Z^2, \quad k^2 S^2 + D^2 = Z^2.$$



Chudnovsky²



Jacobi

2007.01.10, \approx 09:30,

Bernstein–Lange:

Edwards addition law with
standard projective $(X : Y : Z)$,
standard Karatsuba optimization,
common-subexp elimination:

10M + **1S** + **1A**.

Faster than anything seen before!

M: field multiplication.

S: field squaring.

A: multiplication by a .



Karatsuba

Edwards paper: Bulletin AMS
44 (2007), 393–422.

Many papers in 2007, 2008, 2009
have now used Edwards curves
to set speed records
for critical computations
in elliptic-curve cryptography.

Also new speed records
for ECM factorization: see
Lange's talk here on Saturday.

Also expect speedups in verifying
elliptic-curve primality proofs.

Back to B.–L., early 2007.

Edwards $x^2 + y^2 = a^2(1 + x^2y^2)$

doesn't *rationally* include

Euler–Gauss $x^2 + y^2 = 1 - x^2y^2$.

Common generalization,

presumably more curves over \mathbf{Q} ,

presumably more curves over \mathbf{F}_q :

$x^2 + y^2 = c^2(1 + dx^2y^2)$ has

neutral element $(0, c)$, addition

$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{c(1 + dx_1x_2y_1y_2)},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{c(1 - dx_1x_2y_1y_2)}.$$

Convenient to take $c = 1$
for speed, simplicity.

Covers same set of curves
up to birational equivalence:

$$(c, d) \equiv (1, dc^4).$$

$x^2 + y^2 = 1 + dx^2y^2$ has

neutral element $(0, 1)$, addition

$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ with

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

Hmmm, does this really work?

Easiest way to check

the generalized addition law:

pull out the computer!

Pick a prime p ; e.g. 47.

Pick curve param $d \in \mathbf{F}_p$.

Enumerate all affine points

$(x, y) \in \mathbf{F}_p \times \mathbf{F}_p$ satisfying

$$x^2 + y^2 = 1 + dx^2y^2.$$

Use generalized addition law

to make an addition table

for all pairs of points.

Check associativity etc.

Warning: Don't expect
complete addition table.

Addition law works generically
but can fail for some
exceptional pairs of points.

Unified addition law
works for generic additions
and for generic doublings
but can fail for some
exceptional pairs of points.

Basic problem: Denominators
 $1 \pm dx_1x_2y_1y_2$ can be zero.

Even if we switched to projective coordinates, would expect addition law to fail for some points, producing $(0 : 0 : 0)$.

1995 Bosma–Lenstra theorem:
“The smallest cardinality of a complete system of addition laws on E equals two.”



Bosma



Lenstra

Try $p = 47$, $d = 25$:

denominator $1 \pm dx_1x_2y_1y_2$

is nonzero for most points

(x_1, y_1) , (x_2, y_2) on curve.

Edwards addition law is

associative whenever defined.

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Try $p = 47$, $d = -1$:

denominator $1 \pm dx_1x_2y_1y_2$

is nonzero for *all* points

(x_1, y_1) , (x_2, y_2) on curve.

Addition law is a group law!

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Addition law is a group law!



vs.



Z60T

2007 Bernstein–Lange
completeness proof
for all non-square d :

$$\text{If } x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$$
$$\text{and } x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$$
$$\text{and } dx_1 x_2 y_1 y_2 = \pm 1$$

2007 Bernstein–Lange
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$$\text{then } dx_1^2 y_1^2 (x_2 + y_2)^2$$

$$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$$

$$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$$

2007 Bernstein–Lange
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$$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$$

$$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$$

$$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$$

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$$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$$

$$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$$

$$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$$

$$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$$

2007 Bernstein–Lange
completeness proof
for all non-square d :

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$$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$$

$$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$$

$$= x_1^2 + y_1^2 \pm 2x_1 y_1 = (x_1 \pm y_1)^2.$$

2007 Bernstein–Lange
completeness proof
for all non-square d :

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$$\text{and } dx_1 x_2 y_1 y_2 = \pm 1$$

$$\text{then } dx_1^2 y_1^2 (x_2 + y_2)^2$$

$$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$$

$$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$$

$$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$$

$$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$$

$$= x_1^2 + y_1^2 \pm 2x_1 y_1 = (x_1 \pm y_1)^2.$$

Have $x_2 + y_2 \neq 0$ or $x_2 - y_2 \neq 0$;
either way d is a square. Q.E.D.

1995 Bosma–Lenstra theorem:

“The smallest cardinality of a complete system of addition laws on E equals two.”

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“The smallest cardinality of a complete system of addition laws on E equals two.” . . . meaning:

Any addition formula

for a Weierstrass curve E

in projective coordinates

must have exceptional cases

in $E(\bar{k}) \times E(\bar{k})$, where

\bar{k} = algebraic closure of k .

1995 Bosma–Lenstra theorem:

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in $E(\bar{k}) \times E(\bar{k})$, where

\bar{k} = algebraic closure of k .

Edwards addition formula has

exceptional cases for $E(\bar{k})$

. . . but not for $E(k)$.

We do computations in $E(k)$.

Summary: Assume k field;

$2 \neq 0$ in k ; non-square $d \in k$.

Then $\{(x, y) \in k \times k :$

$$x^2 + y^2 = 1 + dx^2y^2\}$$

is a commutative group with

$$(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$$

defined by Edwards addition law:

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

Terminology: “Edwards curves”

allow arbitrary $d \in k^*$; $d = c^4$

are “original Edwards curves”;

non-square d are “complete.”

$d = 0$: “the clock group.”

$x^2 + y^2 = 1$, parametrized

by $(x, y) = (\sin, \cos)$.

Gauss parametrized

$x^2 + y^2 = 1 - x^2y^2$ by

$(x, y) = (\text{“lemn sin”}, \text{“lemn cos”})$.

Abel, Jacobi “sn, cn, dn”

handle all elliptic curves,

but (sn, cn) does *not*

specialize to (lemn sin, lemn cos).

Bad generalization of (sin, cos).

Edwards x is sn;

Edwards y is cn/dn.

Theta view: see Edwards paper.

Every elliptic curve over k
with a point of order 4
is birationally equivalent
to an Edwards curve.

Unique order-2 point \Rightarrow complete.
Convenient for implementors:
no need to worry about
accidentally bumping into
exceptional inputs.

Particularly nice for cryptography:
no need to worry about
attackers manufacturing
exceptional inputs,
hearing case distinctions, etc.

What about elliptic curves
without points of order 4?

What about elliptic curves
over binary fields?

Continuing project (B.–L.):

For *every* elliptic curve E ,
find complete addition law for E
with best possible speeds.

Complete laws are useful
even if slower than Edwards!

2008 B.–Birkner–L.–Peters:

“twisted Edwards curves”

$$ax^2 + y^2 = 1 + dx^2y^2$$

cover all Montgomery curves.

Almost as fast as $a = 1$;

brings Edwards speed

to larger class of curves.

2008 B.–B.–Joye–L.–P.:

every elliptic curve over \mathbf{F}_p

where 4 divides group order

is (1 or 2)-isogenous

to a twisted Edwards curve.

Statistics for many $p \in 1 + 4\mathbf{Z}$,
 \approx number of pairs $(j(E), \#E)$:

Curves	total	odd	2odd	4odd	8odd
orig	$\frac{1}{24}p$	0	0	0	0
compl	$\frac{1}{2}p$	0	0	$\frac{1}{4}p$	$\frac{1}{8}p$
Ed	$\frac{2}{3}p$	0	0	$\frac{1}{4}p$	$\frac{3}{16}p$
twist	$\frac{5}{6}p$	0	0	$\frac{5}{12}p$	$\frac{3}{16}p$
$4\mathbf{Z}$	$\frac{5}{6}p$	0	0	$\frac{5}{12}p$	$\frac{3}{16}p$
all	$2p$	$\frac{2}{3}p$	$\frac{1}{2}p$	$\frac{5}{12}p$	$\frac{3}{16}p$

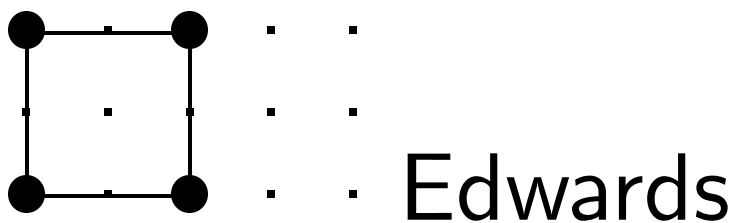
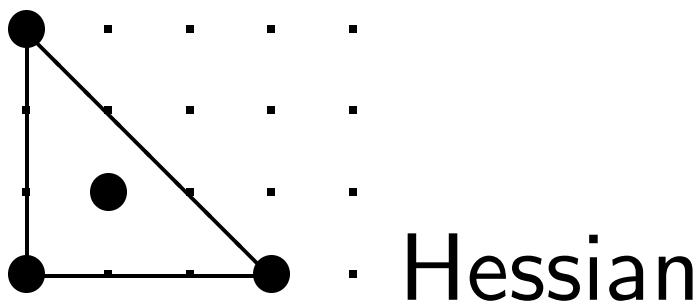
Different statistics for $3 + 4\mathbf{Z}$.

Bad news:

complete twisted Edwards

\equiv complete Edwards!

Some Newton polygons



1893 Baker: genus is generically number of interior points.

2000 Poonen–Rodriguez-Villegas classified genus-1 polygons.

How to generalize Edwards?

Design decision: want quadratic in x and in y .

Design decision: want $x \leftrightarrow y$ symmetry.

$$d_{20} \quad d_{21} \quad d_{22}$$

$$d_{10} \quad d_{11} \quad d_{21}$$

$$d_{00} \quad d_{10} \quad d_{20}$$

Curve shape $d_{00} + d_{10}(x + y) + d_{11}xy + d_{20}(x^2 + y^2) + d_{21}xy(x + y) + d_{22}x^2y^2 = 0$.

Suppose that $d_{22} = 0$:

$$d_{20} \quad d_{21} \quad \cdot$$

$$d_{10} \quad d_{11} \quad d_{21}$$

$$d_{00} \quad d_{10} \quad d_{20}$$

Genus 1 $\Rightarrow (1, 1)$ is an interior point $\Rightarrow d_{21} \neq 0$.

Homogenize:

$$d_{00}Z^3 + d_{10}(X + Y)Z^2 + d_{11}XYZ + d_{20}(X^2 + Y^2)Z + d_{21}XY(X + Y) = 0.$$

Points at ∞ are $(X : Y : 0)$

with $d_{21}XY(X + Y) = 0$: i.e.,

$(1 : 0 : 0)$, $(0 : 1 : 0)$, $(1 : -1 : 0)$.

Study $(1 : 0 : 0)$ by setting

$$y = Y/X, z = Z/X$$

in homogeneous curve equation:

$$d_{00}z^3 + d_{10}(1 + y)z^2 + d_{11}yz + d_{20}(1 + y^2)z + d_{21}y(1 + y) = 0.$$

Nonzero coefficient of y

so $(1 : 0 : 0)$ is nonsingular.

Addition law cannot be complete
(unless k is tiny).

So we require $d_{22} \neq 0$.

Points at ∞ are $(X : Y : 0)$

with $d_{22}X^2Y^2 = 0$: i.e.,

$(1 : 0 : 0), (0 : 1 : 0)$.

Study $(1 : 0 : 0)$ again:

$$d_{00}z^4 + d_{10}(1 + y)z^3 + d_{11}yz^2 + d_{20}(1 + y^2)z^2 + d_{21}y(1 + y)z + d_{22}y^2 = 0.$$

Coefficients of $1, y, z$ are 0

so $(1 : 0 : 0)$ is singular.

Put $y = uz$, divide by z^2
to blow up singularity:

$$d_{00}z^2 + d_{10}(1 + uz)z + d_{11}uz + d_{20}(1 + u^2z^2) + d_{21}u(1 + uz) + d_{22}u^2 = 0.$$

Substitute $z = 0$ to find
points above singularity:

$$d_{20} + d_{21}u + d_{22}u^2 = 0.$$

We require the quadratic

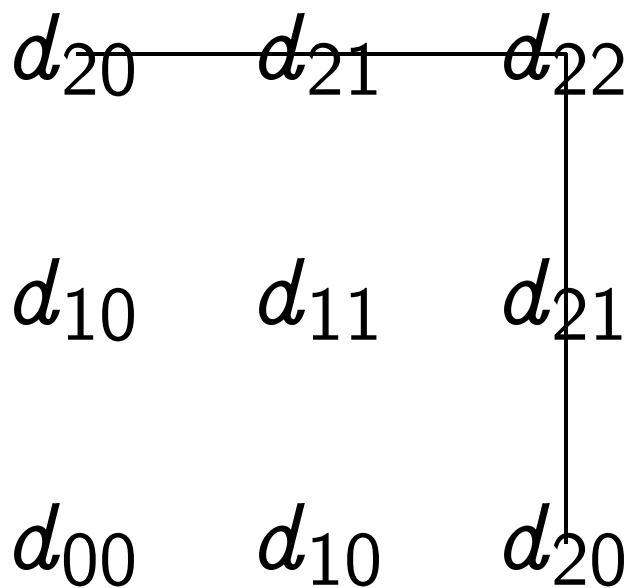
$$d_{20} + d_{21}u + d_{22}u^2$$

to be irreducible in k .

Special case: complete Edwards,

$1 - du^2$ irreducible in k .

In particular $d_{20} \neq 0$:



Design decision: Explore a deviation from Edwards.

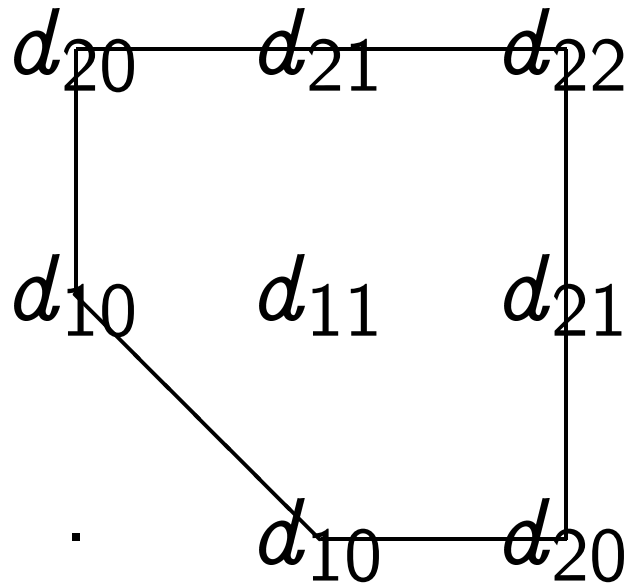
Choose neutral element $(0, 0)$.

$d_{00} = 0$; $d_{10} \neq 0$.

Can vary neutral element.

Warning: bad choice can produce surprisingly expensive negation.

Now have a Newton polygon
for generalized Edwards curves:



By scaling x, y
and scaling curve equation
can limit $d_{10}, d_{11}, d_{20}, d_{21}, d_{22}$
to three degrees of freedom.

2008 B.–L.–Rezaeian Farashahi:
complete addition law for
“binary Edwards curves”

$$d_1(x + y) + d_2(x^2 + y^2) = (x + x^2)(y + y^2).$$

Covers all ordinary elliptic curves
over \mathbf{F}_{2^n} for $n \geq 3$.

Also surprisingly fast,
especially if $d_1 = d_2$.

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Also surprisingly fast,
especially if $d_1 = d_2$.

2009 B.–L.:

complete addition law for
another specialization
covering all the “NIST curves”
over *non-binary* fields.

Consider, e.g., the curve

$$x^2 + y^2 = x + y + txy + dx^2y^2$$

with $d = -1$ and

$$t = \begin{array}{r} 78751018041117252545420999954 \\ 76717646453854506081463020284 \\ 1395651175859201799 \end{array}$$

over \mathbf{F}_p where $p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1$.

Note: d is non-square in \mathbf{F}_p .

Birationally equivalent to
standard “NIST P-256” curve

$$v^2 = u^3 - 3u + a_6 \text{ where}$$

$$a_6 = \begin{array}{r} 41058363725152142129326129780 \\ 04726840911444101599372555483. \\ 5256314039467401291 \end{array}$$

An addition law for

$$x^2 + y^2 = x + y + txy + dx^2y^2,$$

complete if d is not a square:

$$x_3 = \frac{x_1 + x_2 + (t - 2)x_1x_2 + (x_1 - y_1)(x_2 - y_2) + dx_1^2(x_2y_1 + x_2y_2 - y_1y_2)}{1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t - 2)x_2y_2)};$$

$$y_3 = \frac{y_1 + y_2 + (t - 2)y_1y_2 + (y_1 - x_1)(y_2 - x_2) + dy_1^2(y_2x_1 + y_2x_2 - x_1x_2)}{1 - 2dy_1y_2x_2 - dy_1^2(y_2 + x_2 + (t - 2)y_2x_2)}.$$

Note on computing addition laws:
An easy Magma script uses
Riemann–Roch to find addition
law given a curve shape.

Are those laws nice? No!

Find lower-degree laws by
Monagan–Pearce algorithm,
ISSAC 2006; or by evaluation at
random points on random curves.

Are those laws complete? No!

But always seems easy to
find complete addition laws
among low-degree laws where
denominator constant term $\neq 0$.

Birational equivalence from

$$x^2 + y^2 = x + y + txy + dx^2y^2 \text{ to}$$

$$v^2 - (t + 2)uv + dv =$$

$$u^3 - (t + 2)u^2 - du + (t + 2)d$$

$$\text{i.e. } v^2 - (t + 2)uv + dv =$$

$$(u^2 - d)(u - (t + 2)):$$

$$u = (dxy + t + 2)/(x + y);$$

$$v = \frac{((t + 2)^2 - d)x}{(t + 2)xy + x + y}.$$

Assuming $t + 2$ square, d not:

only exceptional point is

$(0, 0)$, mapping to ∞ .

$$\text{Inverse: } x = v/(u^2 - d);$$

$$y = ((t + 2)u - v - d)/(u^2 - d).$$

Completeness

$$x_3 = \frac{x_1 + x_2 + (t - 2)x_1x_2 + (x_1 - y_1)(x_2 - y_2) + dx_1^2(x_2y_1 + x_2y_2 - y_1y_2)}{1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t - 2)x_2y_2)};$$

$$y_3 = \frac{y_1 + y_2 + (t - 2)y_1y_2 + (y_1 - x_1)(y_2 - x_2) + dy_1^2(y_2x_1 + y_2x_2 - x_1x_2)}{1 - 2dy_1y_2x_2 - dy_1^2(y_2 + x_2 + (t - 2)y_2x_2)}.$$

Can denominators be 0?

Only if d is a square!

Theorem: Assume that

k is a field with $2 \neq 0$;

$d, t, x_1, y_1, x_2, y_2 \in k$;

d is not a square in k ;

$27d \neq (2 - t)^3$;

$$x_1^2 + y_1^2 = x_1 + y_1 + tx_1y_1 + dx_1^2y_1^2;$$

$$x_2^2 + y_2^2 = x_2 + y_2 + tx_2y_2 + dx_2^2y_2^2.$$

Then $1 - 2dx_1x_2y_2 -$

$$dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) \neq 0.$$

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Then $1 - 2dx_1x_2y_2 -$

$$dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) \neq 0.$$

By $x \leftrightarrow y$ symmetry

also $1 - 2dy_1y_2x_2 -$

$$dy_1^2(y_2 + x_2 + (t - 2)y_2x_2) \neq 0.$$

Proof: Suppose that

$$1 - 2dx_1x_2y_2 -$$

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Note that $x_1 \neq 0$.

Proof: Suppose that

$$1 - 2dx_1x_2y_2 - dx_1^2(x_2 + y_2 + (t - 2)x_2y_2) = 0.$$

Note that $x_1 \neq 0$.

Use curve equation₂ to see that

$$(1 - dx_1x_2y_2)^2 = dx_1^2(x_2 - y_2)^2.$$

Proof: Suppose that

$$1 - 2dx_1x_2y_2 -$$

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By hypothesis d is non-square

$$\text{so } x_1^2(x_2 - y_2)^2 = 0$$

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Hence $x_2 = y_2$ and $1 = dx_1x_2y_2$.

Curve equation₁ times $1/x_1^2$:

$$1 + y_1^2/x_1^2 =$$

$$1/x_1 + y_1(1/x_1^2 + t/x_1) + dy_1^2.$$

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Substitute $1/x_1 = dx_2^2$:

$$1 + d^2y_1^2x_2^4 =$$

$$dx_2^2 + dy_1(d x_2^4 + x_2^2 t) + dy_1^2.$$

Curve equation₁ times $1/x_1^2$:

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$$1 + d^2y_1^2x_2^4 =$$

$$dx_2^2 + dy_1(dx_2^4 + x_2^2t) + dy_1^2.$$

Substitute $2x_2^2 = 2x_2 + tx_2^2 + dx_2^4$:

$$(1 - dy_1x_2^2)^2 = d(x_2 - y_1)^2.$$

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Hence $1 = dx_2^3$.

Curve equation₁ times $1/x_1^2$:

$$1 + y_1^2/x_1^2 =$$

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Substitute $1/x_1 = dx_2^2$:

$$1 + d^2 y_1^2 x_2^4 =$$

$$dx_2^2 + dy_1(dx_2^4 + x_2^2 t) + dy_1^2.$$

Substitute $2x_2^2 = 2x_2 + tx_2^2 + dx_2^4$:

$$(1 - dy_1 x_2^2)^2 = d(x_2 - y_1)^2.$$

Thus $x_2 = y_1$ and $1 = dy_1 x_2^2$.

Hence $1 = dx_2^3$.

Now $2x_2^2 = 2x_2 + tx_2^2 + x_2$

so $3 = (2-t)x_2$ so $27d = (2-t)^3$.

Contradiction.

What's next?

Make the mathematicians happy:

Prove that all curves
are covered; should be easy
using Weil and rational param.

Make the computer happy:

Find *faster* complete laws.

Latest news, B.–Kohel–L.:

Have complete addition law
for twisted Hessian curves

$$ax^3 + y^3 + 1 = 3dxy$$

when a is non-cube.

Close in speed to Edwards
and covers different curves.