Putnam Mathematical Competition, 4 December 2004

Problem A1

Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

Problem A2

For $i = 1, 2$, let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

Problem A3

Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

Problem A4

Show that for any positive integer n there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers, $-1, 0, 1$.

Problem A5

An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$.We say that two squares, p and q, are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

Problem A6

Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Show that

$$\begin{aligned}
&\int_0^1 \left(\int_0^1 f(x,y)\,dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)\,dy \right)^2 dx \\
&\qquad \leq \left(\int_0^1 \int_0^1 f(x,y)\,dx\,dy \right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2\,dx\,dy.
\end{aligned}$$

Problem B1

Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r, \quad c_n r^2 + c_{n-1} r, \quad c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \quad \ldots, \quad c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

Problem B2

Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

Problem B3

Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y) : 0 \le x \le a, 0 \le y \le f(x)\}$$

has perimeter k units and area k square units for some real number k.

Problem B4

Let n be a positive integer, $n \ge 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then R_2, $\ldots$, then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

Problem B5

Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

Problem B6

Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding x. Let $\mathcal{B}$ denote the set of positive integers b that can be written in the form $b = a - a'$ with $a \in \mathcal{A}$ and $a' \in \mathcal{A}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then $\lim_{x \to \infty} N(x)/x = 0$.

Solutions

D. J. Bernstein, 6 December 2004

Problem A1

Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

Solution: Yes.

By hypothesis $S(N_1) < 0.8N_1$ for some N_1 but $S(N_2) > 0.8N_2$ for some $N_2 > N_1$. Find the smallest $N \geq N_1$ such that $S(N) \geq 0.8N$. Then $N \neq N_1$, so $S(N-1) < 0.8(N-1)$. If she does not make her Nth free throw then $S(N) = S(N-1) < 0.8(N-1) < 0.8N \leq S(N)$, contradiction. If she makes her Nth free throw then $S(N) = S(N-1)+1$ so

$$0 \leq 5S(N) - 4N = 5S(N-1) + 5 - 4N < 4(N-1) + 5 - 4N = 1.$$

The quantity $5S(N) - 4N$ is an integer, so it must be 0; i.e., $S(N) = 0.8N$ as claimed.

Problem A2

For $i = 1, 2$, let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \leq A_2$?

Solution: Yes.

Recall Heron's formula $(4A)^2 = 4a^2b^2 - (a^2 + b^2 - c^2)^2$ for the area A of a triangle with side lengths a, b, c. The derivative of $(4A)^2$ with respect to c is $4c(a^2 + b^2 - c^2)$, which is positive if $c^2 < a^2 + b^2$, i.e., if the angle opposite c is acute. By symmetry, the derivative of $(4A)^2$ with respect to a is positive if the angle opposite a is acute, and the derivative of $(4A)^2$ with respect to b is positive if the angle opposite b is acute.

Find (a, b, c) in the compact set $[a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$ to maximize A. It is not possible to have exactly one of a, b, c smaller than a_2, b_2, c_2 respectively: for example, if $a < a_2$ and $b = b_2$ and $c = c_2$, then $a^2 < a_2^2 < b_2^2 + c_2^2 = b^2 + c^2$ since T_2 is acute, so the angle opposite a is acute, so increasing a increases A, contradiction. Similarly, it is not possible to have two or three of a, b, c smaller than a_2, b_2, c_2 respectively: for example, if $a < a_2$ and $b < b_2$, then at least one angle opposite a or b must be acute, so increasing a or b increases A, contradiction.

Thus $(a, b, c) = (a_2, b_2, c_2)$. In particular, $A_1 \leq A_2$.

Problem A3

Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

Solution: Define $v_0 = 1$, $v_1 = 1$, and $v_n = (n-1)v_{n-2}$ for all $n \geq 2$. Then v_n is an integer.

I claim that $v_n v_{n+1} = n!$ for all $n \geq 0$. Proof: If $n = 0$ then $v_n v_{n+1} = v_0 v_1 = 1 = 0! = n!$ as claimed. If $n \geq 1$ then assume inductively that $v_{n-1} v_n = (n-1)!$. By definition $v_{n+1} = n v_{n-1}$, so $v_n v_{n+1} = n v_{n-1} v_n = n(n-1)! = n!$ as claimed.

I now claim that $u_n = v_n$ for all $n \geq 0$. Proof: If $n = 0$, or $n = 1$, or $n = 2$, then $u_n = 1 = v_n$ as claimed. For $n \geq 3$, assume inductively that $u_{n-3} = v_{n-3}$, that $u_{n-2} = v_{n-2}$, and that $u_{n-1} = v_{n-1}$. By hypothesis $(n-3)! = u_{n-3}u_n - u_{n-1}u_{n-2}$, so

$$\begin{aligned} u_n &= \frac{(n-3)! + u_{n-1}u_{n-2}}{u_{n-3}} \\ &= \frac{(n-3)! + v_{n-1}v_{n-2}}{v_{n-3}} = \frac{v_{n-3}v_{n-2} + (n-2)v_{n-3}v_{n-2}}{v_{n-3}} = (n-1)v_{n-2} = v_n \end{aligned}$$

as claimed.

Hence u_n is an integer.

Problem A4

Show that for any positive integer n there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers, $-1, 0, 1$.

Solution: One can take $N = 2^n$. Specifically, I claim that $x_1 x_2 \cdots x_n$ is the sum of $(a_1 a_2 \cdots a_n / 2^n n!)(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)^n$ over all $(a_1, a_2, \ldots, a_n) \in \{1, -1\}^n$.

Define $P_0(x_1, x_2, \ldots, x_n) = (x_1 + x_2 + \cdots + x_n)^n$. Note for future reference that the coefficient of $x_1 x_2 \cdots x_n$ in P_0 is $n!$.

Define $P_1(x_1, x_2, \ldots, x_n) = P_0(x_1, x_2, \ldots, x_n) - P_0(-x_1, x_2, \ldots, x_n)$. The coefficient of $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ in P_1 is $1 - (-1)^{e_1}$ times the corresponding coefficient in P_0.

Define $P_2(x_1, x_2, \ldots, x_n) = P_1(x_1, x_2, \ldots, x_n) - P_1(x_1, -x_2, \ldots, x_n)$. The coefficient of $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ in P_2 is $1-(-1)^{e_2}$ times the corresponding coefficient in P_1; in other words, $(1-(-1)^{e_1})(1-(-1)^{e_2})$ times the corresponding coefficient in P_0.

Define $P_3, P_4, \ldots, P_n$ similarly. Then the coefficient of $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ in P_n is exactly $(1-(-1)^{e_1})(1-(-1)^{e_2}) \cdots (1-(-1)^{e_n})$ times the corresponding coefficient in P_0. The factor $(1-(-1)^{e_1})(1-(-1)^{e_2}) \cdots (1-(-1)^{e_n})$ is 2^n if $e_1, e_2, \ldots, e_n$ are all odd, otherwise 0. The coefficient in P_0 is 0 unless $e_1 + e_2 + \cdots + e_n = n$. The only way for odd numbers $e_1, e_2, \ldots, e_n$ to have sum n is for all of them to be 1. Hence $P_n(x_1, x_2, \ldots, x_n) = 2^n n! x_1 x_2 \cdots x_n$.

Problem A5

An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$.We say that two squares, p and q, are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

Solution: Sorry, I haven't solved this one yet.

Problem A6

Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)\,dy \right)^2 dx \\ \leq \left(\int_0^1 \int_0^1 f(x,y)\,dx\,dy \right)^2 + \int_0^1 \int_0^1 [f(x,y)]^2\,dx\,dy.$$

Solution: Dave Rusin writes: "Let $F(x, y, z, w) = f(x, y) + f(z, w) - f(x, w) - f(z, y)$; then integrate F^2 over the box $[0, 1]^4$. Done!"

Problem B1

Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r, \quad c_n r^2 + c_{n-1} r, \quad c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \quad \ldots, \quad c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

Solution: Fix $i \in \{0, 1, \ldots, n-1\}$. Write r as u/v where u and v are coprime. Then $c_n(u/v)^n + c_{n-1}(u/v)^{n-1} + \cdots + c_0 = 0$, so $c_n u^n + c_{n-1}u^{n-1}v + \cdots + c_0 v^n = 0$, so $c_n u^n + c_{n-1}u^{n-1}v + \cdots + c_{i+1}u^{i+1}v^{n-i-1} = -c_i u^i v^{n-i} - c_{i-1}u^{i-1}v^{n-i+1} - \cdots - c_0 v^n$ is a multiple of v^{n-i}; so $c_n u^{n-i} + c_{n-1}u^{n-i-1}v + \cdots + c_{i+1}uv^{n-i-1}$ is a multiple of v^{n-i} since u^i and v^{n-i} are coprime; so $c_n(u/v)^{n-i} + c_{n-1}(u/v)^{n-1-i} + \cdots + c_{i+1}(u/v)$ is an integer as claimed.

Problem B2

Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.$$

Solution: Define $g(x) = x \log(1 + 1/x)$ for $x > 0$. Then $g'(x) = \log(1 + 1/x) - (1/x^2)x/(1+1/x) = \log(1+1/x) - 1/(x+1)$, so $g''(x) = (-1/x^2)/(1+1/x) + 1/(x+1)^2 = (1-(x+1)/x)/(x+1)^2 < 0$. The limit of $g'(x)$ as $x \to \infty$ is 0, so $g'(x) > 0$ for all $x > 0$, so g is strictly increasing.

Fix $m \geq 1$. Define $f(n) = (m+n)!m^m n^n/(m+n)^{m+n} m!n!$ for $n \geq 1$. Then $f(1) = m^m/(m+1)^m < 1$, and $f(n+1)/f(n) = (m+n)^{m+n}(n+1)^n/(m+n+1)^{m+n}n^n = g(n)/g(m+n) < 1$, so $f(n) < 1$ for all n.

Alternate proof: Use Stirling's bounds $1 < n!(\exp n)/n^n\sqrt{2\pi n} < \exp(1/12n)$ for $n \geq 1$ to see that $f(n)$ is in $[\exp(-1/12m - 1/12n), \exp(1/12(m+n))]\sqrt{(m+n)/2\pi mn}$ and hence is below $\exp(1/24)\sqrt{1/\pi}$, which is smaller than 1 since $\exp(1/12) < \exp 1 < \pi$.

Alternate proof: The binomial expansion of $(m+n)^{m+n}$ includes at least two terms since the exponent is positive; all the terms are positive, and one of them is $\binom{m+n}{m} m^m n^n$.

Problem B3

Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$

has perimeter k units and area k square units for some real number k.

Solution: The answer is $\{a : a > 2\}$.

For $a > 2$: Define $f(x) = 2a/(a-2)$ and $k = 2a^2/(a-2)$. Then f is nonnegative; f is continuous; the region R is a rectangle of width a and height $2a/(a-2)$; the area of R is $2a^2/(a-2) = k$; and the perimeter of R is $2a+4a/(a-2) = (2a(a-2)+4a)/(a-2) = k$.

For $a \le 2$: Suppose that there exists such a function f. Because f is continuous, it has a maximum value on the compact interval $[0, a]$; say $f(m)$ is the maximum. The area under f is at most $af(m)$. The perimeter of the region is at least $f(m)$, to get from $(0, 0)$ to $(m, f(m))$; plus at least another $f(m)$, to get from $(m, f(m))$ to $(a, 0)$; plus a, to get from $(a, 0)$ to $(0, 0)$; for a total of $a + 2f(m) \ge a + af(m) > af(m)$. Contradiction.

Problem B4

Let n be a positive integer, $n \ge 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k, 0)$ in the xy-plane, for $k = 1, 2, \ldots, n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then R_2, $\ldots$, then R_n. For an arbitrary point (x, y), find, and simplify, the coordinates of $R(x, y)$.

Solution: $R(x, y) = (x + n, y)$.

Put (x, y) into the complex plane as $z = x + iy$, and put $P_1, P_2, \ldots, P_n$ into the complex plane as $1, 2, \ldots, n$. Define $z_0 = z$, and define z_k for $k \ge 1$ as the result of rotating z_{k-1} by θ around k. Then $z_k = \zeta(z_{k-1} - k) + k$ where $\zeta = \cos\theta + i \sin\theta$, so, by induction, $z_k = \zeta^k z - \zeta^k - \zeta^{k-1} - \ldots - \zeta + k$. In particular, $z_n = \zeta^n z - (\zeta^n + \zeta^{n-1} + \ldots + \zeta) + n = z + n$.

Problem B5

Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n} .$$

Solution: The answer is $2/\exp 1$.

The logarithm of $((1 + x^{n+1})/(1 + x^n))^{x^n} = (1 + x^n(x-1)/(1 + x^n))^{x^n}$ is approximately $x^{2n}(x-1)/(1 + x^n)$. There are approximately $\delta/u(1-x)$ nonnegative integers n for which $u \le x^n < u + \delta$. The sum of $x^{2n}(x-1)/(1 + x^n)$ over those n's is approximately $-(\delta/u)u^2/(1 + u) = -(u/(1 + u))\delta$. Hence the sum of $x^{2n}(x-1)/(1 + x^n)$ over all n's is approximately $-\int_0^1 (u/(1 + u))\, du = \log(1 + u) - u|_0^1 = \log 2 - 1$.

To turn this into a complete proof, write down explicit error bounds (using explicit log bounds as in B2) instead of just saying “approximately”; then observe that the error converges to 0 as x approaches 1.

Problem B6

Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding x. Let $\mathcal{B}$ denote the set of positive integers b that can be written in the form $b = a - a'$ with $a \in \mathcal{A}$ and $a' \in \mathcal{A}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then $\lim_{x\to\infty} N(x)/x = 0$.

Solution: Find the smallest positive integer g_1 such that $g_1 \notin \mathcal{B}$.

Find the smallest positive integer h_2 such that $h_2, h_2+1, h_2+2, \ldots, h_2+6g_1 \notin \mathcal{B}$. Define $g_2 = 2g_1(1 + \lceil h_2/2g_1 \rceil)$. Then $g_2 - 2g_1, g_2 - 2g_1 + 1, g_2 - 2g_1 + 2, \ldots, g_2 + 2g_1 \notin \mathcal{B}$, and g_2 is a multiple of $2g_1$.

Find the smallest positive integer h_3 such that $h_3, h_3+1, h_3+2, \ldots, h_3+6g_2 \notin \mathcal{B}$. Define $g_3 = 2g_2(1 + \lceil h_3/2g_2 \rceil)$. Then $g_3 - 2g_2, g_3 - 2g_2 + 1, g_3 - 2g_2 + 2, \ldots, g_3 + 2g_2 \notin \mathcal{B}$, and g_3 is a multiple of $2g_2$.

Similarly define $g_4, g_5, \ldots$.

If k and m are positive integers then, by Lemma 1 below, $\mathcal{A} \cap \{1, 2, \ldots, 2mg_k\}$ has at most $2mg_k/2^k$ elements. Hence $\mathcal{A} \cap [1, x]$ has at most $2g_k + 2\lfloor x/2g_k \rfloor g_k/2^k$ elements; i.e., $N(x) \le 2g_k + x/2^k$. Thus $\limsup_{x\to\infty} N(x)/x \le 1/2^k$. This is true for every k, so $\lim_{x\to\infty} N(x)/x = 0$.

Lemma 1: For all integers $k \ge 1$ and $n \ge 0$, the set $\mathcal{A} \cap \{n+1, n+2, \ldots, n+2g_k\}$ has at most $2g_k/2^k$ elements.

Proof for $k = 1$: If $\mathcal{A}$ has both $n+1$ and $n+g_1+1$ then $g_1 \in \mathcal{B}$, contradiction; thus $\mathcal{A}$ has at most one of $n+1, n+g_1+1$. Similar comments apply to $n+2, n+g_1+2$; $n+3, n+g_1+3$; $\ldots$; $n+g_1, n+2g_1$. Hence $\mathcal{A} \cap \{n+1, n+2, \ldots, n+2g_1\}$ has at most $g_1 = 2g_k/2^k$ elements as claimed.

Proof for $k \ge 2$: Assume inductively that, for all n, the set $\mathcal{A} \cap \{n+1, \ldots, n+2g_{k-1}\}$ has at most $2g_{k-1}/2^{k-1}$ elements.

Consider the sets $S = \{n+1, \ldots, n+2g_{k-1}\}$ and $S' = \{n+g_k+1, \ldots, n+g_k+2g_{k-1}\}$. If $a \in S$ and $a' \in S'$ then $a' - a \in \{g_k - 2g_{k-1} + 1, \ldots, g_k + 2g_{k-1} - 1\}$, so $a' - a \notin \mathcal{B}$ by construction of g_k. Hence $\mathcal{A}$ cannot have elements in common with both S and S'. Furthermore, $\mathcal{A} \cap S$ has at most $2g_{k-1}/2^{k-1}$ elements, and $\mathcal{A} \cap S'$ has at most $2g_{k-1}/2^{k-1}$ elements, so $\mathcal{A} \cap (S \cup S')$ has at most $2g_{k-1}/2^{k-1}$ elements.

Similar comments apply with n shifted by $2g_{k-1}, 4g_{k-1}, \ldots, g_k - 2g_{k-1}$; recall here that g_k is a multiple of $2g_{k-1}$. Hence $\mathcal{A} \cap \{n+1, n+2, \ldots, n+2g_k\}$ has at most $(g_k/2g_{k-1})(2g_{k-1}/2^{k-1}) = 2g_k/2^k$ elements as claimed.