1998 Putnam problems and unofficial solutions

As usual, first come the problems, then the problems with solutions. Comments and criticism at the end.

Send any followup remarks to the `USENET` newsgroup `sci.math`.

Problems

Problem A1

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Problem A2

Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

Problem A3

Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

Problem A4

Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2 A_1 = 10$, $A_4 = A_3 A_2 = 101$, $A_5 = A_4 A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

Problem A5

Let $\mathcal{F}$ be a finite collection of open discs in $\mathbf{R}^2$ whose union contains a set $E \subseteq \mathbf{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supseteq E.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

Problem A6

Let A, B, C denote distinct points with integer coordinates in $\mathbf{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

Problem B1

Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6 + 1/x^6) - 2}{(x+1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

Problem B2

Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

Problem B3

Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P a regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, were A, B, α, and β are real numbers.

Problem B4

Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

Problem B5

Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = \underbrace{1111 \cdots 11}_{1998 \text{ digits}}.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

Problem B6

Prove that, for any integers a, b, c, there exists a positive integer n such that

$$\sqrt{n^3 + an^2 + bn + c}$$

is not an integer.

Unofficial solutions

Problem A1

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Solution: Let x be the side-length of the cube. The top of the cube is at depth $3 - x$ from the top of the cone, and it has a diagonal of length $x\sqrt{2}$ touching opposite sides of the cone, so $3 - x = (3/2)x\sqrt{2}$, so $x = 6/(2 + 3\sqrt{2})$.

Problem A2

Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position, of s.

Solution: Say the arc is from angle α to angle β, with $\alpha < \beta$. Then

$$A + B = \int_{\cos\beta}^{\cos\alpha} y\,dx + \int_{\sin\alpha}^{\sin\beta} x\,dy = \int_{\alpha}^{\beta} \sin^2\theta\,d\theta + \int_{\alpha}^{\beta} \cos^2\theta\,d\theta = \int_{\alpha}^{\beta} 1\,d\theta = \beta - \alpha.$$

Alternative solution: A is the area of the circular sector from α to β, plus the area of the triangle $(0,0)$, $(\cos\alpha, 0)$, $(\cos\alpha, \sin\alpha)$, minus the area of the triangle $(0,0)$, $(\cos\beta, 0)$, $(\cos\beta, \sin\beta)$; B is the area of the same circular sector, plus the area of the triangle $(0,0)$, $(0, \sin\beta)$, $(\cos\beta, \sin\beta)$, minus the area of the triangle $(0,0)$, $(0, \sin\alpha)$, $(\cos\alpha, \sin\alpha)$. Thus $A + B$ is twice the area of the circular sector; the triangle areas cancel.

Problem A3

Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

Solution: If $f(0)f''(0) < 0$ or $f'(0)f'''(0) < 0$ then, by Lemma 1, there exists a with $f(a)f'(a)f''(a)f'''(a) = 0$. Otherwise $f(0)f'(0)f''(0)f'''(0) \geq 0$, so take $a = 0$.

Lemma 1: If g'' is continuous and $g(0)g''(0) < 0$ then there exists a with $g(a) = 0$ or $g'(a) = 0$ or $g''(a) = 0$. Proof: If $g'(0) = 0$ then take $a = 0$. Otherwise define $x = -g(0)/g'(0)$. By the mean value theorem there is some c between 0 and x for which $g'(c)x = g(x) - g(0)$, and there is some b between 0 and c for which $g''(b)c = g'(c) - g'(0)$. Thus $g''(b)cx = g'(c)x - g'(0)x = g(x) - g(0) + g(0) = g(x)$. If $g''(0)g''(b) \leq 0$ then by the intermediate value theorem g'' has a root; otherwise $g(0)g''(b) < 0$ so $g(0)g(x) = g(0)g''(b)cx < 0$ so by the intermediate value theorem g has a root.

Problem A4

Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

Solution: Define d_n as the number of digits in A_n. Then $d_1 = 1$, $d_2 = 1$, and $d_n = d_{n-1} + d_{n-2}$ for $n \geq 3$. By induction d_n mod 2 is $0, 1, 1$ when n mod 3 is $0, 1, 2$.

Now $A_n = 10^{d_{n-2}}A_{n-1} + A_{n-2}$ for $n \geq 3$, so $A_n \equiv (-1)^{d_{n-2}}A_{n-1} + A_{n-2} \pmod{11}$. By induction A_n mod 11 is $1, 0, 1, 10, 2, 1$ when n mod 6 is $0, 1, 2, 3, 4, 5$.

Thus A_n is divisible by 11 if and only if n mod $6 = 1$.

Problem A5

Let $\mathcal{F}$ be a finite collection of open discs in $\mathbf{R}^2$ whose union contains a set $E \subseteq \mathbf{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$\bigcup_{j=1}^{n} 3D_j \supseteq E.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

Solution: Select $D_1 \in \mathcal{F}$ of maximal radius. Select $D_2 \in \mathcal{F}$, not intersecting D_1, of maximal radius. Select $D_3 \in \mathcal{F}$, not intersecting D_1 or D_2, of maximal radius. Continue until every disc in $\mathcal{F}$ intersects one of $D_1, D_2, \ldots, D_n$.

For any point $x \in E$ there is a disc $C \in \mathcal{F}$ with $x \in C$. Find the smallest integer $k \geq 1$ such that C intersects D_k. Then C does not intersect $D_1, D_2, \ldots, D_{k-1}$; by construction of D_k, the radius of C is at most the radius of D_k. The distance from x to the center of D_k is less than twice the radius of C plus the radius of D_k, hence less than three times the radius of D_k; so $x \in 3D_k \subseteq 3D_1 \cup 3D_2 \cup \cdots \cup 3D_n$ as desired.

Problem A6

Let A, B, C denote distinct points with integer coordinates in $\mathbf{R}^2$. Prove that if

$$(|AB| + |BC|)^2 < 8 \cdot [ABC] + 1$$

then A, B, C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

Solution: Define $x = |AB|$ and $y = |BC|$, and let T be the area of the triangle. Then $2T \leq xy$ so $x^2 + y^2 + 4T \leq (x+y)^2 < 8T + 1$ so $x^2 + y^2 < 4T + 1$. Both $x^2 + y^2$ and $4T$ are integers, so $x^2 + y^2 \leq 4T$, so $0 \leq (x-y)^2 \leq 4T - 2xy \leq 0$. Thus $2T = xy$ and $x = y$.

Problem B1

Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

Solution: The ratio is $3(x + 1/x)$, so its minimum value is 6.

Problem B2

Given a point (a, b) with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at (a, b), one on the x-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

Solution: (Thanks to Martin Tangora for pointing out this solution to me.) The perimeter of the triangle (a, b), (d, d), $(c, 0)$ equals the sum of distances from (b, a) to (d, d) to $(c, 0)$ to $(a, -b)$, which is at least the distance from (b, a) to $(a, -b)$, namely $\sqrt{2a^2 + 2b^2}$. This bound can be achieved: the line segment from (b, a) to $(a, -b)$ intersects the lines $y = x$ and $y = 0$ by the hypotheses on a and b.

Problem B3

Let H be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, C the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and P a regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A \sin \alpha + B \cos \beta$, were A, B, α, and β are real numbers.

Solution: The surface area of the hemisphere is 2π. The portion *not* over the pentagon consists of five copies of the surface obtained by rotating $\{(\cos\theta, \sin\theta) : 0 \leq \theta \leq \pi/5\}$ halfway around the x-axis. That surface has area $\pi \int_0^{\pi/5} (\sin\theta) \sqrt{\sin^2\theta + \cos^2\theta}\, d\theta = \pi(1 - \cos(\pi/5))$, so the answer is $5\pi\cos(\pi/5) - 3\pi$. One solution in the stated form is $(A, \alpha, B, \beta) = (-3\pi, \pi/2, 5\pi, \pi/5)$.

Problem B4

Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

Solution: Define $S(m,n) = \sum_{0 \le i < mn} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor}$. Write $\operatorname{ord}_2 t$ for the number of powers of 2 dividing t.

If $\operatorname{ord}_2 m = \operatorname{ord}_2 n$ then $S(m,n) \ne 0$. Indeed, write $m = 2^e u$ and $n = 2^e v$, with u and v odd. Then $S(u,v)$ has an odd number of odd terms, so it is nonzero; and $S(m,n) = 2^{2e} S(u,v)$ by Lemma 1.

If $\operatorname{ord}_2 m \ne \operatorname{ord}_2 n$ then $S(m,n) = 0$. Indeed, write $m = 2^d x$ and $n = 2^e v$, with x and v odd; without loss of generality assume $d > e$. Write $u = 2^{d-e} x$, and substitute $i = uv - 1 - j$ in $S(u,v)$:

$$\begin{aligned} S(u,v) &= \sum_{0 \le j < uv} (-1)^{\lfloor (uv-1-j)/u \rfloor + \lfloor (uv-1-j)/v \rfloor} \\ &= \sum_{0 \le j < uv} (-1)^{u-1+v-1-\lfloor j/u \rfloor - \lfloor j/v \rfloor} \\ &= \sum_{0 \le j < uv} (-1)^{1+\lfloor j/u \rfloor + \lfloor j/v \rfloor} = -S(u,v). \end{aligned}$$

Thus $S(u,v) = 0$; and $S(m,n)$ is a multiple of $S(u,v)$ by Lemma 1.

Lemma 1. $S(bu, bv)$ is a multiple of $S(u,v)$; if $u+v$ is even then $S(bu,bv) = b^2 S(u,v)$. Proof: Write $i = buvq + br + s$, with $0 \le r < uv$ and $0 \le s < b$. Then

$$\begin{aligned} S(bu,bv) &= \sum_{0 \le q < b} \sum_{0 \le r < uv} \sum_{0 \le s < b} (-1)^{\lfloor (buvq+br+s)/bu \rfloor + \lfloor (buvq+br+s)/bv \rfloor} \\ &= \sum_{0 \le q < b} \sum_{0 \le r < uv} \sum_{0 \le s < b} (-1)^{(u+v)q + \lfloor r/u \rfloor + \lfloor r/v \rfloor} \\ &= bS(u,v) \sum_{0 \le q < b} (-1)^{(u+v)q}. \end{aligned}$$

If $u+v$ is even then $\sum_{0 \le q < b} (-1)^{(u+v)q} = b$.

Problem B5

Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = \underbrace{1111 \cdots 11}_{1998 \text{ digits}}.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

Solution: The answer is 1.

More generally, for any positive integer h, write $N = (10^{2h} - 1)/9$. By Lemma 1,

$$\begin{aligned} 10^{h+1}\sqrt{N} &= 10^{2h+1}\sqrt{1 - 10^{-2h}}/3 \\ &= 10^{2h+1}(1 - 10^{-2h}/2 - 10^{-4h}x/4)/3 \\ &= 10^{2h+1}/3 - 10/6 - 10^{1-2h}x/12 \\ &= (10^{2h+1} - 10)/3 + 1 + 2/3 - 10^{1-2h}x/12 \end{aligned}$$

for some x between 0 and 1. Now $0 < 2/3 - 10^{1-2h}x/12 < 1$, and $(10^{2h+1} - 10)/3$ is a multiple of 10, so $\lfloor 10^{h+1}\sqrt{N} \rfloor \bmod 10 = 1$.

For example, $\sqrt{11} \approx 3.3166$, and $\sqrt{1111} \approx 33.3316666$.

Lemma 1. If $0 < d < 1/2$ then $1 - d/2 - d^2/4 < \sqrt{1-d} < 1 - d/2$. Proof: $d + d^2/4 < 1$ so $(1 - d/2 - d^2/4)^2 = 1 - d - (1 - d - d^2/4)d^2/4 < 1 - d < 1 - d + d^2/4 = (1 - d/2)^2$.

Problem B6

Prove that, for any integers a, b, c, there exists a positive integer n such that

$$\sqrt{n^3 + an^2 + bn + c}$$

is not an integer.

Solution: Write $f(n) = n^3 + an^2 + bn + c$. Find an integer m large enough that $8m^3$ exceeds both $(a^2 - 4b)m^2 - am - c + 1/4$ and $(4b - a^2)m^2 - am + c - 1/4$. Then

$$(8m^3 + am - 1/2)^2 < f(4m^2) < (8m^3 + am + 1/2)^2,$$

so if $f(4m^2)$ is a square then $f(4m^2) = (8m^3 + am)^2$, i.e., $(a^2 - 4b)m^2 = c$. This cannot hold for more than one value of m unless $4b = a^2$ and $c = 0$. But then $f(n) = n(n+a/2)^2$; this is not a square if n is a non-square different from $-a/2$.

Comments

In A1, is one required to *prove* that the $6/(2+3\sqrt{2})$ cube fits inside the cone, and that larger cubes do not? In A2, can one assume that the arc length from $(\cos\alpha, \sin\alpha)$ to $(\cos\beta, \sin\beta)$ is $\beta - \alpha$? In A6, can one take "parallelogram with equal sides and with area equal to product of sides" as the definition of a square? In B3, can one use standard facts about areas of portions of a sphere? Contestants should not have to guess how their work will be graded.

The form of answer for B3—"$A \sin\alpha + B \cos\beta$, where A, B, α, and β are real numbers"—was rather strange. Did the question writers mean to say "$A\pi + B\pi\cos(C\pi)$, where A, B, C are rational numbers"?

There are many solutions to B6; what makes the problem difficult is the weakness of its conclusion. Here is a very fast proof by Brian R. Hunt: $f(4k+b+1) - f(4k+b-1) \equiv 2 \pmod 4$ so it is not possible for both $f(4k+b+1) \bmod 4$ and $f(4k+b-1) \bmod 4$ to be in $\{0, 1\}$.

—Daniel J. Bernstein, `djb@cr.yp.to`, 7 December 1998