

**RESEARCH ANNOUNCEMENT:  
FASTER FACTORIZATION  
OF TWO NUMBERS INTO COPRIMES**

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ABSTRACT. This paper presents an algorithm that, given positive integers  $a, b$ , computes the natural coprime base for  $\{a, b\}$  in time  $n(\lg n)^{2+o(1)}$ , where  $n$  is the number of input bits.

My previous paper [1] introduced an algorithm that, given a set  $S$  of positive integers, computes the natural coprime base  $\text{cb} S$  in time  $n(\lg n)^{O(1)}$ , where  $n$  is the number of input bits. I made no attempt in [1] to optimize the exponent of  $\lg n$ .

This paper presents an algorithm that computes  $\text{cb}\{a, b\}$  in time  $n(\lg n)^{2+o(1)}$ . It is reasonable to conjecture that the limiting exponent 2 is optimal (for, e.g., a multitape Turing machine): one has  $\text{cb}\{a, b\} = \{a, b\} - \{1\}$  if and only if  $a, b$  are coprime; the well-known problem of checking coprimality has been stuck at  $n(\lg n)^{2+o(1)}$  for thirty years.

**Step 1.** Swap  $a, b$  if necessary so that  $a \geq b$ . The algorithm will later reduce the input length by at least one third of the length of  $a$ .

If  $a = 1$ , stop.

**Step 2.** Compute  $a_0 = a$ ,  $g_0 = \gcd\{a_0, b\}$ ,  $a_1 = a_0/g_0$ ,  $g_1 = \gcd\{a_1, g_0^2\}$ ,  $a_2 = a_1/g_1$ ,  $g_2 = \gcd\{a_2, g_1^2\}$ , and so on, until  $g_k = 1$ .

For example, if  $a = 2^{100}3^{100}$  and  $b = 2^{137}3^{13}$ , compute  $a_0 = 2^{100}3^{100}$ ,  $g_0 = 2^{100}3^{13}$ ,  $a_1 = 3^{87}$ ,  $g_1 = 3^{26}$ ,  $a_2 = 3^{61}$ ,  $g_2 = 3^{52}$ ,  $a_3 = 3^9$ ,  $g_3 = 3^9$ ,  $a_4 = 1$ ,  $g_4 = 1$ .

Lower level: The gcd inputs  $a_i, g_{i-1}^2$  are often highly unbalanced. To compute  $\gcd\{a_i, g_{i-1}^2\}$ , first divide  $a_i$  by  $g_{i-1}^2$ , and then use any standard fast gcd algorithm to compute  $\gcd\{g_{i-1}^2, a_i \bmod g_{i-1}^2\}$ . The division takes time at most  $n(\lg n)^{1+o(1)}$ ; the gcd takes time at most  $m(\lg m)^{2+o(1)}$  where  $m$  is the length of  $g_{i-1}^2$ .

All the  $g$ 's together have length  $O(n)$ , and  $k$  is at most about  $\lg n$ , so the total time here is at most  $n(\lg n)^{2+o(1)}$ .

**Step 3.** Compute  $x_0 = g_0/\gcd\{g_0, g_1^\infty\}$ ,  $x_1 = g_1/\gcd\{g_1, g_2^\infty\}$ , and so on.

For example, if  $a = 2^{100}3^{100}$  and  $b = 2^{137}3^{13}$ , compute  $x_0 = 2^{100}$ ,  $x_1 = 1$ ,  $x_2 = 1$ ,  $x_3 = 3^9$ .

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Lower level: Compute each  $\gcd\{g_{i-1}, g_i^\infty\}$  as  $\gcd\{g_{i-1}, g_i^{2^{e_i}} \bmod g_{i-1}\}$  where  $e_i$  is the smallest nonnegative integer satisfying  $2^{2^{e_i}} \geq g_{i-1}$ . The repeated squarings and gcd take time at most  $m(\lg m)^{2+o(1)}$  where  $m$  is the total length of  $g_{i-1}, g_i$ . The total time here is at most  $n(\lg n)^{2+o(1)}$ .

**Step 4.** Compute  $y_0 = \gcd\{b, x_0^\infty\}$ ,  $y_1 = \gcd\{g_0, x_1^\infty\}$ ,  $y_2 = \gcd\{b, g_1, x_2^\infty\}$ ,  $y_3 = \gcd\{b, g_2, x_3^\infty\}$ , and so on.

For example, if  $a = 2^{100}3^{100}$  and  $b = 2^{137}3^{13}$ , the algorithm computes  $y_0 = 2^{137}$ ,  $y_1 = 1$ ,  $y_2 = 1$ ,  $y_3 = 3^{13}$ .

Lower level: Compute  $b \bmod g_1, b \bmod g_2, \dots$  with a scaled remainder tree; this takes time  $n(\lg n)^{2+o(1)}$  since  $b, g_1, g_2, \dots$  together have length  $O(n)$ . Then compute  $\gcd\{b, g_1\}$  as  $\gcd\{b \bmod g_1, g_1\}$ ; compute  $\gcd\{b, g_2\}$  as  $\gcd\{b \bmod g_2, g_2\}$ ; and so on.

**Step 5.** Recursively print  $\text{cb}\{x_0, y_0/x_0\}$ ;  $\text{cb}\{x_1, y_1\}$ ;  $\text{cb}\{x_2, y_2\}$ ; and so on. Also print  $\text{cb}\{a'\} = \{a'\} - \{1\}$  and  $\text{cb}\{b'\} = \{b'\} - \{1\}$  where  $a' = a/\gcd\{a, b^\infty\}$  and  $b' = b/\gcd\{b, a^\infty\}$ . Note that  $a'$  has already been computed; it equals  $a_k$ .

For example, if  $a = 2^{100}3^{100}$  and  $b = 2^{137}3^{13}$ , recursively print  $\text{cb}\{2^{100}, 2^{37}\} = \{2\}$  and  $\text{cb}\{3^9, 3^{13}\} = \{3\}$ . Also print  $\text{cb}\{1\} = \{\}$  and  $\text{cb}\{1\} = \{\}$ . The complete output is  $\{2, 3\}$ .

I claim that  $x_0 y_0, x_1 y_1, \dots, a', b'$  are coprime; that  $a = a' x_0 x_1 y_1 x_2 y_2^3 x_3 y_3^7 \dots$ ; that  $b = b' y_0 y_1 y_2 y_3 \dots$ ; and that  $y_0 x_1 y_1 x_2 y_2 \dots$ , the product of inputs to the recursive calls, is at most  $ab/a^{1/3} \leq (ab)^{5/6}$ . Each of these facts can be checked from the following table of  $\text{ord}_p$  values, expressed in terms of  $e = \text{ord}_p a$  and  $f = \text{ord}_p b$ :

$g_0$	$g_1$	$g_2$	$g_3$	$\dots$	$x_0$	$y_0$	$x_1$	$y_1$	$x_2$	$y_2$	$\dots$	$a'$	$b'$	
0	0	0	0	$\dots$	0	0	0	0	0	0	$\dots$	0	$f$	if $e = 0$
$e$	0	0	0	$\dots$	$e$	$f$	0	0	0	0	$\dots$	0	0	if $0 < e \leq f$
$f$	$e-f$	0	0	$\dots$	0	0	$e-f$	$f$	0	0	$\dots$	0	0	if $f < e \leq 3f$
$f$	$2f$	$e-3f$	0	$\dots$	0	0	0	0	$e-3f$	$f$	$\dots$	0	0	if $3f < e \leq 7f$
														$\vdots$
0	0	0	0	$\dots$	0	0	0	0	0	0	$\dots$	$e$	0	if $f = 0 < e$

Consequently the outputs of the algorithm are coprime;  $a$  and  $b$  are products of powers of the outputs; and the recursion multiplies the total time by a bounded factor.

Note that one can easily factor  $a, b$  over  $\text{cb}\{a, b\}$  by tracing the factorizations  $a = a' x_0 x_1 y_1 x_2 y_2^3 x_3 y_3^7 \dots$  and  $b = b' y_0 y_1 y_2 y_3 \dots$  through the recursion.

## REFERENCES

- [1] Daniel J. Bernstein, *Factoring into coprimes in essentially linear time*, to appear, Journal of Algorithms. ISSN 0196-6774. URL: <http://cr.yp.to/papers.html>. ID f32943f0bb67a9317d4021513f9eee5a.

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