NP, coNP and the Nullstellensatz: Lower Bounds for Stable Set and Graph Coloring Nullstellensätze

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1 Key Points

Systems of polynomial equations over the complex numbers can be used to characterize NP-Complete graph-theoretic decision problems. From the point of view of computer algebra and symbolic computation, these are interesting polynomial systems because they are provably hard: solving them is as hard as solving the underlying NP-Complete problem. Furthermore, unless NP $=$ coNP, there must exist infinite instances of these infeasible systems whose Hilbert Nullstellensatz certificates grow with respect to the underlying graphs.

2 Stable Set as a Zero-Dimensional System of Polynomial Equations (L. Lovász)

Given a graph G and an integer k, we construct the following equations:

- For every vertex $i = 1, \ldots, n$, let $x_i^2 - x_i = 0$
- For every edge $(i,j) \in E(G)$, let $x_i x_j = 0$
- Finally, let

$$\left(-k + \sum_{i=1}^{n} x_i \right) = 0$$

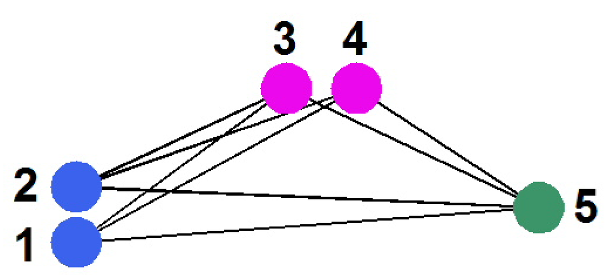
- **Theorem:** Let G be a graph, k an integer, encoded as the above $(n+m+1)$ zero-dimensional system of equations. Then this system has a solution if and only if G has an independent set of size k.

3 Stable Set Nullstellensätze

Theorem 3.1 *Given a graph G, there exists a Nullstellensatz certificate of degree $\alpha(G)$ that certifies the non-existence of a stable set of size $(\alpha(G)+r)$. Moreover, there exist families of graphs for which the minimum degree possible is at least $\alpha(G)/2$.*

Corollary 3.2 *Given a graph G, there exists a Nullstellensatz certificate of degree $\alpha(G)$ certifying the* ***non****-existence of a stable set of size $(\alpha(G)+r)$, where all of the terms in every coefficient is a monomial corresponding to an independent set, and the coefficient for the stable set polynomial contains every independent set.*

4 Turán Graph $T(5,3)$ Nullstellensatz



$$1 = \left(\frac{x_1x_2 + x_3x_4}{12} - \frac{x_1 + x_3 + x_5 + x_2 + x_4}{12} - \frac{1}{4}\right)\left(x_1 + x_3 + x_5 + x_2 + x_4 - 4\right) +$$
$$\left(\frac{x_4}{12} + \frac{x_2}{12} + \frac{1}{6}\right)x_1x_3 + \left(\frac{x_2}{12} + \frac{1}{6}\right)x_1x_4 + \left(\frac{x_2}{12} + \frac{1}{6}\right)x_1x_5 + \left(\frac{x_4}{12} + \frac{1}{6}\right)x_2x_3 +$$
$$\frac{x_2x_4}{6} + \frac{x_2x_5}{6} + \left(\frac{x_4}{12} + \frac{1}{6}\right)x_3x_5 + \frac{x_4x_5}{6} + \left(\frac{x_2}{12} + \frac{1}{12}\right)(x_1^2 - x_1) +$$
$$\left(\frac{x_1}{12} + \frac{1}{12}\right)(x_2^2 - x_2) + \left(\frac{x_4}{12} + \frac{1}{12}\right)(x_3^2 - x_3) + \left(\frac{x_3}{12} + \frac{1}{12}\right)(x_4^2 - x_4) + \frac{x_5^2 - x_5}{12}$$

5 Experimental Results for Minimum-Degree Stable Set Nullstellensätze

Graph	*vertices*	*edges*	$\alpha(G)$	$\deg(\alpha)$
P_3	3	2	2	2
P_5	5	4	3	3
P_{10}	10	9	5	5
$T(5,2)$	5	6	3	3
$T(5,3)$	5	8	2	2
$T(6,2)$	6	9	3	3
$T(6,3)$	6	12	2	2
$T(7,2)$	7	12	4	4
$T(8,2)$	8	16	4	4

Graph	*vertices*	*edges*	$\alpha(G)$	$\deg(\alpha)$
C_3	3	3	1	1
C_5	5	5	2	2
C_{10}	10	10	5	5
$T(3,1)$	3	3	1	1
$T(6,2)$	6	6	2	2
$T(9,3)$	9	9	3	3
$T(12,4)$	12	12	4	4
$T(8,2)$	8	8	2	2
$T(12,3)$	12	12	3	3

6 NP, coNP and the Nullstellensatz

Theorem 6.1 *If NP $\neq$ coNP, then there must exist an infinite family of graphs such that the minimum-degree Nullstellensatz for not-k-colorability or non-existence of a stable set of size k grows with respect to the size of the graph.*

7 Graph Coloring as a Zero-Dimensional System of Polynomial Equations (D. Bayer)

Given a graph G and an integer k, we construct the following equations:

- **vertex polynomials:** For every vertex $i = 1, \ldots, n$,

$$x_i^k - 1 = 0$$

- **edge polynomials:** For every edge $(i,j) \in E(G)$,

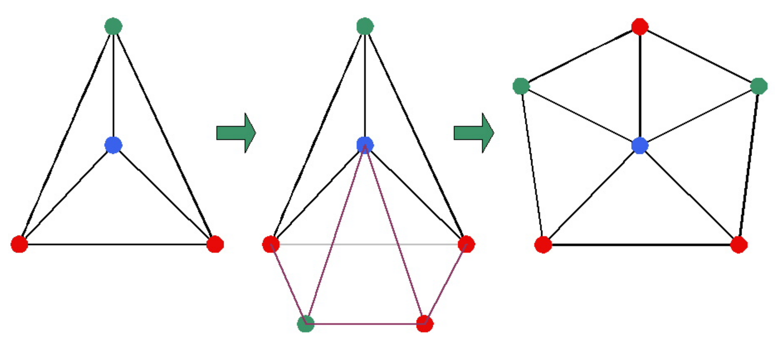
$$\frac{x_i^k - x_j^k}{x_i - x_j} = x_i^{k-1} + x_i^{k-2}x_j + \cdots + x_i x_j^{k-2} + x_j^{k-1} = 0$$

- **Theorem:** Let G be a graph, k an integer, encoded as vertex and edge polynomials. Then this system of equations has a solution if and only if G is k-colorable.

8 Odd Wheels, Cat Ears, Cliques and not-3-colorable Nullstellensätze

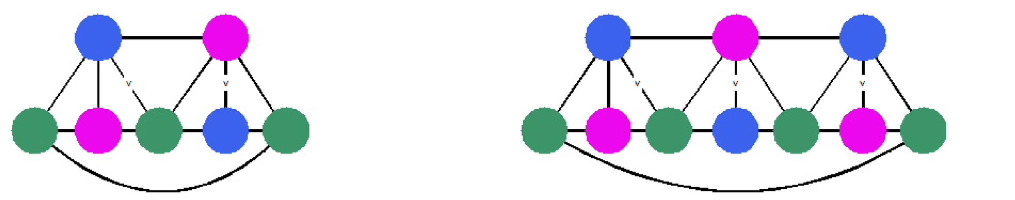
Theorem 8.1 *The minimum degree Nullstellensatz for odd-wheels is four.*

Proof Sketch:



We derive a certificate for the $(n+1)$-odd wheel from the n-th odd wheel, by taking a very particular syzygy on some of the terms from the n-th odd wheel **not** 3-colorable certificate.

Theorem 8.2 *The minimum degree Nullstellensatz for any cat ears graph is four.*



Theorem 8.3 *The minimum degree Nullstellensatz for K_n, $n \geq 4$ is four.*

Theorem 8.4 *The minimum degree Nullstellensatz for any graph is four.*

9 Experimental Results for Minimum-Degree Graph Coloring Nullstellensätze

Kneser graphs, Mycielski graphs, Queen graphs, uniquely-colorable graphs, Mycielski graphs of Myceilski graphs, flowers, assorted triangle-free graphs, Vega graphs, all graphs in six vertices are less. **4**

10 Other Encodings: Hamiltonian Cycle as a Zero-Dimensional System of Polynomial Equations

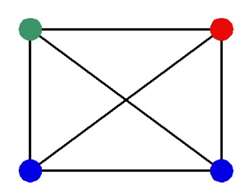
- For every vertex $i = 1, \ldots, n$, we have two equations:

$$\prod_{s=1}^{n}(x_i - s) = 0, \quad \text{and} \quad \prod_{(i,j)\in E(G)}(x_i - x_j + 1)(x_i - x_j - (n-1)) = 0$$

- **Theorem:** A graph G has a hamiltonian cycle if and only if the above zero-dimensional system of $2n$ equations has a solution.
- **Other Encodings:** Longest cycle, graph planarity, max cut and more!

11 Searching for Hilbert Nullstellensatz degrees...

Nullstellensatz: If $V(\langle f_1, \ldots, f_s\rangle) = \varnothing$, then $1 = \sum_{i=1}^{s} \alpha_i f_i$



$$\begin{aligned}1 = &(c_1x + c_2y + c_3z + c_4w + c_5)(x^3 - 1) + (c_6x + c_7y + c_8z + c_9w + c_{10})(y^3 - 1)\\ &+ (c_{11}x + \cdots + c_{15})(z^3 - 1) + (c_{16}x + \cdots + c_{20})(w^3 - 1)\\ &+ (c_{21}x + \cdots + c_{25})(x^2 + xy + y^2) + (c_{26}x + \cdots + c_{30})(x^2 + xz + z^2)\\ &+ (c_{31}x + \cdots + c_{35})(x^2 + xw + w^2) + (c_{36}x + \cdots + c_{40})(y^2 + yz + y^2)\\ &+ (c_{41}x + \cdots + c_{45})(y^2 + yw + w^2) + (c_{46}x + \cdots + c_{50})(z^2 + zw + z^2)\end{aligned}$$

Try a degree for the α polynomials, and construct a *large-scale sparse linear system of equations*. If infeasible, try a larger degree for α. Note: $\deg \alpha$ cannot exceed known upper bounds for Hilbert Nullstellensatz.