Algebraic versions of “P=NP ?”

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Valiant's model : VP$_K$ $=$ VNP$_K$?

– Complexity of a polynomial f measured by number $L(f)$ of arithmetic operations (+,-,$\times$) needed to evaluate f :

$\boxed{\text{L(f) = size of smallest arithmetic circuit computing } f.}$

– $(f_n) \in$ VP if number of variables, $\deg(f_n)$ and $L(f_n)$ are polynomially bounded. For instance, $(X^{2^n}) \notin$VP.

– $(f_n) \in$ VNP if $f_n(\overline{x}) = \sum_{\overline{y}} g_n(\overline{x}, \overline{y})$

for some $(g_n) \in$ VP

(sum ranges over all boolean values of $\overline{y}$).

A typical VNP family : the permanent.

$$\mathrm{per}(X) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} X_{i\sigma(i)}.$$

It is VNP-complete if $\mathrm{char}(K) \neq 2$.

VP and VNP are almost the only classes studied in Valiant's framework.

Sharp contrast with the "complexity theory zoo" of discrete classes (> 400 classes at www.complexityzoo.com).

Some exceptions :

- VQP : $\deg(f_n)$ polynomially bounded and $L(f_n) \leq n^{\text{poly}(\log n)}$.
- Malod (2003) has studied versions of VP and VNP without bound on $\deg(f_n)$: VP$_{nb}$, VNP$_{nb}$; and constant-free classes : VP^0, VNP^0, VP$^0_{nb}$, VNP$^0_{nb}$.
- We will define a class VPSPACE (or VPAR ?) which contains VNP$_{nb}$.

Blum-Shub-Smale model (as presented by Poizat) : $\mathrm{P}_K = \mathrm{NP}_K$?

– Computation model is richer : in addition to $+, -, \times$ gates, circuits may use $=$ and (if K ordered) $\leq$ gates.
  Selection gates :

$$s(x,y,z) = \begin{cases} y \text{ if } x = 0 \\ z \text{ if } x = 1 \end{cases}$$

  We may assume that $x \in \{0,1\}$.
  For instance, $s(x,y,z) = xz + (1-x)y$.
– Focus on decision problems.

Complexity classes

- A problem : $X \subseteq K^\infty = \bigcup_{n\geq 1} K^n$.
- X is P$_K$ if for all $x \in K^n$,

$$x \in X \Leftrightarrow C_n(x_1, \ldots x_n, a_1, \ldots, a_k) = 1$$

  with C_n constructed in polynomial time by a Turing machine.
- X is NP$_K$ if for all $x \in K^n$,

$$x \in X \Leftrightarrow \exists y \in K^{p(n)} \langle x, y\rangle \in Y$$

  with $Y \in$ P$_K$.

  A typical NP$_\mathbb{R}$-complete problem :

  decide whether a polynomial of degree 4 in n variables has a real root.

  Best algorithms to this day are of complexity exponential in n.

Decision is easy if evaluation is easy

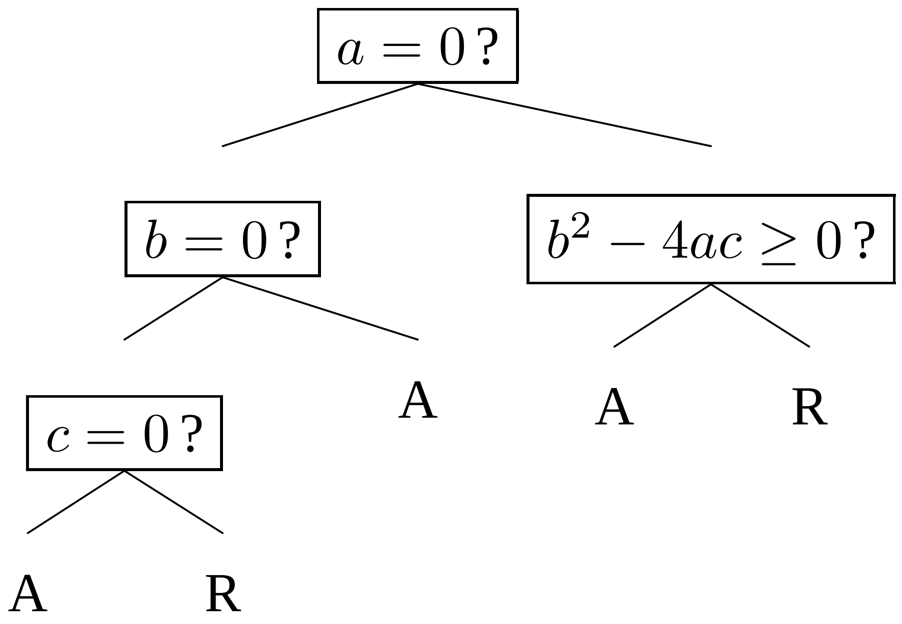
VPAR : Families of polynomials computed by uniform arithmetic circuits of polynomial depth.

Theorem [Koiran-Périfel, STACS 2007] :
Uniform $\mathrm{VP}_{nb} =$ Uniform VPAR $\Rightarrow \mathrm{P}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}} = \mathrm{PAR}_{\mathbb{R}}$.

Several versions (6 ?) of this theorem,
depending on uniformity conditions and the role of constants.

Decision trees

$$\exists x \in \mathbb{R}\ ax^2 + bx + c = 0\ ?$$



Internal nodes labeled by *arbitrary* polynomials.

Complexity $\equiv$ tree depth.

Model is unrealistic :

the complexity of polynomial evaluation should be taken into account !

Circuits versus trees

Circuit with T test ($=, \leq$) or selection gates $\rightarrow$ tree of depth T.

Can $\mathrm{NP}_{\mathbb{R}}$ problems be solved by decision trees of polynomial depth ?
If not, $\mathrm{P}_{\mathbb{R}} \neq \mathrm{NP}_{\mathbb{R}}$!

Similar questions for various structures M, for instance,
$M = (\mathbb{C}, +, -, \times, =),\ (\mathbb{R}, +, -, \leq),\ (\mathbb{R}, +, -, =),\ \{0, 1\}$.

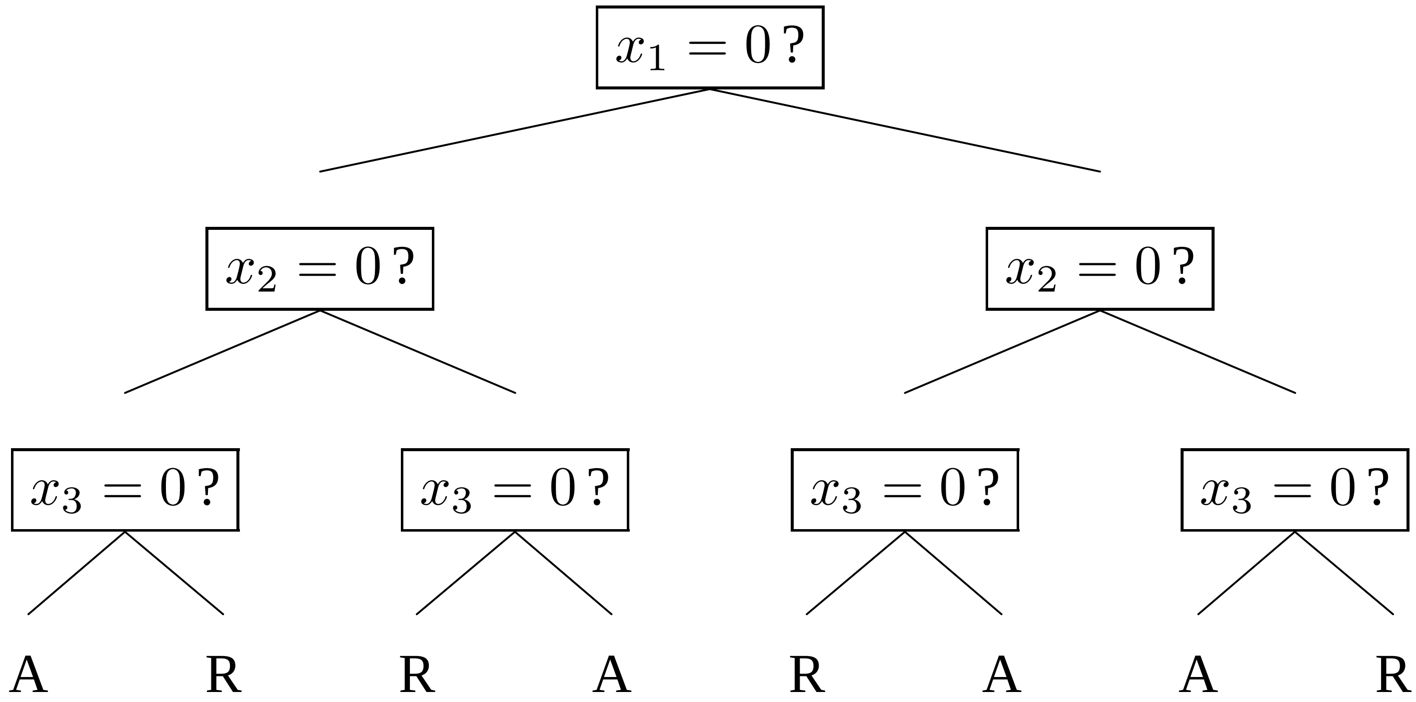
Do NP_M problems have polynomial depth decision trees ?

For $M = \{0, 1\}$, the answer is...

Labels of internal nodes are of the form "$x_i = 0$?".

Do NP$_M$ problems have polynomial depth decision trees ?

For $M = \{0, 1\}$, **Yes.**



Do NP_M problems have polynomial depth decision trees ?

For $M = (\mathbb{R}, +, -, =)$, the answer is...

Internal nodes are of the form :

$$a_1x_1 + \cdots + a_nx_n + b = 0?$$

Do NP_M problems have polynomial depth decision trees ?

For $M = (\mathbb{R}, +, -, =)$, **No.**

Twenty Questions :

INPUT : $x_1, \ldots, x_n$.

QUESTION : $x_1 \in \{0, 1, 2, \ldots, 2^n - 1\}$?

Twenty Questions is in NP_M : guess $y \in \{0, 1\}^n$,
check that $x_1 = \sum_{j=1}^{n} 2^{j-1} y_j$.

A *canonical path argument* shows that its decision tree complexity is 2^n.
Therefore, $\mathrm{P}_M \neq \mathrm{NP}_M$ (Meer).

Conjecture (Shub-Smale) : Twenty Questions is not in $\mathrm{P}_{(\mathbb{C},+,-,\times,=)}$.

Do NP_M problems have polynomial depth decision trees ?

For $M = (\mathbb{R}, +, -, \leq)$, the answer is...

Internal nodes are of the form :

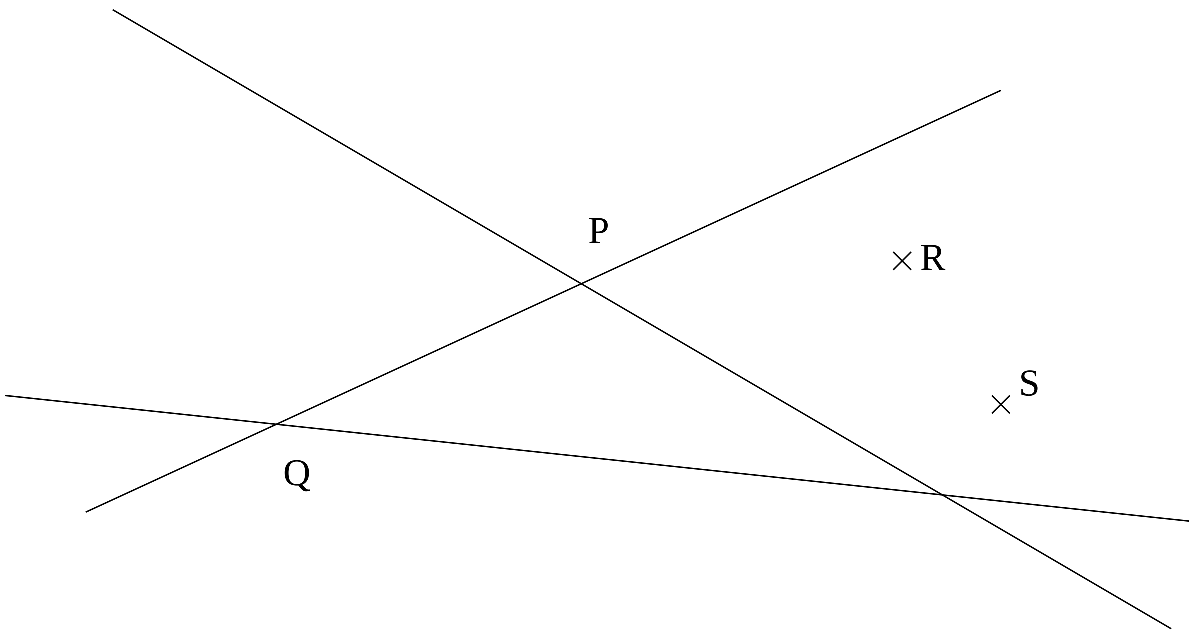
$$a_1 x_1 + \cdots + a_n x_n + b \geq 0?$$

Remark : Twenty Questions *is* in P_M by binary search.

Do NP$_M$ problems have polynomial depth decision trees ?

For $M = (\mathbb{R}, +, -, \leq)$, **Yes.**

Proof based on algorithms for point location in arrangements of hyperplanes.



R, S lie in the same 2-dimensional cell.

$]P, Q[$ is a 1-dimensional cell.

$\{P\}$ and $\{Q\}$ are 0-dimensional cells.

Decision trees for $\mathrm{NP}_{(\mathbb{R},+,-,\leq)}$ problems : the construction

1. $\mathrm{NP}_M \subseteq \mathrm{PAR}_M$: problems solvable in parallel polynomial time (by uniform circuits of possibly exponential size).
2. For inputs in $\mathbb{R}^n$, any PAR_M problem is a union of *cells* of an arrangement of $2^{n^{O(1)}}$ hyperplanes.
3. In this arrangement, point location can be performed in depth $n^{O(1)}$ (Meiser, Meyer auf der Heide). Now, just label the leaves correctly.

Corollary [Fournier-Koiran] : if $\mathrm{P} = \mathrm{NP}$ then $\mathrm{P}_M = \mathrm{NP}_M$.

Proof sketch : with access to an NP oracle,
one can effectively “run” the tree on any input $x \in \mathbb{R}^n$
(i.e., construct the path followed by x from the root to a leaf).

Do NP$_M$ problems have polynomial depth decision trees ?

For $M = (\mathbb{C}, +, -, \times, =)$, the answer is...

Internal nodes are of the form

$$P(x_1, \ldots, x_n) = 0?$$

where P is an arbitrary polynomial.

Do NP_M problems have polynomial depth decision trees ?

For $M = (\mathbb{C}, +, -, \times, =)$, **Yes.**

Not the topic of this talk...

Do NP_M problems have polynomial depth decision trees ?

For $M = (\mathbb{R}, +, -, \times, \leq)$, the answer is...

Internal nodes are of the form

$$P(x_1, \ldots, x_n) \geq 0?$$

where P is an arbitrary polynomial.

Do NP_M problems have polynomial depth decision trees ?

For $M = (\mathbb{R}, +, -, \times, \leq)$, **Yes.**

1. $\mathrm{NP}_{\mathbb{R}} \subseteq \mathrm{PAR}_{\mathbb{R}}$: problems solvable in parallel polynomial time (by uniform circuits of possibly exponential size).
2. For inputs in $\mathbb{R}^n$, any $\mathrm{PAR}_{\mathbb{R}}$ problem is a union of *cells* of an arrangement of $2^{n^{O(1)}}$ hypersurfaces of degree $2^{n^{O(1)}}$.
   Fix polynomials $P_1, \ldots, P_s$.
   Two points x and y are in the same cell if $\mathrm{sign}(P_i(x)) = \mathrm{sign}(P_i(y))$ for all $i = 1, \ldots, s$.
   Here, $sign(a) \in \{-1, 0, 1\}$.
3. In this arrangement, point location can be performed in depth $n^{O(1)}$.
   Now, just label the leaves correctly.

Point location in arrangements of real hypersurfaces

Theorem [Grigoriev] : Point location can be done in depth $O(\log N)$, where N is the number of nonempty cells.

Remark : $N \leq (sd)^{O(n)}$ where $d = \max_{i=1,\ldots,s} \deg(P_i)$.
Hence $\log N = n^{O(1)}$.

Consider inputs x with $P_i(x) \neq 0$ for all i.
Nodes are of the form "$\prod_{j \in F} P_j(x) > 0$?", where F is as follows.

Divide and Conquer Lemma :
Let $X = \{1, \ldots, s\}$ and $F_1, \ldots, F_N$ nonempty subsets of X.
There exists $F \subseteq X$ such that $N/3 \leq |\{F_x;\ |F \cap F_x| \text{ even }\}| \leq 2N/3$.
Apply to sets F_x defined by conditions of the form :

$$j \in F_x \Leftrightarrow P_j(x) < 0.$$

Then $\prod_{j \in F} P_j(x) > 0 \Leftrightarrow |F \cap F_x|$ even.

Improved version of divide and conquer lemma

Theorem [Charbit, Jeandel, Koiran, Périfel, Thomassé] :
The range $[\frac{N}{3}, \frac{2N}{3}]$ can be replaced by $[\frac{N}{2} - \alpha, \frac{N}{2} + \alpha]$ where $\alpha = \sqrt{N}/2$.

Remark : One must have $\alpha = \Omega(\sqrt{N}/(\log N)^{1/4})$.

Probabilistic proof : for a random subset F, let

$$Y_i = 1 \text{ if } |F \cap F_i| \text{ is even, and } Y_i = -1 \text{ otherwise.}$$

Need to show that there exists F such that $Y^2 \leq N$, where $Y = \sum_{i=1}^{N} Y_i$.
This follows from $E[Y^2] = N$:

$$E[Y^2] = E[\sum_{i=1}^{N} Y_i^2 + 2 \sum_{i<j} Y_i Y_j]$$

but $E[Y_i^2] = 1$ and for $i \neq j$, by pairwise independence :

$$E[Y_i Y_j] = E[Y_i]E[Y_j] = 0.$$

This can be turned into a deterministic logspace algorithm.

A remark on derandomization

From Motwani, Naor and Naor 1994 :

"A natural approach towards de-randomizing algorithms is to find a method for searching the associated sample Ω for a good point w with respect to a given input instance I. Given such a point w, the algorithm $\mathcal{A}(I, w)$ is now a deterministic algorithm and it is guaranteed to find a correct solution. The problem faced in searching the sample space is that it is generally exponential in size. The result of Adleman showing that $RP \subseteq P/poly$ implies that the sample space Ω associated with a randomized algorithm always contains a polynomial-sized subspace which has a good point for each possible input instance. However, this result is highly non-constructive and it appears that it cannot be used to actually de-randomize algorithms."

Adleman strikes back

Given s and N, our deterministic logspace algorithm constructs a list of $s^2N^2(N+1)^2$ subsets of $X = \{1, \ldots, s\}$ such that for any input $F_1, \ldots, F_N$:

$$-\frac{\sqrt{N}}{2} \leq |\{F_x;\ |F \cap F_x| \text{ even}\}| - \frac{N}{2} \leq \frac{\sqrt{N}}{2}.$$

holds for some element F of the list.

The deterministic algorithm then performs an exhaustive search in this list.

Effective point location :
Taking the complexity of polynomials into account

For a problem $A \in \mathrm{PAR}_{\mathbb{R}}$, hypersurfaces of the arrangement are defined by polynomials P_i in uniform VPAR :

Families of polynomials computed by uniform arithmetic circuits of polynomial depth.

Nodes of the tree of the form "$\prod_{i \in F} P_i(x) > 0$?" where $F \in$ PSPACE : in Uniform VPAR.

Labels of leaves can be computed in PSPACE.

Theorem [Koiran-Périfel] : If VPAR families have polynomial size circuits, then $\mathrm{PAR}_{\mathbb{R}}$ problems have polynomial size circuits.

Can VPAR families have polynomial size circuits ?

– Very strong hypothesis.
– Admits several versions (6 ?), depending on uniformity conditions and role of constants.

With (polynomially) nonuniform circuits,
and Valiant's convention for constants :

$$\boxed{\begin{array}{rl} \text{(i)} & \mathrm{VPAR} = \mathrm{VP}_{nb}. \\ & \Updownarrow \\ \text{(ii)} & \mathrm{VP} = \mathrm{VNP} \text{ and } \mathrm{PSPACE} \subseteq \mathrm{P}/\mathrm{poly}. \end{array}}$$

$\mathrm{VPAR} = \mathrm{VP}_{nb} \Rightarrow \mathrm{PSPACE} \subseteq \mathrm{P}/\mathrm{poly}$ assumes GRH
(seems necessary to handle arbitrary constants).

Can we refute [VP $=$ VNP and PSPACE $\subseteq$ P$/$poly] ?

To prove that $\neg(A \wedge B)$, one does not always have to prove $\neg A$ or $\neg B$.

For instance, we know that LOGSPACE $\neq$ P or P $\neq$ PSPACE.

It was shown by Bürgisser that (under GRH),
VP $=$ VNP $\Rightarrow$ NP $\subseteq$ NC$/$poly (problems recognized by polynomial size boolean circuits of polylogarithmic depth).

Hence, assuming GRH, the hypothesis implies that PSPACE $\subseteq$ NC$/$poly.

Most uniform version of this hypothesis

$$\boxed{\text{Uniform VPAR}^0 = \text{Uniform VP}^0_{nb} \Rightarrow \text{P-uniform NC} = \text{PSPACE.}}$$

Proof is in two steps. Hypothesis implies :

(i) P = PSPACE.

(ii) P-uniform NC $= \bigoplus$ P.

Proof of (ii) based on $\bigoplus$ P-completeness of $\bigoplus$HAMILTONIAN PATHS.

Note that $\sharp$HAMILTONIAN PATHS is of the form

$$\sum_{\sigma:\ n-\mathrm{cycle}} \prod_{i \neq \mathrm{end}(\sigma)} a_{i\sigma(i)}$$

where (a_{ij}) is the graph's adjacency matrix.

Remark : It is known that LOGSPACE-uniform NC $\neq$ PSPACE.

VPSPACE

Theorem :
A polynomial family $f_n \in \mathbb{Z}[X_1, \ldots, X_{p(n)}]$ is in P-uniform VPAR^0 iff :
(i) $p(n)$ is polynomially bounded.
(ii) $\deg(f_n)$ is exponentially bounded.
(iii) The bit size of the coefficients of f_n is exponentially bounded.
(iv) The map $(1^n, \overline{\alpha}) \mapsto a_{n,\overline{\alpha}}$ is PSPACE computable, where

$$f_n(\overline{X}) = \sum_{\overline{\alpha}} a_{n,\overline{\alpha}} \overline{X}^{\overline{\alpha}}.$$

This characterization is useful in the proof that

$$[\text{VP} = \text{VNP and PSPACE} \subseteq \text{P}/\text{poly}] \Rightarrow \text{VPAR} = \text{VP}_{nb}.$$

A natural example of a VPAR family

Resultants of multivariate polynomial systems form a VPAR family.

Proof sketch :

(i) The *Macaulay matrix* is an exponential size matrix whose non-zero entries are coefficients of the polynomial system.

(ii) Determinants can be computed by arithmetic circuits of polylogarithmic depth.

Outcome of this work

– Focus put back on evaluation problems :
  to show that certain decision problems (in $\mathrm{NP}_{\mathbb{R}}$, or $\mathrm{PAR}_{\mathbb{R}}$) are hard, one must first be able to show that certain evaluation problems (in VPAR) are hard.
– Suggestion of new lower bound problems :
  various versions of "$\mathrm{VP}_{nb} = \mathrm{VPAR}$?".
– Other natural (complete ?) polynomial families in VPAR ?