Polynomial time algorithms to approximate mixed volumes within a simply exponential factor

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Abstract

Let $\mathbf{K} = (K_1 ... K_n)$ be a n-tuple of convex compact subsets in the Euclidean space $\mathbf{R}^n$, and let $V(\cdot)$ be the Euclidean volume in $\mathbf{R}^n$. The Minkowski polynomial $V_{\mathbf{K}}$ is defined as $V_{\mathbf{K}}(x_1, ..., x_n) = V(\lambda_1 K_1 + ... + \lambda_n K_n)$ and the mixed volume $V(K_1, ..., K_n)$ as

$$V(K_1 ... K_n) = \frac{\partial^n}{\partial \lambda_1 ... \partial \lambda_n} V_{\mathbf{K}}(\lambda_1 K_1 + \cdots \lambda_n K_n).$$

The mixed volume is one of the backbones of convexity theory. After **BKH** theorem, the mixed volume(and its generalizations) had become crucially important in computational algebraic geometry.

We present in this talk randomized and deterministic algorithms to approximate the mixed volume of well-presented convex compact sets. Our main result is a poly-time randomized algorithm which approximates $V(K_1, ..., K_n)$ with multiplicative error e^n and with better rates if the affine dimensions of most of the sets K_i are small.
Because of the famous **Barany-Furedi** lower bound, even such rate is not achievable by a poly-time deterministic oracle algorithm.

Our approach is based on the particular, geometric programming, convex relaxation of $\log(V(K_1, ..., K_n))$. We prove the mixed volume analogues of the Van der Waerden and the Schrijver/Valiant conjectures on the permanent. These results , interesting on their own, allow to "justify" the above mentioned convex relaxation, which is solved using the ellipsoid method and a randomized poly-time time algorithm for the approximation of the volume of a convex set.

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