On the number of homotopy types of fibres of a definable map

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Complexity, Coding and Communications Workshop, IMA, April 16, 2007.
(Joint work with N. Vorobjov, University of Bath, England.)

Outline

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1. Introduction
   - Semi-algebraic and semi-Pfaffian sets
   - Topological complexity of semi-algebraic and semi-Pfaffian sets
   - Fibers of a definable map
2. Main Results
   - Main theorems
   - Tightness
   - Some Applications
     - Fewnomials and Polynomials with small additive complexity
     - Metric upper bounds
3. Proofs
   - The special case of a bounded real algebraic variety
   - The general case
4. Open Problems

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Semi-algebraic Sets

- A semi-algebraic set, $S \subset \mathbb{R}^k$, is a subset of $\mathbb{R}^k$ defined by a Boolean formula whose atoms are polynomial equalities and inequalities.
- If all the polynomials involved belong to $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_k]$, we call S a $\mathcal{P}$-semi-algebraic set.
- If the atoms of the Boolean formula are of the form $P \geq 0, P \leq 0, P \in \mathcal{P}$, and there are no negations, then we call S a $\mathcal{P}$-closed semi-algebraic set.

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Pfaffian Functions (following Khovansky)

- A Pfaffian chain, $\mathcal{F}$, of length r, is a sequence $f_1, \ldots, f_r$ of analytic functions defined in some open domain in $U \subset \mathbb{R}^\ell$ and satisfying the following triangular system of differential equations.

$$df_j(\mathbf{x}) = \sum_{i=1}^{n} g_{ij}(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_j(\mathbf{x}))dx_i, 1 \leq j \leq r,$$

where each g_{ij} is a polynomial in $\ell + j$ variables.
- Suppose $\deg(g_{ij}) \leq \alpha$.
- A function $f : U \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = P(\mathbf{x}, f_1(\mathbf{x}), \ldots, f_r(\mathbf{x})),$$

with $\deg(P) \leq \beta$, is called a Pfaffian function of order r and degree (α, β).

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Examples of Pfaffian chains

- A polynomial of degree bounded by d is a Pfaffian function of order 0 and degree $(1, d)$.
- A polynomial

$$P = c_1\mathbf{x}^{\alpha_1} + \cdots + c_m\mathbf{x}^{\alpha_m} \in \mathbb{R}[X_1, \ldots, X_k]$$

having m monomials in its support, is a Pfaffian function of order $k+m$ and degree $(2,1)$, in $(\mathbb{R}\setminus\{0\})^k$ by virtue of the Pfaffian chain,

$$\begin{array}{lcll} dg_i & = & -g_i^2 dx_i, 1 \leq i \leq k, & [g_i(\mathbf{x}) = 1/x_i] \\ df_j & = & \sum_{i=1}^k \alpha_{j,i}\, g_i\, f_j\, dx_i, 1 \leq j \leq m. & [f_j(\mathbf{x}) = \mathbf{x}^{\alpha_j}] \end{array}$$

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Some more examples of Pfaffian chains

- The exponential function $f(x) = e^{ax}$ is Pfaffian of order 1 and degree $(1, 1)$ in $\mathbb{R}$ by virtue of the equation,

$$df(x) = af(x)dx.$$

- The function $f(x) = \cos(x)$ is Pfaffian of order 2 and degree $(2, 1)$ in the domain $U = \mathbb{R} \setminus \cup_{i \in \mathbb{Z}} \{\pi + 2i\pi\}$, by virtue of the equations

$$dg(x) = ((1 + g^2(x))/2)dx, df(x) = -f(x)g(x)dx,$$
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Semi-Pfaffian Sets

- Let $\mathcal{P}$ be a finite set of Pfaffian functions in the open cube $U := (-1, 1)^m \subset \mathbb{R}^m$.
- A set $S \subset U$ is called $\mathcal{P}$-semi-Pfaffian in U if it is defined by a Boolean formula with atoms of the form $P > 0,\ P < 0,\ P = 0$ for $P \in \mathcal{P}$. A $\mathcal{P}$-semi-Pfaffian set S is *restricted* if its closure in U is compact.

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Bounds on Betti Numbers

- For any subset $S \subset \mathbb{R}^k$, we denote by $b_i(S) = \mathrm{rank}(H_i(S))$.
- In the semi-algebraic case: If $S \subset \mathbb{R}^k$ is a $\mathcal{P}$-semi-algebraic set, then (Oleinik, Petrovsky, Thom, Milnor, B., Gabrielov-Vorobjov)

$$\sum_{0 \leq i \leq k} b_i(S) \leq (O(s^2 d))^k$$

where $s = \#(\mathcal{P})$ and $d = \max_{P \in \mathcal{P}} \deg(P)$.
- In the restricted semi-Pfaffian case (Khovansky, Gabrielov-Vorobjov):

$$\sum_{0 \leq i \leq k} b_i(S) \leq s^{2k}\, 2^{\binom{r}{2}}\, O(k\beta + \min(r,k)\alpha)^{k+r}$$

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Fibers of a definable map

- Let $S \subset \mathbb{R}^{m+n}$ be a definable (i.e semi-algebraic or restricted semi-Pfaffian) set, and let $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$ be the projection map on the last n co-ordinates. We denote by $\pi_S = \pi|_S$.
- For $\mathbf{y} \in \mathbb{R}^n$, let $S_\mathbf{y} = S \cap \pi^{-1}(\mathbf{y})$.
- Main question of this talk: How many "topological types" occur amongst the $S_\mathbf{y}$'s as $\mathbf{y}$ varies over $\mathbb{R}^n$?
- As an application: how many topological types occur amongst real or complex hypersurfaces defined by a polynomial of degree d in n variables ?

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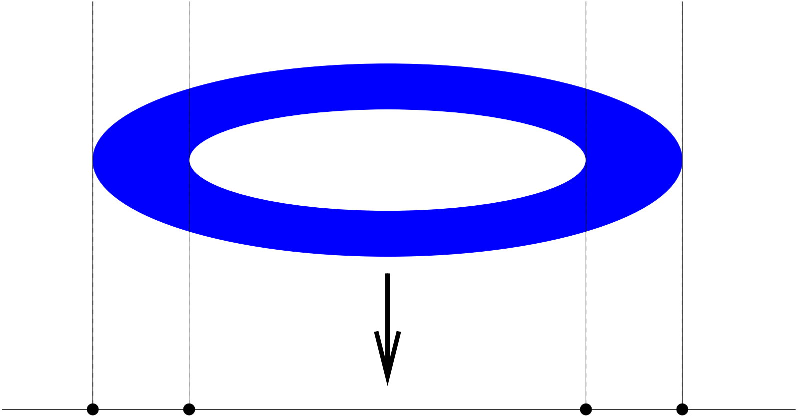
Definable map



Hardt Triviality

Theorem (Hardt, 1980)

Given any definable set $S \subset \mathbb{R}^{m+n}$, there exists a finite partition of $\mathbb{R}^n$ into definable sets $\{T_i\}_{i \in I}$ such that S is definably trivial over each T_i.

This means that for each $i \in I$ and any point $\mathbf{y} \in T_i$, the pre-image $\pi_S^{-1}(T_i)$ is definably homeomorphic to $\pi_S^{-1}(\mathbf{y}) \times T_i$ by a fiber preserving homeomorphism. In particular, for each $i \in I$, all fibers $\pi_S^{-1}(\mathbf{y}), \mathbf{y} \in T_i$ are definably homeomorphic.

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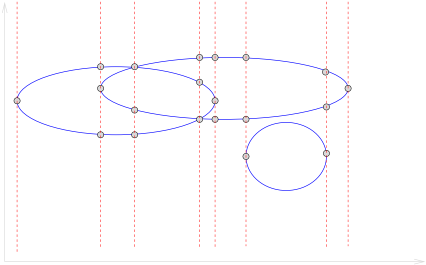
Complexity of the Hardt partition

- Hardt's theorem is a corollary of the existence of *cylindrical cell decompositions* for definable sets.



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- Open problem: prove a single exponential upper bound on the number of homeomorphism types of the fibres of π_S.

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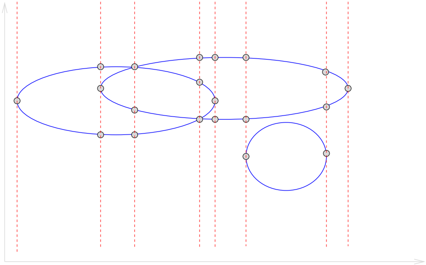

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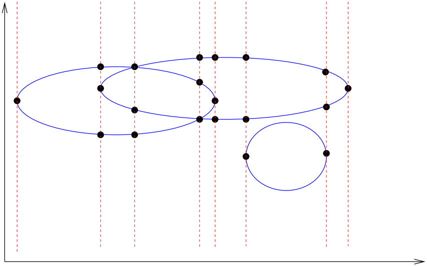

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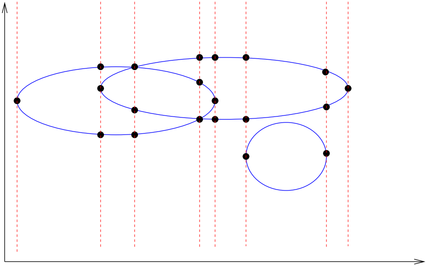

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Semi-algebraic case

Theorem

Let $\mathcal{P} \subset \mathbb{R}[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$, *with* $\deg(P) \leq d$ *for each* $P \in \mathcal{P}$, $\#\mathcal{P} = s$. *Then, there exists a finite set* $A \subset \mathbb{R}^n$, *with*

$$\#A \leq s^{2(m+1)n} \, (2^m nd)^{O(nm)} \; = \; (2^m snd)^{O(nm)},$$

such that for every $\mathbf{y} \in \mathbb{R}^n$ *there exists* $\mathbf{z} \in A$ *such that for every* $\mathcal{P}$*-semi-algebraic set* $S \subset \mathbb{R}^{m+n}$, *the set* $\pi_S^{-1}(\mathbf{y})$ *is semi-algebraically homotopy equivalent to* $\pi_S^{-1}(\mathbf{z})$. *In particular, for any fixed* $\mathcal{P}$*-semi-algebraic set* S, *the number of different homotopy types of fibres* $\pi_S^{-1}(\mathbf{y})$ *for various* $\mathbf{y} \in \pi(S)$ *is also bounded by*

$$s^{2(m+1)n}(2^m nd)^{O(nm)} = (2^m snd)^{O(nm)}.$$

Semi-Pfaffian case

Theorem

Let $\mathcal{P}$ be a finite set of Pfaffian functions defined on the open cube $U := (-1,1)^{m+n} \subset \mathbb{R}^{m+n}$, with $\#\mathcal{P} = s$, and such that all functions in $\mathcal{P}$ have degrees (α, β) and are derived from a common Pfaffian chain of order r. Then, there exists a finite set $A \subset \pi(U)$ with

$$\#A \leq s^{O(nm)} 2^{O(n(m^2+nr^2))} (nm(\alpha+\beta))^{O(n(m+r))},$$

such that for every $\mathbf{y} \in \pi(U)$ there exists $\mathbf{z} \in A$ such that for every $\mathcal{P}$-semi-Pfaffian set $S \subset U$, the set $\pi_S^{-1}(\mathbf{y})$ is homotopy equivalent to $\pi_S^{-1}(\mathbf{z})$.

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Single exponential dependence on m

- Let $P \in \mathbb{R}[X_1, \ldots, X_m] \hookrightarrow \mathbb{R}[X_1, \ldots, X_m, Y]$ be the polynomial defined by

$$P := \sum_{i=1}^{m} \prod_{j=1}^{d} (X_i - j)^2.$$

- Then $\mathrm{Z}(P, \mathbb{R}^{m+1})$ consists of d^m lines all parallel to the Y-axis.
- Consider now the semi-algebraic set $S \subset \mathbb{R}^{m+1}$ defined by

$$(P = 0) \ \wedge \ (0 \leq Y \leq X_1 + dX_2 + d^2 X_3 + \cdots + d^{m-1} X_m).$$

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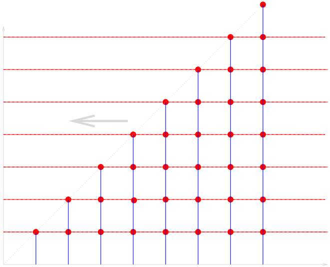
Tightness (cont).

- The fibres $\pi_S^{-1}(y)$, for $y \in \{0, 1, 2, \ldots, d^m - 1\} \subset \mathbb{R}$ are 0-dimensional and of different cardinality.



- There are no examples where the number of homotopy types of the fibres grows with n (with the parameters s, d, and m fixed) since this number can be bounded by a function of s, d and m independent of n.

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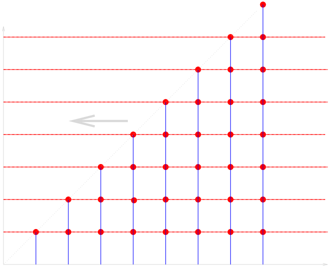

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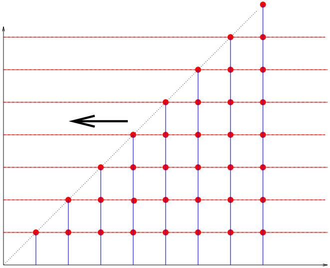

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Outline

A notation

- Let ϕ be a Boolean formula with atoms $\{a_i, b_i, c_i \mid 1 \leq i \leq s\}$. For an ordered list $\mathcal{P} = (P_1, \ldots, P_s)$ of polynomials $P_i \in \mathbb{R}[X_1, \ldots, X_m]$, we denote by $\phi_{\mathcal{P}}$ the formula obtained from ϕ by replacing for each i, $1 \leq i \leq s$, the atom a_i (respectively, b_i and c_i) by $P_i = 0$ (respectively, by $P_i > 0$ and by $P_i < 0$).
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Homotopy types of sets defined by fewnomials

Let $\mathcal{M}_{m,r}$ be the family of ordered lists $\mathcal{P} = (P_1, \ldots, P_s)$ with $P_i \in \mathbb{R}[X_1, \ldots, X_m]$, with the total number of monomials in all polynomials in $\mathcal{P}$ not exceeding r.

Theorem

The number of different homotopy types of ordered lists in $\mathcal{M}_{m,r}$ does not exceed $2^{O(mr)^4}$. In particular, the number of different homotopy types of semi-algebraic sets defined by a fixed formula $\phi_{\mathcal{P}}$, where $\mathcal{P}$ varies over $\mathcal{M}_{m,r}$, does not exceed

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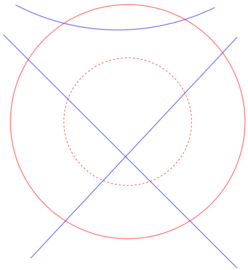
Triviality at infinity

Let $V \subset \mathbb{R}^m$ be a $\mathcal{P}$-semi-algebraic set, where $\mathcal{P} \subset \mathbb{Z}[X_1, \ldots, X_m]$. Let for each $P \in \mathcal{P}$, $\deg(P) < d$, and the maximum of the absolute values of coefficients in P be less than some constant M, $0 < M \in \mathbb{Z}$.



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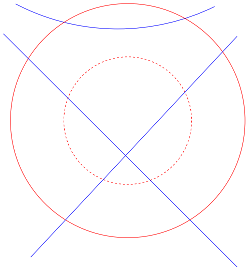

New Theorem

It was known before that there exists a constant $c > 0$, such that for any $R > M^{d^{cm}}$, and for any connected component W of V the intersection $W \cap B_m(0,R)) \neq \emptyset$, and $W \subset B_m(0,R)$ if W is bounded.

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There exists a constant $c > 0$, such that for any $R_1 > R_2 > M^{d^{cm}}$ we have,

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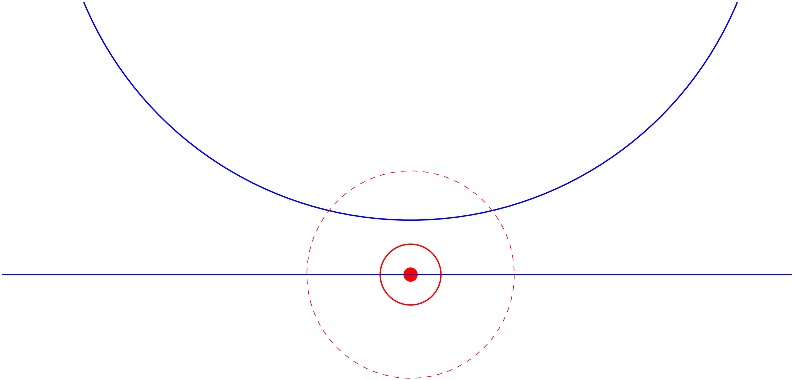
Local Conic Structure

Semi-algebraic sets are locally contractible.



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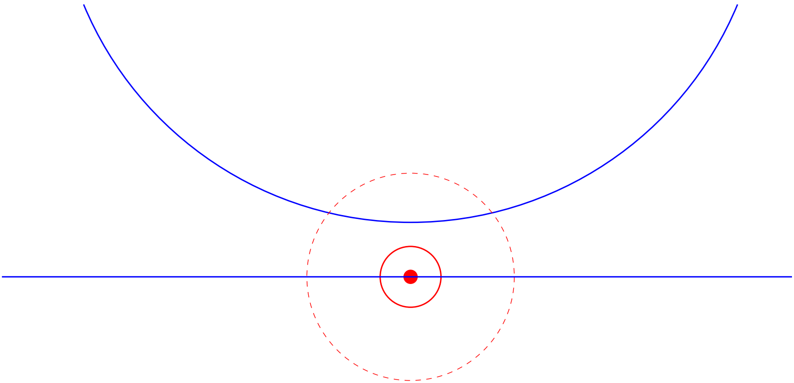

Quantitative Local Contractibility

Theorem

Let $V \subset \mathbb{R}^m$ *be a* $\mathcal{P}$*- semi-algebraic set, with* $\mathcal{P} \subset \mathbb{Z}[X_1, \ldots, X_m]$ *and* $0 \in V$*. Let* $\deg(P) < d$ *for each* $P \in \mathcal{P}$*, and the maximum of absolute values of coefficients of* $P \in \mathcal{P}$ *be less than* M*,* $0 < M \in \mathbb{Z}$*. Then, there exists a constant* $c > 0$ *such that for every* $0 < r < M^{-d^{cm}}$ *the set* $V \cap B_m(0, r)$ *is contractible.*

Outline

Quick Primer on Infinitesimals

- In the proof it will be convenient to use *infinitesimals* instead of sufficiently small elements of the ground field $\mathbb{R}$. We do this by considering non-archimedean extensions of $\mathbb{R}$.
- More precisely, denote by $\mathbb{R}\langle\varepsilon\rangle$ the real closed field of algebraic Puiseux series in ε with coefficients in $\mathbb{R}$.
- Given a semi-algebraic set $S \subset \mathbb{R}^k$, the *extension* of S to $\mathbb{R}'$, denoted $\mathrm{Ext}(S, \mathbb{R}')$, is the semi-algebraic subset of $\mathbb{R}'^k$ defined by the same quantifier free formula that defines S.

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First Ingredient:Thom's Isotopy Lemma

Lemma

Let $S \subset \mathbb{R}^{m+n}$ be a compact, non-singular hypersurface (defined by $Q = 0$) and $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^n$ the projection map on the last n-cordinates. Let $C \subset \mathbb{R}^n$ be a connected subset of $\mathbb{R}^n$ not containing any critical value of π_S. Then, the homeomorphism type of $S_{\mathbf{y}}$ stays the same as $\mathbf{y}$ varies over C.

(Note that, a critical point of π_S is a solution of the system $Q = \frac{\partial Q}{\partial X_1} = \cdots = \frac{\partial Q}{\partial X_m} = 0.$ and a critical value is the image under π of a critical point.)

Second Ingredient: Deformation

- Let ε be an infinitesimal and let,

$$Q_1 = Q^2 - \varepsilon.$$

- Let $T \subset \mathbb{R}\langle\varepsilon\rangle^{m+n}$ denote the set defined by $Q_1 \leq 0$.
- Then, T is bounded by the non-singular hypersurface $\mathrm{Z}(Q_1, \mathbb{R}\langle\varepsilon\rangle^{m+n})$.
- For each fixed $\mathbf{y} \in \mathbb{R}^n$ (Notice: co-ordinates in $\mathbb{R}$),
  1. $\mathrm{Z}(Q_1, \mathbb{R}\langle\varepsilon\rangle^{m+n})_{\mathbf{y}}$ is a non-singular hypersurface in $\mathbb{R}\langle\varepsilon\rangle^m$.
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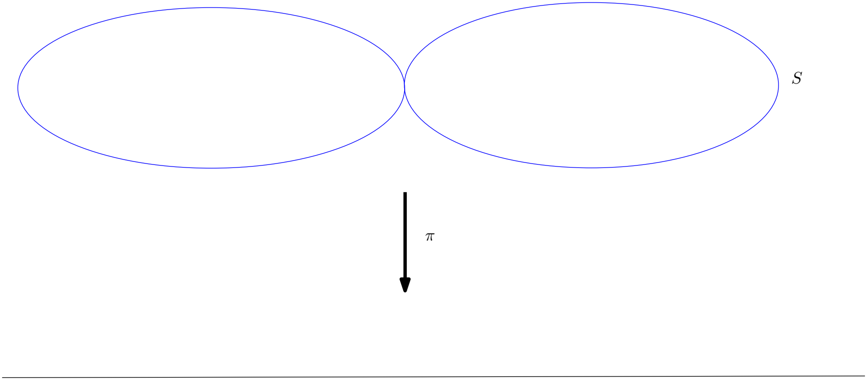

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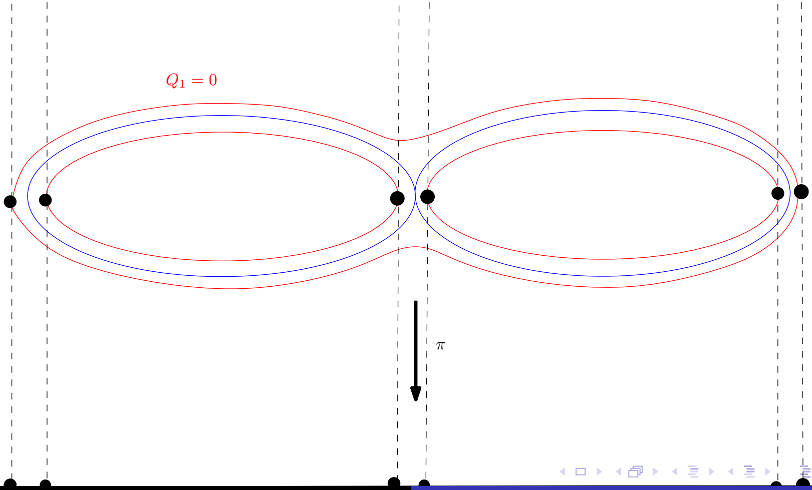

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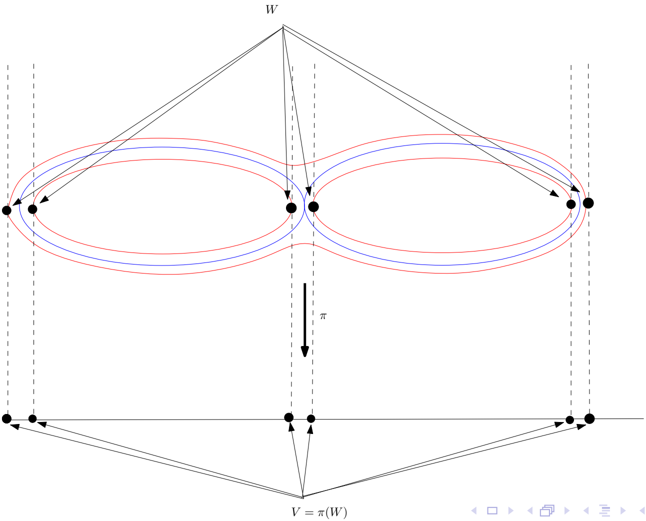

Third Ingredient: Homotopy types of $S_{\mathbf{y}}$ and the number of connected components of regular values

- Let $V \subset \mathbb{R}\langle\varepsilon\rangle^n$ be the set of critical values of π restricted to $\mathrm{Z}(Q_1, \mathbb{R}\langle\varepsilon\rangle^{m+n})$.
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Last Ingredient: Bounding the Betti numbers of projections

- $b_0(\mathbb{R}\langle\varepsilon\rangle^n \setminus V) = b_{n-1}(V),$ by Alexander duality.
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Handling more general situations

- First Ingredient: Thom's Lemma for stratified maps.
- Second Ingredient: Much more complicated scheme of deformation, maintaining the homotopy type of arbitrary semi-algebraic sets defined in terms of the given $\mathcal{P}$. Squares the number of polynomials.
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