

# Smaller decoding exponents: ball-collision decoding

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**Abstract.** Very few public-key cryptosystems are known that can encrypt and decrypt in time  $b^{2+o(1)}$  with conjectured security level  $2^b$  against conventional computers and quantum computers. The oldest of these systems is the classic McEliece code-based cryptosystem.

The best attacks known against this system are generic decoding attacks that treat McEliece's hidden binary Goppa codes as random linear codes. A standard conjecture is that the best possible  $w$ -error-decoding attacks against random linear codes of dimension  $k$  and length  $n$  take time  $2^{\alpha(R,W)+o(1)n}$  if  $k/n \rightarrow R$  and  $w/n \rightarrow W$  as  $n \rightarrow \infty$ .

Before this paper, the best upper bound known on the exponent  $\alpha(R, W)$  was the exponent of an attack introduced by Stern in 1989. This paper introduces “ball-collision decoding” and shows that it has a smaller exponent for each  $(R, W)$ : the speedup from Stern's algorithm to ball-collision decoding is exponential in  $n$ .

**Keywords:** McEliece cryptosystem, Niederreiter cryptosystem, post-quantum cryptography, attacks, information-set decoding, collision decoding

## 1 Introduction

In 1978, McEliece introduced a code-based public-key cryptosystem that has maintained remarkable strength against every proposed attack. The top threats against McEliece's system have always been generic decoding algorithms that decode random linear codes. The standard conjecture is that the best possible generic decoding algorithm takes exponential time for any constant asymptotic code rate  $R$  and constant asymptotic error fraction  $W$ : i.e., time  $2^{\alpha(R,W)+o(1)n}$  for some positive real number  $\alpha(R, W)$  if  $k/n \rightarrow R$  and  $w/n \rightarrow W$  as  $n \rightarrow \infty$ . Here  $n$  is the code length,  $k$  is the code dimension, and  $w$  is the number of errors.

Two decades ago a flurry of fundamental algorithmic improvements produced a new upper bound on the optimal decoding exponent  $\alpha(R, W)$ . The upper bound is the exponent of a 1989 algorithm by Stern [67]. This upper bound arises from an asymptotic binomial-coefficient optimization and does not have a simple formula, but it can be straightforwardly computed to high precision for any particular  $(R, W)$ . For example, for  $W = 0.04$  and  $R = 1 + W \log_2 W + (1 - W) \log_2(1 - W) = 0.7577\dots$ , Stern's algorithm shows that  $\alpha(R, W) \leq 0.0809\dots$

There have also been many polynomial-factor speedups in generic decoding algorithms; there are dozens of papers on this topic, both inside and outside cryptography. Here is an illustration of the cumulative impact of many of the speedups. McEliece's original parameter

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suggestions (“ $n = 1024, k = 524, t = 50$ ”) take about  $524^3 \binom{1024}{50} / \binom{500}{50} \approx 2^{81}$  operations to break by the simple information-set-decoding attack explained in McEliece’s original paper [54, Section 3]. (McEliece estimated the attack cost as  $524^3(1 - 50/1024)^{-524} \approx 2^{65}$ ; this underestimate was corrected by Adams and Meijer in [2, Section 3].) The attack presented by Bernstein, Lange, and Peters in [9], thirty years after McEliece’s paper, builds on several improvements and takes only about  $2^{60.5}$  operations for the same parameters. That attack was carried out successfully, decrypting a challenge ciphertext. More recent improvements include [32] and [59]; see Section 4 for a more comprehensive discussion of the literature.

However, polynomial factors are asymptotically  $2^{o(n)}$ , and thus have no relevance to the exponent  $\alpha(R, W)$  in  $2^{(\alpha(R, W) + o(1))n}$ . The best known upper bound on  $\alpha(R, W)$  has been unchanged since 1989.

**Contents of this paper.** This paper presents smaller upper bounds on the decoding exponent  $\alpha(R, W)$ . Specifically, this paper introduces a generic decoding algorithm and demonstrates that this algorithm is, for every  $(R, W)$ , faster than Stern’s algorithm by a factor exponential in  $n$ . We call this algorithm “ball-collision decoding” because of a geometric interpretation explained in Section 4. The change in the exponent is not very large — for example, this paper uses ball-collision decoding to demonstrate that  $\alpha(R, W) \leq 0.0807\dots$  for the  $(R, W)$  given above — but it is the first exponential improvement in decoding complexity in more than twenty years.

Constant rates  $R$  and constant error fractions  $W$  are traditional in the study of coding-theory asymptotics, but we comment that they are not exactly right in the study of code-based cryptography. Typical code-based cryptosystems limit the error fraction  $W$  to  $(1 - R)/\log_2 n$ , so  $W$  decreases slowly to 0 as  $n \rightarrow \infty$ . (In theory one could replace the Goppa codes in the McEliece system with explicit “asymptotically good” families of codes for which  $k/n$  and  $w/n$  converge to nonzero constants  $R$  and  $W$ ; however, the security of these families has never been evaluated.) It seems reasonable to conjecture that the best possible algorithms take time

$$(1 - R)^{-(1-R)n/\log_2 n + (\beta(R) + o(1))n/(\log_2 n)^2}$$

if  $k/n \rightarrow R$  and  $w/(n/\log_2 n) \rightarrow 1 - R$  as  $n \rightarrow \infty$ ; see generally [10]. Ball-collision decoding produces better upper bounds on  $\beta(R)$  for the same reason that it produces better upper bounds on  $\alpha(R, W)$ . We consider the two-variable function  $\alpha(R, W)$  so that our results can be applied to a wider range of choices of  $W$  and compared to a wider range of previous papers, including papers that make the traditional coding-theory assumption  $R = 1 + W \log_2 W + (1 - W) \log_2(1 - W)$ .

This paper also evaluates the exact cost of ball-collision decoding, using the same bit-operation-counting rules as in the previous literature, and uses this evaluation to illustrate the impact of ball-collision decoding upon cryptographic applications. For example, the parameters  $(6624, 5129, 117)$  were proposed in [9, Section 7] at a 256-bit security level against a state-of-the-art refinement of Stern’s algorithm; this paper shows that ball-collision decoding costs 2.6 times fewer bit operations. At a theoretical 1000-bit security level the improvement grows to 15.5. These concrete figures are consistent with the asymptotic analysis.

Of course, actually breaking these parameters remains very far out of reach, and these results should not be interpreted as damaging the viability of the McEliece cryptosystem. However, these results *do* raise new questions regarding the proper choice of parameters for the McEliece cryptosystem. Appendix B discusses the problem of parameter selection for code-based cryptography.

**Attack model.** “Attacks” above refer only to passive single-target inversion attacks. The original McEliece cryptosystem, like the original RSA cryptosystem, is really just a trapdoor one-way function; when used naively as a public-key cryptosystem it is trivially broken by chosen-ciphertext attacks such as Berson’s attack [11] and the Verheul–Doumen–van Tilborg attack [71].

Protecting the McEliece system against these attacks, to meet the standard notion of IND-CCA2 security for a public-key cryptosystem, requires appropriate padding and randomization, similar to RSA-OAEP. As shown by Kobara and Imai in [48], adding this protection does not significantly increase the cost of the McEliece cryptosystem.

## 2 Review of the McEliece cryptosystem

This section reviews how the McEliece cryptosystem works. Appendix A discusses the bigger picture of McEliece, RSA, ECC, etc.

The public key in the McEliece cryptosystem consists of a random-looking rank- $k$  matrix  $G \in \mathbf{F}_2^{k \times n}$ . The sender encrypts a message  $m$  in  $\mathbf{F}_2^k$  by first multiplying it with the matrix  $G$ , producing  $mG$ ; choosing uniformly at random a word  $e$  in  $\mathbf{F}_2^n$  of Hamming weight  $w$ ; and adding  $e$  to  $mG$ , producing a ciphertext  $mG + e$ . The cryptosystem parameters are  $n, k, w$ .

The legitimate receiver decrypts  $mG + e$  using a secret key which consists of a secret decoding algorithm producing the *error vector*  $e$  given  $mG + e$ . The details are not relevant to the attacks described in this paper and can be found in, e.g., [58].

An attacker is faced with the problem of determining  $e$  given  $G$  and  $mG + e$ . Note that finding  $e$  is equivalent to finding the message  $m$ : subtracting  $e$  from  $mG + e$  produces  $mG$ , and then simple linear transformations produce  $m$ .

The set  $\mathbf{F}_2^k G = \{mG : m \in \mathbf{F}_2^k\}$  is called a *linear code* of length  $n$  and *dimension*  $k$ , specifically the linear code *generated by*  $G$ . The matrix  $G$  is called a *generator matrix* for this code. The elements of  $\mathbf{F}_2^k G$  are called *codewords*. If the linear code  $\mathbf{F}_2^k G$  equals  $\{c \in \mathbf{F}_2^n : Hc = 0\}$  then the matrix  $H$  is called a *parity-check matrix* for the code.

Without loss of generality one can assume that the matrix  $G$  in a CCA2-secure version of the McEliece cryptosystem is given in *systematic form*  $G = (I_k | -A^T)$  where  $I_k$  is a  $k \times k$  identity matrix and  $A$  an  $(n - k) \times k$  matrix. Then the matrix  $H = (A | I_{n-k})$  is a parity-check matrix for the code generated by  $G$ .

An *information set*  $Z$  for  $H$  is a set of  $k$  integers in  $\{1, 2, \dots, n\}$  for which the  $n - k$  columns of  $H$  that are not indexed by  $Z$  are linearly independent. Applying Gaussian elimination to those  $n - k$  columns shows that codewords are determined by their  $Z$ -indexed components. For example,  $\{1, 2, \dots, k\}$  is an information set for  $H = (A | I_{n-k})$ ; codewords are determined by their first  $k$  components.

Let  $c = mG$  for  $m \in \mathbf{F}_2^k$  and  $e \in \mathbf{F}_2^n$  with  $\text{wt}(e) = w$ . Then by linearity one has  $H(c + e) = Hc + He = He$  since  $Hc = 0$ . The result  $s = He$  is called the *syndrome* of  $e$ . It is the sum of the  $w$  columns of  $H$  that are indexed by the positions of 1’s in  $e$ . The attacker’s task is equivalent to finding  $e$  given  $H$  and  $s = He$ .

## 3 The ball-collision-decoding algorithm

This section introduces ball-collision decoding. It first presents a simplified statement of the algorithm and then discusses various optimizations. Section 4 explains how this algorithm relates to previous algorithms.

The algorithm is given a parity-check matrix  $H \in \mathbf{F}_2^{(n-k) \times n}$ , a syndrome  $s \in \mathbf{F}_2^{n-k}$ , and a weight  $w \in \{0, 1, 2, \dots\}$ . The goal of the algorithm is to find a corresponding error vector  $e$ : i.e., a vector  $e \in \mathbf{F}_2^n$  of weight  $w$  such that  $s = He$ .

Ball-collision decoding has its roots in information-set decoding, which was used against the McEliece system in, e.g., [67], [17], [18], and [9]. The previous algorithms select a random information set in the parity-check matrix and then search for vectors having a particular pattern of non-zero entries. Ball-collision decoding is similar but searches for a more complicated, and more likely, pattern. See Section 4 for further discussion of the previous work.

The reader is encouraged to consider, while reading the algorithm, the case that the algorithm is given a matrix  $H$  already in systematic form and that it chooses  $Z = \{1, 2, \dots, k\}$  as information set. The matrix  $U$  in Step 4 is then the identity matrix  $I_{n-k}$ . The algorithm divides  $H$  into blocks, and divides the syndrome  $s$  into corresponding blocks, as specified by algorithm parameters  $\ell_1, \ell_2$ :

$$H = \begin{pmatrix} A_1 & I_1 & 0 \\ A_2 & 0 & I_2 \end{pmatrix}, \quad s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

where  $s_1 \in \mathbf{F}_2^{\ell_1 + \ell_2}$ ,  $s_2 \in \mathbf{F}_2^{n-k-\ell_1-\ell_2}$ ,  $A_1 \in \mathbf{F}_2^{(\ell_1 + \ell_2) \times k}$ ,  $A_2 \in \mathbf{F}_2^{(n-k-\ell_1-\ell_2) \times k}$ , and each  $I_i$  is an identity matrix.

#### One iteration of ball-collision decoding:

CONSTANTS:  $n, k, w \in \mathbf{Z}$  with  $0 \leq w \leq n$  and  $0 \leq k \leq n$ .

PARAMETERS:  $p_1, p_2, q_1, q_2, k_1, k_2, \ell_1, \ell_2 \in \mathbf{Z}$  with  $0 \leq k_1$ ,  $0 \leq k_2$ ,  $k = k_1 + k_2$ ,  $0 \leq p_1 \leq k_1$ ,

$0 \leq p_2 \leq k_2$ ,  $0 \leq q_1 \leq \ell_1$ ,  $0 \leq q_2 \leq \ell_2$ , and  $0 \leq w - p_1 - p_2 - q_1 - q_2 \leq n - k - \ell_1 - \ell_2$ .

INPUT:  $H \in \mathbf{F}_2^{(n-k) \times n}$  and  $s \in \mathbf{F}_2^{n-k}$ .

OUTPUT: Zero or more vectors  $e \in \mathbf{F}_2^n$  with  $He = s$  and  $\text{wt}(e) = w$ .

1. Choose a uniform random information set  $Z$ . Subsequent steps of the algorithm write “ $\mathbf{F}_2^Z$ ” to refer to the subspace of  $\mathbf{F}_2^n$  supported on  $Z$ .
2. Choose a uniform random partition of  $Z$  into parts of sizes  $k_1$  and  $k_2$ . Subsequent steps of the algorithm write “ $\mathbf{F}_2^{k_1}$ ” and “ $\mathbf{F}_2^{k_2}$ ” to refer to the corresponding subspaces of  $\mathbf{F}_2^Z$ .
3. Choose a uniform random partition of  $\{1, 2, \dots, n\} \setminus Z$  into parts of sizes  $\ell_1$ ,  $\ell_2$ , and  $n - k - \ell_1 - \ell_2$ . Subsequent steps of the algorithm write “ $\mathbf{F}_2^{\ell_1}$ ” and “ $\mathbf{F}_2^{\ell_2}$ ” and “ $\mathbf{F}_2^{n-k-\ell_1-\ell_2}$ ” to refer to the corresponding subspaces of  $\mathbf{F}_2^{\{1, 2, \dots, n\} \setminus Z}$ .
4. Find an invertible  $U \in \mathbf{F}_2^{(n-k) \times (n-k)}$  such that the columns of  $UH$  indexed by  $\{1, 2, \dots, n\} \setminus Z$  are an  $(n - k) \times (n - k)$  identity matrix. Write the columns of  $UH$  indexed by  $Z$  as  $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$  with  $A_1 \in \mathbf{F}_2^{(\ell_1 + \ell_2) \times k}$ ,  $A_2 \in \mathbf{F}_2^{(n-k-\ell_1-\ell_2) \times k}$ .
5. Write  $Us$  as  $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$  with  $s_1 \in \mathbf{F}_2^{\ell_1 + \ell_2}$ ,  $s_2 \in \mathbf{F}_2^{n-k-\ell_1-\ell_2}$ .
6. Compute the set  $S$  consisting of all triples  $(A_1 x_0 + x_1, x_0, x_1)$  where  $x_0 \in \mathbf{F}_2^{k_1}$ ,  $\text{wt}(x_0) = p_1$ ,  $x_1 \in \mathbf{F}_2^{\ell_1}$ ,  $\text{wt}(x_1) = q_1$ .
7. Compute the set  $T$  consisting of all triples  $(A_1 y_0 + y_1 + s_1, y_0, y_1)$  where  $y_0 \in \mathbf{F}_2^{k_2}$ ,  $\text{wt}(y_0) = p_2$ ,  $y_1 \in \mathbf{F}_2^{\ell_2}$ ,  $\text{wt}(y_1) = q_2$ .
8. For each  $(v, x_0, x_1) \in S$ :
  - For each  $y_0, y_1$  such that  $(v, y_0, y_1) \in T$ :
    - If  $\text{wt}(A_2(x_0 + y_0) + s_2) = w - p_1 - p_2 - q_1 - q_2$ :
    - Output  $x_0 + y_0 + x_1 + y_1 + A_2(x_0 + y_0) + s_2$ .

Note that Step 8 is a standard “join” operation between  $S$  and  $T$ ; it can be implemented efficiently by sorting, by hashing, or by simple table indexing. Bernstein, Lange, and Peters in [9, Section 6] describe an efficient implementation of essentially the same operation using only about  $2^{\ell_1+\ell_2+1}$  bits of memory. See Sections 5 and 6 for further discussion of arithmetic costs and memory-access costs.

**Theorem 3.1 (Correctness of ball-collision decoding)** *The set of output vectors  $e$  of the ball-collision decoding algorithm is the set of vectors  $e$  that satisfy  $He = s$  and have weights  $p_1, p_2, q_1, q_2, w - p_1 - p_2 - q_1 - q_2$  in  $\mathbf{F}_2^{k_1}, \mathbf{F}_2^{k_2}, \mathbf{F}_2^{\ell_1}, \mathbf{F}_2^{\ell_2}$ , and  $\mathbf{F}_2^{n-k-\ell_1-\ell_2}$  respectively.*

*Proof.* Each element  $(v, x_0, x_1) \in S$  satisfies  $x_0 \in \mathbf{F}_2^{k_1}$  with  $\text{wt}(x_0) = p_1$ ;  $v = A_1x_0 + x_1$  and  $x_1 \in \mathbf{F}_2^{\ell_1}$  with  $\text{wt}(x_1) = q_1$ . Similarly each element  $(v, y_0, y_1) \in T$  satisfies  $y_0 \in \mathbf{F}_2^{k_2}$  with  $\text{wt}(y_0) = p_2$ ;  $v = A_1y_0 + y_1 + s_1$ ;  $y_1 \in \mathbf{F}_2^{\ell_2}$  with  $\text{wt}(y_1) = q_2$ . Now, with  $Z$ -indexed columns visualized as coming before the remaining columns, we have

$$UHe = UH \begin{pmatrix} x_0 + y_0 \\ x_1 + y_1 \\ A_2(x_0 + y_0) + s_2 \end{pmatrix} = \begin{pmatrix} A_1(x_0 + y_0) + x_1 + y_1 \\ A_2(x_0 + y_0) + A_2(x_0 + y_0) + s_2 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = Us$$

so  $He = s$ . Furthermore,  $x_0 + y_0 \in \mathbf{F}_2^{k_1+k_2}$  has weights  $p_1, p_2$  in  $\mathbf{F}_2^{k_1}, \mathbf{F}_2^{k_2}$ ;  $x_1 + y_1 \in \mathbf{F}_2^{\ell_1+\ell_2}$  has weights  $q_1, q_2$  in  $\mathbf{F}_2^{\ell_1}, \mathbf{F}_2^{\ell_2}$ ; and  $\text{wt}(A_2(x_0 + y_0) + s_2) = w - p_1 - p_2 - q_1 - q_2$ .

Conversely, the iteration finds every vector  $e$  having this weight distribution and satisfying  $He = s$ . Indeed, write  $e$  as  $x_0 + y_0 + x_1 + y_1 + e_2$  with  $x_0 \in \mathbf{F}_2^{k_1}, y_0 \in \mathbf{F}_2^{k_2}, x_1 \in \mathbf{F}_2^{\ell_1}, y_1 \in \mathbf{F}_2^{\ell_2}$ , and  $e_2 \in \mathbf{F}_2^{n-k-\ell_1-\ell_2}$ . By hypothesis the weights of  $x_0, y_0, x_1, y_1, e_2$  are  $p_1, p_2, q_1, q_2, w - p_1 - p_2 - q_1 - q_2$ , respectively. Now define  $v = A_1x_0 + x_1$ . The equation  $UHe = Us$  implies  $v = A_1y_0 + y_1 + s_1$ ; and  $e_2 = A_2(x_0 + y_0) + s_2$ . Hence  $(v, x_0, x_1) \in S$  and  $(v, y_0, y_1) \in T$ . Finally  $\text{wt}(A_2(x_0 + y_0) + s_2) = \text{wt}(e_2) = w - p_1 - p_2 - q_1 - q_2$  so the algorithm prints  $e$  as claimed.  $\square$

**Finding an information set.** The simplest way to choose a uniform random information set is to repeatedly choose a uniform random size- $k$  subset  $Z \subseteq \{1, 2, \dots, n\}$  until the  $n - k$  columns of  $H$  indexed by  $\{1, 2, \dots, n\} \setminus Z$  are linearly independent. Standard practice (see, e.g., Stern [67]) is to eliminate the fruitless Gaussian-elimination steps here, at the expense of negligible bias, by assembling the information set one column at a time, ensuring that each newly added column is linearly independent of the previously selected columns. After this optimization there is only one Gaussian-elimination step per iteration.

**Reusing intermediate sums.** Computing the vector  $A_1x_0$  for a weight- $p_1$  word  $x_0$  in  $\mathbf{F}_2^{k_1}$  can be done by adding the specified  $p_1$  columns of  $A_1$  in  $p_1 - 1$  additions in  $\mathbf{F}_2^{\ell_1+\ell_2}$ .

Computing  $A_1x_0$  for *all* the  $\binom{k_1}{p_1}$  vectors  $x_0$  can be done more efficiently than repeating this process for each of them. Start by computing all  $\binom{k_1}{2}$  sums of 2 columns of  $A_1$ ; each sum costs one addition in  $\mathbf{F}_2^{\ell_1+\ell_2}$ . Then compute all  $\binom{k_1}{3}$  sums of 3 columns of  $A_1$  by adding one extra column to the previous results. Proceed in the same way until all  $\binom{k_1}{p_1}$  sums of  $p_1$  columns of  $A_1$  are computed. This produces all required sums in only marginally more than one  $\mathbf{F}_2^{\ell_1+\ell_2}$  addition per sum; see Section 5 for a precise operation count.

**Early abort.** The vector  $A_2(x_0 + y_0) + s_2$  is computed as a sum of  $p_1 + p_2 + 1$  vectors of length  $n - k - \ell_1 - \ell_2$ . Instead of computing the sum on all  $n - k - \ell_1 - \ell_2$  positions one

computes the sum row by row and simultaneously checks the weight. If the weight exceeds  $w - p_1 - p_2 - q_1 - q_2$  one can discard this particular pair  $(x_0, y_0)$ .

We comment that one can further reduce the cost of this step by precomputing sums of smaller sets of columns, but we do not use this idea in our analysis, because it is not critical for the algorithm’s performance.

## 4 Relationship to previous algorithms

This section discusses the relationship of ball-collision decoding to previous information-set-decoding algorithms.

**Collision decoding vs. ball-collision decoding.** We use the name “collision decoding” for the special case  $q_1 = q_2 = 0$  of ball-collision decoding. The idea of collision decoding is more than twenty years old: Stern’s algorithm in [67] is, aside from trivial details, exactly the special case  $q_1 = q_2 = 0$ ,  $p_1 = p_2$ ,  $k_1 \approx k_2$ . Dumer in [27] independently introduced the core idea, although in a more limited form, and in [28] achieved an algorithm similar to Stern’s.

All state-of-the-art decoding attacks since [67] have been increasingly optimized forms of collision decoding. Other approaches to decoding, such as “gradient decoding” ([4]), “supercode decoding” ([5]), and “statistical decoding” (see [3] and [57]), have never been competitive with Stern’s algorithm. This does not mean that those approaches should be ignored; our generalization from collision decoding to ball-collision decoding is inspired by one of the steps in supercode decoding.

Collision decoding searches for collisions in  $\mathbf{F}_2^{\ell_1 + \ell_2}$  between points  $A_1x_0$  and points  $A_1y_0 + s_1$ . Ball-collision decoding expands each point  $A_1x_0$  into a small ball (in the Hamming metric), namely  $\{A_1x_0 + x_1 : x_1 \in \mathbf{F}_2^{\ell_1}, \text{wt}(x_1) = q_1\}$ ; similarly expands each point  $A_1y_0$  into a small ball; and searches for collisions between these balls.

From the perspective of ball-collision decoding, the fundamental disadvantage of collision decoding is that errors are required to avoid an asymptotically quite large stretch of  $\ell_1 + \ell_2$  positions. Ball-collision decoding makes a much more reasonable hypothesis, namely that there are asymptotically increasingly many errors in those positions. It requires extra work to enumerate the points in each ball, but the extra work is only about the square root of the improvement in success probability. The cost ratio is exponential when all parameters are optimized properly; see Section 7.

Collision decoding also has a secondary disadvantage compared to ball-collision decoding: its inner loop is slower, since computing  $A_1x_0$  for a new  $x_0$  is considerably more expensive than adding  $x_1$  for a new  $x_1$ . The cost ratio here is only polynomial, and is not relevant to the exponents (see Section 7), but is accounted for in the bit-operation count (see Section 5). This disadvantage of collision decoding is also visible in the number of memory accesses to  $A_1$  (see Section 6); however, standard practice in the literature on this topic is to count the number of bit operations involved in arithmetic and to ignore the cost of memory access.

**Additional credits.** The simplest form of information-set decoding, introduced by Prange in [61], did not allow errors in the information set. For asymptotic analyses see [54], [1], and [2].

The idea of allowing errors was published by Lee and Brickell in [50], by Leon in [52], and by Krouk in [49], but without Stern’s collision idea; in the terminology of ball-collision decoding, with  $p_2 = 0$ ,  $q_1 = q_2 = 0$ , and  $\ell_2 = 0$ . For each pattern of  $p_1$  errors in  $k$  columns, Lee and Brickell checked the weight of the remaining  $n - k$  columns; Leon and Krouk required  $\ell_1$

columns to have weight 0, and usually checked only those columns. For asymptotic analyses see [49], [24], and [25].

Overbeck and Sendrier [58] give a visual comparison of the algorithms by comparing to which interval they restrict how many errors. Figure 4.1 extends their picture to include ball-collision decoding. It shows that the new algorithm allows errors in an interval that had to be error-free in Leon’s and Stern’s algorithms.

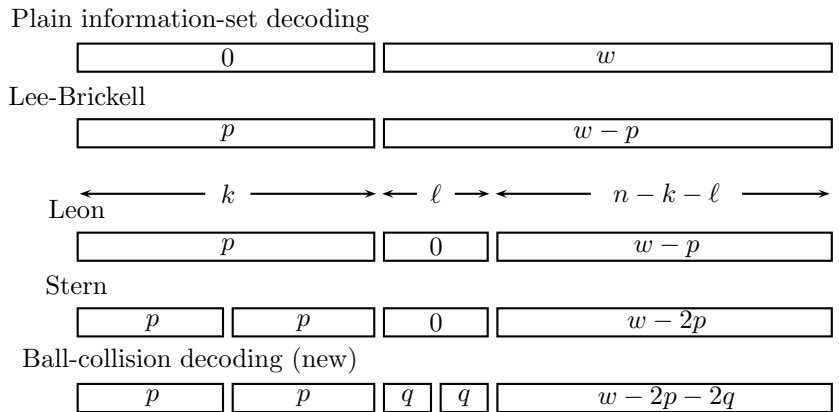


Fig. 4.1. Error positions hypothesized by various decoding algorithms.

The idea of allowing errors everywhere can be extracted, with considerable effort, from the description of supercode decoding in [5]. After a detailed analysis we have concluded that the algorithm in [5] is much slower than collision decoding. The same algorithm is claimed in [5] to have smaller exponents than collision decoding (with astonishing gaps, often 15% or more), but this claim is based on a chain of exponentially large inaccuracies in the algorithm analysis in [5]. The starting point of the chain is [5, “Corollary 12”], which claims size  $\binom{k}{e_1} \binom{y}{e_2} / 2^{by}$  for lists that actually have size  $\binom{k}{e_1} \binom{y}{e_2}^b / 2^{by}$ .

The idea of allowing errors everywhere can also be found in the much more recent paper [32], along with a polynomial-factor “birthday” speedup obtained by dropping Stern’s left-right separation. The algorithm analysis by Finiasz and Sendrier in [32] concludes that the overall “gain compared with Stern’s algorithm” is a constant times  $\sqrt[4]{\pi p/2}$ , which is bounded by a polynomial in  $n$ . Our own assessment is that if parameters had been chosen more carefully then the algorithm of [32] would have led to an exponential improvement over collision decoding, contrary to the conclusions in [32]. This algorithm would still have retained the secondary disadvantage described above, and therefore would not have been competitive with ball-collision decoding.

A more detailed analysis of the “birthday” speedup in collision decoding appeared in [59] along with an optimized generalization to  $\mathbf{F}_q$ . These modifications can be adapted to ball-collision decoding but would complicate the algorithm statement and analysis without changing the exponent of binary decoding; we have skipped these modifications for simplicity.

One way to speed up Gaussian elimination is to change only one information-set element in each iteration. This idea was introduced by Omura, according to [22, Section 3.2.4]. It was applied to increasingly optimized forms of information-set decoding by van Tilburg in [68] and [69], by Chabanne and Courteau in [19], by Chabaud in [20], by Canteaut and Chabanne

in [16], by Canteaut and Chabaud in [17], and by Canteaut and Sendrier in [18]. Bernstein, Lange, and Peters in [9] improved the balance between Gaussian-elimination cost and error-searching cost by changing  $c$  information-set elements in each iteration for an optimized value of  $c$ . The ideas of reusing sums and aborting weight calculations also appeared in [9], in the context of an improved collision-decoding algorithm.

## 5 Complexity analysis

This section analyzes the complexity of ball-collision decoding. In particular, this section analyzes the success probability of each iteration and the number of bit operations needed for each iteration.

**Success probability.** Assume that  $e$  is a uniform random vector of weight  $w$ . One iteration of ball-collision decoding finds  $e$  exactly if it has the right weight distribution, namely weight  $p_1$  in the first  $k_1$  positions specified by the information set, weight  $p_2$  in the remaining  $k_2$  positions specified by the information set, weight  $q_1$  on the first  $\ell_1$  positions outside the information set, and weight  $q_2$  on the next  $\ell_2$  positions outside the information set.

The probability that  $e$  has this weight distribution is, by a simple counting argument, exactly

$$b(p_1, p_2, q_1, q_2, \ell_1, \ell_2) = \binom{n}{w}^{-1} \binom{n - k - \ell_1 - \ell_2}{w - p_1 - p_2 - q_1 - q_2} \binom{k_1}{p_1} \binom{k_2}{p_2} \binom{\ell_1}{q_1} \binom{\ell_2}{q_2}.$$

The expected number of iterations of the outer loop is, for almost all  $H$ , very close to the reciprocal of the success probability of a single iteration. We explicitly disregard, without further comment, the extremely unusual codes for which the average number of iterations is significantly different from the reciprocal of the success probability of a single iteration. For further discussion of this issue and how unusual it is see, e.g., [25] and [10].

**Gaussian elimination.** There are several ways to speed up Gaussian elimination, as discussed in Section 4, and implementors are encouraged to use those optimizations. However, in this paper we will be satisfied with a quite naive form of Gaussian elimination, taking  $(1/2)(n - k)^2(n + k)$  bit operations; our interest is in large input sizes, and Gaussian elimination takes negligible time for those sizes.

**Building the set  $\mathcal{S}$ .** The total cost of computing  $A_1x_0$  for all  $x_0$  of Hamming weight  $p_1$ , using intermediate sums as explained in Section 3, is

$$(\ell_1 + \ell_2) \left( \binom{k_1}{2} + \binom{k_1}{3} + \cdots + \binom{k_1}{p_1} \right).$$

Using  $L(k, p) = \sum_{i=1}^p \binom{k}{i}$  as a shorthand, the costs can be written as  $(\ell_1 + \ell_2) (L(k_1, p_1) - k_1)$ . The  $\ell_1 + \ell_2$  factor is the number of bit operations to compute  $A_1x_0$  from  $A_1x'_0$  where  $x_0$  extends  $x'_0$  by a single bit.

Then for each  $x_0$  all  $\binom{\ell_1}{q_1}$  possible words  $x_1$  in  $\mathbf{F}_2^{\ell_1}$  of weight  $q_1$  are added to  $A_1x_0$ , producing  $A_1x_0 + x_1$ . For  $x_1$ , as for  $x_0$ , we loop over the possible sets of indices, and reuse sums obtained from subsets. This slightly increases the number of sums up to  $L(\ell_1, q_1)$ , but decreases the cost of each sum down to a single bit operation, computing  $A_1x_0 + x_1$  from  $A_1x_0 + x'_1$ . Overall this step takes  $\min\{1, q_1\} \binom{k_1}{p_1} L(\ell_1, q_1)$  bit operations; note that for  $q_1 = 0$  the cost of this step is indeed 0.



Each choice of  $(x_0, x_1)$  adds one element to  $S$ . Hence, the number of elements in  $S$  equals exactly the number of choices for  $x_0$  and  $x_1$ , i.e.  $\#S = \binom{k_1}{p_1} \binom{\ell_1}{q_1}$ .

**Building the set  $T$ .** The set  $T$  is built similarly to the set  $S$ . The only difference is that the expression  $A_1y_0 + y_1 + s_1$  involves adding  $s_1$  and thus the single columns (corresponding to weight-1 words  $y_0$ ) already cost  $(\ell_1 + \ell_2) \binom{k_2}{1}$  bit operations. In total this step takes  $(\ell_1 + \ell_2)L(k_2, p_2) + \min\{1, q_2\} \binom{k_2}{p_2} L(\ell_2, q_2)$ .

The set  $T$  contains exactly  $\#T = \binom{k_2}{p_2} \binom{\ell_2}{q_2}$  elements.

**Checking collisions.** The last step does one check for every  $(x_0, x_1, y_0, y_1)$  satisfying the equation  $A_1x_0 + x_1 = A_1y_0 + y_1 + s_1$ . There are  $\binom{k_1}{p_1} \binom{k_2}{p_2} \binom{\ell_1}{q_1} \binom{\ell_2}{q_2}$  choices of  $(x_0, x_1, y_0, y_1)$ .

If the vectors  $v$  appearing in  $S$  and  $T$  were uniformly distributed among the  $2^{\ell_1 + \ell_2}$  possible values then on average  $\#S \cdot \#T \cdot 2^{-\ell_1 - \ell_2}$  checks would be done. The expected number of checks is extremely close to this for almost all  $H$ ; as above we disregard the extremely unusual codes with different behavior.

Each check consists of computing  $\text{wt}(A_2(x_0 + y_0) + s_2)$  and testing whether it equals  $w - p_1 - p_2 - q_1 - q_2$ . When using the early-abort weight calculation, on average only  $2(w - p_1 - p_2 - q_1 - q_2 + 1)$  bits of the result are computed before the weight is found too high. Each bit of the result costs  $p_1 + p_2$  bit operations because  $x_0 + y_0$  has weight  $p_1 + p_2$ .

**Cost of one iteration.** To summarize, the total cost per iteration of the inner loop with parameters  $p_1, p_2, q_1, q_2, \ell_1, \ell_2$  amounts to

$$\begin{aligned} c(p_1, p_2, q_1, q_2, \ell_1, \ell_2) &= \frac{1}{2}(n - k)^2(n + k) + (\ell_1 + \ell_2)(L(k_1, p_1) + L(k_2, p_2) - k_1) \\ &\quad + \min\{1, q_1\} \binom{k_1}{p_1} L(\ell_1, q_1) + \min\{1, q_2\} \binom{k_2}{p_2} L(\ell_2, q_2) \\ &\quad + 2(w - p_1 - p_2 - q_1 - q_2 + 1)(p_1 + p_2) \binom{k_1}{p_1} \binom{k_2}{p_2} \binom{\ell_1}{q_1} \binom{\ell_2}{q_2} 2^{-\ell_1 - \ell_2}. \end{aligned}$$

## 6 Concrete parameter examples

This section considers concrete examples in order to show the speedup gained by ball-collision decoding in comparison to collision decoding. The first parameters were previously proposed to achieve 256-bit security against current attacks. We designed the second parameters according to similar rules to achieve a 1000-bit security level against current attacks. We do not mean to suggest that 1000-bit security is of any real-world relevance; we consider it to illustrate the asymptotic superiority of ball-collision decoding.

Finiasz and Sendrier in [32] presented “lower bounds on the effective work factor of existing real algorithms, but also on the future improvements that could be implemented”; and said that beating these bounds would require the introduction of “new techniques, never applied to code-based cryptosystems”. For each set of parameters we evaluate the Finiasz–Sendrier lower bound and the costs of three algorithms:

- (1) collision decoding ( $q_1 = q_2 = 0$ ),
- (2) collision decoding using the birthday trick from [32] as analyzed in [59], and
- (3) ball-collision decoding.

Ball-collision decoding beats the Finiasz–Sendrier lower bound in both of these examples. The main reason for this is that ball-collision decoding dodges the secondary disadvantage described in Section 4; the lower bound assumes that each new vector requires  $\ell_1 + \ell_2$  bit operations to update  $A_1x_0$ , but in ball-collision decoding each new vector requires just 1 bit operation to update  $x_1$ .

We emphasize that all of these costs and bounds use the same model of computation, counting the number of bit operations for arithmetic and disregarding costs of memory access, copies, etc. A table-indexing join operation can easily be carried out for free in this model. We would prefer a more carefully defined model of computation that includes realistic memory-access costs, such as the Brent–Kung circuit model [13], but the bit-operation model is simpler and is standard in papers on this topic. Note that any reasonable accounting for memory-access costs would need at least one memory access for each new  $x_1$  in ball-collision decoding (for the join) but would need at least *two* memory accesses for each new  $A_1x_0$  in the Finiasz–Sendrier lower bound (one for  $A_1$  and one for the join).

**256-bit security revisited.** According to [9, Section 7] a binary code with length  $n = 6624$ ,  $k = 5129$ ,  $w = 117$  achieves 256-bit security. The best collision-decoding parameters are actually slightly below  $2^{256}$  bit operations: they use  $2^{181.4928}$  iterations (on average), each taking  $2^{74.3741}$  bit operations, for a total of  $2^{255.8669}$  bit operations.

Collision decoding with the birthday trick takes, with optimal parameters,  $2^{255.54880}$  bit operations. The birthday trick increases the cost per iteration by a factor of 2.2420 compared to the classical collision-decoding algorithm, to  $2^{75.5390}$  bit operations. However, the trick increases the chances of finding the desired error vector noticeably, reducing the number of iterations by a factor of 2.7951, to  $2^{180.0099}$ . Thus the birthday trick yields an overall  $1.2467\times$  speedup.

The Finiasz–Sendrier lower bound is  $2^{255.1787}$  bit operations,  $1.6112\times$  smaller than the cost of collision decoding.

Ball-collision decoding with parameters  $k_1 = 2565$ ,  $k_2 = 2564$ ,  $\ell_1 = \ell_2 = 47$ ,  $p_1 = p_2 = 8$ , and  $q_1 = q_2 = 1$  needs only  $2^{254.1519}$  bit operations to attack the same system. On average the algorithm needs  $2^{170.6473}$  iterations each taking  $2^{83.5046}$  bit operations.

Ball-collision decoding thus costs  $3.2830\times$  less than collision decoding,  $2.6334\times$  less than collision decoding with the birthday trick, and  $2.0375\times$  less than the Finiasz–Sendrier lower bound.

**1000-bit security.** Attacking a system based on a code of length  $n = 30332$ ,  $k = 22968$ ,  $w = 494$  requires  $2^{1000.9577}$  bit operations using collision decoding with the optimal parameters  $k_1 = k_2 = 11484$ ,  $\ell_1 = \ell_2 = 140$ ,  $p_1 = p_2 = 27$  and  $q_1 = q_2 = 0$ .

The birthday trick reduces the cost by a factor of 1.7243, to  $2^{1000.1717}$  bit operations. This means that this system offers 1000-bit security against all previously known attacks.

The Finiasz–Sendrier lower bound is  $2^{999.45027}$  bit operations,  $2.8430\times$  smaller than the cost of collision decoding and  $1.6488\times$  smaller than the cost of collision decoding with the birthday trick.

Ball-collision decoding with parameters  $k_1 = k_2 = 11484$ ,  $\ell_1 = \ell_2 = 156$ ,  $p_1 = p_2 = 29$ , and  $q_1 = q_2 = 1$  needs only  $2^{996.21534}$  bit operations. This is  $26.767\times$  smaller than the cost of collision decoding,  $15.523\times$  smaller than the cost of collision decoding with the birthday trick, and  $9.415\times$  smaller than the Finiasz–Sendrier lower bound.

## 7 Asymptotic complexity of ball-collision decoding

This section analyzes the asymptotic behavior of the cost of ball-collision decoding, and shows that it always has a smaller asymptotic exponent than the cost of collision decoding.

**Input sizes.** Fix a real number  $W$  with  $0 < W < 1/2$ , and fix a real number  $R$  with  $-W \log_2 W - (1 - W) \log_2(1 - W) \leq 1 - R < 1$ .

Consider codes and error vectors of very large length  $n$ , where the codes have dimension  $k \approx Rn$ , and the error vectors have weight  $w \approx Wn$ . More precisely, fix functions  $k, w : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$  that satisfy  $\lim_{n \rightarrow \infty} k(n)/n = R$  and  $\lim_{n \rightarrow \infty} w(n)/n = W$ ; more concisely,  $k/n \rightarrow R$  and  $w/n \rightarrow W$ .

**Attack parameters.** Fix real numbers  $P, Q, L$  with  $0 \leq P \leq R/2$ ,  $0 \leq Q \leq L$ , and  $0 \leq W - 2P - 2Q \leq 1 - R - 2L$ . Fix ball-collision parameters  $p_1, p_2, q_1, q_2, k_1, k_2, \ell_1, \ell_2$  with  $p_i/n \rightarrow P$ ,  $q_i/n \rightarrow Q$ ,  $k_i/n \rightarrow R/2$ , and  $\ell_i/n \rightarrow L$ .

We have also analyzed more general asymptotic parameter spaces, for example splitting  $P$  into  $P_1, P_2$  where  $p_i/n \rightarrow P_i$ . Balanced parameters always turned out to be asymptotically optimal (as one would expect), so this section focuses on the parameter space  $(P, Q, L)$  stated above. Note that the asymptotic optimality of  $P_1 = P_2$  does *not* imply the concrete optimality of  $p_1 = p_2$ ; for example,  $(p_1, p_2) = (2, 1)$  appears to be optimal for some small input sizes.

In the formulas below, expressions of the form  $x \log_2 x$  are extended (continuously but not differentially) to 0 at  $x = 0$ . For example, the expression  $P \log_2 P$  means 0 if  $P = 0$ .

**Success probability.** We repeatedly invoke the standard asymptotic formula for binomial coefficients, namely

$$\frac{1}{n} \log_2 \binom{(\alpha + o(1))n}{(\beta + o(1))n} \rightarrow \alpha \log_2 \alpha - \beta \log_2 \beta - (\alpha - \beta) \log_2(\alpha - \beta),$$

to compute the asymptotic exponent of the success probability of a single iteration of ball-collision decoding:

$$\begin{aligned} B(P, Q, L) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 \left( \binom{n}{w}^{-1} \binom{n - k - \ell_1 - \ell_2}{w - p_1 - p_2 - q_1 - q_2} \binom{k_1}{p_1} \binom{k_2}{p_2} \binom{\ell_1}{q_1} \binom{\ell_2}{q_2} \right) \\ &= W \log_2 W + (1 - W) \log_2(1 - W) \\ &\quad + (1 - R - 2L) \log_2(1 - R - 2L) - (W - 2P - 2Q) \log_2(W - 2P - 2Q) \\ &\quad - (1 - R - 2L - (W - 2P - 2Q)) \log_2(1 - R - 2L - (W - 2P - 2Q)) \\ &\quad + R \log_2(R/2) - 2P \log_2 P - (R - 2P) \log_2(R/2 - P) \\ &\quad + 2L \log_2 L - 2Q \log_2 Q - 2(L - Q) \log_2(L - Q). \end{aligned}$$

The success probability of a single iteration is asymptotically  $2^{n(B(P,Q,L)+o(1))}$ .

**Iteration cost.** We similarly compute the asymptotic exponent of the cost of an iteration:

$$\begin{aligned} C(P, Q, L) &= \lim_{n \rightarrow \infty} \frac{1}{n} \log_2 \left( \binom{k_1}{p_1} \binom{\ell_1}{q_1} + \binom{k_2}{p_2} \binom{\ell_2}{q_2} + \binom{k_1}{p_1} \binom{\ell_1}{q_1} \binom{k_2}{p_2} \binom{\ell_2}{q_2} 2^{-\ell_1 - \ell_2} \right) \\ &= \max\{ (R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P) \\ &\quad + L \log_2 L - Q \log_2 Q - (L - Q) \log_2(L - Q), \\ &\quad R \log_2(R/2) - 2P \log_2 P - (R - 2P) \log_2(R/2 - P) \\ &\quad + 2L \log_2 L - 2Q \log_2 Q - 2(L - Q) \log_2(L - Q) - 2L \}. \end{aligned}$$

The cost of a single iteration is asymptotically  $2^{n(C(P,Q,L)+o(1))}$ . Note that we have simplified the iteration cost to  $\binom{k_1}{p_1}\binom{\ell_1}{q_1} + \binom{k_2}{p_2}\binom{\ell_2}{q_2} + \binom{k_1}{p_1}\binom{\ell_1}{q_1}\binom{k_2}{p_2}\binom{\ell_2}{q_2}2^{-\ell_1-\ell_2}$ . The cost is actually larger than this, but only by a factor  $\leq \text{poly}(n)$ , which we are free to disregard since  $\frac{1}{n}\log_2 \text{poly}(n) \rightarrow 0$ . We also comment that the bounds are valid whether or not  $q_i = 0$ .

**Overall attack cost.** The overall asymptotic ball-collision-decoding-cost exponent is the difference  $D(P, Q, L)$  of the iteration-cost exponent  $C(P, Q, L)$  and the success-probability exponent  $B(P, Q, L)$ , thus

$$\begin{aligned} D(P, Q, L) = \max\{ & -(R/2)\log_2(R/2) + P\log_2 P + (R/2 - P)\log_2(R/2 - P) \\ & - L\log_2 L + Q\log_2 Q + (L - Q)\log_2(L - Q), -2L\} \\ & - W\log_2 W - (1 - W)\log_2(1 - W) \\ & - (1 - R - 2L)\log_2(1 - R - 2L) + (W - 2P - 2Q)\log_2(W - 2P - 2Q) \\ & + (1 - R - 2L - (W - 2P - 2Q))\log_2(1 - R - 2L - (W - 2P - 2Q)). \end{aligned}$$

Example: Take  $W = 0.04$  and  $R = 1 + W\log_2 W + (1 - W)\log_2(1 - W) = 0.7577078109\dots$ . Choose  $P = 0.004203556640625$ ,  $Q = 0.000192998046875$ , and  $L = 0.017429431640625$ ; we use very high precision here to simplify verification. The success-probability exponent is  $-0.0458435310\dots$ , and the iteration-cost exponent is  $0.0348588632\dots$ , so the ball-collision decoding exponent is  $0.0807023942\dots$ . Ball-collision decoding with these parameters therefore costs  $2^{(0.0807023942\dots+o(1))n}$  to correct  $(0.04 + o(1))n$  errors in a code of rate  $0.7577078109\dots + o(1)$ .

**Collision-decoding cost and the lower bound.** Traditional collision decoding is the special case  $p_1 = p_2$ ,  $k_1 = k_2$ ,  $\ell_1 = \ell_2$ ,  $q_1 = q_2 = 0$  of ball-collision decoding. Its asymptotic cost exponent is the case  $Q = 0$  of the ball-collision decoding exponent stated above.

Consider again  $W = 0.04$  and  $R = 1 + W\log_2 W + (1 - W)\log_2(1 - W)$ . Choosing  $P = 0.00415087890625$ ,  $Q = 0$ , and  $L = 0.0164931640625$  achieves decoding exponent  $0.0809085120\dots$ . We partitioned the  $(P, L)$  space into small intervals and performed interval-arithmetic calculations to show that  $Q = 0$  cannot do better than  $0.0809$ ; ball-collision decoding therefore has a slightly smaller exponent than collision decoding in this case.

We performed similar calculations for other pairs  $(W, R)$  and in each case found that the infimum of all collision-decoding-cost exponents was beaten by a ball-collision-decoding-cost exponent. Ball-collision decoding therefore has a smaller exponent than collision decoding, as stated in the introduction of this paper.

**The case  $Q = 0$  is always suboptimal.** The interval-arithmetic calculations described above are proofs of the suboptimality of  $Q = 0$  for some specific pairs  $(W, R)$ . These proofs have the advantage of computing explicit bounds on the collision-decoding-cost exponents for those pairs  $(W, R)$ , but the proofs have two obvious disadvantages.

The first disadvantage is that these proofs do not cover *all* pairs  $(W, R)$ ; they leave open the possibility that ball-collision decoding has the same exponent as collision decoding for other pairs  $(W, R)$ . The second disadvantage is that the proofs are much too long to verify by hand. The first disadvantage could perhaps be addressed by much more extensive interval-arithmetic calculations, partitioning the space of pairs  $(W, R)$  into boxes so small that, within each box, the ball-collision-decoding exponent is uniformly better than the minimum collision-decoding exponent; but this would exacerbate the second disadvantage.

To address both of these disadvantages we give, in Appendix C, a hand-verifiable proof that  $Q = 0$  is always suboptimal: for *every*  $(W, R)$ , ball-collision decoding has a smaller asymptotic

cost exponent than collision decoding. Specifically, we prove the following theorem about the overall asymptotic cost exponent:

**Theorem 7.1** *For each  $R, W$  it holds that*

$$\begin{aligned} & \min\{D(P, 0, L) : 0 \leq P \leq R/2, 0 \leq W - 2P \leq 1 - R - 2L\} \\ & > \min\{D(P, Q, L) : 0 \leq P \leq R/2, 0 \leq Q \leq L, 0 \leq W - 2P - 2Q \leq 1 - R - 2L\}. \end{aligned}$$

Note that  $\{(P, 0, L)\}$  and  $\{(P, Q, L)\}$  are compact sets, and  $D$  is continuous, so we are justified in writing “min” rather than “inf”. The proof strategy analyzes generic perturbations of  $D$  and combines all necessary calculations into a small number of elementary inequalities in the proofs of Lemmas C.2 and C.3.

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## A Long-term cryptography

McEliece’s cryptosystem is not as well known as RSA or ECC, because of a specific problem discussed below; but it also has several attractive features. The attractive features appear to be of increasing importance, while the problem appears to be of decreasing importance. To provide context for this paper we briefly review both the features and the problem.

**Main features.** McEliece’s cryptosystem provides an excellent asymptotic speed/security tradeoff. Specifically, straightforward encryption algorithms take time just  $b^{2+o(1)}$  when parameters are chosen to provide  $b$ -bit security against the best attacks known. This asymptotic formula  $b^{2+o(1)}$  arises as follows:

- The best attacks known have always been generic decoding algorithms, and those algorithms have (for any constant  $R$ ) always taken time  $2^{\Theta(n/\log_2 n)}$ . Security therefore requires code length  $n \in \Theta(b \log_2 b)$ .
- The bottleneck in encryption is a simple matrix-vector product where the matrix has  $\Theta(n^2) = \Theta(b^2(\log_2 b)^2)$  bits.



Encryption also takes time  $b^{2+o(1)}$  when parameters are chosen to provide  $b$ -bit security against the best *quantum* attack known today. Decryption is more complicated than encryption but asymptotically even faster.

For comparison, low-exponent RSA encryption takes time  $b^{3+o(1)}$  when RSA moduli are chosen to provide  $b$ -bit security against the best attack known today. This asymptotic formula arises as follows:

- The best attacks known have always been factorization algorithms. Modern algorithms to factor  $n$ -bit integers take time  $2^{n^{1/3+o(1)}}$ . Security therefore requires key size  $b^{3+o(1)}$ . Key size  $b^{2+o(1)}$  would have sufficed against Schroepel’s linear sieve, the best factorization algorithm mentioned in the original 1978 RSA paper [63], but the introduction of the number-field sieve in the early 1990s [51] forced asymptotically much larger key sizes.
- Low-exponent encryption using an asymptotically fast FFT-based multiplication algorithm takes time essentially linear in the key size, and therefore time  $b^{3+o(1)}$ .

Even more time is required for RSA *decryption*, Diffie–Hellman key exchange (in its original form using multiplicative groups), etc. Furthermore, the introduction of Shor’s algorithm in 1994 [65] showed that RSA would need encryption time at least  $2^{(1/2+o(1))b}$  to provide  $b$ -bit security against quantum attacks.

Elliptic-curve cryptography is a much closer competitor: it provides  $b^{2+o(1)}$  encryption time and decryption time and has not suffered any serious attacks since its introduction in 1985. However, a closer look suggests that —asymptotically and in practice— the  $o(1)$  for ECC is not competitive with the  $o(1)$  for McEliece. Furthermore, like RSA, ECC will not survive quantum computers.

Code-based cryptography is not the only candidate for post-quantum public-key encryption. Lattice-based cryptography, specifically NTRU and the closely related Ring-LWE, appears to provide a reasonable speed/security tradeoff, even against quantum computers. However, NTRU will be patented until 2017, and Ring-LWE is less attractive than the McEliece cryptosystem for users concerned with the length and depth of cryptanalytic scrutiny.

RSA plays a dominant role today in applications such as HTTPS. However, it is clear that continued increases in the computer power available to attackers are drawing attention to alternative cryptosystems that provide better scalability than RSA; at the same time, continued progress towards quantum computing is drawing attention to post-quantum cryptography. The McEliece cryptosystem is ideally placed in light of these two trends. We can imagine the cryptosystem taking a much more prominent role in future cryptography than it does in current cryptography. The complexity of generic decoding will be important for implementors selecting parameters for the McEliece cryptosystem, the same way that the complexity of factorization (see, e.g., [47]) is important for implementors selecting RSA key sizes today, that the complexity of discrete logarithms (see, e.g., [42] and [43]) is important for implementors selecting DSA/DH key sizes today, etc.

The exponent of ball-collision decoding is not as low as the exponent reported by Bernstein in [7] for *quantum* information-set decoding. Construction of quantum computers might show that the cost of a quantum computer is close to linear in the number of qubits; if so, then all sufficiently large generic decoding problems will be more efficiently solved by quantum information-set decoding than by ball-collision decoding, and it might turn out that “sufficiently large” includes the parameter sizes of interest in cryptography, superseding ball-collision decoding as a cryptanalytic tool. However, it might also turn out that quantum

computers and quantum decoding algorithms scale somewhat less smoothly, leaving non-quantum algorithms as the top decoding threat even if quantum computers are powerful enough to break RSA. It seems prudent to continue investigating both the complexity of quantum algorithms, as in [7], and the complexity of non-quantum algorithms, as in this paper.

**The problem.** As mentioned above, there is one problem with the McEliece cryptosystem: namely, key size. The McEliece key size  $b^{2+o(1)}$  is *asymptotically* smaller than the RSA key size  $b^{3+o(1)}$ , but the RSA key size is smaller for all practical values of  $b$ . The ECC key size is even smaller, just  $2b$  bits. For example, the smallest McEliece key size proposed in [9, Section 7] for  $b = 128$  was 192192 *bytes*, while RSA keys at the same security level are just 3072 bits, and ECC keys are just 256 bits.

The relatively large key size has made the McEliece cryptosystem unsuitable for many applications over the past thirty years. On the other hand, it seems obvious that network bandwidth, storage space, and the number of messages exchanged between users are all increasing more quickly than the number of users. A modern 1.5-terabyte hard drive costing \$80 can store several million 192192-byte keys, and a server storing millions of keys can use those keys to protect much larger volumes of network traffic. At the low end, Eisenbarth, Güneysu, Heyse, and Paar at CHES 2009 [30] reported successfully implementing the McEliece cryptosystem (at a somewhat lower security level) on an AVR microcontroller and a Spartan FPGA.

There have been many proposals that reduce the McEliece key size by deviating in various ways from McEliece’s original selection of random binary Goppa codes as error-correcting codes. Several of these proposals have been broken by “structural attacks” that exploit non-randomness in the public key. For example, [34] and [31] broke many, but not all, cases of [55]. Perhaps some proposals will survive, but the maximum security that any designer can hope to achieve is limited by the generic decoding attacks analyzed in this paper.

## B Choosing McEliece parameters

The traditional approach to selecting cryptosystem parameters is as follows:

- Consider the fastest known attacks against the system. For example, in the case of RSA, consider the latest refinements [47] of the number-field sieve.
- Restrict attention to parameters for which these attacks take time at least  $2^{b+\delta}$ . Here  $b$  is the desired security level, and  $\delta$  is a “security margin” meant to protect against the possibility of further improvements in the attacks.
- Within the remaining parameter space, choose the most efficient parameters. The definition of efficiency depends on the target application: it could mean minimal key size, for example, or minimum decryption time.

This approach does not make clear how to choose the security margin  $\delta$ . Some applications have ample time and space for cryptography, and can simply increase  $\delta$  to the maximum value for which the costs of cryptography are still insignificant; but in some applications cryptography is an important bottleneck, and users insist on minimizing  $\delta$  for the sake of performance.

Finiasz and Sendrier in [32] identified a bound on “future improvements” in attacks against the McEliece cryptosystem, and suggested that designers use this bound to “choose durable

parameters". The general idea of identifying bottlenecks in any possible attack, and of using those bottlenecks to systematically choose  $\delta$ , is quite natural and attractive, and has been used successfully in many contexts. However, as discussed in Section 6, ball-collision decoding disproves the specific bound in [32], violating one of the assumptions in [32] and raising the question of how many more assumptions can be violated.

We propose replacing the bound in [32] with the simpler bound

$$\min \left\{ \frac{1}{2} \binom{n}{w} \binom{n-k}{w-p}^{-1} \binom{k}{p}^{-1/2} : p \geq 0 \right\};$$

i.e., choosing the code length  $n$ , code rate  $k/n$ , and error fraction  $w/n$  so that this bound is at least  $2^b$ . As usual, implementors can exploit the remaining flexibility in parameters to optimize decryption time, compressed key size  $k(n-k)$ , or efficiency in any other metric of interest.

This bound has several attractive features. It is easy to estimate via standard binomial-coefficient approximations. It is easy to compute exactly. It covers a very wide class of attacks, as we explain in a moment. It is nevertheless in the same ballpark as the cost of known attacks: for example, it is  $2^{49.69}$  for the original parameters  $(n, k, w) = (1024, 524, 50)$ , and  $2^{236.49}$  for  $(n, k, w) = (6624, 5129, 117)$ . Note that these numbers give lower bounds on the cost of the attack. Parameters protecting against this bound pay only about a 20% performance penalty at high security levels, compared to parameters that merely protect against known attacks.

The reader can easily verify that parameters  $(n, k, w) = (3178, 2384, 68)$  achieve 128-bit security against this bound. For 256-bit security  $(n, k, w) = (6944, 5208, 136)$  are recommended.

Here is the class of attacks mentioned above. Assume that each iteration of the attack chooses an information set, hoping for exactly  $p$  errors in the set; that the choices of information sets are independent of the target syndrome; that each iteration considers at least  $\binom{k}{p}^{1/2}$  error patterns within the information set; and that testing each pattern costs at least 1. The  $\binom{k}{p}^{1/2}$  iterations model the cost of a birthday-type attack on all vectors of length  $k$  with Hamming weight  $p$ .

For each  $\epsilon \geq 0$ , a cost bound of  $\epsilon \binom{n}{w} \binom{n-k}{w-p}^{-1} \binom{k}{p}^{-1/2}$  allows at most  $\epsilon \binom{n}{w} \binom{n-k}{w-p}^{-1} \binom{k}{p}^{-1}$  iterations, and each iteration covers at most  $\binom{n-k}{w-p} \binom{k}{p}$  patterns of  $w$  errors, so overall the iterations cover at most  $\epsilon \binom{n}{w}$  possible patterns; i.e., the attack succeeds with probability at most  $\epsilon$ . The average attack time is therefore at least  $\frac{1}{2} \binom{n}{w} \binom{n-k}{w-p}^{-1} \binom{k}{p}^{-1/2}$ . Note that batching attacks, i.e., attacking multiple targets at once, does not provide any benefits in this approach. Thus the Johansson–Jonsson speedups for attacking batches of McEliece ciphertexts [41] are subject to the same bound, as are the Fossorier–Kobara–Imai speedups [33].

One can object that this class does not include, e.g., attacks that hope for *at most*  $p$  errors in the information set, or attacks that consider fewer error patterns per iteration at the expense of success probability. One can object, in the opposite direction, that the conditional success probability per error pattern inspected is actually a constant factor smaller than the  $\binom{k}{p}^{-1/2}$  hypothesized above; see generally [32, Appendix A]. A more complicated bound that accounts for these variations and limitations would be slightly larger than the bound stated above but would also be more difficult to compute; our view is that a simpler, slightly smaller bound is more useful. In any event, it is clear that beating this bound would be an astonishing breakthrough.

## C Proof of suboptimality of $Q = 0$ (Theorem 7.1)

This appendix shows that, for each pair  $(W, R)$  within the range considered in Section 7, there are asymptotic parameters  $(P, Q, L)$  for ball-collision decoding whose cost exponent is smaller than the minimum collision-decoding-cost exponent, i.e., smaller than the minimum cost exponent for parameters  $(P, 0, L)$ .

**Input space and parameter space.** Throughout this appendix  $W$  and  $R$  are real numbers with  $0 < W < 1/2$  and  $-W \log_2 W - (1 - W) \log_2(1 - W) \leq 1 - R < 1$ .

The *parameter space* is the set of vectors  $(P, Q, L)$  of real numbers satisfying  $0 \leq P \leq R/2$ ,  $0 \leq Q \leq L$ , and  $0 \leq W - 2P - 2Q \leq 1 - R - 2L$ . This parameter space depends implicitly on  $W$  and  $R$ .

Section 7 considers codes of length  $n$  and dimension  $k$ , and errors of weight  $w$ , where  $n \rightarrow \infty$ ,  $k/n \rightarrow R$ , and  $w/n \rightarrow W$ . The ball-collision parameters  $p_1, p_2, q_1, q_2, k_1, k_2, \ell_1, \ell_2$  satisfy  $p_i/n \rightarrow P$ ,  $q_i/n \rightarrow Q$ ,  $k_i/n \rightarrow R/2$ , and  $\ell_i/n \rightarrow L$ . The proof does not rely on this coding-theoretic interpretation of  $W, R, P, Q, L$ , but readers already familiar with collision decoding may find the interpretation helpful in understanding Lemma C.1 below.

**Cost exponent for collision decoding.** Most of the proof consists of analyzing the asymptotic cost exponent  $D(P, 0, L)$  for collision decoding, namely

$$\begin{aligned} & \max\{- (R/2) \log_2(R/2) + P \log_2 P + (R/2 - P) \log_2(R/2 - P), -2L\} \\ & - W \log_2 W - (1 - W) \log_2(1 - W) - (1 - R - 2L) \log_2(1 - R - 2L) \\ & + (W - 2P) \log_2(W - 2P) + (1 - R - 2L - (W - 2P)) \log_2(1 - R - 2L - (W - 2P)). \end{aligned}$$

As mentioned earlier,  $D(P, 0, L)$  is a continuous function of the parameters  $(P, 0, L)$ , and the parameter space is compact, so there exist optimal collision-decoding parameters  $(P, 0, L)$ , i.e., parameters that achieve the infimum of collision-decoding costs. This does not imply, and the proof of Theorem 7.1 does not use, *uniqueness* of the optimal parameters.

**Optimal collision-decoding parameters.** The proof that ball-collision decoding beats collision decoding relies on the following three facts about optimal collision-decoding parameters  $(P, 0, L)$ :

**Lemma C.1** *If  $(P, L)$  are optimal collision-decoding parameters then*

$$0 < L; \quad 0 < W - 2P; \quad \text{and} \quad W - 2P < (1 - R - 2L)/2.$$

In other words, the collision space  $\mathbf{F}_2^{\ell_1 + \ell_2}$  is asymptotically quite large, and the uncontrolled  $n - k_1 - k_2 - \ell_1 - \ell_2$  positions include asymptotically many error positions, although asymptotically more non-error positions than error positions.

We do not claim that the three facts in Lemma C.1 are news to the many authors who have written previous papers on collision decoding. However, we have not found *proofs* of these facts in the literature, so for completeness we include proofs here.

The proofs do not require any background in coding theory. The main tool is nothing more than basic calculus. In order to study the growth of the collision-cost exponent induced by an increase or decrease in the values of  $P$  and  $L$ , we use the Taylor-series expansion of the logarithm function: for example, a term such as  $(L + \epsilon) \log_2(L + \epsilon)$  has series expansion  $L \log_2 L + \epsilon \log_2(eL) + O(\epsilon^2)$  around  $\epsilon = 0$ . Here  $e = \exp(1)$  is the base of natural logarithms.

Beware that extra work is required in moving away from corners of the parameter space: for example,  $L \log_2 L$  is not differentiable at  $L = 0$ .

**How to improve upon optimal collision-decoding parameters.** Before proving the three parts of Lemma C.1 we show how to deduce Theorem 7.1 from Lemma C.1. The proof is constructive, showing how to slightly adjust optimal parameters for collision decoding to obtain better parameters for ball-collision decoding.

*Proof (of Theorem 7.1).* Start with optimal collision-decoding parameters  $(P, 0, L)$ . Now consider the impact of increasing  $Q$  from 0 to  $\delta$  and increasing  $L$  by  $-(1/2)\delta \log_2 \delta$ , for very small  $\delta$ . Of course, the increase in  $Q$  requires generalizing from collision decoding to ball-collision decoding. Lemma C.1 says that optimal collision-decoding parameters  $(P, 0, L)$  must have  $0 < L$  and  $0 < W - 2P < (1 - R - 2L)/2$ ; consequently the parameter space has room for  $Q$  and  $L$  to increase.

The quantity  $L \log_2 L - Q \log_2 Q - (L - Q) \log_2(L - Q)$  increases by  $-\delta \log_2 \delta + O(\delta)$ , and  $2L \log_2 L - 2Q \log_2 Q - 2(L - Q) \log_2(L - Q) - 2L$  also increases by  $-\delta \log_2 \delta + O(\delta)$ . The iteration-cost exponent therefore increases by  $-\delta \log_2 \delta + O(\delta)$ . The success-probability exponent increases by  $\delta \log_2 \delta \log_2(e(1 - R - 2L)) - \delta \log_2 \delta \log_2(e(1 - R - 2L - (W - 2P))) - 2\delta \log_2 \delta + O(\delta)$ . The total cost exponent therefore increases by  $(\delta \log_2 \delta)(1 + \log_2(1 - R - 2L - (W - 2P))) - \log_2(1 - R - 2L) + O(\delta)$ .

Rewrite  $W - 2P < (1 - R - 2L)/2$  as  $1 + \log_2(1 - R - 2L - (W - 2P)) - \log_2(1 - R - 2L) > 0$ , and deduce that the increase in the cost exponent is negative for all sufficiently small  $\delta > 0$ ; note here that  $\log_2 \delta$  is negative, and that  $O(\delta)/(\delta \log_2 \delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . Consequently the optimal collision-decoding parameters  $(P, 0, L)$  are beaten by  $(P, \delta, L - (1/2)\delta \log_2 \delta)$  for all sufficiently small  $\delta > 0$ .  $\square$

**How to optimize collision decoding.** We build up to Lemma C.1 using several lemmas. The first two lemmas require the most difficult calculations, establishing a useful inequality. The next three lemmas show that optimal collision-decoding parameters  $(P, L)$  can never have  $L = 0$ : Lemma C.4 covers the case  $P = 0$ ; Lemma C.5 covers the case  $P = R/2$ ; Lemma C.6 covers the intermediate cases  $0 < P < R/2$ . Each of the proofs is constructive, showing how to move from  $(P, 0)$  to better collision-decoding parameters.

The next two lemmas show similarly that optimal collision-decoding parameters  $(P, L)$  cannot have  $0 = W - 2P = 1 - R - 2L$ , and cannot have  $0 = W - 2P < 1 - R - 2L$ , so they must have  $0 < W - 2P$ . Proving Lemma C.1 then boils down to proving  $W - 2P < (1 - R - 2L)/2$ ; that proof concludes the appendix.

If  $(P', 0, L')$  and  $(P, 0, L)$  are in the parameter space and  $D(P', 0, L') < D(P, 0, L)$  then we say that  $(P', L')$  *improves upon*  $(P, L)$ . We also say that  $(P', L')$  improves upon  $(P, L)$  in the vacuous case that  $(P, 0, L)$  is not in the parameter space.

**Lemma C.2** Write  $G = (1 + W \log_2 W + (1 - W) \log_2(1 - W))/2$  and

$$X = \frac{1 - 2G - G \log_2 G + (W/2) \log_2(W/2) + (G - (W/2)) \log_2(G - (W/2))}{W}.$$

If  $0 < W \leq 0.099$  then  $X > 2$ .

*Proof.* We begin by commenting that simply graphing the function  $W \mapsto X$  for  $0 < W < 0.2$  suggests that  $X$  drops to a minimum close to 2.8 for  $W \approx 0.13$ . However, most graphing

programs are easily fooled; a graph is not the same as a proof! It is easy to imagine a better graphing program that uses interval arithmetic to automatically prove a lower bound on  $X$  for a *closed* interval of  $W$ , but handling  $W \rightarrow 0$  would still take more work, such as the work shown in Case 4 below.

The proof begins by rewriting

$$X = \frac{-\log_2 W}{2} + \frac{-2(1-W)\log_2(1-W) + 2G\log_2(1-W/(2G)) - W\log_2(2G-W)}{2W}.$$

Note that  $G$  is a monotonic function of  $W$  for  $0 < W < 0.5$ , decreasing from 0.5 down to 0.

Case 1:  $0.079 \leq W \leq 0.099$ . Evaluating  $G$  for  $W = 0.079$  and  $W = 0.099$  shows that  $G$  is in the interval  $[0.267090, 0.300678]$ . Continuing in the same way through interval evaluations of  $W/(2G)$ ,  $\log_2(1-W/(2G))$ ,  $2G\log_2(1-W/(2G))$ , etc. eventually shows that  $X \in [2.21, 3.66]$ .

Case 2:  $0.057 \leq W \leq 0.079$ . The same sequence of calculations shows first that  $G \in [0.300676, 0.342291]$  and then that  $X \in [2.17, 4.03]$ .

Case 3:  $0.036 \leq W \leq 0.057$ . Here  $G \in [0.342289, 0.388180]$  and  $X \in [2.21, 4.56]$ .

Case 4:  $0 < W \leq 0.036$ . This is the heart of the proof.

Write  $E = 1 - 2G = -W\log_2 W - (1-W)\log_2(1-W)$ . Then  $E$  and  $E+W$  are increasing functions of  $W$ ; evaluating them for  $W = 0.036$  shows that  $0 < E < E+W < 1$  for  $0 < W \leq 0.036$ .

The Taylor-series expansion  $\log(1-t) = -t - t^2/2 - t^3/3 - t^4/4 - t^5/5 - \dots$ , where  $\log$  denotes the natural logarithm, implies  $\log(1-t) \leq -t$  and  $\log(1-t) \geq -t - t^2 - t^3 - t^4 - \dots = -t/(1-t)$  for  $0 < t < 1$ . In other words, for each  $t$  with  $0 < t < 1$  there exists  $u \in [0, 1]$  such that  $(1-t)\log(1-t) = -t + ut^2$ . In particular,  $(1-W)\log_2(1-W) = (-W + \alpha W^2)/\log 2$  for some  $\alpha \in [0, 1]$ ;  $(1-E)\log_2(1-E) = (-E + \beta E^2)/\log 2$  for some  $\beta \in [0, 1]$ ; and  $(1-E-W)\log_2(1-E-W) = -(E+W) + \gamma(E+W)^2/\log 2$  for some  $\gamma \in [0, 1]$ .

Now  $E = -W(\log W)/\log 2 - (-W + \alpha W^2)/\log 2$  so

$$\begin{aligned} \frac{X}{-\log_2 W} &= \frac{1}{2} - \frac{(1-W)\log(1-W)}{-W\log W} - \frac{(1-E)\log(1-E)}{-2W\log W} + \frac{(1-E-W)\log(1-E-W)}{-2W\log W} \\ &= \frac{1}{2} + \frac{-W + \alpha W^2}{W\log W} + \frac{-E + \beta E^2}{2W\log W} - \frac{-E - W + \gamma(E+W)^2}{2W\log W} \\ &= \frac{1}{2} + \frac{-1 + 2\alpha W}{2\log W} + \frac{\beta E^2 - \gamma(E+W)^2}{2W\log W} \\ &= \frac{1}{2} + \frac{-1 + 2\alpha W}{2\log W} - \frac{\gamma W(\log 2 - 2\log W + 2(1 - \alpha W))}{2\log 2 \log W} \\ &\quad + \frac{(\beta - \gamma)W(1 + \log^2 W - 2\alpha W + \alpha^2 W^2 - 2(1 - \alpha W)\log W)}{2\log^2 2 \log W} \end{aligned}$$

so

$$\begin{aligned}
 X &= \frac{1}{2\log 2} - \frac{\log W}{2\log 2} - \frac{\alpha W}{\log 2} + \frac{\gamma W(\log 2 - 2\log W + 2(1 - \alpha W))}{2\log^2 2} \\
 &\quad - \frac{(\beta - \gamma)W(1 + \log^2 W - 2\alpha W + \alpha^2 W^2 - 2(1 - \alpha W)\log W)}{2\log^3 2} \\
 &= \frac{1}{2\log 2} - \frac{\log W}{2\log 2} - \frac{(\beta - \gamma)W \log^2 W}{2\log^3 2} \\
 &\quad + \frac{(\beta - \gamma(1 + \log 2))W \log W}{\log^3 2} + \frac{(\gamma(\log^2 2 + 2\log 2 + 1) - \alpha 2\log^2 2 - \beta)W}{2\log^3 2} \\
 &\quad - \frac{(\beta - \gamma)\alpha W^2 \log W}{\log^3 2} + \frac{(\beta - \gamma(1 + \log 2))\alpha W^2}{\log^3 2} - \frac{(\beta - \gamma)\alpha^2 W^3}{2\log^3 2} \\
 &\in \frac{1}{2\log 2} - \frac{\log W}{2\log 2} - \frac{[-1, 1]W \log^2 W}{2\log^3 2} \\
 &\quad + \frac{[-1 - \log 2, 1]W \log W}{\log^3 2} + \frac{[-1 - 2\log^2 2, \log^2 2 + 2\log 2 + 1]W}{2\log^3 2} \\
 &\quad - \frac{[-1, 1]W^2 \log W}{\log^3 2} + \frac{[-1 - \log 2, 1]W^2}{\log^3 2} - \frac{[-1, 1]W^3}{2\log^3 2}.
 \end{aligned}$$

We now put a separate lower bound on each term, using the positivity of  $W$ ,  $-W \log W$ , etc.:

$$\begin{aligned}
 X &\geq \frac{1}{2\log 2} - \frac{\log W}{2\log 2} - \frac{W \log^2 W}{2\log^3 2} + \frac{W \log W}{\log^3 2} \\
 &\quad + \frac{(-1 - 2\log^2 2)W}{2\log^3 2} - \frac{-W^2 \log W}{\log^3 2} + \frac{(-1 - \log 2)W^2}{\log^3 2} - \frac{W^3}{2\log^3 2}.
 \end{aligned}$$

Each term here is monotonically decreasing for  $0 < W \leq 0.036$ : for example,  $W \log^2 W$  has derivative  $2\log W + \log^2 W$ , which is zero only for  $W = 1/e^2 > 0.1$ . Each quantity is therefore bounded below by its value at 0.036: i.e.,

$$\begin{aligned}
 X &\geq \frac{1}{2\log 2} - \frac{\log 0.036}{2\log 2} - \frac{0.036 \log^2 0.036}{2\log^3 2} + \frac{0.036 \log 0.036}{\log^3 2} \\
 &\quad + \frac{(-1 - 2\log^2 2)0.036}{2\log^3 2} - \frac{-0.036^2 \log 0.036}{\log^3 2} + \frac{(-1 - \log 2)0.036^2}{\log^3 2} - \frac{0.036^3}{2\log^3 2} \\
 &\geq 2.02
 \end{aligned}$$

for  $0 < W \leq 0.036$ . □

**Lemma C.3** *Each  $(P, 0, L)$  in the parameter space satisfies*

$$1 - R - ((R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P)) > 2W.$$

*Proof.* Case 1:  $0.099 < W < 0.5$ .

Recall that  $1 - R \geq -W \log_2 W - (1 - W) \log_2(1 - W)$ . Hence  $1 - (3/2)R - 2W \geq -(1/2) - 2W - (3/2)W \log_2 W - (3/2)(1 - W) \log_2(1 - W)$ . The values of this lower bound at 0.099, 0.3, 0.5 are  $\approx 0.000726, \approx 0.2218, 0$  respectively; the derivative of the lower bound is  $-2 - (3/2) \log_2(eW) + (3/2) \log_2(e(1 - W))$ , which has a unique zero at  $W = 1/(1 +$

$2^{4/3}) \approx 0.2841$ ; so the lower bound is positive for all  $W$  with  $0.099 < W < 0.5$ . In particular  $2W < 1 - (3/2)R$  if  $0.099 < W < 0.5$ .

The maximum possible value of  $(R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P)$  is  $(R/2) \log_2(R/2) - 2(R/4) \log_2(R/4) = R/2$ , so  $1 - R - ((R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P)) \geq 1 - (3/2)R > 2W$  as claimed.

Case 2:  $0 < W \leq 0.099$ .

Define  $G$  and  $X$  as in Lemma C.2; then  $X > 2$ . Note that  $G - W$  has derivative  $(1/2) \log_2(eW) - (1/2) \log_2(e(1 - W)) - 1 < 0$  for  $0 < W \leq 0.099$ , and value  $0.168091 \dots > 0$  at  $W = 0.099$ , so  $G > W$  for  $0 \leq W \leq 0.099$ .

Furthermore  $R/2 \leq G$  by definition of  $G$  and the parameter space. So  $(R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P) \leq G \log_2 G - P \log_2 P - (G - P) \log_2(G - P)$ ; note that for any fixed  $c > 0$  and  $x > c$  the function  $x \log_2 x - (x - c) \log_2(x - c)$  is increasing (check its derivative).

Note also that the function  $x \log_2 x + (c - x) \log_2(c - x)$  is decreasing for  $x < c/2$ . The parameter space forces  $P \leq W/2 < G/2$  so  $G \log_2 G - P \log_2 P - (G - P) \log_2(G - P) \leq G \log_2 G - (W/2) \log_2(W/2) - (G - (W/2)) \log_2(G - (W/2))$ .

Combining all of these inequalities produces  $1 - R - ((R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P)) \geq 1 - R - (G \log_2 G - (W/2) \log_2(W/2) - (G - (W/2)) \log_2(G - (W/2))) \geq XW > 2W$  as claimed.  $\square$

**Lemma C.4** *There is a real number  $\delta > 0$  such that  $(\delta, -(1/2)\delta \log_2 \delta)$  improves upon  $(0, 0)$ .*

*Proof.* For all sufficiently small real numbers  $\delta \geq 0$  the parameters  $(P, L) = (\delta, -(1/2)\delta \log_2 \delta)$  satisfy the constraints  $0 \leq P \leq R/2$ ,  $0 \leq L$ , and  $0 \leq W - 2P \leq 1 - R - 2L$  since  $0 < R$  and  $0 < W$ . The collision-cost exponent is  $\max\{\delta \log_2 \delta + O(\delta), \delta \log_2 \delta\} - (1 - W) \log_2(1 - W) - (1 - R) \log_2(1 - R) + (1 - R - W) \log_2(1 - R - W) - (\delta \log_2 \delta) \log_2(e(1 - R)) + (\delta \log_2 \delta) \log_2(e(1 - R - W)) + O(\delta) = -(1 - W) \log_2(1 - W) - (1 - R) \log_2(1 - R) + (1 - R - W) \log_2(1 - R - W) + (\delta \log_2 \delta)(1 + \log_2(1 - R - W) - \log_2(1 - R)) + O(\delta)$ . The inequality  $2W < 1 - R$  implies  $1 + \log_2(1 - R - W) - \log_2(1 - R) > 0$ , so  $(\delta \log_2 \delta)(1 + \log_2(1 - R - W) - \log_2(1 - R)) + O(\delta)$  is negative for all sufficiently small  $\delta > 0$ , improving upon  $(0, 0)$ .  $\square$

**Lemma C.5** *There is a real number  $\delta > 0$  such that  $(R/2 - \delta, -(1/2)\delta \log_2 \delta)$  improves upon  $(R/2, 0)$ .*

*Proof.* If  $W < R$  then  $(R/2, 0, 0)$  is outside the parameter space so the conclusion is vacuously satisfied. Assume from now on that  $W \geq R$ .

For all sufficiently small  $\delta \geq 0$  the parameters  $(P, L) = (R/2 - \delta, -(1/2)\delta \log_2 \delta)$  satisfy the constraints  $0 \leq P \leq R/2$ ,  $0 \leq L$ , and  $0 \leq W - 2P \leq 1 - R - 2L$ . The iteration-cost exponent is  $\max\{\delta \log_2 \delta + O(\delta), \delta \log_2 \delta\} - W \log_2 W - (1 - R) \log_2(1 - R) + (W - R + 2\delta) \log_2(W - R + 2\delta) - (\delta \log_2 \delta) \log_2(e(1 - R)) + (\delta \log_2 \delta) \log_2(e(1 - W)) + O(\delta) = -W \log_2 W - (1 - R) \log_2(1 - R) + (W - R + 2\delta) \log_2(W - R + 2\delta) + (\delta \log_2 \delta)(1 + \log_2(1 - W) - \log_2(1 - R)) + O(\delta)$ .

If  $W = R$  then  $(W - R + 2\delta) \log_2(W - R + 2\delta) + (\delta \log_2 \delta)(1 + \log_2(1 - W) - \log_2(1 - R)) + O(\delta) = 3\delta \log_2 \delta + O(\delta)$ . This is negative for all sufficiently small  $\delta > 0$ .

Otherwise  $W > R$  so the difference between  $(W - R + 2\delta) \log_2(W - R + 2\delta) + (\delta \log_2 \delta)(1 + \log_2(1 - W) - \log_2(1 - R)) + O(\delta)$  and  $(W - R) \log_2(W - R)$  is  $(\delta \log_2 \delta)(1 + \log_2(1 - W) - \log_2(1 - R)) + O(\delta)$ . This difference is also negative for all sufficiently small  $\delta > 0$ : recall that  $2(1 - W) > 1 > 1 - R$ , so the coefficient  $1 + \log_2(1 - W) - \log_2(1 - R)$  is positive.

Either way  $(P, L) = (R/2 - \delta, -(1/2)\delta \log_2 \delta)$  improves upon  $(R/2, 0)$ .  $\square$



**Lemma C.6** *If  $0 < P < R/2$  then there is a real number  $\delta > 0$  such that  $(P, \delta)$  improves upon  $(P, 0)$ .*

*Proof.* Consider the impact of changing  $L$  from 0 to  $\delta$ . The quantity  $-(R/2) \log_2(R/2) + P \log_2 P + (R/2 - P) \log_2(R/2 - P)$  is negative and unchanged, and  $-2L$  changes from 0 to  $-2\delta$ , so  $\max\{-(R/2) \log_2(R/2) + P \log_2 P + (R/2 - P) \log_2(R/2 - P), -2L\}$  increases by  $-2\delta$  if  $\delta$  is sufficiently small. The quantity  $-(1 - R - 2L) \log_2(1 - R - 2L)$  increases by  $2\delta \log_2(e(1 - R)) + O(\delta^2)$ . The quantity  $(1 - R - 2L - (W - 2P)) \log_2(1 - R - 2L - (W - 2P))$  increases by  $-2\delta \log_2(e(1 - R - (W - 2P))) + O(\delta^2)$ .

The total collision-cost exponent increases by  $2\delta(-1 + \log_2(1 - R) - \log_2(1 - R - (W - 2P))) + O(\delta^2)$ . The coefficient  $-1 + \log_2(1 - R) - \log_2(1 - R - (W - 2P))$  is negative since  $W - 2P < (1 - R)/2$ . Hence  $(P, \delta)$  improves upon  $(P, 0)$  for all sufficiently small  $\delta > 0$ .  $\square$

**Lemma C.7** *There is a real number  $c \geq 2$  satisfying the following condition: if  $W < R$  then  $c \log_2 c - (c - 1) \log_2(c - 1) > (1/2)(\log_2(R - W) - \log_2 W)$ . For any such  $c$  there is a real number  $\delta > 0$  such that  $((W - \delta)/2, (1 - R - c\delta)/2)$  improves upon  $(W/2, (1 - R)/2)$ .*

*Proof.* If  $W > R$  then  $(W/2, (1 - R)/2)$  is outside the parameter space and the conclusions are vacuously satisfied for, e.g.,  $c = 2$  and  $\delta = 1$ . Assume from now on that  $W \leq R$ .

Choose a real number  $c$  large enough to meet both of the following constraints: first,  $c \geq 2$ ; second, if  $W < R$  then  $c \log_2 c - (c - 1) \log_2(c - 1) > (1/2)(\log_2(R - W) - \log_2 W)$ . This can always be done:  $c \log_2 c - (c - 1) \log_2(c - 1) \rightarrow \infty$  as  $c \rightarrow \infty$ .

Consider the impact of changing  $L$  from  $(1 - R)/2$  to  $(1 - R - c\delta)/2$ , and at the same time changing  $P$  from  $W/2$  to  $(W - \delta)/2$ . This change fits the parameter constraints for sufficiently small  $\delta > 0$ .

The quantity  $-(1 - R - 2L) \log_2(1 - R - 2L)$  changes from 0 to  $-c\delta \log_2(c\delta)$ . The quantity  $(W - 2P) \log_2(W - 2P)$  changes from 0 to  $\delta \log_2 \delta$ . The quantity  $(1 - R - 2L - (W - 2P)) \log_2(1 - R - 2L - (W - 2P))$  changes from 0 to  $(c - 1)\delta \log_2((c - 1)\delta)$ . The quantity  $\max\{-(R/2) \log_2(R/2) + P \log_2 P + (R/2 - P) \log_2(R/2 - P), -2L\}$  is dominated by its first term since  $2L = 1 - R > 2W + ((R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P))$  by Lemma C.3. It thus increases by  $((W - \delta)/2) \log_2((W - \delta)/2) - (W/2) \log_2(W/2) + ((R - W + \delta)/2) \log_2((R - W + \delta)/2) - ((R - W)/2) \log_2((R - W)/2)$  if  $\delta$  is sufficiently small.

The total cost exponent increases by  $\delta((c - 1) \log_2(c - 1) - c \log_2 c) + ((W - \delta)/2) \log_2((W - \delta)/2) - (W/2) \log_2(W/2) + ((R - W + \delta)/2) \log_2((R - W + \delta)/2) - ((R - W)/2) \log_2((R - W)/2)$ .

If  $W = R$  then this increase is  $(\delta/2) \log_2(\delta/2) + O(\delta)$  and is therefore negative for all sufficiently small  $\delta > 0$ .

If  $W < R$  then this increase is  $\delta((c - 1) \log_2(c - 1) - c \log_2 c + (1/2)(\log_2(e(R - W)/2) - \log_2(eW/2))) + O(\delta^2)$ . The coefficient of  $\delta$  is negative by choice of  $c$ , so the increase is negative for all sufficiently small  $\delta > 0$ .

In all cases  $((W - \delta)/2, (1 - R - c\delta)/2)$  improves upon  $(W/2, (1 - R)/2)$ .  $\square$

**Lemma C.8** *Assume that  $0 < 1 - R - 2L$ . Then there is a real number  $\delta > 0$  such that  $((W - \delta)/2, L)$  improves upon  $(W/2, L)$ .*

*Proof.* Consider collision-decoding parameters  $(P, L)$  with  $0 = W - 2P < 1 - R - 2L$ . If  $W > R$  then  $(W/2, 0, L)$  is outside the parameter space so the conclusion is vacuously satisfied. Assume from now on that  $W \leq R$ .

Consider the impact of changing  $P$  from  $W/2$  to  $(W - \delta)/2$ . This change fits the parameter constraints for sufficiently small  $\delta > 0$ .

The quantity  $(W - 2P) \log_2(W - 2P)$  increases by  $\delta \log_2 \delta$ . The quantity  $(1 - R - 2L - (W - 2P)) \log_2(1 - R - 2L - (W - 2P))$  increases by  $O(\delta)$ . The quantity

$$\max\{-(R/2) \log_2(R/2) + P \log_2 P + (R/2 - P) \log_2(R/2 - P), -2L\}$$

increases by a value between 0 and  $((W - \delta)/2) \log_2((W - \delta)/2) - (W/2) \log_2(W/2) + ((R - W + \delta)/2) \log_2((R - W + \delta)/2) - ((R - W)/2) \log_2((R - W)/2)$ , which is  $(\delta/2) \log_2(\delta/2) + O(\delta)$  if  $W = R$  and  $O(\delta)$  if  $W < R$ . The total increase in the cost is between  $\delta \log_2 \delta + O(\delta)$  and  $(3/2)\delta \log_2 \delta + O(\delta)$ , and is therefore negative for all sufficiently small  $\delta > 0$ .  $\square$

*Proof (of Lemma C.1).* The hypothesis is that  $(P, L)$  minimizes  $D(P, 0, L)$ , i.e., that nothing improves upon  $(P, L)$ .

The definition of the parameter space implies  $L \geq 0$ . Suppose that  $L = 0$ . Then  $P < 0$  would contradict the definition of the parameter space;  $P = 0$  would contradict Lemma C.4;  $0 < P < R/2$  would contradict Lemma C.6;  $P = R/2$  would contradict Lemma C.5; and  $P > R/2$  would contradict the definition of the parameter space. Hence  $L > 0$ .

The definition of the parameter space also implies  $0 \leq W - 2P$ . Suppose that  $0 = W - 2P$ . Then  $0 = 1 - R - 2L$  would force  $(P, L) = (W/2, (1 - R)/2)$ , contradicting Lemma C.7;  $0 < 1 - R - 2L$  would contradict Lemma C.8; and  $0 > 1 - R - 2L$  would contradict the definition of the parameter space. Hence  $0 < W - 2P$ .

Suppose that  $2L > (R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P)$ . Consider the impact of decreasing  $L$  by  $\delta$ . The quantity

$$\max\{-(R/2) \log_2(R/2) + P \log_2 P + (R/2 - P) \log_2(R/2 - P), -2L\}$$

is dominated by the first term, so it is unchanged for sufficiently small  $\delta$ . The total cost decreases by  $(2 \log_2(1 - R - 2L) - 2 \log_2(1 - R - 2L - (W - 2P)))\delta + O(\delta^2)$ , contradicting the optimality of  $(P, L)$ ; note that the coefficient  $2 \log_2(1 - R - 2L) - 2 \log_2(1 - R - 2L - (W - 2P))$  is positive since  $W - 2P > 0$ .

Therefore  $2L \leq (R/2) \log_2(R/2) - P \log_2 P - (R/2 - P) \log_2(R/2 - P)$ , and  $1 - R - 2L > 2W \geq 2(W - 2P)$  as claimed.  $\square$