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COMPUTING LOGARITHM INTERVALS WITH THE ARITHMETIC-GEOMETRIC-MEAN ITERATION

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ABSTRACT. This paper presents a fast algorithm that, given a tight interval around a positive real number x , computes a tight interval around $\log x$. To obtain p bits of precision for typical values of x , the algorithm uses about $2 \lg p$ square roots and about $5 \lg p$ multiplications (or fewer for subsequent logarithms). Here \log is the natural logarithm, and \lg is the base-2 logarithm. This paper also presents short proofs of all necessary properties of complete elliptic integrals.

1. INTRODUCTION

This paper presents an algorithm that, given a positive real number x to high accuracy, computes $\log x$ to high accuracy. Here \log is the natural logarithm. The algorithm has several useful features:

- It is extremely fast. For p bits of precision, and for x between (e.g.) 2 and 2^p , the algorithm time is dominated by about $7 \lg p$ operations on p -bit numbers, specifically $2 \lg p$ square roots and $5 \lg p$ multiplications. Here \lg is the base-2 logarithm.
- It computes π to high precision, practically for free, as a side effect of computing \log .
- It computes subsequent logarithms at even higher speed: asymptotically, per logarithm, about $2 \lg p$ square roots and about $2 \lg p$ multiplications.
- Given a tight *interval* containing x , it computes tight intervals containing $\log x$ and π . See [4] for an application. For large p , the extra cost of handling intervals is negligible.

See Section 5 for details of the algorithm.

The algorithm relies on various properties of complete elliptic integrals and the arithmetic-geometric-mean (AGM) iteration. This paper presents streamlined proofs of those properties. Section 2 introduces the elliptic integrals I and I_1 used throughout the paper; Section 3 relates I and I_1 to logarithms; Section 4 introduces the AGM iteration.

Previous work. Salamin pointed out more than thirty years ago that the elliptic integral I could be used to quickly compute high-precision approximations to $\log x$ and π ; see [2, Item 143]. Salamin's algorithm is dominated by $\Theta(\lg p)$ high-precision operations, with larger constant factors than the algorithm here.

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Subsequent refinements of Salamin's algorithm included almost all of the ideas necessary for minimizing the constant factors, but the ideas were never combined properly. See Section 6 for a survey of the literature.

Brent in [9, Section 6] introduced another method of computing high-precision exponentials and logarithms, without relying on elliptic integrals. See [3, Sections 12–16] for further discussion. Brent's method may be faster than the algorithm here for small p , but for large p it is slower by a factor roughly proportional to $\lg p$.

Logarithm algorithms are occasionally presented with explicit bounds suitable for interval computation. See [14] and [13].

The results of Sections 2, 3, and 4 (modulo language, and modulo the difference between explicit inequalities and order-of-magnitude estimates) have been known for two centuries, and are a small fraction of the wealth of material collected in [8]. However, I have not seen such short proofs of the I_1 properties, particularly Theorem 2.6 and Theorem 4.3.

2. ELLIPTIC INTEGRALS: BASIC PROPERTIES

For positive real numbers a, b define $I(a, b) = \int_0^\infty (x^2 + a^2)^{-1/2} (x^2 + b^2)^{-1/2} dx$. Also define $I_1(a, b) = \partial I(a, b) / \partial a = \int_0^\infty -a(x^2 + a^2)^{-3/2} (x^2 + b^2)^{-1/2} dx$.

Integrability, differentiation under the integral sign, etc. follow from the fact that all the integrands in this section are constant in sign, continuous in x , and differentiable in a, b .

Theorem 2.1. $\pi/2a \leq I(a, b) \leq \pi/2b$ if $b \leq a$.

Proof. $x^2 + b^2 \leq x^2 + a^2$ so $\int_0^\infty (x^2 + a^2)^{-1} dx \leq I(a, b) \leq \int_0^\infty (x^2 + b^2)^{-1} dx$. \square

Theorem 2.2. $0 \leq -aI_1(a, b) \leq I(a, b)$.

Proof. $0 \leq a^2 \leq x^2 + a^2$. \square

Theorem 2.3. $I(a, b) = I(1, b/a)/a$.

Proof. Substitute $x = au$: $aI(a, b) = \int_0^\infty a^2(a^2u^2 + a^2)^{-1/2}(a^2u^2 + b^2)^{-1/2} du = \int_0^\infty (u^2 + 1)^{-1/2}(u^2 + b^2/a^2)^{-1/2} du = I(1, b/a)$. \square

Theorem 2.4. $I(a, b) + aI_1(a, b) + b(\partial I(a, b))/\partial b = 0$.

Proof. Apply $a(\partial/\partial a) + b(\partial/\partial b)$ to Theorem 2.3. \square

Theorem 2.5. $I(a, b) = I(\frac{a+b}{2}, \sqrt{ab})$.

Proof. Substitute $u = (x - ab/x)/2$, using $(2u)^2 + (a+b)^2 = (x^2 + a^2)(x^2 + b^2)/x^2$ and $x du = (u^2 + ab)^{1/2} dx$: $I(a, b) = \int_{-\infty}^\infty ((2u)^2 + (a+b)^2)^{-1/2} (u^2 + ab)^{-1/2} du = 2 \int_0^\infty (2^2(u^2 + (\frac{a+b}{2})^2))^{-1/2} (u^2 + ab)^{-1/2} du = I(\frac{a+b}{2}, \sqrt{ab})$. \square

Theorem 2.6. $I(a, b) + 2aI_1(a, b) = I_1(\frac{a+b}{2}, \sqrt{ab})(a-b)/2$.

Proof. Apply $a(\partial/\partial a) - b(\partial/\partial b)$ to Theorem 2.5: $aI_1(a, b) - b(\partial I(a, b)/\partial b) = I(\frac{a+b}{2}, \sqrt{ab})(a-b)/2$. By Theorem 2.4, $I(a, b) + aI_1(a, b) + b(\partial I(a, b)/\partial b) = 0$. \square

Theorem 2.7. $I(1, b) = 2 \int_0^{\sqrt{b}} (x^2 + 1)^{-1/2} (x^2 + b^2)^{-1/2} dx$.

Proof. $\int_{\sqrt{b}}^\infty (x^2 + 1)^{-1/2} (x^2 + b^2)^{-1/2} dx = \int_0^{\sqrt{b}} ((\frac{b}{u})^2 + 1)^{-1/2} ((\frac{b}{u})^2 + b^2)^{-1/2} \frac{b}{u^2} du = \int_0^{\sqrt{b}} (b^2 + u^2)^{-1/2} (1 + u^2)^{-1/2} du$. \square

Theorem 2.8. $(1+b)^{-1} + I(1, b) + I_1(1, b) = 2 \int_0^{\sqrt{b}} b^2(x^2+1)^{-1/2}(x^2+b^2)^{-3/2} dx$.

Proof. By Theorem 2.4, $I(1, b) + I_1(1, b) + b(\partial I(1, b)/\partial b) = 0$. By Theorem 2.7, $\partial I(1, b)/\partial b = b^{-1/2}(b+1)^{-1/2}(b+b^2)^{-1/2} + 2 \int_0^{\sqrt{b}} -b(x^2+1)^{-1/2}(x^2+b^2)^{-3/2} dx$. \square

Theorem 2.9. $a^2 I(a, b) + (a^2 - b^2)aI_1(a, b) = \int_0^\infty a^2(x^2 + a^2)^{-3/2}(x^2 + b^2)^{1/2} dx$.

Proof. $x^2 + a^2 - (a^2 - b^2) = x^2 + b^2$. \square

Theorem 2.9 explains why $I(a, b)$, $a^2 I(a, b) + (a^2 - b^2)aI_1(a, b)$, etc. are called **elliptic integrals**, and why related algebraic objects are called **elliptic curves**: substitute $x = a \tan \theta$ to see that $a^2 I(a, b) + (a^2 - b^2)aI_1(a, b)$ is the arc length of one quadrant of the ellipse $\theta \mapsto (a \cos \theta, b \sin \theta)$. Theorem 2.9 is not used elsewhere in this paper.

3. ELLIPTIC INTEGRALS: LOGARITHM BOUNDS

This section presents precise bounds along the following lines: “If $b \approx 0$ then $I(1, b) \approx \log(4/b)$ and $I(1, b) + I_1(1, b) \approx 1$.” Write $L(b) = \log(\sqrt{b^{-1}} + \sqrt{b^{-1} + 1})$.

The simple proof technique used here is not new, and explicit bounds in this context are not new, but I am not aware of previous simple proofs of explicit bounds. For inexplicit bounds using the same proof technique, see [18, page 522] and [15]; for explicit bounds with longer proofs, see [7, pages 355–356] and [8, Theorem 7.2].

Theorem 3.1. *If $0 < b \leq 1$ then $(2 + \frac{1}{2}b^2)L(b) - (\frac{1}{2}b)(1+b)^{1/2} \leq I(1, b) \leq (2 + \frac{1}{2}b^2 + \frac{9}{32}b^4)L(b) - (\frac{1}{2}b - \frac{3}{16}b^2 + \frac{9}{32}b^3)(1+b)^{1/2}$.*

The difference between lower and upper bounds is on the scale of b^2 . The bounds can easily be made tighter by the same technique.

Proof. $I(1, b) = 2 \int_0^{\sqrt{b}} (1+x^2)^{-1/2}(x^2+b^2)^{-1/2} dx$ by Theorem 2.7. Recall that $(1+x^2)^{-1/2} \geq 1-x^2/2$ for $0 \leq x \leq 1$, since $1 \geq 1-(3-x^2)x^4/4 = (1-x^2/2)^2(1+x^2)$. Thus

$$\begin{aligned} I(1, b) &\geq \int_0^{\sqrt{b}} (2-x^2)(x^2+b^2)^{-1/2} dx \\ &= (2 + \frac{1}{2}b^2) \log(x + (x^2 + b^2)^{1/2}) - (\frac{1}{2}x)(x^2 + b^2)^{1/2} \Big|_0^{\sqrt{b}} \\ &= (2 + \frac{1}{2}b^2) \log \left(\frac{\sqrt{b} + \sqrt{b + b^2}}{\sqrt{0} + \sqrt{0 + b^2}} \right) - (\frac{1}{2}\sqrt{b})(b + b^2)^{1/2} \\ &= (2 + \frac{1}{2}b^2)L(b) - (\frac{1}{2}b)(1+b)^{1/2}. \end{aligned}$$

Similarly, $(1+x^2)^{-1/2} \leq 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4$ for $0 \leq x$, so

$$\begin{aligned} I(1, b) &\leq \int_0^{\sqrt{b}} (2 - x^2 + \frac{3}{4}x^4)(x^2 + b^2)^{-1/2} dx \\ &= (2 + \frac{1}{2}b^2 + \frac{9}{32}b^4) \log(x + (x^2 + b^2)^{1/2}) - (\frac{1}{2} - \frac{3}{16}x^2 + \frac{9}{32}b^2)x(x^2 + b^2)^{1/2} \Big|_0^{\sqrt{b}} \\ &= (2 + \frac{1}{2}b^2 + \frac{9}{32}b^4)L(b) - (\frac{1}{2}b - \frac{3}{16}b^2 + \frac{9}{32}b^3)(1+b)^{1/2}. \end{aligned}$$

\square

Theorem 3.2. *If $0 < b \leq 1$ then $(2 + b^2)(1 + b)^{-1/2} - b^2 L(b) \leq (1 + b)^{-1} + I(1, b) + I_1(1, b) \leq (2 + b^2 + \frac{3}{8}b^3 + \frac{9}{8}b^4)(1 + b)^{-1/2} - (b^2 + \frac{9}{8}b^4)L(b)$.*

Proof. $(1 + b)^{-1} + I(1, b) + I_1(1, b) = 2 \int_0^{\sqrt{b}} b^2 (x^2 + 1)^{-1/2} (x^2 + b^2)^{-3/2} dx$ by Theorem 2.8. Recall that $(1 + x^2)^{-1/2} \geq 1 - x^2/2$ for $0 \leq x \leq 1$, as in Theorem 3.1, so

$$\begin{aligned} (1 + b)^{-1} + I(1, b) + I_1(1, b) &\geq \int_0^{\sqrt{b}} b^2 (2 - x^2)(x^2 + b^2)^{-3/2} dx \\ &= (2 + b^2)x(x^2 + b^2)^{-1/2} - b^2 \log(x + (x^2 + b^2)^{1/2}) \Big|_0^{\sqrt{b}} \\ &= (2 + b^2)(1 + b)^{-1/2} - b^2 L(b). \end{aligned}$$

Similarly, $(1 + x^2)^{-1/2} \leq 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4$ for $0 \leq x$, so

$$\begin{aligned} (1 + b)^{-1} + I(1, b) + I_1(1, b) &\leq \int_0^{\sqrt{b}} b^2 (2 - x^2 + \frac{3}{4}x^4)(x^2 + b^2)^{-3/2} dx \\ &= (2 + b^2 + \frac{3}{8}b^2x^2 + \frac{9}{8}b^4)x(x^2 + b^2)^{-1/2} - (b^2 + \frac{9}{8}b^4) \log(x + (x^2 + b^2)^{1/2}) \Big|_0^{\sqrt{b}} \\ &= (2 + b^2 + \frac{3}{8}b^3 + \frac{9}{8}b^4)(1 + b)^{-1/2} - (b^2 + \frac{9}{8}b^4)L(b). \end{aligned}$$

□

4. ELLIPTIC INTEGRALS: AGM ITERATION

Throughout this section, a_0 and b_0 are positive real numbers; a_n and b_n are defined recursively by $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = \sqrt{a_n b_n}$. As n increases, both a_n and b_n converge rapidly to $\pi/2I(a_0, b_0)$, the **arithmetic-geometric mean** of a_0 and b_0 ; see Theorems 4.2 and 4.5.

Theorem 4.1. *If $n \geq 1$ then $a_n \geq b_n$.*

Proof. The geometric mean cannot exceed the arithmetic mean: if $n \geq 0$ then $a_{n+1}^2 - b_{n+1}^2 = (a_n - b_n)^2/4 \geq 0$ so $a_{n+1} \geq b_{n+1}$. □

Theorem 4.2. $\pi/2a_n \leq I(a_0, b_0) \leq \pi/2b_n$ for $n \geq 1$.

Proof. By Theorem 2.5, $I(a_0, b_0) = I(a_1, b_1) = I(a_2, b_2) = \dots = I(a_n, b_n)$. By Theorem 4.1, $a_n \geq b_n$. Apply Theorem 2.1. □

Theorem 4.3. *Define $\epsilon_n = 2^n(a_n^2 - b_n^2)(-a_n(I_1/I)(a_n, b_n))$. Then $0 \leq \epsilon_n \leq 2^n(a_n^2 - b_n^2)$ for $n \geq 1$, and $(a_0^2 - b_0^2)(-a_0(I_1/I)(a_0, b_0)) = \epsilon_n + \sum_{0 \leq i < n} 2^{i-1}(a_i^2 - b_i^2)$.*

Proof. $0 < b_n \leq a_n$ for $n \geq 1$, so $0 \leq -a_n(I_1/I)(a_n, b_n) \leq 1$ by Theorem 2.2; i.e., $0 \leq \epsilon_n \leq 2^n(a_n^2 - b_n^2)$.

Substitute $(a_{n+1}^2 - b_{n+1}^2)a_{n+1} = (a_n^2 - b_n^2)(a_n - b_n)/8$:

$$\epsilon_{n+1} = 2^n(a_n^2 - b_n^2) \left(\frac{-(a_n - b_n)}{4} \left(\frac{I_1}{I} \right) (a_{n+1}, b_{n+1}) \right).$$

Then apply Theorems 2.6 and 2.5:

$$\epsilon_{n+1} = 2^n(a_n^2 - b_n^2) \left(\frac{-1}{2} - a_n \left(\frac{I_1}{I} \right) (a_n, b_n) \right) = -2^{n-1}(a_n^2 - b_n^2) + \epsilon_n.$$

Thus $\epsilon_0 = \epsilon_n + \sum_{0 \leq i < n} 2^{i-1}(a_i^2 - b_i^2)$ by induction. □

Theorem 4.4. *If $1 \leq a_0/b_0 \leq 1 + 2^{2^m}$ then $1 \leq a_n/b_n \leq 1 + 2^{2^m - n}$ for $n \geq 0$.*

Proof. If $1 \leq a_n/b_n \leq 1 + 2^{2^m - n}$ then $a_{n+1}/b_{n+1} = (\sqrt{a_n/b_n} + \sqrt{b_n/a_n})/2 \leq \sqrt{a_n/b_n} \leq \sqrt{1 + 2^{2^m - n}} \leq 1 + 2^{2^m - n - 1}$. \square

Theorem 4.5. *If $m \geq 0$ and $1 \leq a_0/b_0 \leq 1 + 2^{2^m}$ then $1 \leq a_n/b_n \leq 1 + 2^{3 - 2^{n+1} - m}$ for $n \geq m$.*

Proof. The base case $n = m$ follows from Theorem 4.4 since $2^{2^m - m} = 2^{3 - 2^{m+1} - m}$.

If $a_n/b_n = 1 + \epsilon$ with $0 \leq \epsilon \leq 2^{3 - 2^{n+1} - m}$ then $4 + 4\epsilon + \epsilon^2 \leq (4 + \epsilon^2 + \epsilon^4/16)(1 + \epsilon)$, so $(1 + \epsilon) + 2 + 1/(1 + \epsilon) \leq (2 + \epsilon^2/4)^2$, so $\sqrt{a_n/b_n} + \sqrt{b_n/a_n} \leq 2 + \epsilon^2/4$, so $a_{n+1}/b_{n+1} \leq 1 + \epsilon^2/8 \leq 1 + 2^{3 - 2^{n+2} - m}$. \square

5. COMPUTING LOGARITHM INTERVALS

This section presents several algorithms that, given an interval containing x , compute an interval containing $\log x$. See [6] for fast subroutines to compute sums, differences, products, quotients, and square roots of intervals.

These algorithms use a parameter p to decide when to stop. For the “arbitrary x ” algorithms, the output interval has approximately p bits of precision if the input interval does. For the “super-size x ” algorithms, the output precision is limited to about $4 \lg x$ bits, even if p is much larger. The “super-size x ” algorithms also slow down as $\lg \lg x$ grows past $\lg p$; if $\lg \lg x$ is much larger than $\lg p$, one should use the “arbitrary x ” algorithms instead.

Beware that, as discussed in [5], the usual algorithms for arithmetic operations such as square root—and for sequences of arithmetic operations—contain many redundancies that can be eliminated. A sequence of AGM steps can be made almost three times faster than reported in [11, Theorem 9.1], for example. Note also that Borwein and Borwein in [8, page 222] observed speed improvements from a “quartic AGM” in which one computes $\sqrt{a_{n+2}}$ and $\sqrt{b_{n+2}}$ directly from $\sqrt{a_n}$ and $\sqrt{b_n}$; my impression is that these speedups become slowdowns when the square-root algorithms are properly optimized, but I will leave experiments to the reader.

Computing $\log x$ for a super-size x . Let $x > 4$ be a real number. Define $a_0 = 1$ and $b_0 = b = (2x/(x^2 - 1))^2$. Note that $\sqrt{b^{-1}} + \sqrt{b^{-1} + 1} = x$, so $L(b) = \log x$, where L is defined in Section 3; note also that $0 < b < 1/2$. Define $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = \sqrt{a_n b_n}$ for $n \geq 0$, as in Section 4. Define $c = 1 + (I_1/I)(1, b)$; note that $0 \leq c \leq 1$ by Theorem 2.2.

Compute an interval containing b . Successively compute intervals containing $a_1, b_1, a_2, b_2, \dots, a_n, b_n$, stopping when $a_n - b_n$ is no longer clearly larger than $1/2^p$. The number of steps n is at most about $\lg \lg x + \lg p$ by Theorem 4.5.

Compute an interval containing $[0, 2^n(a_n^2 - b_n^2)] + \sum_{0 \leq i < n} 2^{i-1}(a_i^2 - b_i^2)$. By Theorem 4.3, this interval contains $(1 - b^2)(1 - c)$. Divide by an interval containing $1 - b^2$, and subtract from 1, to obtain an interval containing c . Finally, compute an interval containing

$$\left[\frac{(\frac{1}{2}b - \frac{3}{16}b^2 + \frac{9}{32}b^3)c(1+b)^{1/2} - (1+b)^{-1} + (2+b^2)(1+b)^{-1/2}}{(2 + \frac{1}{2}b^2 + \frac{9}{32}b^4)c + b^2}, \frac{(\frac{1}{2}b)c(1+b)^{1/2} - (1+b)^{-1} + (2+b^2 + \frac{3}{8}b^3 + \frac{9}{8}b^4)(1+b)^{-1/2}}{(2 + \frac{1}{2}b^2)c + (b^2 + \frac{9}{8}b^4)} \right].$$

By Theorems 3.1 and 3.2, this interval contains $L(b) = \log x$. Note that terms such as $\frac{9}{8}b^4$ have very little effect on the output and can be replaced by crude bounds.

At this point one can also compute an interval containing π with a few additional operations: the interval

$[(2 + \frac{1}{2}b^2)L(b) - (\frac{1}{2}b)(1+b)^{1/2}, (2 + \frac{1}{2}b^2 + \frac{9}{32}b^4)L(b) - (\frac{1}{2}b - \frac{3}{16}b^2 + \frac{9}{32}b^3)(1+b)^{1/2}]$ contains $I(1, b)$ by Theorem 3.1, and the interval $[2b_n I(1, b), 2a_n I(1, b)]$ contains π by Theorem 4.2. This very fast computation of π will be exploited later.

Numerical stability: The computation of c as $1 - (1 - c)$ loses about $\lg \log(4/b) \approx \lg \lg x$ bits of precision, since $c \approx 1/\log(4/b)$. The other arithmetic operations are stable, each losing only a bounded number of bits of precision. However, the intervals in Theorems 3.1 and 3.2 are inherently limited to about $\lg(1/b^2) \approx 4 \lg x$ bits of precision.

Computing $\log x$ for several super-size x 's. After computing a super-size \log as explained above, one can compute another super-size \log at somewhat higher speed, by taking advantage of the π interval obtained from the first computation.

Starting from x , define and compute $b, a_1, b_1, \dots, a_n, b_n$ as above, but skip the computation of $\sum_i 2^{i-1}(a_i^2 - b_i^2)$. Compute an interval containing $[\pi/2a_n, \pi/2b_n]$; by Theorem 4.2, this interval contains $I(1, b)$. Compute an interval containing

$$\left[\frac{I(1, b) + (\frac{1}{2}b - \frac{3}{16}b^2 + \frac{9}{32}b^3)(1+b)^{1/2}}{2 + \frac{1}{2}b^2 + \frac{9}{32}b^4}, \frac{I(1, b) + (\frac{1}{2}b)(1+b)^{1/2}}{2 + \frac{1}{2}b^2} \right];$$

by Theorem 3.1, this interval contains $L(b) = \log x$.

A further improvement is available in applications that compute logarithms only to divide them by each other. The above method is to use I_1/I to compute $\log x$ and π , then use I to compute $\pi/\log y$, then divide to obtain $(\log y)/\log x$; it is somewhat faster to use I twice to compute $\pi/\log x$ and $\pi/\log y$, then divide to obtain $(\log y)/\log x$.

Computing $\log x$ for an arbitrary x . Given an interval containing a positive real number x , find an integer k such that $2^k x$ is larger, but not much larger, than $2^{p/4}$. Then use the algorithm above to compute intervals containing the logs of the super-size numbers $2^k x$ and $2^{\lceil p/4 \rceil}$. Extract intervals containing $\log 2 = (\log 2^{\lceil p/4 \rceil})/\lceil p/4 \rceil$ and $\log x = \log 2^k x - k \log 2$.

The time taken by this algorithm is practically independent of x . However, readers trying to formulate a theorem along these lines are cautioned to consider the possibility that k has many digits.

When x is between (for example) 2 and 2^p , the following algorithm is faster: select an integer $m > \lg p - 2 - \lg \lg x$, compute an interval containing the super-size number x^{2^m} by repeated squaring, compute an interval containing $\log x^{2^m}$, and divide by 2^m . Beware that this approach is slow when x is very close to 1.

Computing $\log x$ for several arbitrary x 's. The general problem of computing logarithms of ℓ numbers can be reduced to the problem of computing logarithms of $\ell + 1$ super-size numbers, one of which is a power of 2, as in the case $\ell = 1$ above.

When all the numbers are in reasonable ranges, and when ℓ is small, it is faster to repeatedly square each number, obtaining ℓ super-size numbers. However, for large ℓ , this is slower than the power-of-2 method.

6. PREVIOUS LOGARITHM ALGORITHMS USING ELLIPTIC INTEGRALS

In the following survey, the word “optimal” means “as fast as possible among all the techniques that I know.” It is not meant to exclude the possibility of future improvements.

Salamin, as reported in [2, Item 143], proposed using the AGM iteration to compute $\pi/\log x$ for super-size x , and thus to compute $\log x$ using π . This is the optimal strategy when x is super-size and π is already known.

Salamin proposed computing π as follows: compute $\exp 1$ by a different method, then compute $(\exp 1)^{2^m} = \exp 2^m$ by repeated squaring, then use the AGM iteration to compute $\pi/\log \exp 2^m = \pi/2^m$. Schroepel in [2, Item 144] proposed using 2^{2^m} and $2^{2^m} \exp 1$ instead of $\exp 2^m$. Both methods are considerably slower than optimal.

For x in a reasonable range such as [2, 4], Salamin proposed computing $\log x$ by computing $\log x^{2^m}$. This is the optimal strategy when only one log is to be computed, although it is not optimal when many logs are to be computed.

Brent in [10] proposed a different log algorithm using incomplete elliptic integrals. Brent’s algorithm is somewhat slower than optimal: it saves half the AGM steps by focusing on moderate values of b , but it works with more than two values of b .

Brent, and independently Salamin in [16], also proposed using the Legendre-Gauss formula $I(1, 2^{-1/2})(I(1, 2^{-1/2}) + I_1(1, 2^{-1/2})) = \pi/2$ to compute π . This is faster than Salamin’s previous method of computing π , but in the context of log computation it is not optimal. The same comment applies to several subsequent methods of computing π , not cited here.

Brent in [11, Section 9] proposed computing $\log x$ for arbitrary x by computing $\log 2^k x$ where k is chosen so that $2^k x$ is super-size. This is the optimal strategy when many logs are to be computed.

Borwein and Borwein in [7, Section 4] proposed another log algorithm in the spirit of [10] but relying solely on complete elliptic integrals. The strategy is somewhat slower than optimal.

Newman in [15] proposed computing π and $\log x$ for super-size x by using the AGM iteration to compute $(\log x)/\pi$, using it again to compute $(\log(x+1))/\pi \approx (\log x + 1/x)/\pi$, and subtracting. Beware that the subtraction loses about $\lg x$ bits of precision. A better (but still suboptimal) strategy is to use $x \exp 1$ in place of $x + 1$, with $\exp 1$ computed by a different method.

Newman’s goal was not maximum speed, but maximum simplicity. In particular, Newman avoided the Legendre-Gauss formula and any other use of I_1 . However, I disagree with Newman’s view of “Landen’s transformation law” (Theorem 2.6 here) as “heavy use of elliptic function theory”; I hope I have convinced the reader that the basic properties of I_1 can be established just as easily as the basic properties of I . (The Legendre-Gauss formula is not much more difficult.)

Borwein and Borwein in [8, Algorithm 7.2] proposed the following method of computing $\log x$ for, e.g., x between 2 and 4: use I_1/I to compute two super-size logarithms, namely $\log 2^k$ and $\log 2^k x$ for an appropriate k . The use of I_1/I is optimal for computing one super-size logarithm, but it is suboptimal for computing two or more.

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