Integer factorization

Daniel J. Bernstein

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1 Introduction

1.1 Factorization problems. "The problem of distinguishing prime numbers from composite numbers, and of resolving the latter into their prime factors, is known to be one of the most important and useful in arithmetic," Gauss wrote in his *Disquisitiones Arithmeticae* in 1801. "The dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated."

But what exactly *is* the problem? It turns out that there are many different factorization problems, as discussed below.

1.2 Recognizing primes; finding prime factors. Do we want to distinguish prime numbers from composite numbers? Or do we want to find all the prime factors of composite numbers?

These are quite different problems. Imagine, for example, that someone gives you a 10000-digit composite number. It turns out that you can use "Artjuhov's generalized Fermat test"—the "sprp test"—to quickly write down a reasonably short proof that the number is in fact composite. However, unless you're extremely lucky, you won't be able to find all the prime factors of the number, even with today's state-of-the-art factorization methods.

1.3 Proven output; correct output. Do we care whether the answer is accompanied by a proof? Is it good enough to have an answer that's always correct but not accompanied by a proof? Is it good enough to have an answer that has never been observed to be incorrect?

Consider, for example, the "Baillie-Pomerance-Selfridge-Wagstaff test": if $n \in 3 + 40\mathbf{Z}$ is a prime number then $2^{(n-1)/2} + 1$ and $x^{(n+1)/2} + 1$ are both zero in the ring $(\mathbf{Z}/n)[x]/(x^2 - 3x + 1)$. Nobody has been able to find a composite $n \in 3+40\mathbf{Z}$ satisfying the same condition, even though such n's are conjectured to exist. (Similar comments apply to arithmetic progressions other than $3 + 40\mathbf{Z}$.) If both $2^{(n-1)/2} + 1$ and $x^{(n+1)/2} + 1$ are zero, is it unacceptable to claim that n is prime?

1.4 All prime divisors; one prime divisor; etc. Do we actually want to find *all* the prime factors of the input? Or are we satisfied with *one* prime divisor? Or *any* factorization?

More than 70% of all integers n are divisible by 2 or 3 or 5, and are therefore very easy to factor if we're satisfied with *one* prime divisor. On the other hand, some integers n have the form pq where p and q are primes; for these integers n, finding one factor is just as difficult as finding the complete factorization.

1.5 Small prime divisors; large prime divisors. Do we want to be able to find the

prime factors of every integer n? Or are we satisfied with an algorithm that gives up when n has large prime factors?

Some algorithms don't seem to care how large the prime factors are; they aren't the best at finding small primes but they compensate by finding large primes at the same speed. Example: "The Pollard-Buhler-Lenstra-Pomerance-Adleman number-field sieve" is conjectured to find the prime factors of n using $\exp((64/9 + o(1))^{1/3}(\log n)^{1/3}(\log\log n)^{2/3})$ bit operations.

Other algorithms are much faster at finding small primes—let's say primes in the range $\{2, 3, \ldots, y\}$, where y is a parameter chosen by the user. Example: "Lenstra's elliptic-curve method" is conjectured to find all of the small prime factors of n using $\exp\sqrt{(2 + o(1))(\log y)\log\log y}$ multiplications (and similar operations) of integers smaller than n. See Section 11.1.

1.6 Typical inputs; worst-case inputs; etc. More generally, consider an algorithm that tries to find the prime factors of n, and that has different performance (different speed; different success chance) for different inputs n. Are we interested in the algorithm's performance for *typical* inputs n? Or its *average* performance over all inputs n? Or its performance for *worst-case* inputs n, such as integers chosen by cryptographers to be difficult to factor?

Consider, as an illustration, the "Schnorr-Lenstra-Shanks-Pollard-Atkin-Rickert class-group method." This method was originally conjectured to find all the prime factors of n using $\exp\sqrt{(1 + o(1))(\log n)\log\log n}$ bit operations. The conjecture was later modified: the method seems to run into all sorts of trouble when n is divisible by the square of a large prime.

1.7 Conjectured speeds; proven bounds. Do we compare algorithms according to their *conjectured* speeds? Or do we compare them according to *proven* bounds?

The "Schnorr-Seysen-Lenstra-Lenstra-Pomerance class-group method" developed in [125], [129], [79], and [87] has been *proven* to find all prime factors of n using at most $\exp\sqrt{(1 + o(1))(\log n)\log\log n}$ bit operations. The number-field sieve is *conjectured* to be much faster, once n is large enough, but we don't even know how to prove that the number-field sieve *works* for every n, let alone that it's fast.

1.8 Serial; parallel. How much parallelism do we allow in our algorithms?

One variant of the number-field sieve uses $L^{1.18563\ldots+o(1)}$ seconds on a machine costing $L^{0.79042\ldots+o(1)}$ dollars. Here $L = \exp((\log n)^{1/3}(\log\log n)^{2/3})$. The machine contains $L^{0.79042\ldots+o(1)}$ tiny parallel CPUs carrying out a total of $L^{1.97605\ldots+o(1)}$ bit operations. The price-performance ratio of this computation is $L^{1.97605\ldots+o(1)}$ dollar-seconds. This variant is designed to minimize price-performance ratio.

Another variant—watch the first two exponents of L here—uses $L^{2.01147\ldots+o(1)}$ seconds on a *serial* machine costing $L^{0.74884\ldots+o(1)}$ dollars. The machine contains

$L^{0.74884...+o(1)}$ bytes of memory and a single serial CPU carrying out $L^{2.01147...+o(1)}$ bit operations. The price-performance ratio of this computation is $L^{2.76031...+o(1)}$ dollar-seconds. This variant is designed to minimize price-performance ratio for serial computations.

Another variant—watch the last two exponents of L here—uses $L^{1.90188...+o(1)}$ seconds on a serial machine costing $L^{0.95094...+o(1)}$ dollars. The machine contains $L^{0.95094...+o(1)}$ bytes of memory and a single serial CPU carrying out $L^{1.90188...+o(1)}$ bit operations. The price-performance ratio of this computation is $L^{2.85283...+o(1)}$ dollar-seconds. This variant is designed to minimize the number of bit operations.

1.9 One input; multiple inputs. Do we want to find the prime factors of just a single integer n? Or do we want to solve the same problem for many integers $n_1, n_2, \ldots$? Or do we want to solve the same problem for *as many of* $n_1, n_2, \ldots$ *as possible*?

One might guess that the most efficient way to handle many inputs is to handle each input separately. But "sieving" finds the small factors of many consecutive integers $n_1, n_2, \ldots$ with far fewer bit operations than handling each integer separately; see Section 12.1. Furthermore, recent factorization methods have removed the words "consecutive" and "small"; see, e.g., Sections 14.1 and 15.8.

2 Overview

2.1 The primary factorization problem. Even a year-long course can't possibly cover all the interesting factorization methods in the literature. In my lectures at the Arizona Winter School I'm going to focus on "congruence-combination" factorization methods, specifically the number-field sieve, which holds the speed records for real-world factorizations of worst-case inputs such as RSA moduli.

Here's how this fits into the spectrum of problems considered in Section 1:

- I don't merely want to know that the input n is composite; I want to know its prime factors.
- I'm much less concerned with proving the primality of the factors than with finding the factors in the first place.
- I want an algorithm that works well for every n—in particular, I don't want to give up on n's with large prime factors.
- I want to factor n as quickly as possible. I'm willing to sacrifice proven bounds on performance in favor of reasonable conjectures.
- I'm interested in parallelism to the extent that I have parallel machines.
- I might be interested in speedups from factoring many integers at once, but my primary concern is the already quite difficult problem of factoring a single large integer.

Sections 15 through 18 in these notes discuss congruence-combination methods, starting with the **Q** sieve and culminating with the number-field sieve.

2.2 The secondary factorization problem. A secondary factorization problem of a quite different flavor turns out to be a critical subroutine in "congruence-combination" algorithms. What these algorithms do is

- write down many "congruences" related to the integer n being factored;
- search for "fully factored" congruences; and then
- combine the fully factored congruences into a "congruence of squares" used to factor n.

The secondary factorization problem is the middle step here, the search for "fully factored" congruences. This step involves many inputs, not just one; it involves inputs typically much smaller than n; inputs with large prime factors can be, and are, thrown away; the problem is to find small factors as quickly as possible.

In my lectures at the Arizona Winter School I'll present the state of the art in small-factors algorithms, including the elliptic-curve method and a newer method relying on the Schönhage-Strassen FFT-based algorithm for multiplying billion-digit integers.

The student project attached to this course at the Arizona Winter School is to actually use computers to factor a bunch of integers! The project will take the secondary perspective: there are many integers to factor; integers with large prime factors—integers that aren't "smooth"—are thrown away; the problem is to find small factors as quickly as possible. We're going to write real programs, see which programs are fastest, and see which programs are most successful at finding factors.

Sections 6 through 14 in these notes discuss methods to find small factors of many inputs. One approach is to handle each input separately; Sections 6 through 11 discuss methods to find small factors of one input. Sections 12 through 14 discuss methods that gain speed by handling many inputs simultaneously.

2.3 Other resources. There are several books presenting integer-factorization algorithms: Knuth's *Art of computer programming*, Section 4.5.4; Cohen's *Course in computational algebraic number theory*, Chapter 10; Riesel's *Prime numbers and computer methods for factorization*; and, newest and generally most comprehensive, the Crandall-Pomerance book *Prime numbers: a computational perspective*. All of these books also cover the problems of recognizing prime numbers and proving primality. The Crandall-Pomerance book is highly recommended.

The costs of basic arithmetic

3 Multiplication

3.1 Bit operations. One can multiply two b-bit integers using $\Theta(b \lg 2b \lg\lg 4b)$ bit operations. Often I'll state speeds in less detail to focus on the exponent of b: multiplication uses $b^{1+o(1)}$ bit operations.

More precisely: For each integer $b \geq 1$, there is a circuit using $\Theta(b \lg 2b \lg\lg 4b)$ bit operations to multiply two integers in $\{0, 1, \ldots, 2^b - 1\}$, when each input is represented in the usual way as a b-bit string and the output is represented in the usual way as a $2b$-bit string. A "bit operation" means, by definition, the 2-bit-to-1-bit function $x, y \mapsto 1 - xy$; a "circuit" is a chain of bit operations.

This means that multiplication is not much slower than addition. Addition also uses $b^{1+o(1)}$ bit operations: more precisely, $\Theta(b)$ bit operations. For comparison, the most obvious multiplication methods use $\Theta(b^2)$ bit operations; multiplication methods using $\Theta(b \lg 2b \lg\lg 4b)$ bit operations are often called "fast multiplication."

The bound $b^{1+o(1)}$ for multiplication was first achieved by Toom in [139]. The bound $O(b \lg 2b \lg\lg 4b)$ was first achieved by Schönhage and Strassen in [128], using the fast Fourier transform (FFT). See my paper [20, Sections 2–4] for an exposition of the Schönhage-Strassen circuit.

3.2 Instructions. One can view the multiplication circuit in Section 3.1 as a series of $\Theta(b \lg 2b \lg\lg 4b)$ instructions for a machine that reads bits from various memory locations, performs simple operations on those bits, and writes the bits back to various memory locations. The circuit is "uniform": this means that one can, given b, easily compute the series of instructions.

The instruction counts in these notes don't take account of the possibility of changing the representation of integers to exploit more powerful instructions. For example, in many models of computation, a single instruction can operate on a w-bit "word" with $w \geq \lg 2b$; this instruction reduces the costs of both addition and multiplication by a factor of w if integers are appropriately recoded. Machine-dependent speedups often save time in real factorizations.

3.3 Serial price-performance ratio. The $b^{1+o(1)}$ instructions in Section 3.2 take $b^{1+o(1)}$ seconds on a serial machine costing $b^{1+o(1)}$ dollars for $b^{1+o(1)}$ bits of memory; the price-performance ratio is $b^{2+o(1)}$ dollar-seconds. More precisely, the $\Theta(b \lg 2b \lg\lg 4b)$ instructions take $\Theta(b \lg 2b \lg\lg 4b)$ seconds on a serial machine with $\Theta(b \lg 2b)$ bits of memory; the price-performance ratio is $\Theta(b^2 (\lg 2b)^2 \lg\lg 4b)$ dollar-seconds.

One can and should object to the notion that an instruction finishes in $\Theta(1)$ seconds. In any realistic model of a two-dimensional computer, $b^{1+o(1)}$ bits of memory are laid out in a $b^{0.5+o(1)} \times b^{0.5+o(1)}$ mesh, and random access to the mesh has a latency of $b^{0.5+o(1)}$ seconds; transmitting signals across a distance of $b^{0.5+o(1)}$ meters necessarily takes $b^{0.5+o(1)}$ seconds. However, in this computation, the memory accesses can be pipelined: $b^{0.5+o(1)}$ memory accesses take place during the same $b^{0.5+o(1)}$ seconds, hiding the latency of each access.

3.4 Parallel price-performance ratio. A better-designed multiplication machine of size $b^{1+o(1)}$ takes just $b^{0.5+o(1)}$ seconds, asymptotically much faster than any serial multiplication machine of the same size. The price-performance ratio is $b^{1.5+o(1)}$ dollar-seconds.

The critical point here is that a properly designed machine of size $b^{1+o(1)}$ can carry out $b^{1+o(1)}$ instructions in parallel. In $b^{0.5+o(1)}$ seconds, one can do much more than access $b^{0.5+o(1)}$ memory locations: all $b^{1+o(1)}$ parallel processors can exchange data with each other. See Section 5.4 below.

Details of a b-bit multiplication machine achieving this performance, size $b^{1+o(1)}$ and $b^{0.5+o(1)}$ seconds, were published by Brent and Kung in [33, Section 4]. Brent and Kung also proved that the limiting exponent 0.5 is optimal for an extremely broad class of 2-dimensional multiplication machines of size $b^{1+o(1)}$.

4 Division and greatest common divisor

4.1 Instructions. One can compute the quotient and remainder of two b-bit integers using $b^{1+o(1)}$ instructions: more precisely, $\Theta(b \lg 2b \lg \lg 4b)$ instructions, within a constant factor of the cost of multiplication.

One can compute the greatest common divisor $\gcd\{x, y\}$ of two b-bit integers x, y using $b^{1+o(1)}$ instructions: more precisely, $\Theta(b(\lg 2b)^2 \lg \lg 4b)$ instructions, within a logarithmic factor of the cost of multiplication.

Recall that the multiplication instructions described in Section 3.2 simulate the bit operations in a circuit. The same is not true for division and gcd: the instructions for division and gcd use not only bit operations but also data-dependent branches.

The bound $b^{1+o(1)}$ for division was first achieved by Cook in [46, pages 77–86]. The bound $\Theta(b \lg 2b \lg \lg 4b)$ was first achieved by Brent in [28]. See my paper [20, Sections 6–7] for an exposition.

The bound $b^{1+o(1)}$ for gcd was first achieved by Knuth in [76]. The bound $\Theta(b(\lg 2b)^2 \lg \lg 4b)$ was first achieved by Schönhage in [127]. See my paper [20, Sections 21–22] for an exposition.

4.2 Bit operations. One can compute the quotient, remainder, and greatest common divisor of two b-bit integers using $b^{1+o(1)}$ bit operations. I'm not aware of any literature stating, let alone minimizing, a more precise upper bound.

4.3 Serial price-performance ratio. Division and gcd, like multiplication, use $b^{1+o(1)}$ bits of memory. The $b^{1+o(1)}$ memory accesses in a serial computation can be pipelined, with more effort than in Section 3.3, to take $b^{1+o(1)}$ seconds. The price-performance ratio is $b^{2+o(1)}$ dollar-seconds.

4.4 Parallel price-performance ratio. Division, like multiplication, can be heavily parallelized. A properly designed division machine of size $b^{1+o(1)}$ takes just $b^{0.5+o(1)}$ seconds, asymptotically much faster than any serial division machine of the same size. The price-performance ratio is $b^{1.5+o(1)}$ dollar-seconds.

Parallelizing a gcd computation is a famous open problem. Every polynomial-sized gcd machine in the literature takes at least $b^{1+o(1)}$ seconds. The best known price-performance ratio is $b^{2+o(1)}$ dollar-seconds.

5 Sorting

5.1 Bit operations. One can sort n integers, each having b bits, into increasing order using $\Theta(bn \lg 2n)$ bit operations: more precisely, using a circuit of $\Theta(n \lg 2n)$ comparisons, each comparison consisting of $\Theta(b)$ bit operations.

History: Two sorting circuits with $n^{1+o(1)}$ comparisons, specifically $\Theta(n(\lg 2n)^2)$ comparisons, were introduced by Batcher in [15]; an older circuit, "Shell sort," was later proven to achieve the same $\Theta(n(\lg 2n)^2)$ comparisons. A circuit with $\Theta(n \lg 2n)$ comparisons was introduced by by Ajtai, Komlos, and Szemeredi in [6].

5.2 Instructions. One can sort n integers, each having b bits, into increasing order using $\Theta(bn)$ instructions. Note the lg factor improvement compared to Section 5.1; instructions are slightly more powerful than bit operations because they can use variables as memory addresses.

History: "Merge sort," "heap sort," et al. use $O(n \lg 2n)$ compare-exchange steps, where each compare-exchange step uses $\Theta(b)$ instructions. "Radix-2 sort" is not comparison-based and uses only $\Theta(bn)$ instructions. All of these algorithms are standard.

5.3 Serial price-performance ratio. The $\Theta(bn)$ instructions in Section 5.2 are easily pipelined to take $\Theta(bn)$ seconds on a serial machine with $\Theta(bn)$ bits of memory. The price-performance ratio is $\Theta(b^2n^2)$ dollar-seconds.

5.4 Parallel price-performance ratio. A better-designed sorting machine of size $\Theta(bn)$ takes just $\Theta(bn^{1/2})$ seconds, much faster than any serial sorting machine of the same size. The price-performance ratio is $\Theta(b^2n^{3/2})$ seconds.

History: Thompson and Kung in [138] showed that an $n^{1/2} \times n^{1/2}$ mesh of small processors can sort n numbers with $\Theta(n^{1/2})$ adjacent compare-exchange steps. Schnorr and Shamir in [126] presented an algorithm using $(3 + o(1))n^{1/2}$ adjacent compare-exchange steps. Schimmler in [124] presented a simpler algorithm using $(8 + o(1))n^{1/2}$ adjacent compare-exchange steps. I recommend that you start with Schimmler's algorithm if you're interested in learning parallel sorting algorithms.

Finding small factors of one integer

6 Trial division

6.1 Introduction. "Trial division" tries to factor a positive integer n by checking whether n is divisible by 2, checking whether n is divisible by 3, checking whether n is divisible by 4, checking whether n is divisible by 5, and so on. Trial division stops after checking whether n is divisible by y; here y is a positive integer chosen by the user.

If n is not divisible by $2, 3, 4, \ldots, d-1$, but turns out to be divisible by d, then trial division prints d as output and recursively attempts to factor n/d. The divisions by $2, 3, 4, \ldots, d-1$ can be skipped inside the recursion: n/d is not divisible by $2, 3, 4, \ldots, d-1$.

6.2 Example. Consider the problem of factoring $n = 314159265$. Trial division with $y = 10$ performs the following computations:

- 314159265 is not divisible by 2;
- 314159265 is divisible by 3, so replace it with $314159265/3 = 104719755$ and print 3;
- 104719755 is divisible by 3, so replace it with $104719755/3 = 34906585$ and print 3 again;
- 34906585 is not divisible by 3;
- 34906585 is not divisible by 4;
- 34906585 is divisible by 5, so replace it with 6981317 and print 5;
- 6981317 is not divisible by 5;
- 6981317 is not divisible by 6;
- 6981317 is not divisible by 7;
- 6981317 is not divisible by 8;
- 6981317 is not divisible by 9;
- 6981317 is not divisible by 10.

This computation has revealed that 314159265 is 3 times 3 times 5 times 6981317; the factorization of 6981317 is unknown.

6.3 Improvements. Each divisor d found by trial division is prime. Trial division will never output 6, for example. One can exploit this by skipping composites and checking only divisibility by primes; there are only about $y/\log y$ primes in $\{2, 3, 4, \ldots, y\}$.

If n has no prime divisor in $\{2, 3, \ldots, d-1\}$, and $1 < n < d^2$, then n must be prime; there is no point in checking divisibility of n by d or anything larger. One easy way to exploit this is by stopping trial division—and printing n as output—if the quotient $\lfloor n/d \rfloor$ is smaller than d. One can also use more sophisticated tests to check for primes n larger than d^2.

6.4 Speed. Trial division performs at most $y - 1 + r$ divisions if it discovers r factors of n. Each division is a division of n by an integer in $\{2, 3, 4, \ldots, y\}$. One can safely ignore the r here: r cannot be larger than $\lg n$, and normally y is chosen much larger than $\lg n$.

Skipping composites incurs the cost of enumerating the primes below y—using, for example, the techniques discussed in Section 12.3—but reduces the number of divisions to about $y/\log y$.

Stopping trial division at d when $\lfloor n/d \rfloor < d$ can eliminate many more divisions, depending on n, at the minor cost of a comparison.

Checking for a larger prime n can also eliminate many more divisions. The time for a primality test is, for an average n, comparable to roughly $\lg n$ divisions, so it doesn't add a noticeable cost if it's carried out after, e.g., $10 \lg n$ divisions.

6.5 Effectiveness. Trial division—with the $\lfloor n/d \rfloor < d$ improvement—is guaranteed to find the complete factorization of n if $y^2 > n$. In other words, trial division—with the $\lfloor n/d \rfloor < d$ improvement, and skipping composites—is guaranteed to succeed with at most about $\sqrt{n}/\log \sqrt{n}$ divisions. For example, if $n \approx 2^{100}$, then trial division is guaranteed to succeed with at most about $2^{44.9}$ divisions.

Often trial division succeeds with a much smaller value of y. For example, if $y = 10$, then trial division will completely factor any integer n of the form $2^a 3^b 5^c 7^d$; there are about $(\log x)^4/24(\log 2)(\log 3)(\log 5)(\log 7)$ such integers n in the range $[1, x]$. Trial division will also completely factor any integer n of the form $2^a 3^b 5^c 7^d p$ where p is one of the 21 primes between 11 and 100; there are about $21(\log x)^4/24(\log 2)(\log 3)(\log 5)(\log 7)$ such integers n in the range $[1, x]$. There are, for example, millions of integers n below 2^{100} that will be factored by trial division with $y = 10$.

6.6 Definition of smoothness. Integers n that factor completely into primes $\leq y$ (equivalently, that factor completely into integers $\leq y$; equivalently, that have no prime divisors $> y$) are called "y-smooth."

Integers n that factor completely into primes $\leq y$, except possibly for one prime $\leq y^2$, are called "(y, y^2)-semismooth." These integers are completely factored by trial division up through y.

There's a series of increasingly incomprehensible notations for more complicated factorization conditions.

6.7 The standard smoothness estimate. There are roughly x/u^u positive integers $n \leq x$ that are $x^{1/u}$-smooth, i.e., that are products of powers of primes $\leq x^{1/u}$.

For example, there are roughly $2^{90}/3^3$ positive integers $n \leq 2^{90}$ that are 2^{30}-smooth, and there are roughly $2^{300}/10^{10}$ positive integers $n \leq 2^{300}$ that are 2^{30}-smooth. All of these integers are completely factored by trial division with $y = 2^{30}$, i.e., with about $2^{30}/\log 2^{30} \approx 2^{25.6}$ divisions.

The u^u formula isn't extremely accurate. More careful calculations show that a uniform random positive integer $n \leq 2^{300}$—"uniform" means that each possibility is chosen with the same probability—has only about 1 chance in $3.3 \cdot 10^{10}$ of being 2^{30}-smooth, not 1 chance in 10^{10}. But the u^u formula is in the right ballpark.

7 Early aborts

7.1 Introduction. Here is a method that tries to completely factor n into small primes:

- Use trial division with primes $\leq 2^{15}$ to find some small factors of n, along with the unfactored part $n_1 = n/$those factors.
- Give up if $n_1 > 2^{100}$. This is an example of an "early abort."
- Use trial division with primes $\leq 2^{20}$ to find small factors of n_1.
- Give up if n_1 is not completely factored. This is a "final abort."

Of course, the second trial-division step can skip primes $\leq 2^{15}$.

More generally, given parameters $y_1 \leq y_2 \leq \cdots \leq y_k$ and $A_1 \geq A_2 \geq \cdots \geq A_{k-1}$, one can try to completely factor n as follows:

- Use trial division with primes $\leq y_1$ to find some small factors of n, along with the unfactored part n_1.
- Give up if $n_1 > A_1$.
- Use trial division with primes $\leq y_2$ to find some small factors of n_1, along with the unfactored part n_2.
- Give up if $n_2 > A_2$.
- ...
- Use trial division with primes $\leq y_k$ to find some small factors of n_{k-1}, along with the unfactored part n_k.
- Give up if n_k is not completely factored.

This is called "trial division with $k-1$ early aborts."

In subsequent sections we'll see several alternatives to trial division: the rho method, the $p-1$ method, etc. Early aborts combine all of these methods, with various parameter choices, into a grand unified method. What's interesting is that

the cost-effectiveness curve of the grand unified method is considerably better than all of the curves for the original methods.

7.2 Speed. There's a huge parameter space for trial division with multiple early aborts. How do we choose the prime bounds $y_1, y_2, \ldots$ and the aborts $A_1, A_2, \ldots$?

Define $T(y)$ as the average cost of trial division with primes $\leq y$. The average cost of multiple-early-abort trial division is then $T(y_1) + p_1 T(y_2) + p_1 p_2 T(y_3) + \cdots$ where p_1 is the probability that n survives the first abort, $p_1 p_2$ is the probability that n survives the first two aborts, etc.

One can choose the aborts to roughly balance the summands $T(y_1)$, $p_1 T(y_2)$, $p_1 p_2 T(y_3)$, etc.: e.g., to have $p_1 \approx T(y_1)/2T(y_2)$, to have $p_2 \approx T(y_2)/2T(y_3)$, etc., so that the total cost is only about $2T(y_1)$. Given many n's, one can simply look at the smallest n_i's and keep a fraction $p_i \approx T(y_i)/2T(y_{i+1})$ of them; one does not need to choose A_i in advance.

What about the prime bounds $y_1, y_2, \ldots$? An easy answer is to choose the prime bounds in geometric progression: $y_1 = y^{1/k}$, $y_2 = y^{2/k}$, $y_3 = y^{3/k}$, and so on through $y_k = y^{k/k} = y$. The total cost is then about $2T(y^{1/k})$. The user simply needs to choose y, the final prime bound, and k, the number of aborts.

What about the alternatives to trial division that we'll see later: rho, $p - 1$, etc.? We can simply redefine $T(y)$ as the average cost of our favorite method to find primes $\leq y$, and then proceed as above. The total cost will still be approximately $2T(y^{1/k})$, but most p_i's will be larger, allowing more n's to survive the aborts and improving the effectiveness of the algorithm.

These choices of y_i and A_i are certainly not optimal, but Pomerance's analysis in [110, Section 4], and the empirical analysis in [122], shows that they're in the right ballpark.

7.3 Effectiveness. Assume that n is a uniform random integer in the range $[1, x]$. Recall from Section 6.7 that n is y-smooth with probability roughly $1/u^u$ if $y = x^{1/u}$. The cost-effectiveness ratio for trial division with primes $\leq y$, without any early aborts, is roughly $u^u T(y)$.

The chance of n being completely factored by multiple-early-abort trial division, with the parameter choices discussed in Section 7.2, is smaller than $1/u^u$ by a factor of roughly $\sqrt{T(y_k)/T(y_1)} = \sqrt{T(y)/T(y^{1/k})}$. But the cost is reduced by a much larger factor, namely $T(y)/2T(y^{1/k})$. The cost-effectiveness ratio improves from roughly $u^u T(y)$ to roughly $2u^u \sqrt{T(y)T(y^{1/k})}$.

The ideas behind these estimates are due to Pomerance in [110, Section 4]. I've gone to some effort to extract comprehensible formulas out of Pomerance's complicated formulas.

7.4 Punctured early aborts. Recall the first example of an early abort from Section 7.1: trial-divide n with primes $\leq 2^{15}$; abort if the remaining part n_1 exceeds 2^{100};

trial-divide n_1 with primes $\leq 2^{20}$.

There is no point in keeping n_1 if it is between 2^{20} and 2^{30}, or between 2^{40} and 2^{45}: if n is 2^{20}-smooth then n_1 is a product of primes between 2^{15} and 2^{20}. There's also very little point in keeping n_1 if, for example, it's only slightly below 2^{40}. The lack of prime factors below 2^{15} means that the *smallest* integers are not the integers *most likely to be smooth*.

I'm not aware of any analysis of the speedup produced by this idea.

8 The rho method

8.1 Introduction. Define $\rho_1, \rho_2, \rho_3, \ldots$ by the recursion $\rho_{i+1} = \rho_i^2 + 10$, starting from $\rho_1 = 1$. The "rho method" tries to find a nontrivial factor of n by computing

$$\gcd\{n, (\rho_2 - \rho_1)(\rho_4 - \rho_2)(\rho_6 - \rho_3) \cdots (\rho_{2z} - \rho_z)\};$$

here z is a parameter chosen by the user.

There's nothing special about the number 10. The "randomized rho method" replaces 10 with a uniform random element of $\{3, 4, \ldots, n-3\}$.

We'll see that the rho method with $z \approx \sqrt{y}$ is about as effective as trial division with all primes $\leq y$; see Section 8.5. We'll also see that the rho method takes about $4z \approx 4\sqrt{y}$ multiplications modulo n; see Section 8.4. This is faster than $y/\log y$ divisions if y is not very small.

The rho method was introduced by Pollard in [107]. The name "rho" comes from the shape of the Greek letter ρ; the shape is often helpful in visualizing the graph with edges $\rho_i \bmod p \mapsto \rho_{i+1} \bmod p$, where p is a prime.

8.2 Example. The rho method, with $z = 3$, calculates $\rho_2 - \rho_1 = 10$; $\rho_4 - \rho_2 = 17160$; $\rho_6 - \rho_3 = 86932542660248880$; and finally $\gcd\{n, 10 \cdot 17160 \cdot 86932542660248880\}$.

To understand why this is effective, observe that $10 \cdot 17160 \cdot 86932542660248880 = 2^8 \cdot 3^5 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 53 \cdot 71 \cdot 101 \cdot 191 \cdot 1553$. The differences $\rho_{2i} - \rho_i$ constructed in the rho method have a surprisingly large number of small prime factors. See Section 8.5 for further discussion.

8.3 Improvements. Brent in [29] pointed out that one can reduce the cost of the rho method by about 25% by replacing the sequence $(2, 1), (4, 2), (6, 3), \ldots$ with another sequence. Montgomery in [92, Section 3] saved another 15% with a more complicated function of the ρ_i's.

Brent and Pollard in [34] replaced squaring with kth powering to save another constant factor in finding primes in $1 + k\mathbf{Z}$. Brent and Pollard used this method to find the factor 1238926361552897 of $2^{256} + 1$, completing the factorization of $2^{256} + 1$.

The gcd computed by the rho method could be equal to n. If $n = pq$, where p and q are primes, then the function $z \mapsto \gcd\{n, (\rho_2 - \rho_1) \cdots (\rho_{2z} - \rho z)\}$ typically increases from 1 to p to pq, or from 1 to q to pq. A binary search, evaluating the function at a logarithmic number of values of z, will quickly locate p or q. It's possible, but rare, for the function to instead jump directly from 1 to pq; one can react to this by restarting the randomized rho method with a new random number. Similar comments apply to integers n with more prime factors: the rho method could produce a composite gcd, but that gcd is reasonably easy to factor.

8.4 Speed. The integers ρ_i rapidly become very large: ρ_{2z} has approximately 2^{2z} digits. However, one can replace ρ_i with the much smaller integer $\rho_i \bmod n$, which satisfies the recursion $\rho_{i+1} \bmod n = ((\rho_i \bmod n)^2 + 10) \bmod n$.

Computing $\rho_1 \bmod n, \rho_2 \bmod n, \ldots, \rho_z \bmod n$ thus takes z squarings modulo n. Simultaneously computing $\rho_2 \bmod n, \rho_4 \bmod n, \ldots, \rho_{2z} \bmod n$ takes another $2z$ squarings modulo n. Computing $\rho_2 - \rho_1 \bmod n, \rho_4 - \rho_2 \bmod n, \ldots, \rho_{2z} - \rho_z \bmod n$ takes an additional z subtractions modulo n. Computing the product of these integers modulo n takes $z-1$ multiplications modulo n. And then there is one final gcd, taking negligible time if z is not very small.

All of this takes very little memory. One can save z squarings at the expense of $\Theta(z \lg n)$ bits of memory by storing $\rho_1 \bmod n, \rho_2 \bmod n, \ldots, \rho_z \bmod n$; this is slightly beneficial if you're counting instructions, but a disaster if you're measuring price-performance ratio.

8.5 Effectiveness. Consider an odd prime number p dividing n. What's the chance of a collision among the quantities $\rho_1 \bmod p, \rho_2 \bmod p, \ldots, \rho_z \bmod p$: an equation $\rho_i \bmod p = \rho_j \bmod p$ with $1 \le i < j \le z$?

There are $(p+1)/2$ possibilities for $\rho_i \bmod p$, namely the squares plus 10 modulo p. Choosing z independent uniform random numbers from a set of size $(p+1)/2$ produces a collision with high probability for $z \approx \sqrt{p}$, and a collision with probability approximately z^2/p for smaller z. The quantities $\rho_i \bmod p$ in the randomized rho method are not independent uniform random numbers, but experiments show that they collide about as often. See [13] if you're interested in what can be proven along these lines.

If a collision $\rho_i \bmod p = \rho_j \bmod p$ does occur then one also has $\rho_{i+1} \bmod p = \rho_{j+1} \bmod p$, $\rho_{i+2} \bmod p = \rho_{j+2} \bmod p$, etc. Consequently $\rho_k \bmod p = \rho_{2k} \bmod p$ for any integer $k \ge i$ that's a multiple of $j - i$. There's a good chance that $j - i$ has a multiple $k \in \{i, i+1, \ldots, z\}$; if so then p divides $\rho_{2k} - \rho_k$, and hence divides the gcd computed by the rho method.

One can thus reasonably conjecture that the gcd computed by the randomized rho method, after $\Theta(z)$ multiplications modulo n, has chance $\Theta(z^2/p)$ of being divisible by p. This doesn't mean that the gcd is equal to p, but one can factor a

composite gcd as discussed in Section 8.3, or one can observe that the gcd is usually equal to p for typical inputs.

For example, an average prime $p \leq 2^{50}$ will be found by the rho method with $z \approx 2^{25}$, using only about 2^{27} multiplications modulo n, much faster than trial division.

8.6 The fast-factorials method. The "fast-factorials method" was introduced by Strassen in [135], simplifying a method introduced by Pollard in [106, Section 2]. This method is *proven* to find all primes $\leq y$ using $y^{1/2+o(1)}(\log n)^{1+o(1)}$ instructions. But this method uses much more memory than the ρ method, and a more detailed speed analysis shows that it's slower than the ρ method. Skip this method unless you're interested in provable speeds.

9 The $p-1$ method

9.1 Introduction. The "$p-1$ method" tries to find a nontrivial factor of n by computing $\gcd\{n, 2^{E(y,z)} - 1\}$. Here y and z are parameters chosen by the user, and $E(y,z)$ is a product of powers of primes $\leq z$, the powers being chosen to be as large as possible while still being $\leq y$; in other words, $E(y,z)$ is the least common multiple of all z-smooth prime powers $\leq y$.

We'll see that the $p-1$ method finds *certain* primes $p \leq y$ at surprisingly high speed: specifically, if the multiplicative group $\mathbf{F}_p^*$ has z-smooth order, then p divides $2^{E(y,z)} - 1$. See Section 9.4.

The $p-1$ method is generally quite incompetent at finding other primes, so it doesn't very quickly factor typical smooth integers; see [121] for an asymptotic analysis. But later we'll see a broader spectrum of methods—first the $p+1$ method, then the elliptic-curve method with various parameters—that in combination can find all primes.

The $p-1$ method was introduced by Pollard in [106, Section 4].

9.2 Example. If $y = 10$ and $z = 10$ then $E(y,z) = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 8 \cdot 9 \cdot 5 \cdot 7 = 2520$. The $p-1$ method tries to find a nontrivial factor of n by computing $\gcd\{n, 2^{2520} - 1\}$.

To understand why this is effective, look at how many primes divide $2^{2520} - 1$: 3, 5, 7, 11, 13, 17, 19, 29, 31, 37, 41, 43, 61, 71, 73, 109, 113, 127, 151, 181, 211, 241, 281, 331, 337, 421, 433, 631, 1009, 1321, 1429, 2521, 3361, 5419, 14449, 21169, 23311, 29191, 38737, 54001, 61681, 86171, 92737, etc.

9.3 Speed. In the $p-1$ method, as in Section 8.4, one can replace $2^{E(y,z)}$ by $2^{E(y,z)} \bmod n$. Computing a qth power modulo n takes only about $\lg q = (\log q)/\log 2$ multiplications modulo n. so computing an $E(y,z)$th power takes only about $\lg E(y,z)$ multiplications modulo n.

Each prime $\leq z$ contributes a power $\leq y$ to $E(y,z)$, so $\lg E(y,z)$ is at most about $z(\lg y)/\log z$. The $p-1$ method thus takes only about $z(\lg y)/\log z$ multiplications modulo n. The final gcd has negligible cost if y and z are not very small.

9.4 Effectiveness. If p is an odd prime, $p-1 \leq y$, and $p-1$ is z-smooth, then p divides $2^{E(y,z)}-1$. The point is that $E(y,z)$ is a multiple of $p-1$: every prime power in $p-1$ is a power $\leq p-1 \leq y$ of a prime $\leq z$, and is therefore a divisor of $E(y,z)$. By Fermat's little theorem, p divides $2^{p-1}-1$, so p divides $2^{E(y,z)}-1$.

In particular, assume that p is an odd prime divisor of n, that $p-1 \leq y$, and that $p-1$ is z-smooth. Then p divides the gcd computed by the $p-1$ method. This does not necessarily mean that the gcd equals p, but a composite gcd can easily be handled as in Section 8.3.

The $p-1$ method often finds other primes, but let's uncharitably pretend that it's limited to the primes described above. How many odd primes $p \leq y+1$ have $p-1$ being z-smooth?

Typically z is chosen as roughly $\exp\sqrt{(1/2)\log y \log\log y}$: for example, $z \approx 2^6$ when $y \approx 2^{20}$, and $z \approx 2^8$ when $y \approx 2^{30}$. A uniform random integer in $[1,y]$ then has chance roughly $1/z$ of being z-smooth; this follows from the standard smoothness estimate in Section 6.7. A uniform random number $p-1$ in the same range is not a uniform random integer, but experiments show that it has approximately the same smoothness probability, roughly $1/z$.

To recap: There are roughly $y/(\lg y)\exp\sqrt{(1/2)\log y\log\log y}$ primes $\leq y$ that—if they divide n—can be found with roughly $\sqrt{(\lg y)}\exp\sqrt{(1/2)\log y\log\log y}$ multiplications modulo n.

For example, there are roughly 2^{76} primes $\leq 2^{100}$ that can be found with the $p-1$ method with 2^{24} multiplications. For comparison, the rho method finds only about 2^{44} different primes $\leq 2^{100}$ with 2^{24} multiplications.

9.5 Improvements. One can replace $E(y,z)$ by $E(z,z)$, reducing the number of multiplications to about z, without much loss of effectiveness: for typical pairs (y,z), most z-smooth numbers $\leq y$ have no prime powers above z. Actually, for the same $\approx z$ multiplications, one can afford $E(z^{3/2},z)$ or larger. I'm not aware of any serious analysis of the change in effectiveness: most of the literature simply uses $E(z,z)$.

One can multiply $2^{E(y,z)}-1$ by $2^{E(y,z)q}-1$ for several primes q above z. For example, one can multiply $2^{2520}-1$ by $2^{2520\cdot 11}-1$, $2^{2520\cdot 13}-1$, $2^{2520\cdot 17}-1$, and $2^{2520\cdot 19}-1$, before computing the gcd with n. Using this "second stage" for all primes q between z and $z\log z$ costs an extra $\approx z$ multiplications but finds many more primes p.

The "FFT continuation" in [97] and [93] pushes the upper limit $z\log z$ up to $z^{2+o(1)}$, still with the same number of instructions as performing z multiplications

modulo n. The point is that, within that number of instructions, FFT-based fast-multiplication techniques can evaluate the product modulo n of $2^{E(y,z)s} - 2^{E(y,z)t}$ over all $z^{2+o(1)}$ pairs $(s,t) \in S \times T$, where S and T each have size $z^{1+o(1)}$. By choosing S and T sensibly one can arrange for the differences $s - t$ to cover all primes q up to $z^{2+o(1)}$, forcing the product to be divisible by each $2^{E(y,z)q} - 1$. Beware that this computation needs much more memory than the usual second stage, so it is a disaster from the perspective of price-performance ratio.

10 The $p + 1$ method

10.1 Introduction. The "$p + 1$ method" tries to find a nontrivial factor of n by computing $\gcd\{n, (\alpha^{E(y,z)})_0 - 1\}$. Here α is a formal variable satisfying $\alpha^2 = 10\alpha - 1$; $(u + v\alpha)_0$ means u; $E(y,z)$ is defined as in the $p - 1$ method, Section 9.1; y and z are again parameters chosen by the user.

The $p + 1$ method has the same basic characteristics as the $p - 1$ method: it finds *certain* primes $p \leq y$ at surprisingly high speed. Specifically, p has a good chance of dividing $(\alpha^{E(y,z)})_0 - 1$ if the "twisted multiplicative group of $\mathbf{F}_p$," the set of norm-1 elements in a quadratic extension of $\mathbf{F}_p$, has smooth order. See Section 10.4.

What makes the $p + 1$ method interesting is that it finds *different* primes from the $p - 1$ method. Feeding n to both the $p - 1$ method and the $p + 1$ method is more effective than feeding n to either method alone; it is also more expensive than either method alone, but the increase in effectiveness is useful.

There's nothing special about the number 10 in the $p + 1$ method. One can try to further increase the effectiveness by trying several numbers in place of 10; but this idea has limited benefit. We'll see much larger gains in effectiveness from the elliptic-curve method; see Section 11.1.

The $p + 1$ method was introduced by Williams in [145].

10.2 Example. If $y = 10$ and $z = 10$ then $E(y,z) = 2520$. The $p+1$ method tries to find a nontrivial factor of n by computing $\gcd\{n, (\alpha^{2520})_0 - 1\}$. One has $\alpha^2 = 10\alpha - 1$, $\alpha^4 = (10\alpha - 1)^2 = 100\alpha^2 - 20\alpha + 1 = 980\alpha - 99$, etc. The prime divisors of $(\alpha^{2520})_0$ are 2, 3, 5, 7, 11, 13, 17, 19, 29, 41, 43, 59, 71, 73, 83, 89, 97, 109, 179, 211, 241, 251, 337, 419, 587, 673, 881, 971, 1009, 1259, 1901, 2521, 3361, 3779, 4549, 5881, 6299, 8641, 8819, 9601, 12889, 17137, 24571, 32117, 35153, 47251, 91009, etc.

10.3 Speed. The powers of α in the $p + 1$ method can be reduced modulo n, just like the powers of 2 in the $p - 1$ method. The $p + 1$ method requires about $z(\lg y)/\log z$ multiplications in the ring $(\mathbf{Z}/n)[\alpha]/(\alpha^2 - 10\alpha + 1)$. Each multiplication in that

ring takes a few multiplications in $\mathbf{Z}/n$, making the $p+1$ method a few times slower than the $p-1$ method.

10.4 Effectiveness. The $p+1$ method often finds the same primes as the $p-1$ method, namely primes p where $p-1$ is smooth. What is new is that it often finds primes p where $p+1$ is smooth.

Consider an odd prime p such that 10^2-4 is not a square modulo p; half of all primes p satisfy this condition. If $p+1 \leq y$ and $p+1$ is z-smooth then $p+1$ divides $E(y,z)$ so p divides $\alpha^{E(y,z)}-1$. The point is that the ring $\mathbf{F}_p[\alpha]/(\alpha^2-10\alpha+1)$ is a field; the pth power of α in that field is the conjugate of α, namely $10-\alpha$, so the $p+1$st power is the norm $\alpha(10-\alpha)=1$.

There are about as many primes p with $p+1$ smooth as primes p with $p-1$ smooth, so the $p+1$ method has about the same effectiveness as the $p-1$ method. What's important is that the $p-1$ method and the $p+1$ method find many *different* primes; one can profitably apply both methods to the same n.

10.5 Improvements. All of the improvements to the $p-1$ method from Section 9.5 can be applied analogously to the $p+1$ method.

There are also constant-factor speed improvements for exponentiation in the ring $(\mathbf{Z}/n)[\alpha]/(\alpha^2-10\alpha+1)$.

10.6 Other cyclotomic methods. One can generalize the $p-1$ and $p+1$ methods to a $\Phi_k(p)$ method that works with a kth-degree algebraic number α. Here Φ_k is the kth cyclotomic polynomial: $\Phi_3(p)=p^2+p+1$, for example, and $\Phi_4(p)=p^2+1$. See [14] for details.

Unfortunately, p^2+p+1 and p^2+1 and so on have much smaller chances of being smooth than $p-1$ and $p+1$, so the higher-degree cyclotomic methods are much less effective than the $p-1$ method and the $p+1$ method. In contrast, the elliptic-curve method—see Section 11.1—is just as effective and only slightly slower.

11 The elliptic-curve method

11.1 Introduction. The "elliptic-curve method" defines integers $x_1, d_1, x_2, d_2, \ldots$ as follows:

$$\begin{aligned} x_1 &= 2 \\ d_1 &= 1 \\ x_{2i} &= (x_i^2 - d_i^2)^2 \\ d_{2i} &= 4x_i d_i (x_i^2 + a x_i d_i + d_i^2) \\ x_{2i+1} &= 4(x_i x_{i+1} - d_i d_{i+1})^2 \\ d_{2i+1} &= 8(x_i d_{i+1} - d_i x_{i+1})^2 \end{aligned}$$

Here $a \in \{6, 10, 14, 18, \ldots\}$ is a parameter chosen by the user.

The elliptic-curve method tries to find a nontrivial factor of n by computing $\gcd\{n, d_{E(y,z)}\}$. Here y, z are parameters chosen by the user, and $E(y, z)$ is defined as in Section 9.1.

For any particular choice of a, the elliptic-curve method has about the same effectiveness as the $p - 1$ method and the $p + 1$ method. It finds *certain* primes $p \leq y$ at surprisingly high speed. Specifically, if the "group of $\mathbf{F}_p$-points on the elliptic curve $(4a+10)y^2 = x^3+ax^2+x$" has z-smooth order, then p divides $d_{E(y,z)}$. See Sections 11.3 and 11.5.

What's interesting about the elliptic-curve method is that the group order varies in a seemingly random fashion as a changes, while always staying close to p. One can find most—conjecturally all—primes $\leq y$ by trying a moderate number of different values of a.

The elliptic-curve method was introduced by Lenstra in [85]. I use a variant introduced by Montgomery in [92]; the variant has the benefits of being faster (by a small constant factor) and allowing a shorter description (again by a small constant factor). Various implementation results appear in [30], [31], [8], [132], [27], and [32]; for example, [32] reports the use of the elliptic-curve method to find the 132-bit factor 4659775785220018543264560743076778192897 of $2^{1024} + 1$, completing the factorization of $2^{1024} + 1$.

11.2 Example. If $a = 6$ then the elliptic-curve method computes $(x_1, d_1) = (2, 1)$; $(x_2, d_2) = (9, 136)$; $(x_4, d_4) = (339112225, 126909216)$; etc. The prime divisors of d_{2520} are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 37, 47, 59, 61, 71, 79, 83, 89, 101, 107, 127, 137, 139, 157, 167, 179, 181, 193, 197, 233, 239, 251, 263, 353, 359, 397, 419, 503, 577, 593, 677, 719, 761, 769, 839, 1009, 1259, 1319, 1399, 1549, 1559, 1741, 1877, 1933, 1993, 2099, 2447, 3137, 3779, 5011, 5039, 5209, 6073, 6337, 6553, 6689, 7057, 7841, 7919, 8009, 8101, 9419, 9661, 10259, 13177, 14281, 14401, 14879, 15877, 16381, 16673, 17029, 18541, 19979, 20719, 21313, 27827, 29759, 32401, 32957, 35449, 35617, 37489, 38921, 42299, 46273, 62207, 62273, 70393, 70729, 79633, 87407, 92353, 93281, 93493, etc.

11.3 Algebraic structure. Let p be an odd prime not dividing $(a^2 - 4)(4a + 10)$. Consider the set $\{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p : (4a + 10)y^2 = x^3 + ax^2 + x\} \cup \{\infty\}$. There is a unique group structure on this set satisfying the following conditions:

- $\infty = 0$ in the group.
- $(x, y) + (x, -y) = 0$ in the group.
- If $P, Q, R \in \{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p : (4a + 10)y^2 = x^3 + ax^2 + x\}$ are distinct and collinear then $P + Q + R = 0$ in the group.
- If $P, Q \in \{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p : (4a + 10)y^2 = x^3 + ax^2 + x\}$ are distinct, and the tangent line at P passes through Q, then $2P + Q = 0$ in the group.

This group is the "group of $\mathbf{F}_p$-points on the elliptic curve $(4a+10)y^2 = x^3 + ax^2 + x$." The group order depends on a but is always between $p+1-2\sqrt{p}$ and $p+1+2\sqrt{p}$.

The recursive computation of x_i, d_i in the elliptic-curve method is tantamount to a computation of the ith multiple of the element $(2,1)$ of this group: if p does not divide d_i then the ith multiple of $(2,1)$ is $(x_i/d_i, y_i/d_i)$ for some y_i. What's important is the contrapositive: p has to divide d_i if i is a multiple of the order of $(2,1)$ in the elliptic-curve group. It's also possible (as far as I know), but rare, for p to divide d_i in other cases.

11.4 Speed. As in previous methods, one can replace x_i by $x_i \bmod n$ and replace d_i by $d_i \bmod n$, so each multiplication is a multiplication modulo n.

Assume that $(a-2)/4$ is small, so that multiplication by $(a-2)/4$ has negligible cost. The formulas

$$x_{2i} = (x_i - d_i)^2 (x_i + d_i)^2$$
$$d_{2i} = ((x_i + d_i)^2 - (x_i - d_i)^2) \left((x_i + d_i)^2 + \frac{a-2}{4} ((x_i + d_i)^2 - (x_i - d_i)^2) \right)$$

then produce (x_{2i}, d_{2i}) from (x_i, d_i) using 2 squarings and and 2 additional multiplications. The formulas

$$x_{2i+1} = ((x_i - d_i)(x_{i+1} + d_{i+1}) + (x_i + d_i)(x_{i+1} - d_{i+1}))^2$$
$$d_{2i+1} = 2((x_i - d_i)(x_{i+1} + d_{i+1}) - (x_i + d_i)(x_{i+1} - d_{i+1}))^2$$

produce (x_{2i+1}, d_{2i+1}) from $(x_i, d_i), (x_{i+1}, d_{i+1})$ using 2 squarings and 2 additional multiplications. Recursion produces any $(x_i, d_i), (x_{i+1}, d_{i+1})$ from $(x_1, d_1), (x_2, d_2)$ with 4 squarings and 4 additional multiplications for each bit of i.

In particular, the elliptic-curve method obtains $d_{E(y,z)}$ with 4 squarings and 4 additional multiplications for each bit of $E(y,z)$. For comparison, recall from Section 9.3 that the $p-1$ method uses about 1 squaring for each bit.

11.5 Effectiveness. The elliptic-curve method with any particular a—for example, with the elliptic curve $34y^2 = x^3 + 6x^2 + x$—has about the same effectiveness as the $p-1$ method. The number of primes $p \le y$ with a smooth group order of the curve $34y^2 = x^3 + 6x^2 + x$ is—conjecturally, and in experiments—about the same as the number of primes $p \le y$ with $p-1$ smooth.

What's important is that one can profitably use many elliptic curves with many different choices of a. The cost grows linearly with the number of choices, but the effectiveness also grows linearly until *most* primes $p \le y$ are found. By trying roughly $\exp \sqrt{(1/2) \log y \log \log y}$ choices of a one finds most—conjecturally all—primes $\le y$ with roughly $\exp \sqrt{2 \log y \log \log y}$ multiplications modulo n.

The obvious conjectures regarding the effectiveness of the elliptic-curve method can to some extent be proven when a is chosen randomly. See, e.g., [85], [73], and [87, Section 7].

11.6 Improvements. All of the improvements to the $p-1$ method from Section 9.5 can be applied analogously to the elliptic-curve method.

The elliptic-curve method, for one a, is several times slower than the $p-1$ method, so one should try the $p-1$ method and perhaps the $p+1$ method before trying the elliptic-curve method.

Using several elliptic curves $y^2 = x^3 - 3x + a$, with "affine coordinates" and the batch-inversion idea of [92, Section 10.3.1], takes only $7 + o(1)$ multiplications per bit rather than 8. On the other hand, only 1 of the 7 multiplications is a squaring, and the $o(1)$ is quite noticeable; there doesn't seem to be any improvement in bit operations, or instructions, or price-performance ratio.

11.7 The hyperelliptic-curve method. The hyperelliptic-curve method, introduced by Lenstra, Pila, and Pomerance in [86], is *proven* to find all primes $\leq y$ with overwhelming probability using $f(y)$ multiplications modulo n, where f is a particular subexponential function. The hyperelliptic-curve method is much slower than the elliptic-curve method; skip it unless you're interested in provable speeds.

Finding small factors of many integers

12 Consecutive integers: sieving

12.1 Introduction. "Sieving" tries to factor positive integers $n+1, n+2, \ldots, n+m$ by generating in order of p, and then sorting in order of i, all pairs (i,p) such that $p \leq y$ is prime, $i \in \{1,2,\ldots,m\}$, and p divides $n+i$. Here y is a parameter chosen by the user.

Generating the pairs (i,p) for a single p is a simple matter of writing down $(p-(n \bmod p),p)$ and $(2p-(n \bmod p),p)$ and $(3p-(n \bmod p),p)$ and so on until the first component exceeds m.

The sorted list shows all the primes $\leq y$ dividing $n+1$, then all the primes $\leq y$ dividing $n+2$, then all the primes $\leq y$ dividing $n+3$, etc. One can compute the unfactored part of each $n+i$ by division, with extra divisions to check for repeated factors.

Sieving challenges the notion that m input integers should be handled by m separate computations. Sieving using all primes $\leq y$ produces the same results for each of the m input integers as trial division using all primes $\leq y$; but for large m sieving uses fewer bit operations, and fewer instructions, than any known method to handle each input integer separately. See Section 12.5. On the other hand, sieving is much less impressive from the perspective of price-performance ratio. See Section 12.6.

12.2 Example. The chart below shows how sieving, with $y = 10$, tries to factor the positive integers $1001, 1002, 1003, \ldots, 1020$. One generates the pairs $(2,2),(4,2),\ldots$ for $p=2$, then the pairs $(2,3),(5,3),\ldots$ for $p=3$, then the pairs $(5,5),(10,5),\ldots$ for $p=5$, then the pairs $(1,7),(8,7),\ldots$ for $p=7$. Sorting the pairs by the first component produces

$$(1,7),(2,2),(2,3),(4,2),(5,3),(5,5),(6,2),(8,2),(8,3),(8,7),\ldots,$$

revealing (for example) that 1008 is divisible by $2,3,7$. Dividing 1008 repeatedly by 2, then 3, then 7, reveals that $1008 = 2^4 \cdot 3^2 \cdot 7$.

1001				$(1,7)$
1002	$(2,2)$	$(2,3)$		
1003				
1004	$(4,2)$			
1005		$(5,3)$	$(5,5)$	
1006	$(6,2)$			
1007				
1008	$(8,2)$	$(8,3)$		$(8,7)$
1009				
1010	$(10,2)$		$(10,5)$	
1011		$(11,3)$		
1012	$(12,2)$			
1013				
1014	$(14,2)$	$(14,3)$		
1015			$(15,5)$	$(15,7)$
1016	$(16,2)$			
1017		$(17,3)$		
1018	$(18,2)$			
1019				
1020	$(20,2)$			

12.3 The sieve of Eratosthenes. One common use of sieving is to generate a list of primes in an interval.

Choose $y > \sqrt{n+m}$; then an integer in $\{n+1, n+2, \ldots, n+m\}$ is prime if and only if it has no prime divisors $\leq y$. Sieve the integers $\{n+1, n+2, \ldots, n+m\}$ using all primes $\leq y$. The output shows whether $n+1$ is prime, whether $n+2$ is prime, etc.

In this application, one can skip the final divisions in sieving. One can also eliminate the second component of the pairs being sorted. One can also replace $\{n+1, n+2, \ldots, n+m\}$ by the subset of odd integers, or the subset of integers coprime to 6, or more generally the subset of integers coprime to w, where w is a product of very small primes; this idea saves a logarithmic factor in cost when w is optimized. One can enumerate "norms from quadratic number fields," rather than "norms from $\mathbf{Q} \times \mathbf{Q}$," to save another logarithmic factor; see [9].

12.4 Sieving for smooth numbers. Another common use of sieving is to find y-smooth integers in an interval. Recall that 1008 was detected as being 10-smooth in the example in Section 12.2.

One can sieve with all prime powers $\leq y^2$ of primes $\leq y$, rather than just sieving with the primes. The benefit is that most of the final divisibility tests are

eliminated: if $n+i$ has the pair (i,p) but not (i,p^2) then one does not need to try dividing $n+i$ by p^2. The cost is small: there are only about twice as many prime powers as primes.

One can skip all of the final divisibility tests and simply print the integers $n+i$ that have collected enough pairs. Imagine, for example, sorting all $(i,p^j,\lg p)$, and then adding the numbers $\lg p$ for each i: the sum of $\lg p$ is $\lg(n+i)$ if and only if $n+i$ is y-smooth. One can safely use low-precision approximations to the logarithms here: the sum of $\lg p$ is below $\lg(n+i)-\lg y$ if $n+i$ is not y-smooth. An integer $n+i$ divisible by a high power of a prime may be missed if one limits the exponent j used in sieving, but this loss of effectiveness is usually outweighed by the gain in speed.

12.5 Speed, counting bit operations. The interval $n+1,n+2,\ldots,n+m$ has about $m/2$ multiples of 2, about $m/3$ multiples of 3, about $m/5$ multiples of 5, etc. The total number of pairs to be sorted is about $\sum_{p\le y}(m/p) \approx m \log\log y$. Each pair has $\lg ym$ bits. Sorting all the pairs takes $\Theta(m \lg(m\log\log y)\lg ym\log\log y)$ bit operations; i.e., $\Theta(\lg(m\log\log y)\lg ym\log\log y)$ bit operations per input. One can safely assume that m is at most $y^{O(1)}$—otherwise the input integers can be profitably partitioned—so the sorting uses $\Theta((\lg y)^2\log\log y)$ bit operations per input.

The initial computation of $n \bmod p$ for each prime $p \le y$ uses as many bit operations as worst-case trial division, but this computation is independent of m and becomes unnoticeable as m grows past $y^{1+o(1)}$.

Final divisions, to compute the unfactored part of each input, can be done in $(\lg(n+m))^{1+o(1)}$ bit operations per input, by techniques similar to those explained in Section 14.3. Typically $\lg(n+m) \in (\lg y)^{2+o(1)}$, so sieving for large m uses a total of $(\lg y)^{2+o(1)}$ bit operations per input.

For comparison, recall from Section 6.4 that trial division often performs about $y/\log y$ divisions for a single input. Trial division can finish sooner, but for typical inputs it doesn't. Each division uses $(\lg n)^{1+o(1)}$ bit operations, so the total number of bit operations per input is $(y/\log y)(\lg n)^{1+o(1)}$; i.e., $y^{1+o(1)}$, when y is not very small. The rho method takes only $y^{1/2+o(1)}$ bit operations per input, and the elliptic-curve method takes only $\exp\sqrt{(2+o(1))\log y\log\log y}$ bit operations per input, but no known single-input method takes $(\lg y)^{O(1)}$ bit operations per average input.

12.6 Speed, measuring price-performance ratio. Sieving handles m inputs in only $m^{1+o(1)}$ bit operations when m is large, but it also uses $m^{1+o(1)}$ bits of memory, for a price-performance ratio of $m^{2+o(1)}$ dollar-seconds.

One can do much better with a parallel machine, as in Section 5.4. A sieving machine costing $m^{1+o(1)}$ dollars can generate the $m^{1+o(1)}$ pairs in parallel in $m^{0+o(1)}$

seconds, and can sort them in $m^{0.5+o(1)}$ seconds, for a price-performance ratio of $m^{1.5+o(1)}$ dollar-seconds.

For comparison, a machine applying the elliptic-curve method to each input separately, with m parallel elliptic-curve processors, costs $m^{1+o(1)}$ dollars and takes only $m^{0+o(1)}$ seconds, for a price-performance ratio of $m^{1+o(1)}$ dollar-seconds.

From this price-performance perspective, sieving is a very bad idea when m is large. It is useful for very small m and very small y, finding very small primes more quickly than separate trial divisions, but the elliptic-curve method is better at finding larger primes.

12.7 Comparing cost measures. Sections 12.5 and 12.6 used different cost measures and came to quite different conclusions about the value of sieving. Section 12.5 counted bit operations, and concluded that sieving was much better than the elliptic-curve method for finding large primes. Section 12.6 measured price-performance ratio, and concluded that sieving was much worse than the elliptic-curve method for finding large primes.

The critical difference between these cost measures—at least for a sufficiently parallelizable computation—is in the cost that they assign to long wires between bit operations; equivalently, in the cost that they assign to instructions that access large arrays. Measuring price-performance ratio makes each access to an array of size $m^{1+o(1)}$ look as expensive as $m^{0.5+o(1)}$ bit operations. Counting instructions makes each access look as inexpensive as a single bit operation.

Readers who grew up in a simple instruction-counting world might wonder whether price-performance ratio is a useful concept. Do real computers actually take longer to access larger arrays? The answer, in a nutshell, is yes. Counting instructions is a useful simplification for small computers carrying out small computations, but it is misleading for large computers carrying out large computations, as the following example demonstrates.

Today's record-setting computations are carried out on a very large, massively parallel computer called the "Internet." The machine has about 2^{60} bytes of storage spread among millions of nodes (often confusingly also called "computers"). A node can randomly access a small array in a few nanoseconds but needs much longer to access larger arrays. A typical node, one of the "Athlons" in my office in Chicago, has

- 2^{16} bytes of "L1 cache" that can stream data at about 2^0 nanoseconds/byte with a latency of 2^2 nanoseconds;
- 2^{18} bytes of "L2 cache" that can stream data at about 2^1 nanoseconds/byte with a latency of 2^4 nanoseconds;
- 2^{29} bytes of "dynamic RAM" that can stream data at about 2^4 nanoseconds/byte with a latency of 2^7 nanoseconds;

- 2^{36} bytes of "disk" that can stream data at about 2^7 nanoseconds/byte with a latency of 2^{23} nanoseconds;
- a "local network" providing relatively slow access to 2^{40} bytes of storage on nearby nodes; and
- a "wide-area network" providing even slower access to the entire Internet.

Sieving is the method of choice for finding primes $\approx 2^{10}$, is not easy to beat for finding primes $\approx 2^{20}$, and remains tolerable for primes $\approx 2^{30}$, but is essentially never used to find primes $\approx 2^{40}$. Record-setting factorizations use sieving for small primes and then switch to other methods, such as the elliptic-curve method, to find larger primes.

13 Consecutive polynomial values: sieving revisited

13.1 Introduction. Sieving can be generalized from consecutive integers to consecutive values of a low-degree polynomial. For example, one can factor positive integers $(n+1)^2 - 10, (n+2)^2 - 10, (n+3)^2 - 10, \ldots, (n+m)^2 - 10$ by generating in order of p, and then sorting in order of i, all pairs (i, p) such that $p \leq y$ is prime, $i \in \{1, 2, \ldots, m\}$, and p divides $(n+i)^2 - 10$. As usual, y is a parameter chosen by the user.

The point is that, for any particular prime p, the pairs (i, p) are a small union of easily computed arithmetic progressions modulo p. Consider, for example, the prime $p = 997$. The polynomial $x^2 - 10$ in $\mathbf{F}_p[x]$ factors as $(x - 134)(x - 863)$; so 997 divides $(n+i)^2 - 10$ if and only if $n + i \bmod 997 \in \{134, 863\}$.

13.2 Speed. Sieving m values of an arbitrary polynomial takes almost as few bit operations as sieving m consecutive integers. Factoring the polynomial modulo each prime $p \leq y$ takes more work than a single division by p, but this extra work becomes unnoticeable as m grows.

14 Arbitrary integers: exploiting fast multiplication

14.1 Introduction. Sieving finds small factors of a batch of m numbers more quickly than finding small factors of each number separately—if the numbers are consecutive integers, as in Section 12.1, or more generally consecutive values of a low-degree polynomial, as in Section 13.1. What about numbers that don't have such a simple relationship?

It turns out that, by taking advantage of fast multiplication, one can find small factors of m *arbitrary* integers more quickly than finding small factors of each

integer separately. Specifically, one can find all primes $\leq y$ dividing each integer in just $(\lg y)^{O(1)}$ bit operations, if there are at least $y/(\lg y)^{O(1)}$ integers, each having $(\lg y)^{O(1)}$ bits.

This batch-factorization algorithm starts from a set P of primes and a nonempty set S of nonzero integers to be factored. It magically figures out the set Q of primes in P relevant to S. If $\#S = 1$ then the algorithm is done at this point: it prints Q, S and stops. Otherwise the algorithm splits S into two halves, say T and $S - T$; it recursively handles Q, T; and it recursively handles $Q, S - T$.

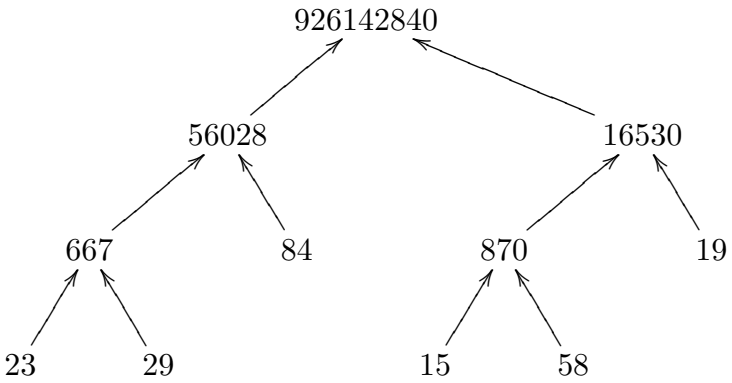
How does the algorithm magically figure out which primes in P are relevant to S? Answer: The algorithm uses a "product tree" to quickly compute a huge number, namely the product of all the elements of S. It then uses a "remainder tree" to quickly compute this product modulo each element of P. Computing Q is now a simple matter of sorting: an element of P is relevant to S if and only if the corresponding remainder is 0.

If S has y bits then the product-tree computation takes $y(\lg y)^{2+o(1)}$ bit operations; see Section 14.2. The remainder-tree computation takes $y(\lg y)^{2+o(1)}$ bit operations; see Section 14.3. The batch-factorization algorithm takes $y(\lg y)^{3+o(1)}$ bit operations; see Section 14.4. An alternate algorithm uses only $y(\lg y)^{2+o(1)}$ bit operations to figure out which integers are y-smooth, or more generally to compute the y-smooth part of each integer; see Section 14.5.

I introduced the batch-factorization algorithm in [21]. The improved algorithm for smoothness is slightly tweaked from an algorithm of Franke, Kleinjung, Morain, and Wirth; see [22]. There are many more credits that I should be giving here—for example, fast remainder trees were introduced in the 1970s by Fiduccia, Borodin, and Moenck—but I'll simply point you to the "history" parts of my paper [20].

14.2 Product trees. The "product tree of $x_1, x_2, \ldots, x_m$" is a tree whose root is the product $x_1 x_2 \cdots x_m$; whose left subtree, for $m \geq 2$, is the product tree of $x_1, x_2, \ldots, x_{\lceil m/2 \rceil}$; and whose right subtree, for $m \geq 2$, is the product tree of $x_{\lceil m/2 \rceil + 1}, \ldots, x_{m-1}, x_m$.

For example, here is the product tree of $23, 29, 84, 15, 58, 19$:

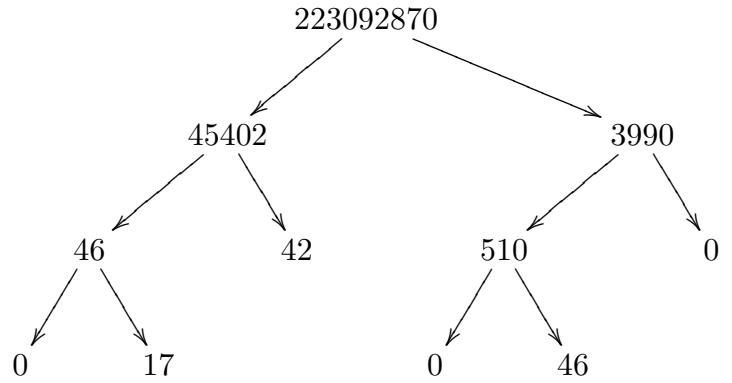


One can compute each non-leaf node by multiplying its children. There are approximately $\lg 2m$ levels in the tree, and the multiplications at each level use $O(b \lg 2b \lg\lg 4b)$ bit operations if $x_1, x_2, \ldots, x_m$ have b bits in total, so the product-tree computation uses $O(b \lg 2m \lg 2b \lg\lg 4b)$ bit operations.

A recent idea of Robert Kramer, which I call "FFT doubling," eliminates $1/3 + o(1)$ of the cost of computing a product tree.

14.3 Remainder trees. The "remainder tree for $Y, x_1, x_2, \ldots, x_m$" has one node Y mod X for each node X in the product tree of $x_1, x_2, \ldots, x_m$.

For example, here is the remainder tree of $223092870, 23, 29, 84, 15, 58, 19$:



After computing the product tree, one can compute each non-root node of the remainder tree by reducing its parent modulo the corresponding node of the product tree. Overall the remainder-tree computation uses $O(b \lg 2m \lg 2b \lg\lg 4b)$ bit operations if there are b bits total in $Y, x_1, x_2, \ldots, x_m$.

There are several ways to save time in computing remainder trees. A "scaled remainder tree," described in [23], is only $3/2 + o(1)$ times as expensive as a product tree, if the product tree is already computed.

14.4 Speed of batch factorization. Let's say there are b bits in the set S to be factored. The sets $Q, T, S - T$ that appear at the next level of recursion together

have at most about $2b$ bits. The crucial point is that the elements of Q are coprime and divide the product of the elements of S, so the product of the elements of Q divides the product of the elements of S, so the total length of Q is at most about the total length of S.

The sets that appear at the next level of recursion also have at most about $2b$ bits: the sets that appear in handling Q, T have at most about twice as many bits as T, and the sets that appear in handling $Q, S - T$ have at most about twice as many bits as $S - T$.

Similar comments apply at each level of recursion. There are about $\lg 2m$ levels of recursion, each handling a total of about $2b$ bits, for a total of $2b \lg 2m$ bits. The product-tree and remainder-tree computations use $O(\lg 2my \lg 2b \lg \lg 4b)$ bit operations for each bit they're given, for a total of $O(b \lg 2m \lg 2my \lg 2b \lg \lg 4b)$ bit operations.

There are several constant-factor improvements here. The product tree of S can be reused for T and $S - T$. One can split S into more than two pieces; this is particularly helpful when Q is noticeably smaller than S. One should also remove extremely small primes from S in advance by a different method.

14.5 Faster smoothness detection. Define Y as the product of all primes $\leq y$. One can compute the y-smooth part of n as $\gcd\{n, (Y \bmod n)^{2^k} \bmod n\}$ where $k = \lceil \lg \lg n \rceil$. In particular, n is y-smooth if and only if $(Y \bmod n)^{2^k} \bmod n = 0$.

Computing Y with a product tree takes $y(\lg y)^{2+o(1)}$ bit operations. Computing $Y \bmod n$ for many integers n with a remainder tree takes $y(\lg y)^{2+o(1)}$ bit operations if the n's together have approximately y bits. The final k squarings modulo n, and the gcd, take negligible time if each n is small.

Optional extra step: If there are not very many smooth integers then finding their prime factors, by feeding them to the batch factorization algorithm, takes negligible extra time.

14.6 Combining methods. Recall from Sections 12.6 and 12.7 that sieving has an unimpressive price-performance ratio as y grows—it uses a great deal of memory, making it asymptotically more expensive than the elliptic-curve method. The same comment applies to the batch-factorization algorithm.

Recall from Section 7 that many methods can be combined into a unified method that's better than any of the original methods. One could, for example, use trial division to find primes $\leq 2^3$; then use sieving—if the inputs are sieveable—to find primes $\leq 2^{15}$; then abort large results; then use batch factorization to find primes $\leq 2^{28}$; then abort large results; then use rho to find primes $\leq 2^{32}$; then abort large results; then use $p - 1$, $p + 1$, and the elliptic-curve method to find primes $\leq 2^{40}$. Actually, I speculate that batch factorization should be used for considerably larger primes, but I still have to finish my software for disk-based integer multiplication.

Finding large factors of one integer

15 The Q sieve

15.1 Introduction. The "**Q** sieve" tries to find a nontrivial factor of n by computing $\gcd\{n, \prod_{i\in S} i - \sqrt{\prod_{i\in S} i(n+i)}\}$. Here S is a nonempty set of positive integers such that $\prod_{i\in S} i(n+i)$ is a square. I'll refer to the integer $i(n+i)$ as a "congruence," alluding to the simple congruence $i \equiv n + i \pmod{n}$.

To find qualifying sets S—i.e., to find square products of congruences—the **Q** sieve looks at many congruences, finds some that are easily factored into primes, and then performs linear algebra on exponent vectors modulo 2 to find a square product of fully factored congruences.

Section 15.3 discusses the number of congruences that need to be considered before a square product appears. Section 15.4 discusses the cost of finding fully factored congruences. Section 15.5 discusses the cost of combining fully factored congruences by linear algebra. The final computation of $\prod_{i\in S} i - \sqrt{\prod_{i\in S} i(n+i)}$ can be done with a product tree as described in Section 14.2, or with the help of the known factorizations of the congruences $i(n+i)$; either way, it has negligible cost and isn't worth discussing further.

Sometimes the final gcd is 1 or n. In this case the **Q** sieve throws away one of the i's in S (to guarantee that S won't be seen again), looks at some new i's, finds a new set S, and tries again. One can reasonably conjecture that every odd integer n will be completely factored into prime powers by a small number of sets S. The point is that each odd prime divisor of n has to divide $\prod i - \sqrt{\prod i(n+i)}$ or $\prod i + \sqrt{\prod i(n+i)}$; it can't divide both (unless it happens to divide one of the i's); the decision between $-$ and $+$ appears to be sufficiently unpredictable that two different primes have a good chance of making different decisions.

The congruence-combination algorithms discussed in subsequent sections can be viewed as refinements of the **Q** sieve, adding complications for the sake of speed.

The **Q** sieve was introduced by Schnorr in [125, Section 3], refining an idea of Miller. The **Q** sieve was not the first congruence-combination factoring method, and it is much slower than most of its predecessors, but it is useful as an introduction to congruence-combination algorithms.

15.2 Example. The following tables show small factors of the integers $1, 2, 3, \ldots, 20$ and $612, 613, 614, \ldots, 631$.

1							
2	2						
3			3				
4	2	2					
5					5		
6	2		3				
7							7
8	2	2	2				
9			3	3			
10	2				5		
11							
12	2	2		3			
13							
14	2						7
15			3		5		
16	2	2	2	2			
17							
18	2		3	3			
19							
20	2	2			5		

612	2	2		3	3		
613							
614	2						
615				3		5	
616	2	2	2				7
617							
618	2			3			
619							
620	2	2				5	
621				3	3	3	
622	2						
623							7
624	2	2	2	2	3		
625						5	5 5 5
626	2						
627				3			
628	2	2					
629							
630	2			3	3	5	7
631							

The tables show a fully factored congruence $14 \cdot 625 = 2^1 3^0 5^4 7^1$. Continuing the computation I also found $64 \cdot 675 = 2^6 3^3 5^2 7^0$ and $75 \cdot 686 = 2^1 3^1 5^2 7^3$. Multiplying all three of these congruences produces a square $14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686 = 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$. Computing $\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$ reveals a factor of 611, namely 47.

15.3 Effectiveness. Choose an integer y close to $\exp \sqrt{(1/2) \log n \log \log n}$. Then a uniform random integer in $[1, n]$ has chance roughly $1/y$ of being y-smooth, as in Section 9.4.

It is conjectured that, among the first $y^{2+o(1)} = \exp \sqrt{(2 + o(1)) \log n \log \log n}$ congruences $i(n + i)$, there are more y-smooth integers than primes $\leq y$, forcing a subset to have square product. Of course, the congruences are not uniform random integers in $[1, n]$, but they seem to be smooth almost as often.

More generally, there must be at least k independent square products if the number of y-smooth congruences is k more than the number of primes $\leq y$. For example, achieving $k \approx \lg n$ is conjectured to take negligibly more i's than achieving $k = 1$.

Of course, one can choose a larger y, increasing the fraction of y-smooth integers and decreasing the number of congruences that need to be considered before a square product appears. The increase in effectiveness is limited; an extremely large prime is unlikely to appear twice; Pomerance has proven for uniform random

integers that one expects to look at $\exp \sqrt{(2+o(1)) \log n \log\log n}$ integers no matter how large y is; see [118]. On the other hand, this $o(1)$ is not the same as the previous $o(1)$, and experiments indicate that there is some benefit in using fairly large primes.

15.4 Finding fully factored congruences. How do we find completely factored integers among the congruences $1(n+1), 2(n+2), 3(n+3), \ldots$?

One answer—the answer that accounts for the word "sieve" in the name of the **Q** sieve—is to choose y, sieve the integers $1, 2, 3, \ldots$ using the primes $\leq y$, and sieve the integers $n+1, n+2, n+3, \ldots$ using the primes $\leq y$; see Section 12.1. If $y \approx \exp \sqrt{(1/2) \log n \log\log n}$ then conjecturally $y^{2+o(1)}$ integers suffice, as discussed in Section 15.3. Sieving $y^{2+o(1)}$ consecutive integers using primes $\leq y$ uses $y^{2+o(1)} = \exp \sqrt{(2+o(1)) \log n \log\log n}$ bit operations. Actually, within $y^{2+o(1)}$ bit operations, one can afford to sieve using primes $\leq y^{2+o(1)}$.

See Section 15.6 for a discussion of price-performance ratio.

15.5 Combining fully factored congruences. Define the "exponent vector" of a positive y-smooth integer $2^{e_2} 3^{e_3} 5^{e_5} \cdots$ as the vector $(e_2, e_3, e_5, \ldots)$, with one entry for each prime $\leq y$.

Given the exponent vectors of some congruences, one can find a nonempty subset with square product by finding a nonempty subset of the exponent vectors adding up to 0 modulo 2; i.e., by finding a nonzero element of the kernel of the exponent matrix over $\mathbf{F}_2$. Similarly, one can find k independent square products by finding k independent elements of the kernel. The kernel is guaranteed to be this large if the number of integers is k larger than the number of primes $\leq y$.

Recall that sieving is conjectured to find $y^{1+o(1)}$ smooth congruences with $y^{2+o(1)}$ bit operations if $y \approx \exp \sqrt{(1/2) \log n \log\log n}$. The resulting $y^{1+o(1)} \times y^{1+o(1)}$ exponent matrix has $y^{1+o(1)}$ nonzero entries. An algorithm by Wiedemann in [144] finds uniform random kernel elements using $y^{2+o(1)}$ bit operations.

One can speed up Wiedemann's algorithm by first looking for primes p that appear in only a few congruences: if p doesn't appear at all then it can be removed from the congruences; if p appears in only one congruence then it can be removed, along with that congruence; if p appears in a few congruences then it can be removed after those congruences are appropriately merged. The resulting algorithm still appears to take time $y^{2+o(1)}$ but with a noticeably smaller $o(1)$. There is much more to say about linear algebra; see [119], [77], [49], [47], [84], [95], [58], and [16].

One can always save time in linear algebra by reducing y at the expense of sieving. However, experiments suggest that linear algebra takes considerably fewer bit operations than sieving, when y is chosen to minimize bit operations in sieving alone.

See Section 15.6 for a discussion of price-performance ratio.

15.6 Price-performance ratio. Sieving has an unimpressive price-performance ratio, taking $y^{1.5+o(1)}$ seconds on a parallel machine of size $y^{1+o(1)}$ to find $y^{1+o(1)}$ smooth congruences out of $y^{2+o(1)}$ congruences. For comparison, the elliptic-curve method takes only $y^{1+o(1)}$ seconds on a parallel machine of size $y^{1+o(1)}$. Record-setting factorizations (even those labelled as "sieve" factorizations) use early aborts to combine sieving with other methods of finding small primes.

Linear algebra has an equally unimpressive price-performance ratio: it takes $y^{1.5+o(1)}$ seconds on a parallel machine of size $y^{1+o(1)}$. There does not seem to be an alternative. Overall the **Q** sieve takes $\exp\sqrt{(9/8+o(1))\log n\log\log n}$ seconds on a parallel machine of size $\exp\sqrt{(1/2+o(1))\log n\log\log n}$. The machine performs $\exp\sqrt{(2+o(1))\log n\log\log n}$ bit operations with a price-performance ratio of $\exp\sqrt{(25/8+o(1))\log n\log\log n}$ dollar-seconds.

To reduce the cost of linear algebra, bringing it into balance with sieving, one can reduce y from $\exp\sqrt{(1/2+o(1))\log n\log\log n}$ to $\exp\sqrt{(1/3+o(1))\log n\log\log n}$. Each congruence then has, conjecturally, chance roughly $1/y^{1.5}$ of being y-smooth. With this revised choice of y, the **Q** sieve takes $\exp\sqrt{(3/4+o(1))\log n\log\log n}$ seconds on a parallel machine of size $\exp\sqrt{(1/3+o(1))\log n\log\log n}$, performing $\exp\sqrt{(25/12+o(1))\log n\log\log n}$ bit operations with a price-performance ratio of $\exp\sqrt{(25/12+o(1))\log n\log\log n}$ dollar-seconds.

15.7 Sublattices. If i is in the arithmetic progression $-n+1000003^2\mathbf{Z}$ then $i(n+i)$ is divisible by 1000003^2. Instead of considering consecutive integers i in $\mathbf{Z}$, one can consider consecutive integers i in this arithmetic progression.

More generally, one can choose q, choose an arithmetic progression of integers i for which the congruences $i(n+i)$ are divisible by q, and look for fully factored congruences within that progression. It isn't necessary for q to be the square of a prime, but if q is a prime above y then q needs to be incorporated into the exponent vectors.

Varying q allows a practically unlimited supply of new congruences $i(n+i)/q$ on the scale of n, whereas the original congruences $i(n+i)$ grow with i and have only a limited supply on the scale of n. Multiple q's won't have much overlap if they're all reasonably large.

Consider, as an extreme case, taking just one i for each q, namely $i=q\lceil n/q\rceil-n$. One hopes that the quotient $i(n+i)/q=(q\lceil n/q\rceil-n)\lceil n/q\rceil$ is smooth. This quotient is, on average, around $n/2$.

Consider, as an intermediate possibility, taking an arithmetic progression of length $y^{1+o(1)}$ for each q. This allows sieving with all primes $\le y$, just like the original **Q** sieve; but all of the congruences are bounded by $y^{1+o(1)}n$, whereas the original **Q** sieve uses congruences as large as $y^{2+o(1)}n$.

An additional advantage of forcing $n+i$ to be divisible by a large q is that it

helps balance the pair $i, (n+i)/q$. Recall from Section 6.7 that a uniform random integer in $[1, n]$ has chance roughly u^{-u} of being $n^{1/u}$-smooth. An integer in $[1, n]$ that's a product of two independent uniform random integers in $[1, n^{1/2}]$ has chance roughly $(u/2)^{-u/2}(u/2)^{-u/2} = u^{-u}2^u$ of being $n^{1/u}$-smooth. The extra factor 2^u is quite noticeable.

In subsequent sections we'll replace the polynomial $i(n+i)$ by more subtle choices of polynomials, producing smaller congruences; but there will always be a limited supply of polynomial values close to the minimum. Forcing factors of q, by considering congruences in sublattices, fixes this problem, producing a practically unlimited supply of polynomial values close to the minimum. Often it also allows better balancing of products.

There are many different names for minor variations of the same sublattice idea: for example, the "special-q variation," the "multiple-polynomial quadratic sieve," and the "lattice sieve."

15.8 Finding large factors of many integers. If n' is very close to n then there is a large overlap between $n'+1, n'+2, \ldots, n'+y^2$ and $n+1, n+2, \ldots, n+y^2$. Consequently n and n' can share sieving effort in the **Q** sieve. This idea was introduced by Schnorr in [125, page 117].

Linear algebra might seem to become a bottleneck when this idea is applied to many integers; but one can always reduce y to reduce the cost of linear algebra, as in Section 15.6.

15.9 Negative integers. One can expand the **Q** sieve to allow $i(n+i)$ with $-n/2 < i < 0$; smaller values of i are redundant. Negative integers then require one extra component in exponent vectors: a factorization looks like $(-1)^{e_{-1}}2^{e_2}3^{e_3}\cdots$, with exponent vector $(e_{-1}, e_2, e_3, \ldots)$. Similar comments apply to the algorithms in subsequent sections.

16 The linear sieve

16.1 Introduction. The "linear sieve" tries to find a nontrivial factor of n by computing $\gcd\left\{n, \prod_{(i,j)\in S}(r+i)(r+j) - \sqrt{\prod_{(i,j)\in S}(r+i)(r+j)((r+i)(r+j)-n)}\right\}$. Here $r = \lfloor\sqrt{n}\rfloor$, and S is a nonempty set of nondecreasing pairs of positive integers such that the product $\prod_{(i,j)\in S}(r+i)(r+j)((r+i)(r+j)-n)$ is a square.

In other words, the linear sieve is just like the **Q** sieve, except that it has a different choice of congruences: namely, the integers $(r+i)(r+j)((r+i)(r+j)-n)$, alluding to the congruences $(r+i)(r+j) \equiv (r+i)(r+j) - n \pmod{n}$.

The linear sieve allows exponent vectors $(e_2, e_3, e_5, \ldots, f_1, f_2, \ldots)$ representing $2^{e_2}3^{e_3}5^{e_5}\cdots(r+1)^{f_1}(r+2)^{f_2}\cdots$. This means that a congruence is fully factored if

and only if the integer $(r+i)(r+j)-n$ is fully factored. That integer is bounded by $n^{1/2+o(1)}$ for reasonable ranges of i, j, whereas the **Q** sieve needed smoothness of much larger integers.

Schroeppel introduced the linear sieve in 1977, along with the general idea of congruence-combination algorithms that define congruences as consecutive values of polynomials so that a batch of congruences can be factored by sieving.

16.2 Effectiveness. Choose an integer y close to $\exp\sqrt{(1/4)\log n \log\log n}$. Then a uniform random integer in $[1, n^{1/2}]$ has chance roughly $1/y$ of being y-smooth, as in Section 15.3.

Choose z much smaller than n, and consider integers $i \in \{1, 2, 3, \ldots, z\}$. These integers form $z(z-1)/2$ nondecreasing pairs (i, j) and thus $z(z-1)/2$ congruences $(r+i)(r+j)((r+i)(r+j)-n)$. The integers $(r+i)(r+j)-n$ are not much larger than $n^{1/2}$; conjecturally they have chance $1/y^{1+o(1)}$ of being y-smooth, producing $z^2/y^{1+o(1)}$ fully factored congruences. The length of the exponent vectors is only about $z+y/\log y$; choosing a sufficiently large z in $y^{1+o(1)}$ will ensure that $z^2/y^{1+o(1)}$ exceeds $z + y/\log y$.

The linear sieve is thus conjectured to find squares within the first $y^{2+o(1)} = \exp\sqrt{(1+o(1))\log n \log\log n}$ congruences.

17 The quadratic sieve

17.1 Introduction. The "quadratic sieve" tries to find a nontrivial factor of n by computing $\gcd\{n, \prod_{i\in S}(r+i) - \sqrt{\prod_{i\in S}((r+i)^2-n)}\}$. Here $r = \lfloor\sqrt{n}\rfloor$, and S is a nonempty set of positive integers such that $\prod_{i\in S}((r+i)^2-n)$ is a square.

The quadratic sieve can be viewed as a special case of the linear sieve, always taking $i = j$; see Section 16.1. The requirement $i = j$ makes $(r+i)(r+j)$ a square, meaning that the quadratic sieve doesn't need to expand its exponent vectors in the way that the linear sieve did. Another advantage of the quadratic sieve over the linear sieve is that the quadratic sieve allows sublattices as in Section 15.7, producing a practically unlimited supply of small congruences $((r+i)^2-n)/q$.

The quadratic sieve was introduced by Pomerance shortly after the linear sieve. It was used to set many factorization records. For further information see [110], [67], [133], [53], [111], [56], [54], [130], [40], [120], [55], [136], [83], [12], [114], [137], [131], [104], [57], [7], [10], [26], and [45]. The quadratic sieve remains one of the most popular methods to factor integers below about 2^{300}, although for larger sizes it has been supplanted by the number-field sieve.

17.2 Effectiveness. The quadratic sieve, like the linear sieve, is conjectured to find squares within the first $y^{2+o(1)} = \exp\sqrt{(1+o(1))\log n \log\log n}$ congruences.

An interesting feature of the quadratic sieve is that only half of all primes p are potential divisors of the congruences, namely those for which n is a square modulo p. Integers n that are squares modulo many small primes are factored somewhat more quickly than other integers n. Consequently one can often speed up the quadratic sieve by replacing n with a small multiple such as $2n$ or $3n$.

17.3 The continued-fraction method. The "continued-fraction method" produces small squares modulo n by squaring numerators of truncations ("convergents") of the continued fraction for $\sqrt{n}$. Each numerator is congruent modulo n to an integer between $-2\sqrt{n}$ and $2\sqrt{n}$.

Consider, for example, $n = 314159265358979323$. The truncations of the continued fraction for $\sqrt{n}$ are good approximations to $\sqrt{n}$: namely, 560499122/1, 1120998243/2, 1681497365/3, 6165490338/11, etc. The numerators of the truncations have small squares modulo n: $560499122^2 \equiv 403791561$, $1120998243^2 \equiv -626830243$, $1681497365^2 \equiv 271129318$, $6165490338^2 \equiv -465143839$, etc.

The continued-fraction method is of historical interest as the first congruence-combination factoring method. It was introduced by Lehmer and Powers in [78]. After it was refined decades later by Morrison and Brillhart in [98] it was successfully used to factor a wide variety of integers. See [147], [110], [102], [122], [133], [148], [146], and [143].

The quadratic sieve replaced the continued-fraction method in the early 1980s, for reasons that seem quaint in retrospect. Subroutines at the time for linear algebra and for smoothness detection had larger cost exponents than today's fastest subroutines. Pomerance in [110] conjectured that the continued-fraction method used $\exp \sqrt{(1.118\ldots + o(1)) \log n \log\log n}$ bit operations while the quadratic sieve used $\exp \sqrt{(1.020\ldots + o(1)) \log n \log\log n}$ bit operations. The congruences in the quadratic sieve had a small disadvantage, namely being somewhat larger than $\sqrt{n}$, but they had a larger advantage, namely sieveability.

The advantage of the continued-fraction method became obsolete: sublattices allowed the quadratic sieve to generate consistently small congruences. See Section 15.7. The advantage of the quadratic sieve also became obsolete: smoothness detection for arbitrary integers became almost as fast as sieving. See, e.g., Section 14.1. The continued-fraction method conjecturally uses $\exp \sqrt{(1 + o(1)) \log n \log\log n}$ bit operations when it is combined with modern subroutines for smoothness detection and linear algebra. My impression is that the continued-fraction method is slightly slower than the quadratic sieve, but I haven't seen a careful analysis.

17.4 The random-squares method. One can simply consider $i^2 \bmod n$ for random integers i, throwing away both sieveability and continued fractions. The resulting algorithm has the virtue of *provably* factoring n into prime powers in subexponential average time.

On the other hand, the congruences $i^2 \bmod n$ are on the scale of n rather than $n^{1/2}$, so this algorithm is much slower than the linear sieve, the quadratic sieve, and the continued-fraction method; it's about as slow as the **Q** sieve. A more complicated idea achieves intermediate performance, with congruences on the scale of $n^{2/3}$.

If you're interested in these provably fast "random squares" methods, see [59], [112], and [141].

18 The number-field sieve

Not yet ready. Some references: [108], [82], [81], [4], [39], [109], [51], [24], [36], [115], [94], [69], [18], [60], [117], [61], [62], [63], [65], [52], [64], [96], [99], [103], and [100]. Also [48] for a better exponent.

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