Hash-based signatures I
Hash functions and one-time signatures

Daniel J. Bernstein\textsuperscript{124} and Tanja Lange\textsuperscript{34}

\textsuperscript{1}University of Illinois at Chicago
\textsuperscript{2}Ruhr University Bochum
\textsuperscript{3}Eindhoven University of Technology
\textsuperscript{4}Academia Sinica

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Python snippets for this talk:
https://cr.yp.to/talks/2022.04.01-2/ots-20220401.tar.gz
The SHA-256 cryptographic hash function

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$ echo hello
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The `sha256sum` program computes the SHA-256 hash function. This is a function $H : \{0, 1\}^* \rightarrow \{0, 1\}^{256}$. Each output is 32 bytes.

(Actually, SHA-256 requires input to be at most $2^{64} - 1$ bits. 
Exercise: Compute # years for today's fastest CPU to reach this limit.)
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c316678498bdf2a77d64e1f3af0cdc6e943234d19ce38034e24ccf98a5ab5901 -
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The SHA-256 cryptographic hash function in Python 3

```python
>>> import hashlib
>>> def sha256(x):
...     h = hashlib.sha256()
...     h.update(x)
...     return h.digest()
...
>>> print(sha256(b'hello').hex())
2cf24dba5fb0a30e26e83b2ac5b9e29e1b161e5c1fa7425e73043362938b9824
>>> print(sha256(b'hello
').hex())
5891b5b522d5df086d0ff0b110fbd9d21bb4fc7163af34d08286a2e846f6be03
>>> print(sha256(b'hello
'*1000000).hex())
1a2cce61984891495b00826ef591104a34ff35766bbbcaaff965f766154812ab
```
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>>> print(sha256(b'hello\n'*1000000).hex())
1a2cce61984891495b00826ef591104a34ff35766bcbcaaff965f766154812ab
```
Goals of cryptographic hash functions

What do we want from a hash function $H : \{0, 1\}^* \rightarrow \{0, 1\}^n$?

For any string $x$, think of $H(x)$ as an $n$-bit fingerprint of $x$.

Goals:

▶ $H(x)$ looks totally random;
▶ nobody can find two different strings $x, x'$ with $H(x) = H(x')$;
▶ any tiny change from $x$ to $x'$ makes a totally new $H(x')$;
▶ nobody can compute $H(x)$ without knowing all of $x$;
▶ nobody can compute a secret $x$ given only $H(x)$;
▶ . . .

Warning: Some hash goals are difficult to mathematically define.
Generic hardness of preimage resistance

Goal: Given $y \in H(\{0, 1\}^*)$, finding $x \in \{0, 1\}^*$ with $H(x) = y$ is hard.

Here $y$ is given, and is known to be the image of some $x \in \{0, 1\}^*$. Typically there are many such $x$, but it should be computationally hard to find any.
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Generic attack: Try $\approx 2^n$ random choices of $x$. If the output of $H$ is distributed uniformly then each $x$ has a $1/2^n$ chance of $H(x) = y$.

e.g. $\approx 2^{128}$ tries if $n = 128$: very expensive.
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Exercise: Given $y_1, y_2, \ldots, y_{2^{20}}$, how long does it take to find $x_1, x_2, \ldots, x_{2^{20}}$ such that $H(x_1) = y_1$ and $H(x_2) = y_2$ and $\ldots$ and $H(x_{2^{20}}) = y_{2^{20}}$?
Generic hardness of second-preimage resistance

Goal: Given \( x \in \{0, 1\}^* \), finding \( x' \in \{0, 1\}^* \) with \( x \neq x' \) and \( H(x') = H(x) \) is hard.

Here \( x \) is given, determining \( y = H(x) \).
Typically there are many other \( x' \neq x \) with the same image,
but it should be computationally hard to find any.
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Typically there are many other \( x' \neq x \) with the same image, but it should be computationally hard to find any.

Generic attack: Try \( \approx 2^n \) random choices of \( x' \neq x \).
Same speed as for first preimages.
Generic hardness of collision resistance

Goal: Finding \( x, x' \in \{0, 1\}^* \) with \( x \neq x' \) and \( H(x') = H(x) \) is hard.

Attacker has full flexibility to choose any output \( y \).
It should still be computationally hard to find two different strings \( x, x' \) with the same output.
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It should still be computationally hard
to find two different strings $x, x'$ with the same output.

Generic attack: Try $\approx 2^{n/2}$ random choices of $x$. This number is much lower than the other two because there is no restriction on the target.

The “birthday paradox”: if one draws $\approx 1.17 \sqrt{m}$ elements at random from a set of $m$ elements, then with 50% probability one has picked one element twice.
Weaknesses in common cryptographic hash functions

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Some hash functions take $n$ large enough
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Practical attack, chosen-prefix collision (2020):
https://sha-mbles.github.io/
The NSA view of cryptographic standardization

“The NSA view of cryptographic standardization

“Narrowing the encryption problem to a single, influential algorithm might drive out competitors, and that would reduce the field that NSA had to be concerned about. Could a public encryption standard be made secure enough to protect against everything but a massive brute force attack, but weak enough to still permit an attack of some nature using very sophisticated (and expensive) techniques?”

(Emphasis added.)

This quote is from an internal NSA history book.
Some unbroken hash functions

SHA-256 (2001 NSA): \( n = 256 \), so \( 2^{128} \) generic collision attack.

SHA-512 (2001 NSA): \( n = 512 \).

“SHA-2” refers to SHA-256, SHA-512, etc.
Some unbroken hash functions

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SHA3-512 (2015 Bertoni–Daemen–Peeters–van Assche): \( n = 512 \).
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Another popular SHA-3 finalist, faster than SHA-3 in software: BLAKE.
Hash-based signatures

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Using secret key, signer can sign any message $m$, producing a signed message $(m, s)$.

Everyone can verify $(m, s)$ using signer’s public key.
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Attacker looks at public key and at signed messages. Tries modifying the signed messages or creating new messages.
A signature scheme for empty messages: key generation

```python
import os, hashlib

def sha3_256(x):
    h = hashlib.sha3_256()
    h.update(x)
    return h.digest()

def keypair():
    secret = sha3_256(os.urandom(32))
    public = sha3_256(secret)
    return public, secret

>>> import signempty
>>> pk, sk = signempty.keypair()
>>> pk.hex()
'61ba682f03259a276dc2d790ed4863113d5559ad7cdd3c282083b9aa6b170ff8'
>>> sk.hex()
'4645dd39db47dd18b646a34b8f2dc6afd7fa62cc6faafc2ad3426dc943943355'
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'4645dd39db47dd18b646a34b8f2dc6afd7fa62cc6faafc2ad3426dc94394356'
```
def sign(message, secret):
    if not isinstance(message, bytes):
        raise TypeError('message must be a byte string')
    if message != b'':
        raise ValueError('message must be empty')
    signedmessage = secret
    return signedmessage

def open(signedmessage, public):
    if len(signedmessage) != 32:
        raise ValueError('bad signature')
    if sha3_256(signedmessage) != public:
        raise ValueError('bad signature')
    message = b''
    return message
Signing and verifying empty messages

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def open(signedmessage, public):
    if len(signedmessage) != 32:
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    if sha3_256(signedmessage) != public:
        raise ValueError('bad signature')
    message = b''
    return message

>>> sm = signempty.sign(b'', sk)
>>> signempty.open(sm, pk)
b''
import signempty

def keypair():
    p0, s0 = signempty.keypair()
    p1, s1 = signempty.keypair()
    return (p0, p1), (s0, s1)

def sign(message, secret):
    if not isinstance(message, bytes):
        raise TypeError('message must be a byte string')
    if message == b'0':
        return message, signempty.sign(b'', secret[0])
    if message == b'1':
        return message, signempty.sign(b'', secret[1])
    raise ValueError("message must be b'0' or b'1'")
A signature scheme for 1-bit messages: verification

def open(signedmessage, public):
    if not isinstance(signedmessage[0], bytes):
        raise TypeError('message must be a byte string')
    if signedmessage[0] == b'0':
        signempty.open(signedmessage[1], public[0])
        return b'0'
    if signedmessage[0] == b'1':
        signempty.open(signedmessage[1], public[1])
        return b'1'
    raise ValueError('bad signature')
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        signempty.open(signedmessage[1], public[1])
        return b'1'
    raise ValueError('bad signature')
```

```python
>>> import signbit
>>> pk, sk = signbit.keypair()
>>> sm = signbit.sign(b'1', sk)
>>> signbit.open(sm, pk)
b'1'
```
A signature scheme for 4-bit messages: key generation

import signbit

def keypair():
    p0,s0 = signbit.keypair()
    p1,s1 = signbit.keypair()
    p2,s2 = signbit.keypair()
    p3,s3 = signbit.keypair()
    return (p0,p1,p2,p3),(s0,s1,s2,s3)

def sign(m,secret):
    if not isinstance(m,bytes):
        raise TypeError('message must be a byte string')
    if len(m) != 4:
        raise ValueError('message must have length 4')
    sm0 = signbit.sign(m[0:1],secret[0])
    sm1 = signbit.sign(m[1:2],secret[1])
    sm2 = signbit.sign(m[2:3],secret[2])
    sm3 = signbit.sign(m[3:4],secret[3])
    return sm0,sm1,sm2,sm3
A signature scheme for 4-bit messages: sign & verify

def open(sm, public):
    if len(sm) != 4:
        raise ValueError('signed message must have length 4')
    m0 = signbit.open(sm[0], public[0])
    m1 = signbit.open(sm[1], public[1])
    m2 = signbit.open(sm[2], public[2])
    m3 = signbit.open(sm[3], public[3])
    return m0 + m1 + m2 + m3
Do not use one secret key to sign two messages!

```python
>>> import sign4bits
>>> pk, sk = sign4bits.keypair()
>>> sm0111 = sign4bits.sign(b'0111', sk)
>>> sign4bits.open(sm0111, pk)

b'0111'

>>> sm1101 = sign4bits.sign(b'1101', sk)
>>> sign4bits.open(sm1101, pk)

b'1101'

>>> forgery = sm1101[:2]+sm0111[2:]
>>> sign4bits.open(forgery, pk)

b'1111'
```
Do not use one secret key to sign two messages!

```python
>>> import sign4bits
>>> pk, sk = sign4bits.keypair()
>>> sm0111 = sign4bits.sign(b'0111', sk)
>>> sign4bits.open(sm0111, pk)
'b'0111'
>>> sm1101 = sign4bits.sign(b'1101', sk)
>>> sign4bits.open(sm1101, pk)
'b'1101'

>>> forgery = sm1101[:2] + sm0111[2:]
>>> sign4bits.open(forgery, pk)
'b'1111'
```
Lamport’s 1-time signature system

Sign arbitrary-length message by signing its 256-bit hash:

def hashbits(message):
    h = sha3_256(message)
    return [(b'0',b'1')[1&(h[i//8]>>(i%8))] for i in range(256)]

def keypair():
    keys = [signbit.keypair() for n in range(256)]
    return zip(*keys)

def sign(message,secret):
    hbits = hashbits(message)
    sigs = [signbit.sign(hbits[i],secret[i]) for i in range(256)]
    return sigs,message

def open(sm,public):
    if len(sm[0]) != 256:
        raise ValueError('wrong signature length')
    message = sm[1]
    hbits = hashbits(message)
    for i in range(256):
        if hbits[i] != signbit.open(sm[0][i],public[i]):
            raise ValueError('bit %d of hash does not match'%i)
    return message
Can we build shorter signatures?

How big are Lamport’s signatures?

- Each Lamport signature has 256 signbit signatures.
- Each signbit signature has 1 signempty signature.
- Each signempty signature has one hash output (32 bytes).

Total 256 hash outputs (8192 bytes).

For a 4-bit message: 4 hash outputs (128 bytes).
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How big are Lamport’s signatures?
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- Each \texttt{signbit} signature has 1 \texttt{signempty} signature.
- Each \texttt{signempty} signature has one hash output (32 bytes).
Total 256 hash outputs (8192 bytes).

For a 4-bit message: 4 hash outputs (128 bytes).

Idea for doing better, just 1 hash output for a 4-bit message:
- Define
  \[ H^i(x) = H(H^{i-1}(x)) = H(H(...(H(x)))) \]
  \[ i \text{ times} \]
- Pick random \texttt{sk}, compute \texttt{pk} = H^{16}(\texttt{sk}).
- For message \( m \in \{0, 1, \ldots, 15\} \) reveal \( s = H^m(\texttt{sk}) \) as signature.
- To verify check that \texttt{pk} = \( H^{16-m}(s) \).
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- Each `signbit` signature has 1 `signempty` signature.
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- Define

  $H^i(x) = H(H^{i-1}(x)) = H(H(\cdots(H(x)))$)

  \(i\) times

- Pick random sk, compute pk = $H^{16}(sk)$.
- For message $m \in \{0, 1, \ldots, 15\}$ reveal $s = H^m(sk)$ as signature.
- To verify check that pk = $H^{16-m}(s)$.

This is the weak Winternitz signature system.
Weak Winternitz

def keypair():
    secret = sha3_256(os.urandom(32))
    public = secret
    for i in range(16): public = sha3_256(public)
    return public,secret

def sign(m, secret):
    if not isinstance(m, int) or m<0 or m>15:
        raise ValueError('message must be in \{0,1,...,15\}')
    s = secret
    for i in range(m): s = sha3_256(s)
    return s, m

def open(sm, public):
    if not isinstance(sm[1], int) or sm[1]<0 or sm[1]>15:
        raise ValueError('message must be in \{0,1,...,15\}')
    c = sm[0]
    for i in range(16-sm[1]): c = sha3_256(c)
    if c != public: raise ValueError('bad signature')
    return sm[1]
Why this is "weak" Winternitz

This is insecure even if you sign only 1 message!

```python
>>> import weak_winternitz
>>> pk, sk = weak_winternitz.keypair()
>>> sm7 = weak_winternitz.sign(7, sk)
>>> H = weak_winternitz.sha3_256
>>> weak_winternitz.open(sm7, pk)
7
>>> forgery = H(sm7[0]), 8
>>> weak_winternitz.open(forgery, pk)
8
>>> forgery2 = H(forgery[0]), 9
>>> weak_winternitz.open(forgery2, pk)
9
>>> 
```
Why this is “weak” Winternitz

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```python
>>> import weak_winternitz
>>> pk, sk = weak_winternitz.keypair()
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>>> weak_winternitz.open(forgery, pk)
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>>> forgery2 = H(forgery[0]), 9
>>> weak_winternitz.open(forgery2, pk)
9
```  

Fix: Strong Winternitz uses weak Winternitz twice, running one chain forward, one chain backward.  
(Exercise: this is safe with $H^{15}$ instead of $H^{16}$ in weak Winternitz.)
import weak_winternitz

def keypair():
    keys = [weak_winternitz.keypair() for n in range(2)]
    return zip(*keys)

def sign(m, secret):
    if not isinstance(m, int) or m < 0 or m > 15:
        raise ValueError('message must be in \{0,1,\ldots,15\}\)')
    sign0 = weak_winternitz.sign(m, secret[0])
    sign1 = weak_winternitz.sign(15-m, secret[1])
    return sign0[0], sign1[0], m

def open(sm, public):
    if not isinstance(sm[2], int) or sm[2] < 0 or sm[2] > 15:
        raise ValueError('message must be in \{0,1,\ldots,15\}\)')
    weak_winternitz.open((sm[0], sm[2]), public[0])
    weak_winternitz.open((sm[1], 15-sm[2]), public[1])
    return sm[2]
The complete Winternitz system

Define parameter $w$. Each chain will run for $2^w$ steps.

For signing a 256-bit hash this needs $t_1 = \lceil 256/w \rceil$ chains.
Write $m$ in base $2^w$ (integers of $w$ bits):

$$m = (m_{t_1-1}, \ldots, m_1, m_0)$$

(zero-padding if necessary).

Put

$$c = \sum_{i=0}^{t_1-1} (2^w - m_i)$$

Note that $c \leq t_1 2^w$.

The checksum $c$ gets larger if $m_i$ is smaller.

Write $c$ in base $2^w$. This takes $t_2 = 1 + \lceil [(\log_2 t_1) + 1)/w \rceil$ $w$-bit integers

$$c = (c_{t_2-1}, \ldots, c_1, c_0).$$

Publish $t_1 + t_2$ public keys, sign with chains of lengths

$$m_{t_1-1}, \ldots, m_1, m_0, c_{t_2-1}, \ldots, c_1, c_0.$$
The complete Winternitz system for $w = 8$

Define parameter $w = 8$. Each chain will run for $2^8 = 256$ steps.

For signing a 256-bit hash this needs $t_1 = \lceil 256/8 \rceil = 32$ chains.

Write $m$ in base $2^8$ (integers of 8 bits):

$$m = (m_{31}, \ldots, m_1, m_0)$$

(zero-padding if necessary).

Put

$$c = \sum_{i=0}^{31} (2^8 - m_i)$$

Note that $c \leq 32 \cdot 2^8 = 2^{13}$.

The checksum $c$ gets larger if $m_i$ is smaller.

Write $c$ in base $2^8$. This takes $t_2 = 1 + \lceil (5 + 1)/8 \rceil = 2$ 8-bit integers

$$c = (c_1, c_0).$$

Publish $t_1 + t_2 = 34$ public keys, sign with chains of lengths

$$m_{31}, \ldots, m_1, m_0, c_1, c_0.$$
Exercise

How does Winternitz with $w = 5$ compare to Winternitz with $w = 8$ for signing a 256-bit hash? Efficiency metrics:

- How many bytes are in the signature?
- How many bytes are in the public key?
- How many bytes are in the secret key?
- How many hash-function computations are needed in signing?
- How many hash-function computations are needed in verifying?

Remember that you also need to sign the checksum component!