Quantum walks

Daniel J. Bernstein
University of Illinois at Chicago

Focusing on quantum walks
as an algorithm-design tool:
e.g. Grover's algorithm.
e.g. Ambainis's algorithm.

Can also study quantum walks
on much more general graphs.
2008 Childs, 2009 Lovett–
Cooper–Everitt–Trevers–Kendon:
Can view, e.g., Shor's algorithm
as quantum walk on Shor graph.

Examples of applications to crypto

Minimum asymptotic ops known, assuming plausible heuristics:

pre-q	post-q	problem
1	0.5	cipher
ρ	$\rho/2$	McEliece
$0.791\ldots$	$0.462\ldots$	MQ
$0.290\ldots$	$0.241\ldots$	subset sum

"Pre-q" e: as $n \to \infty$, $2^{(e+o(1))n}$ simple non-quantum ops.

"Post-q" e: as $n \to \infty$, $2^{(e+o(1))n}$ simple quantum ops.

"Cipher": find n-bit cipher key.
0.5: 1996 Grover.

"McEliece": in linear code of length $(1 + o(1))n \log_2 n$ and dimension $(R + o(1))n \log_2 n$, decode $(1 - R + o(1))n$ errors.

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0.462: 2017 Bernstein–Yang (via Grover), independently 2017 Faugère–Horan–Kahrobaei–Kaplan–Kashefi–Perret.

"Subset sum" ("hard" case):
find $S \subseteq \{1, 2, \ldots, n\}$ given
$x_1, x_2, \ldots, x_n \in \{0, 1, \ldots, 2^n - 1\}$
and $\sum_{i \in S} x_i$.

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0.291: 2011 Becker–Coron–Joux.

0.241: 2013 Bernstein–Jeffery–Lange–Meurer, using HGJ and quantum walks (not just Grover).

Grover's algorithm

Assume: unique $s \in \{0,1\}^n$ has $f(s) = 0$.

Traditional algorithm to find s: compute f for many inputs, hope to find output 0.
Success probability is very low until #inputs approaches 2^n.

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Grover's algorithm takes only $2^{n/2}$
reversible computations of f.
Typically: reversibility overhead
is small enough that this easily
wins for all sufficiently large n.

Start from uniform superposition a over $q \in \{0,1\}^n$: $a_q = 2^{-n/2}$.

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This is fast.

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Negate a around its average.
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Repeat Step 1 + Step 2
about $0.58 \cdot 2^{n/2}$ times.

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Negate a around its average.
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Repeat Step 1 + Step 2
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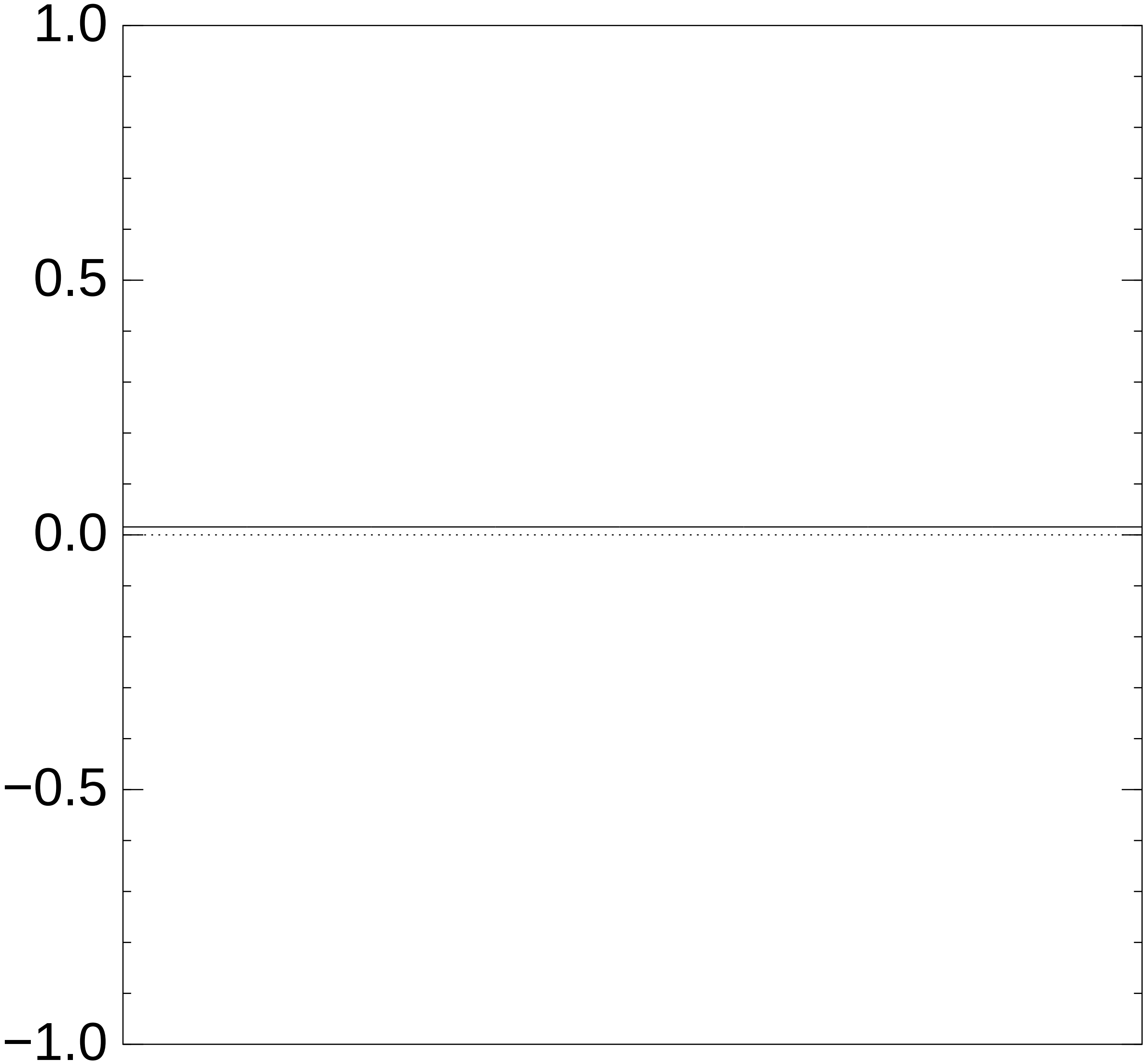
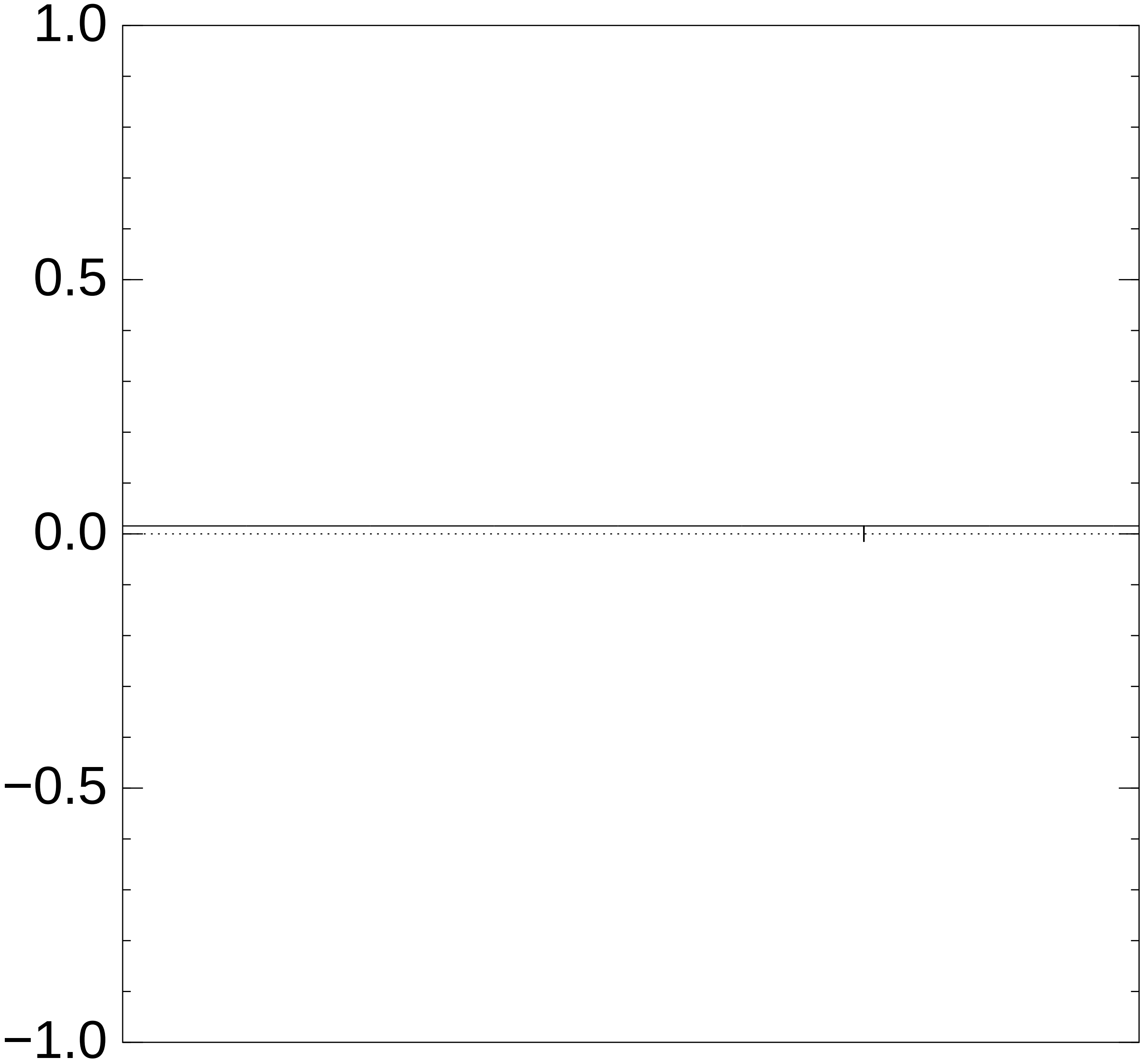
Measure the n qubits.
With high probability this finds s.

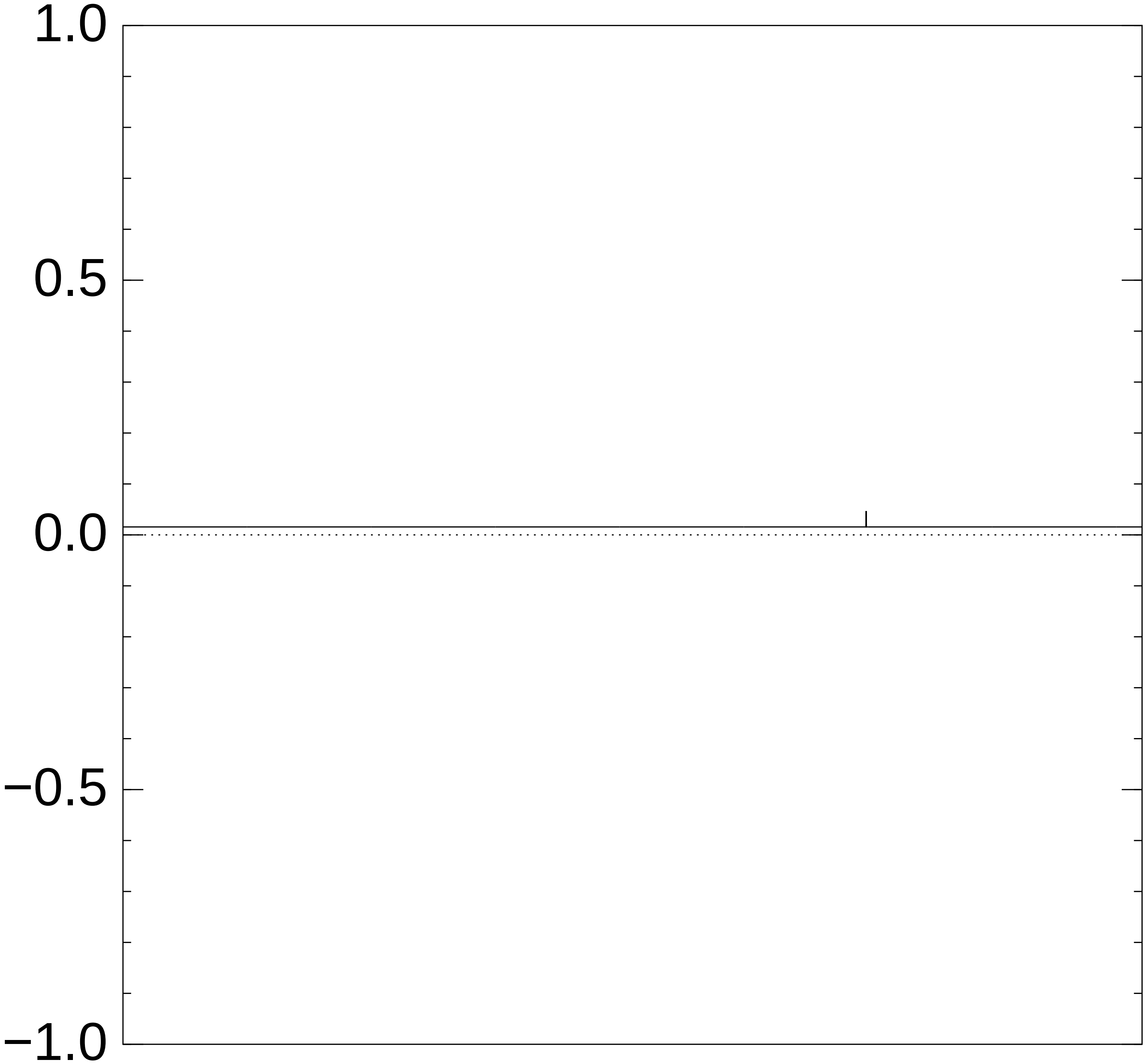
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after 0 steps:



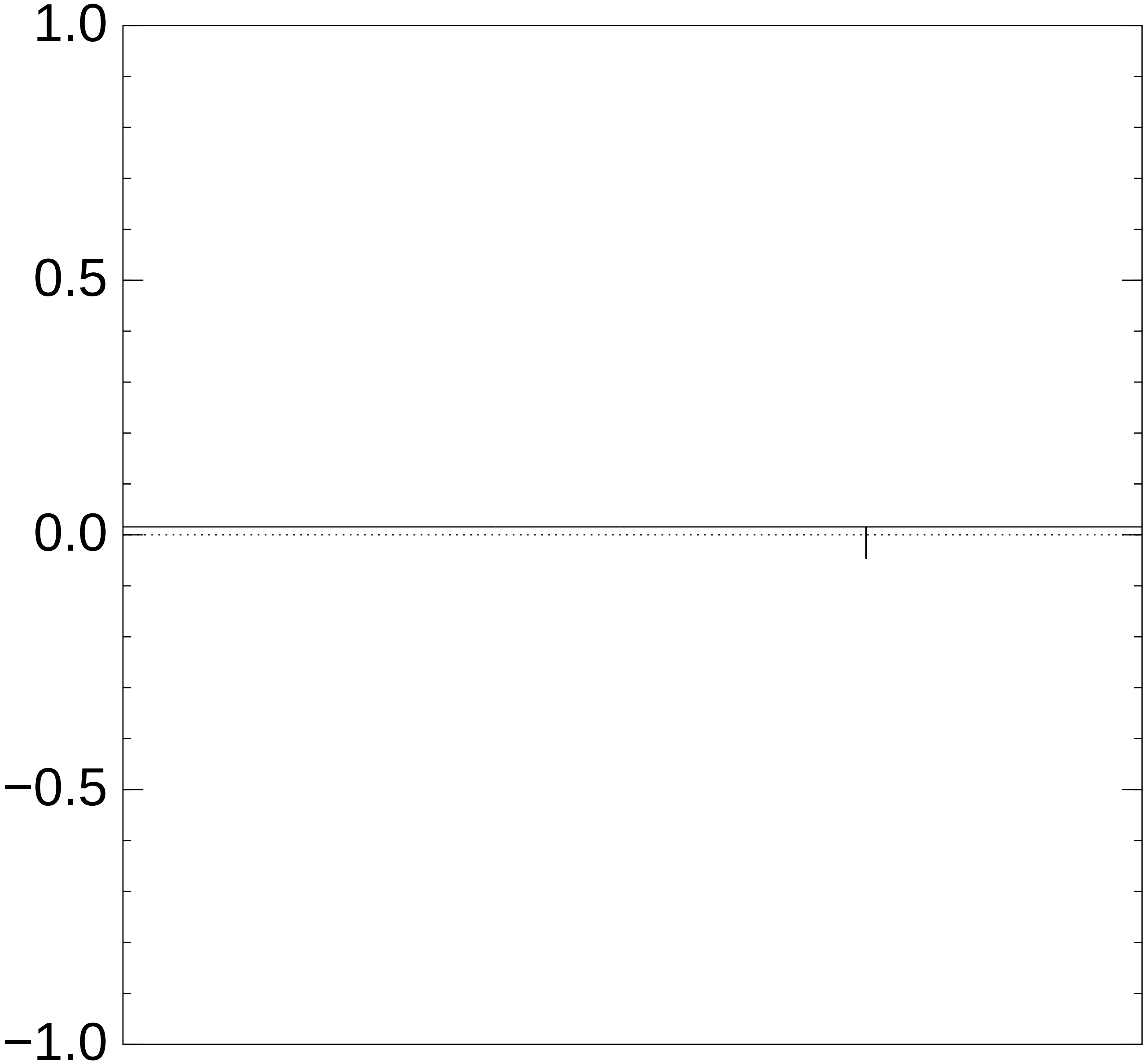
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after Step 1:



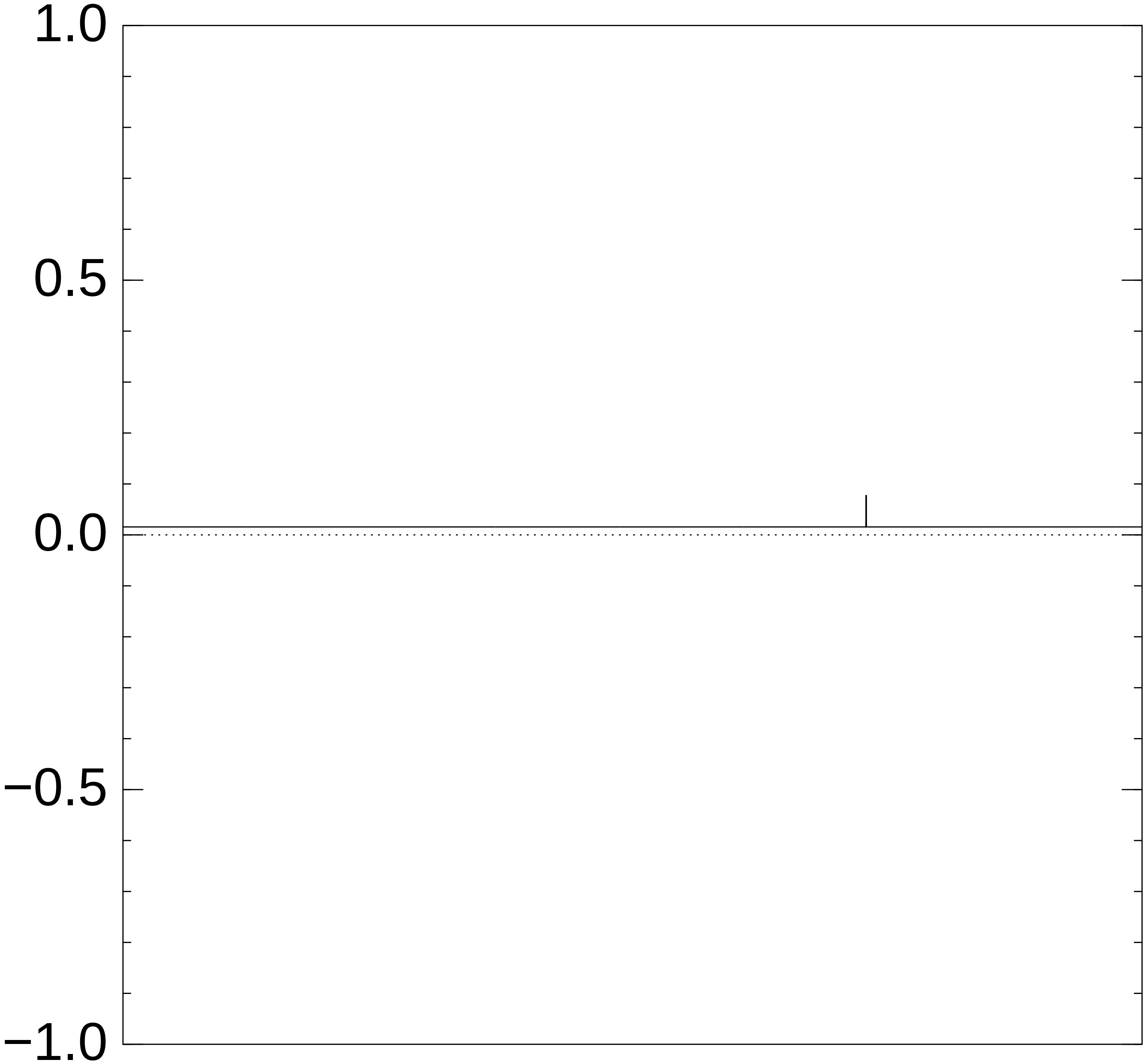
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after Step 1 + Step 2:



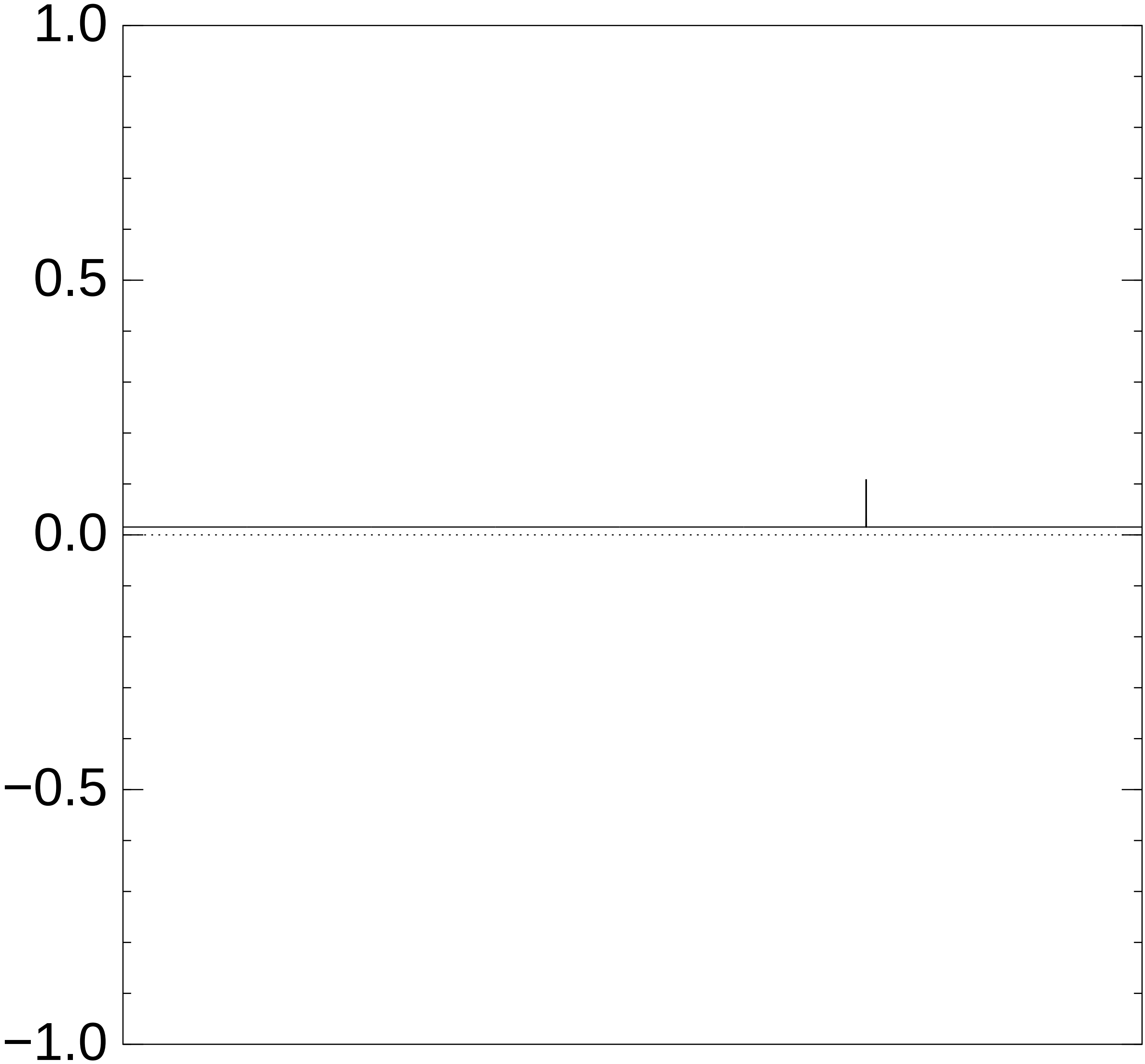
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after Step $1 +$ Step $2 +$ Step 1:



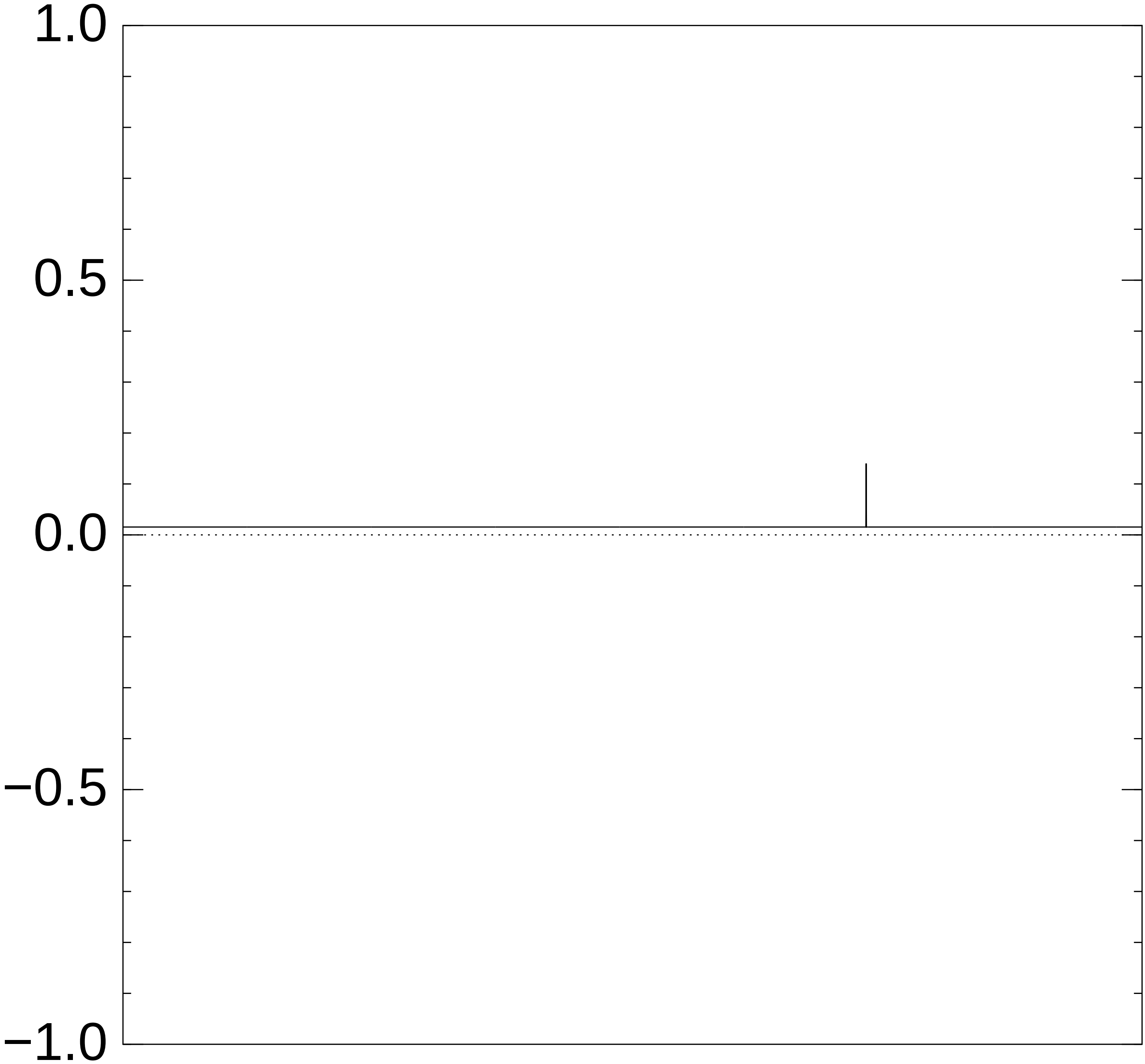
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $2 \times (\text{Step } 1 + \text{Step } 2)$:



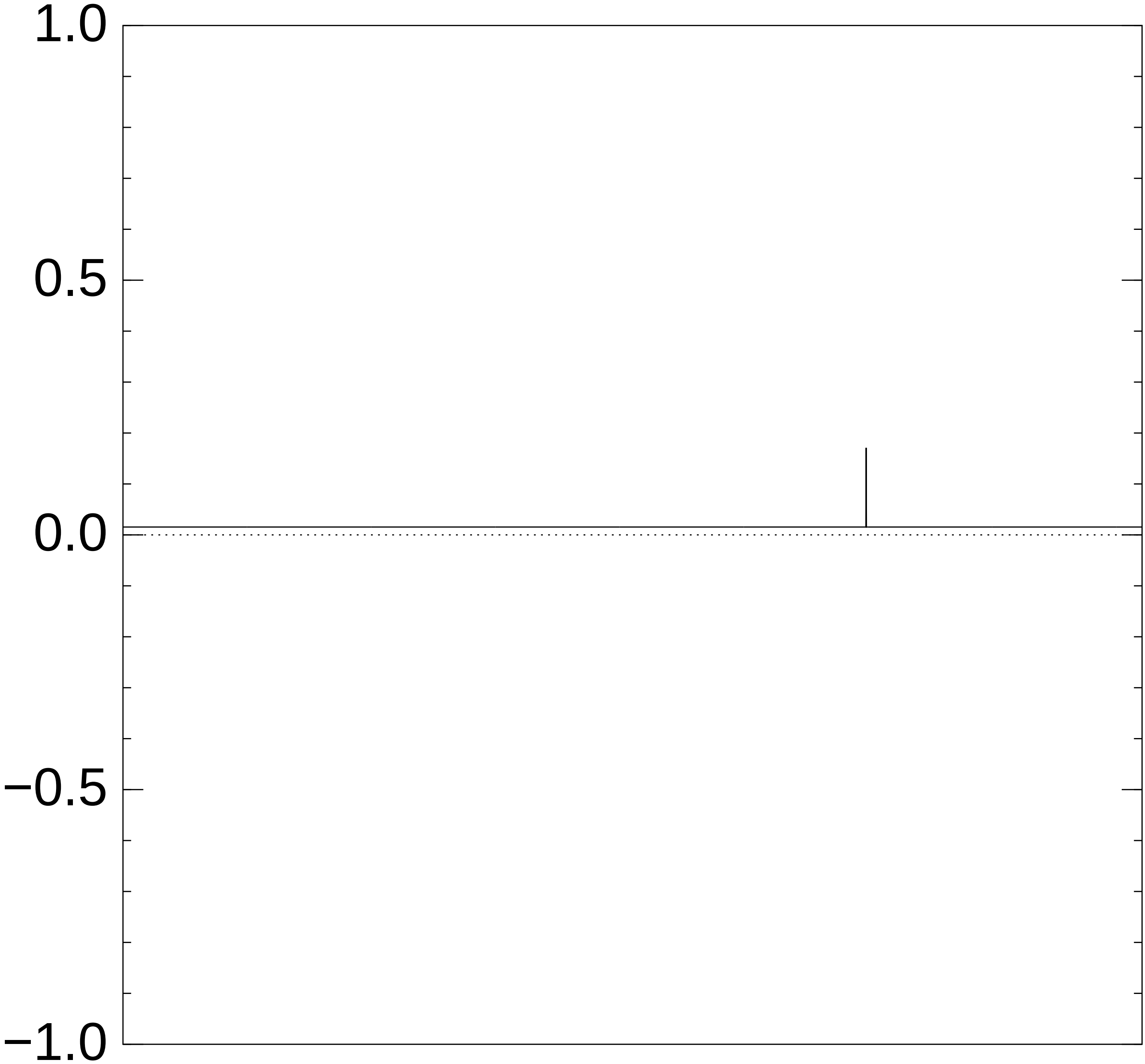
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $3 \times (\text{Step } 1 + \text{Step } 2)$:



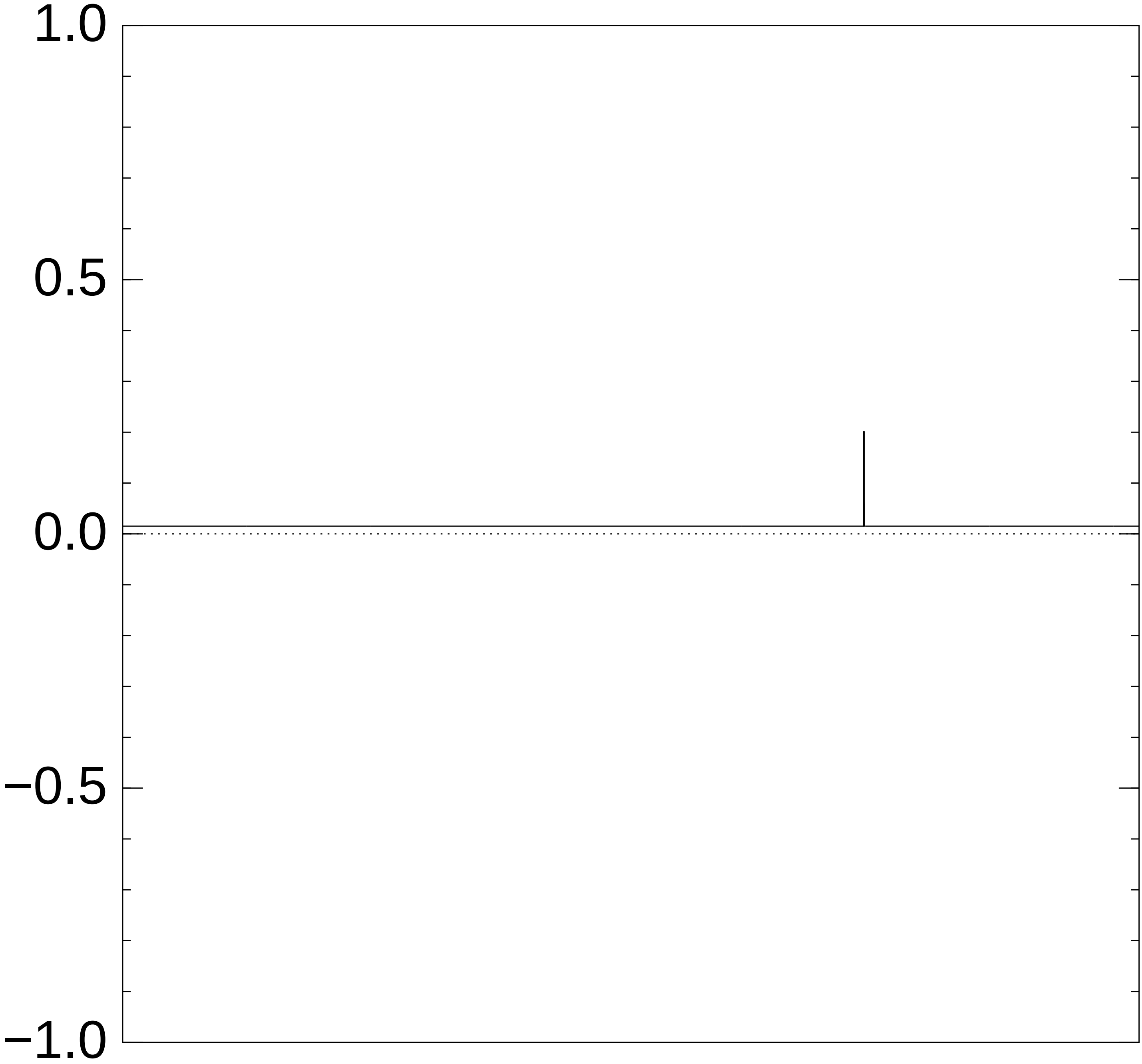
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $4 \times (\text{Step } 1 + \text{Step } 2)$:



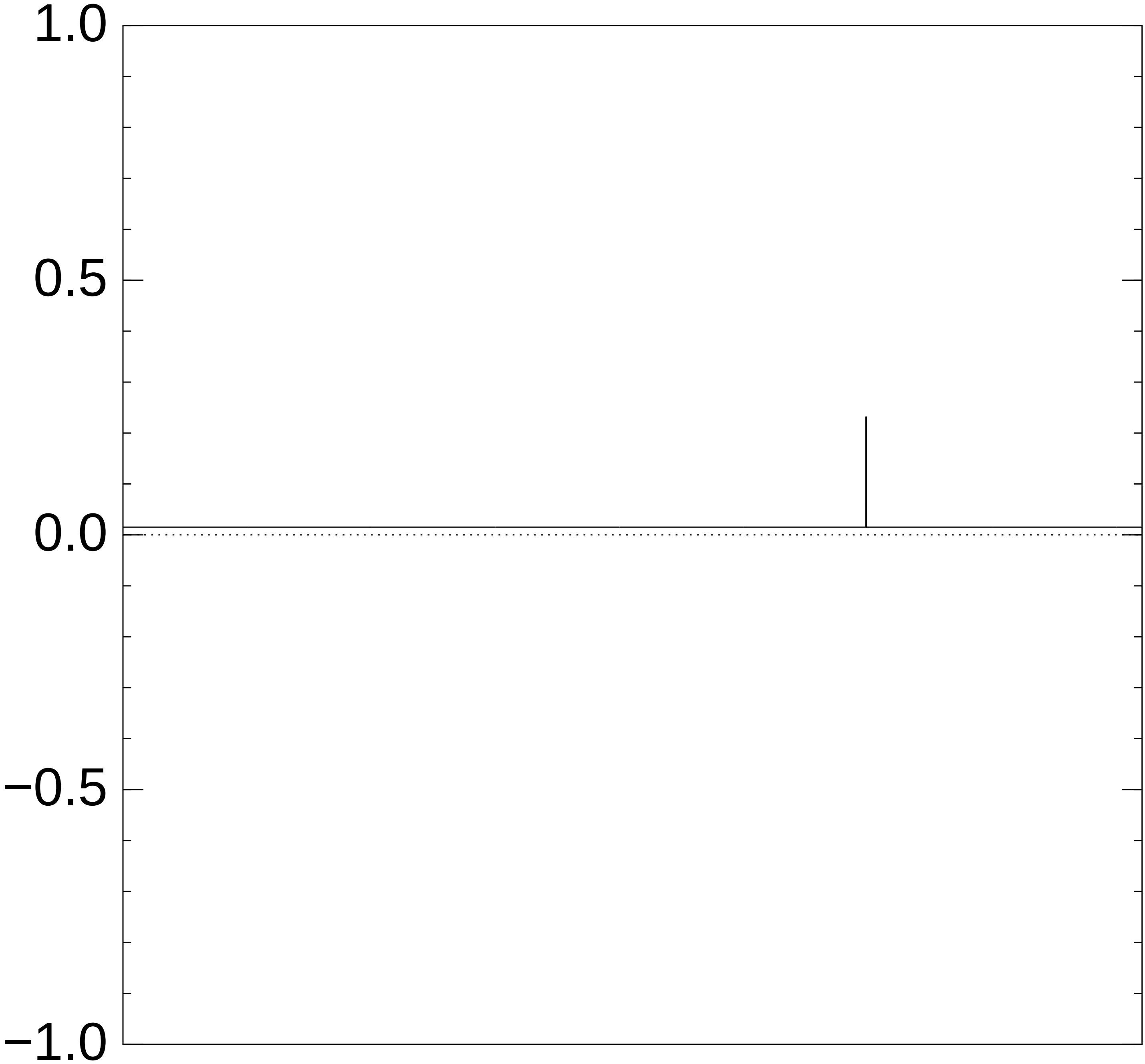
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $5 \times (\text{Step } 1 + \text{Step } 2)$:



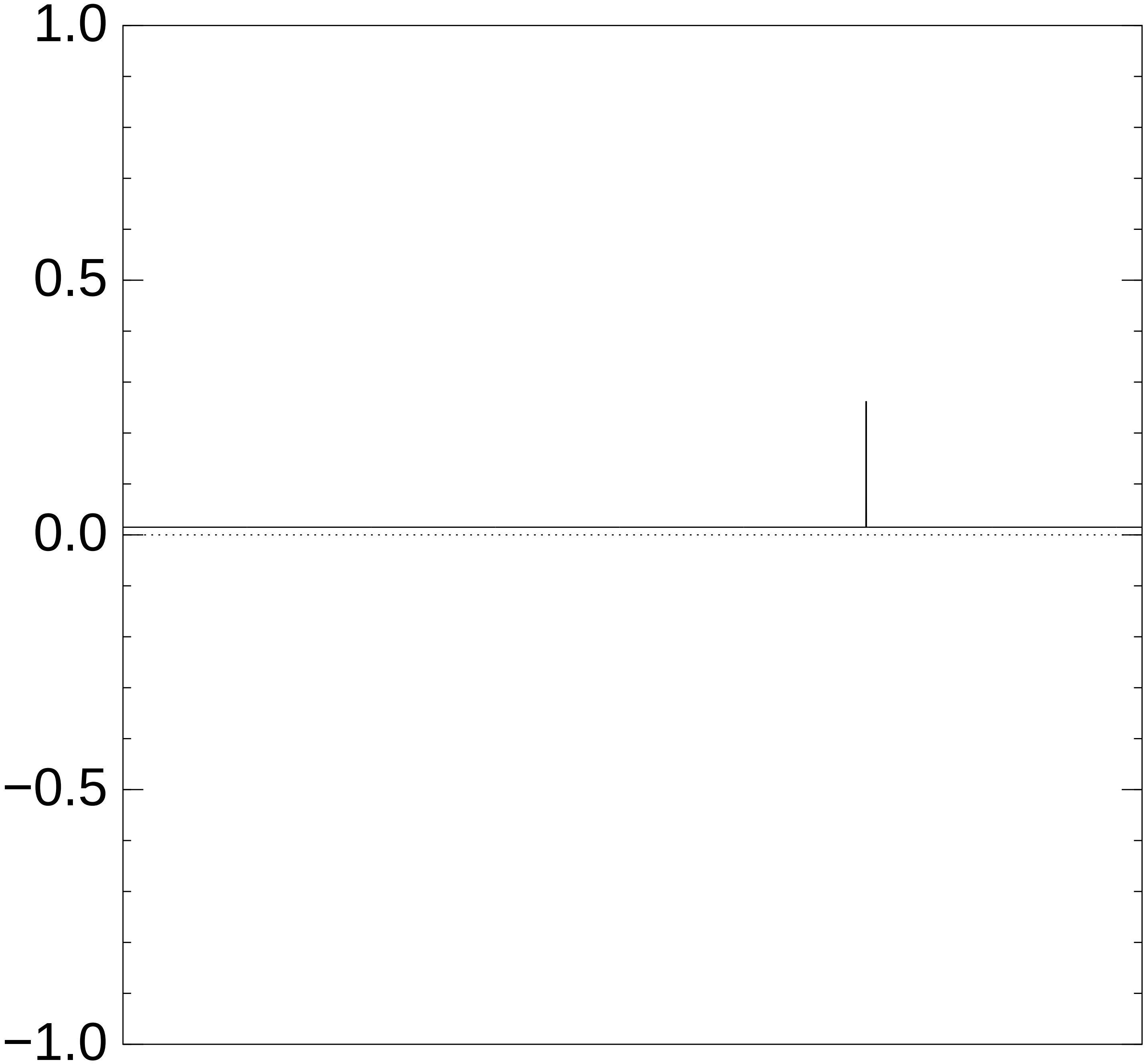
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $6 \times$ (Step $1 +$ Step 2):



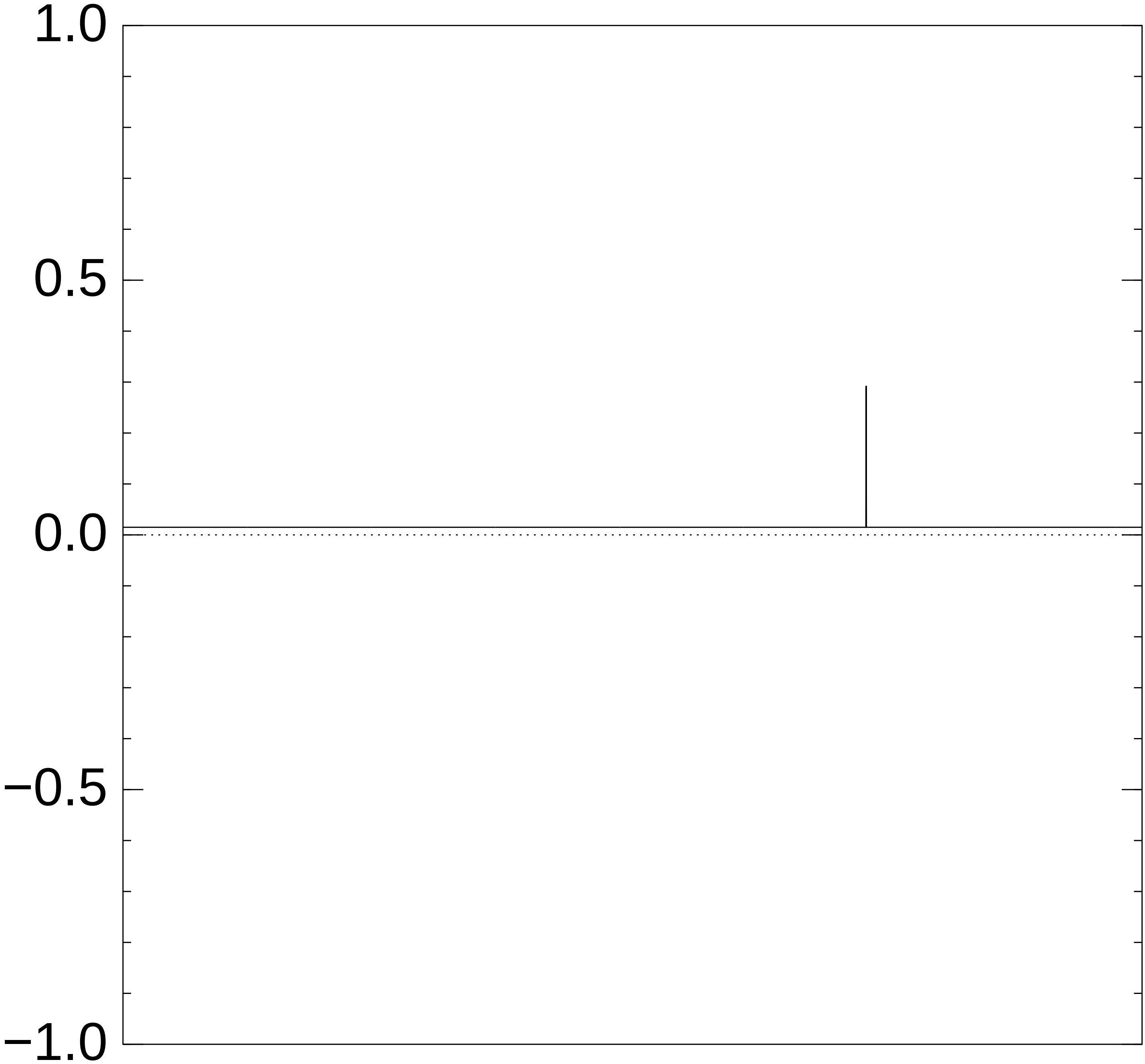
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $7 \times (\text{Step } 1 + \text{Step } 2)$:



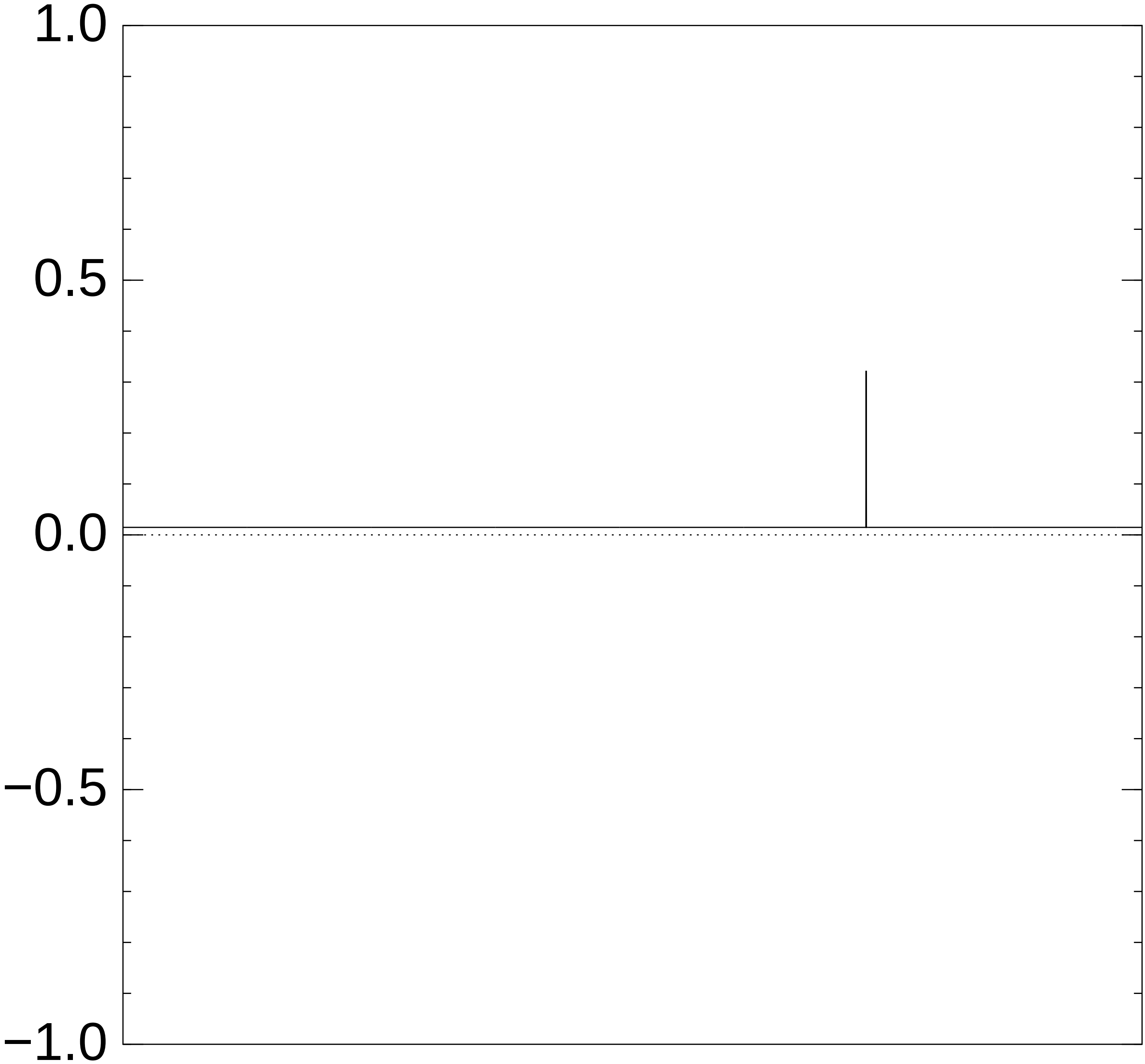
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $8 \times (\text{Step } 1 + \text{Step } 2)$:



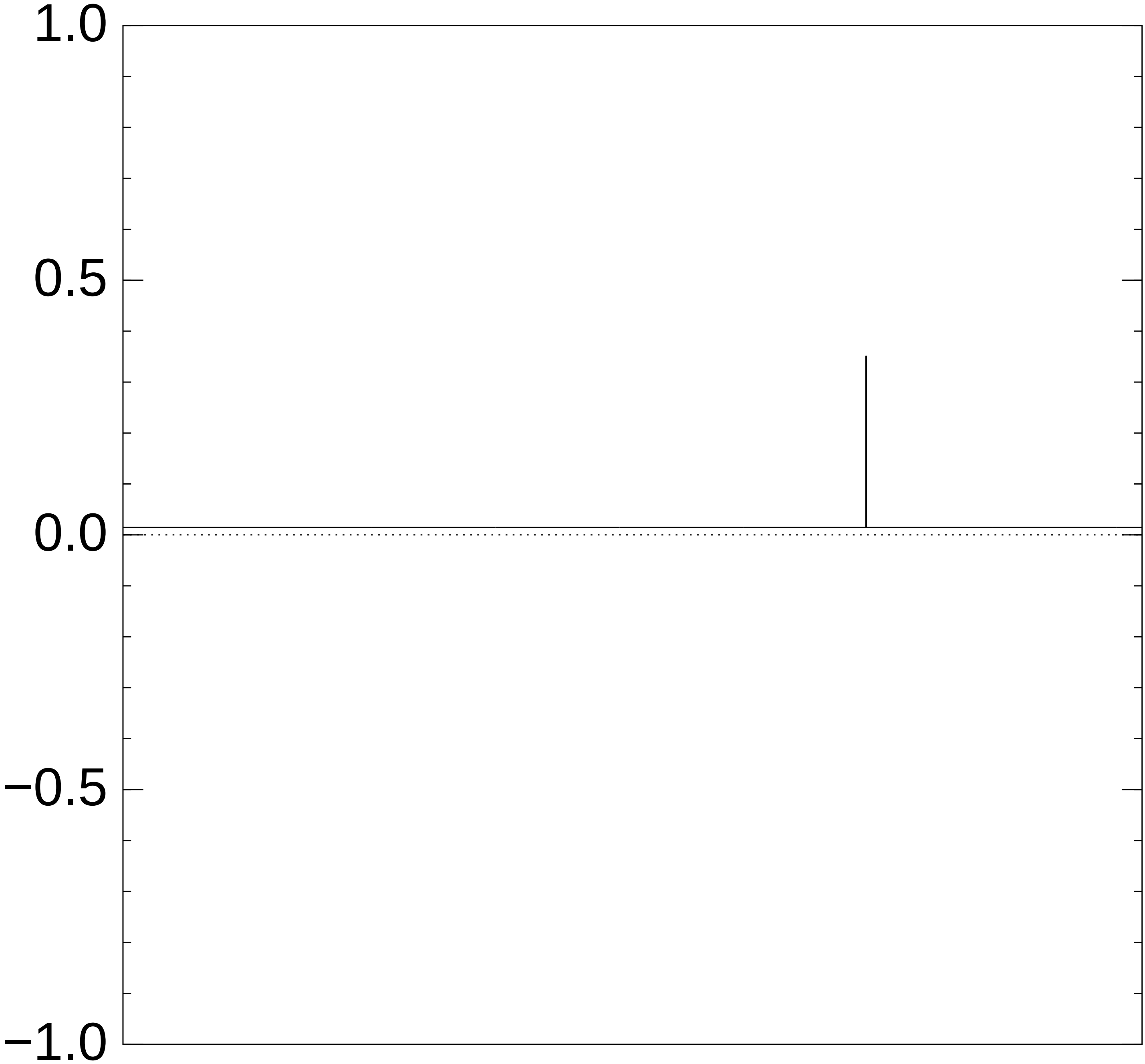
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $9 \times$ (Step $1 +$ Step 2):



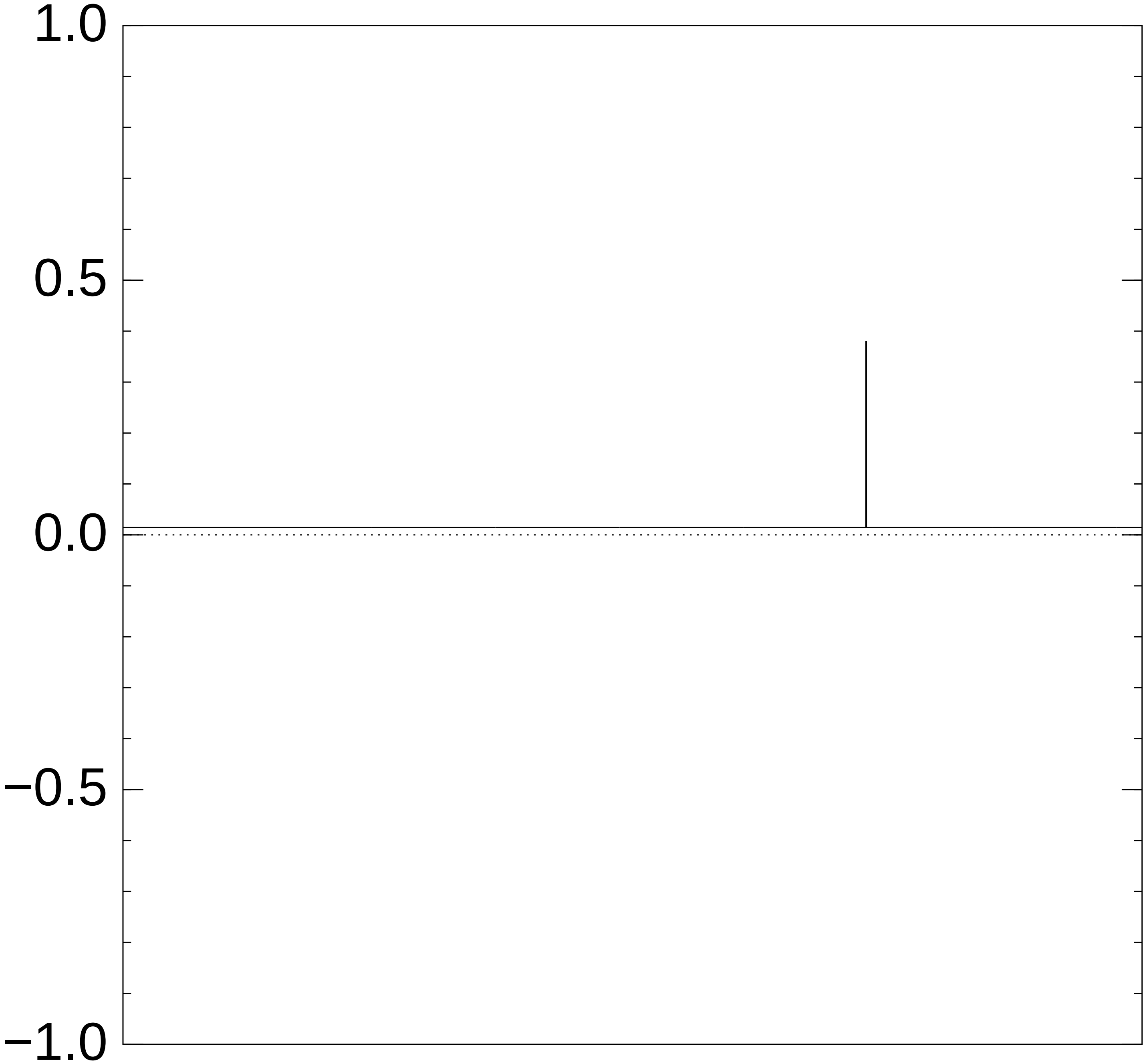
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $10 \times (\text{Step } 1 + \text{Step } 2)$:



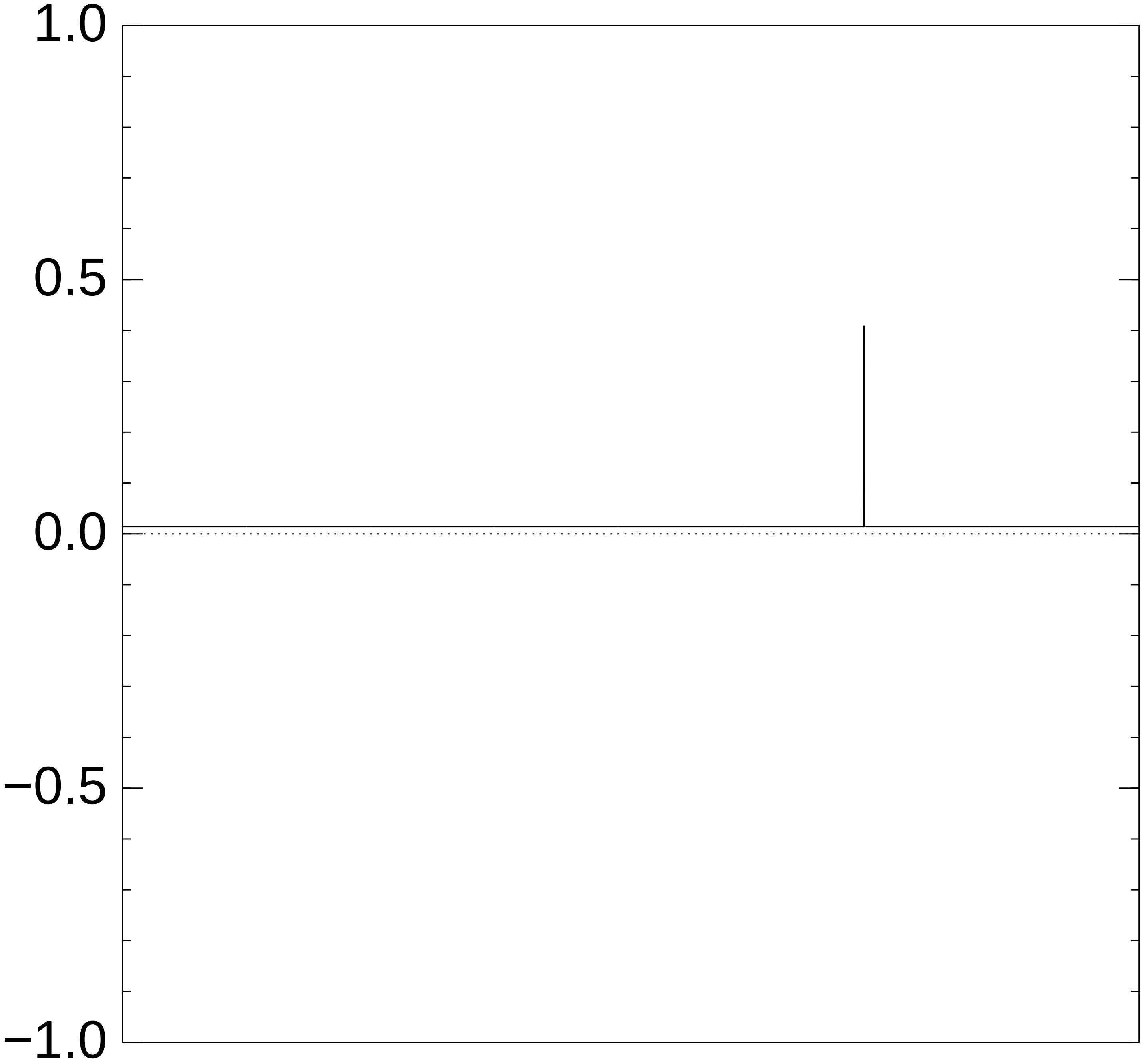
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $11 \times$ (Step $1 +$ Step 2):



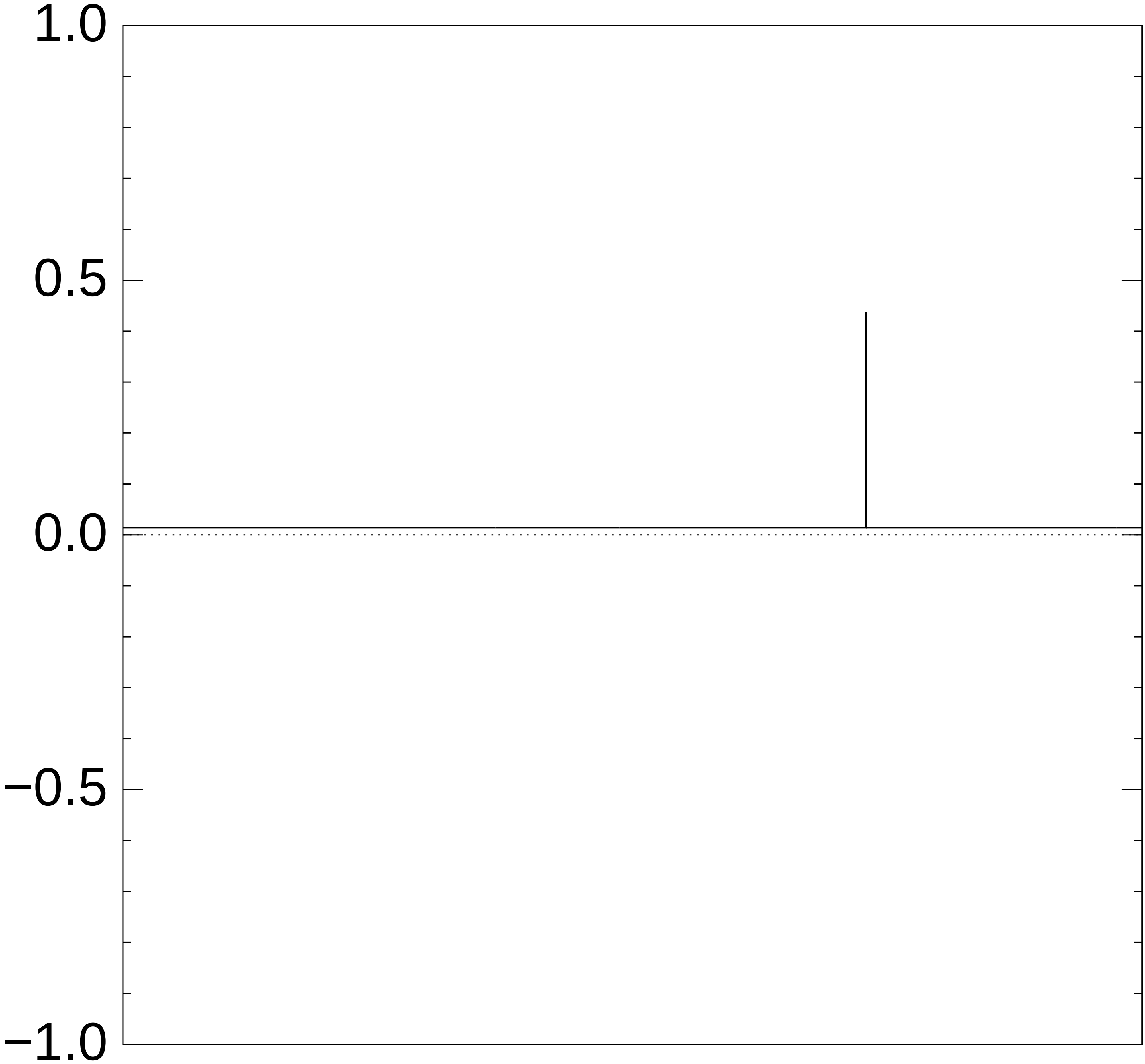
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $12 \times (\text{Step } 1 + \text{Step } 2)$:



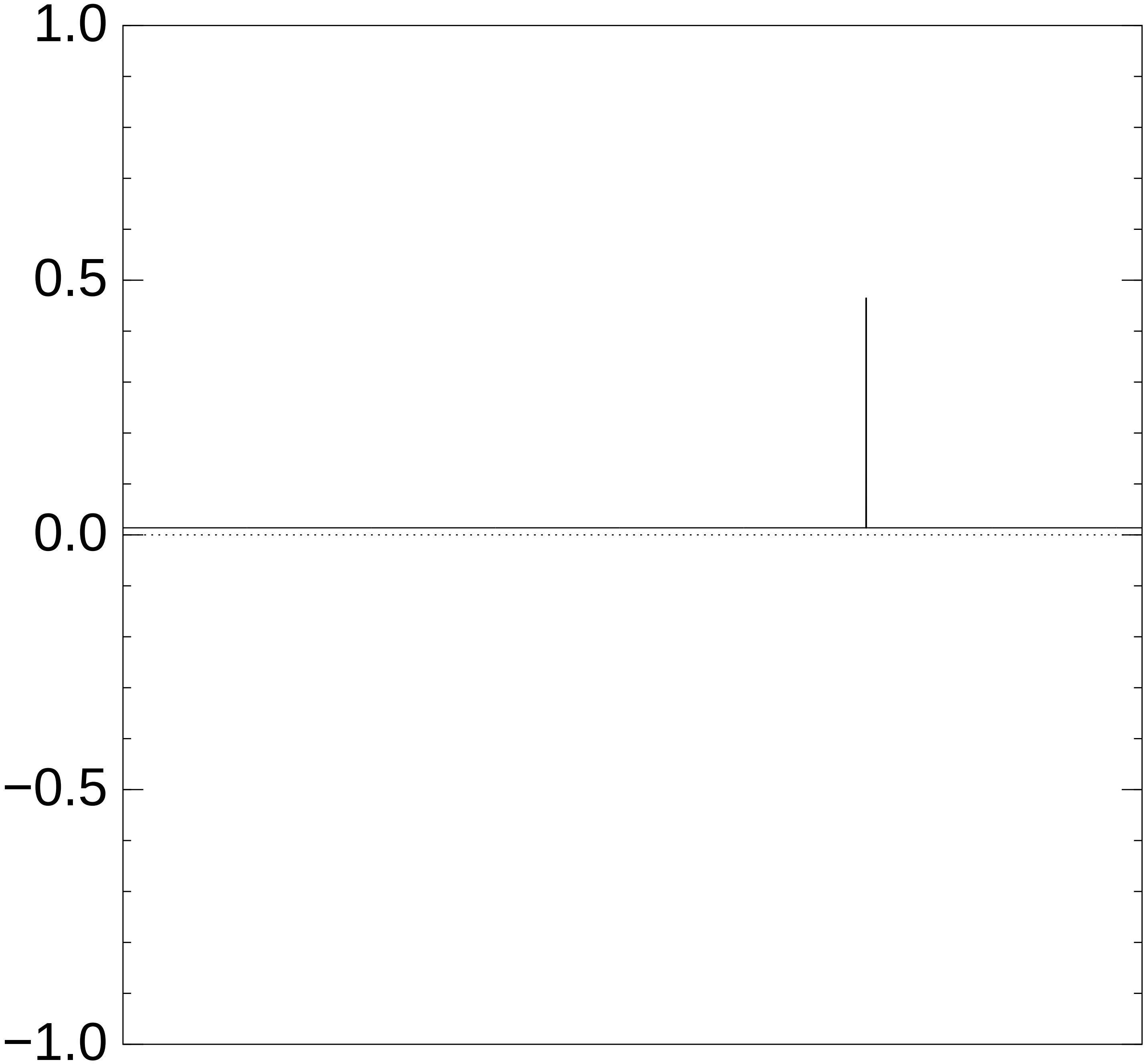
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $13 \times$ (Step 1 + Step 2):



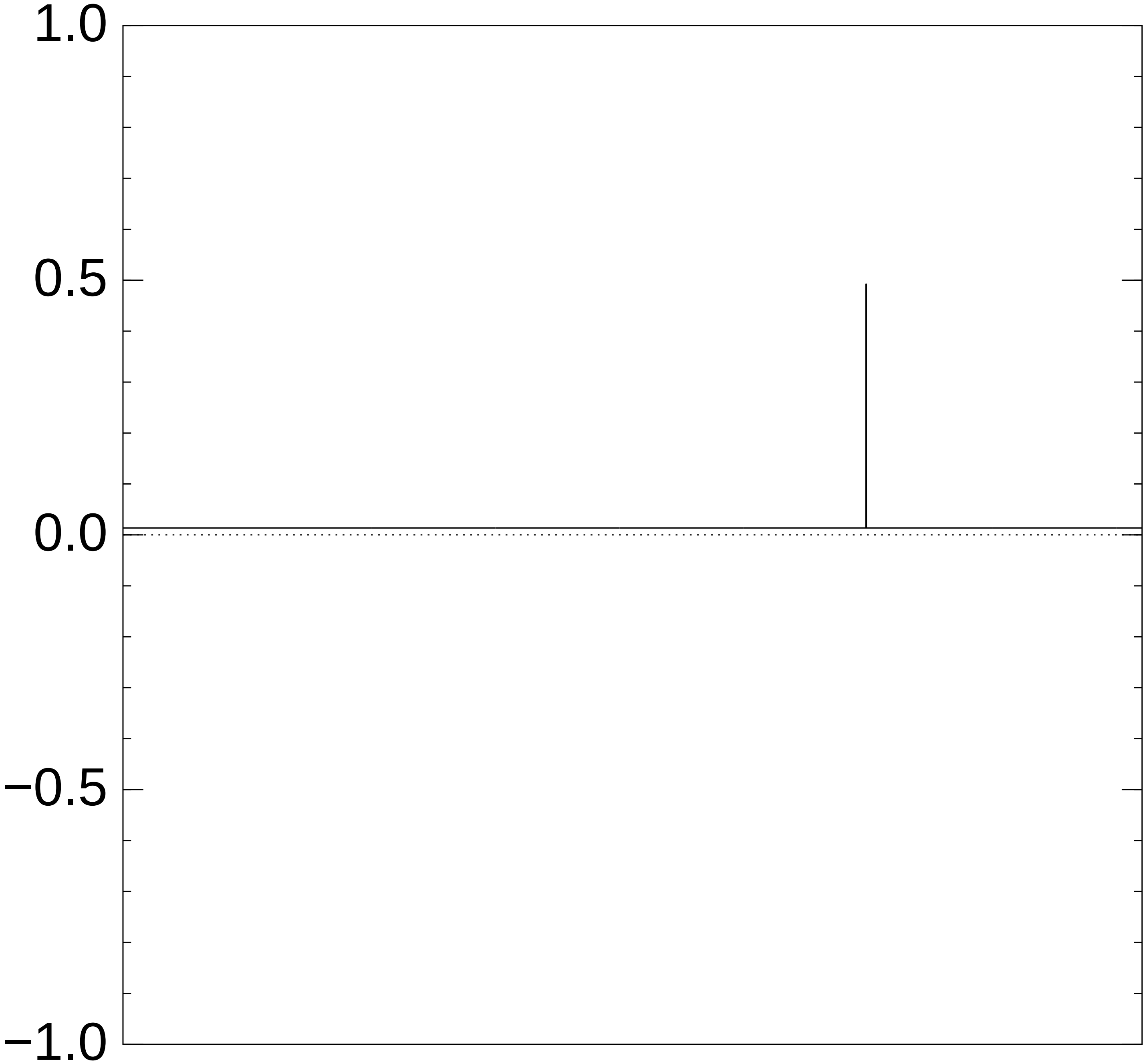
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $14 \times$ (Step $1 +$ Step 2):



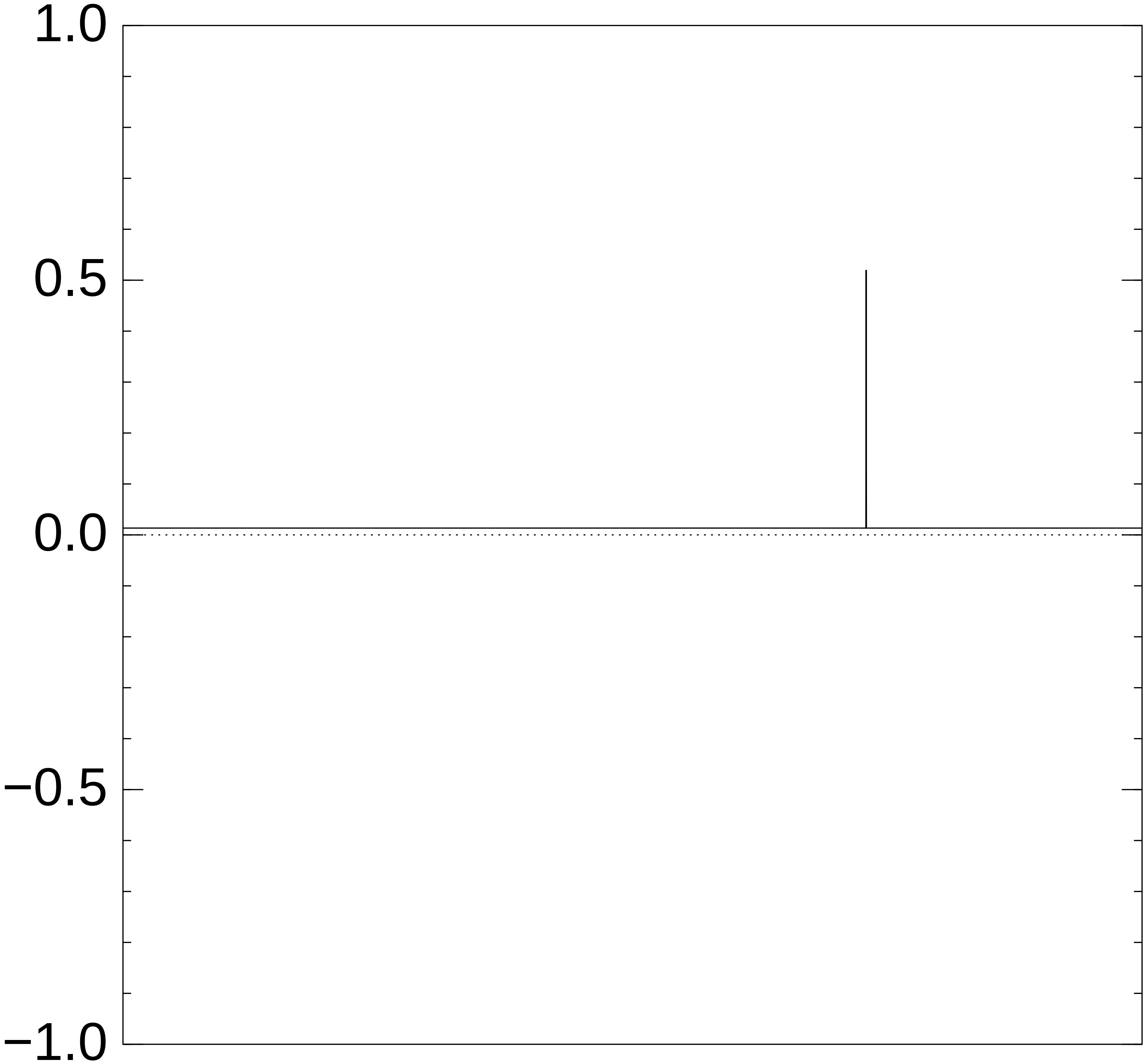
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $15 \times$ (Step $1 +$ Step 2):



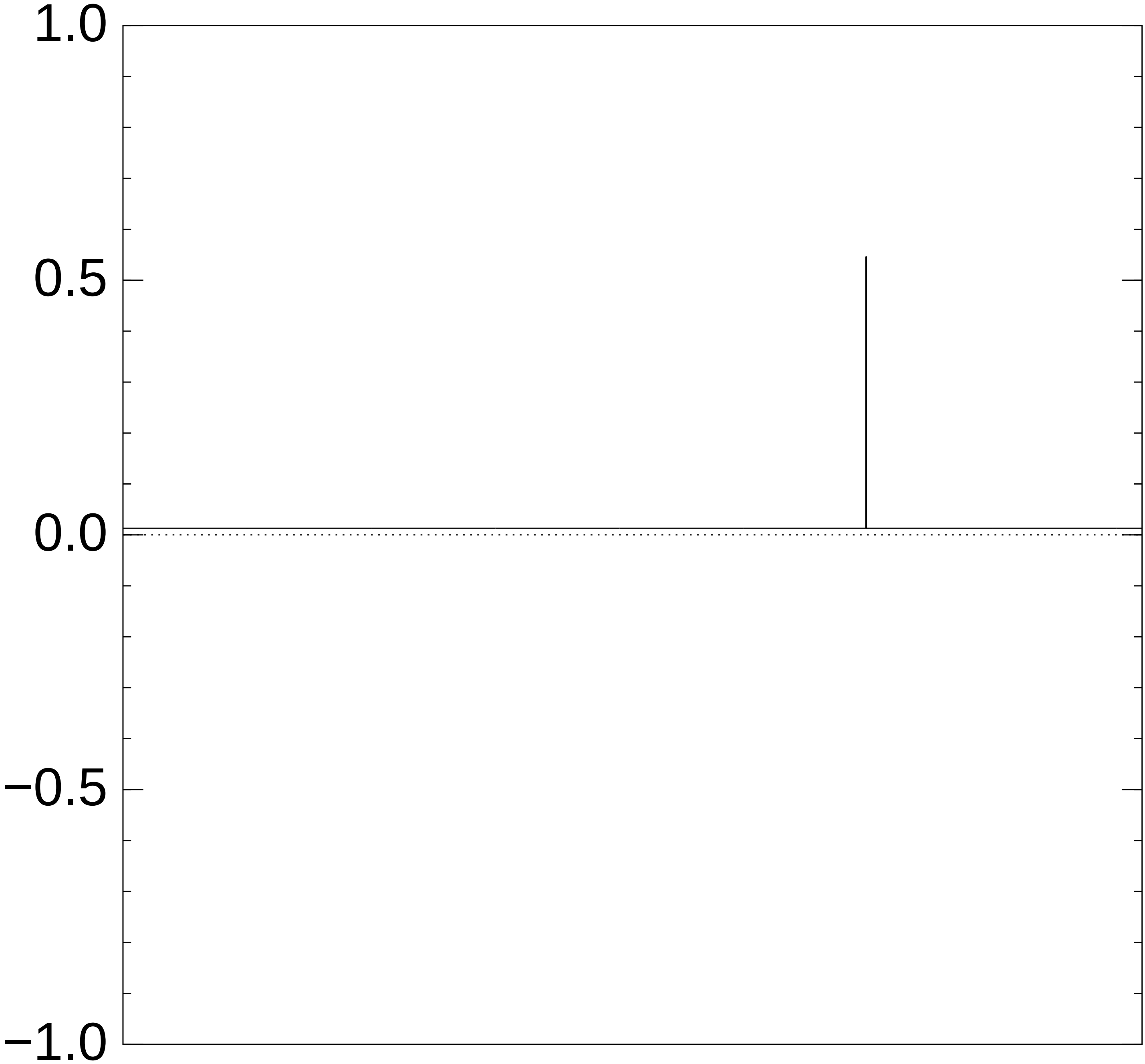
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $16 \times$ (Step $1 +$ Step 2):



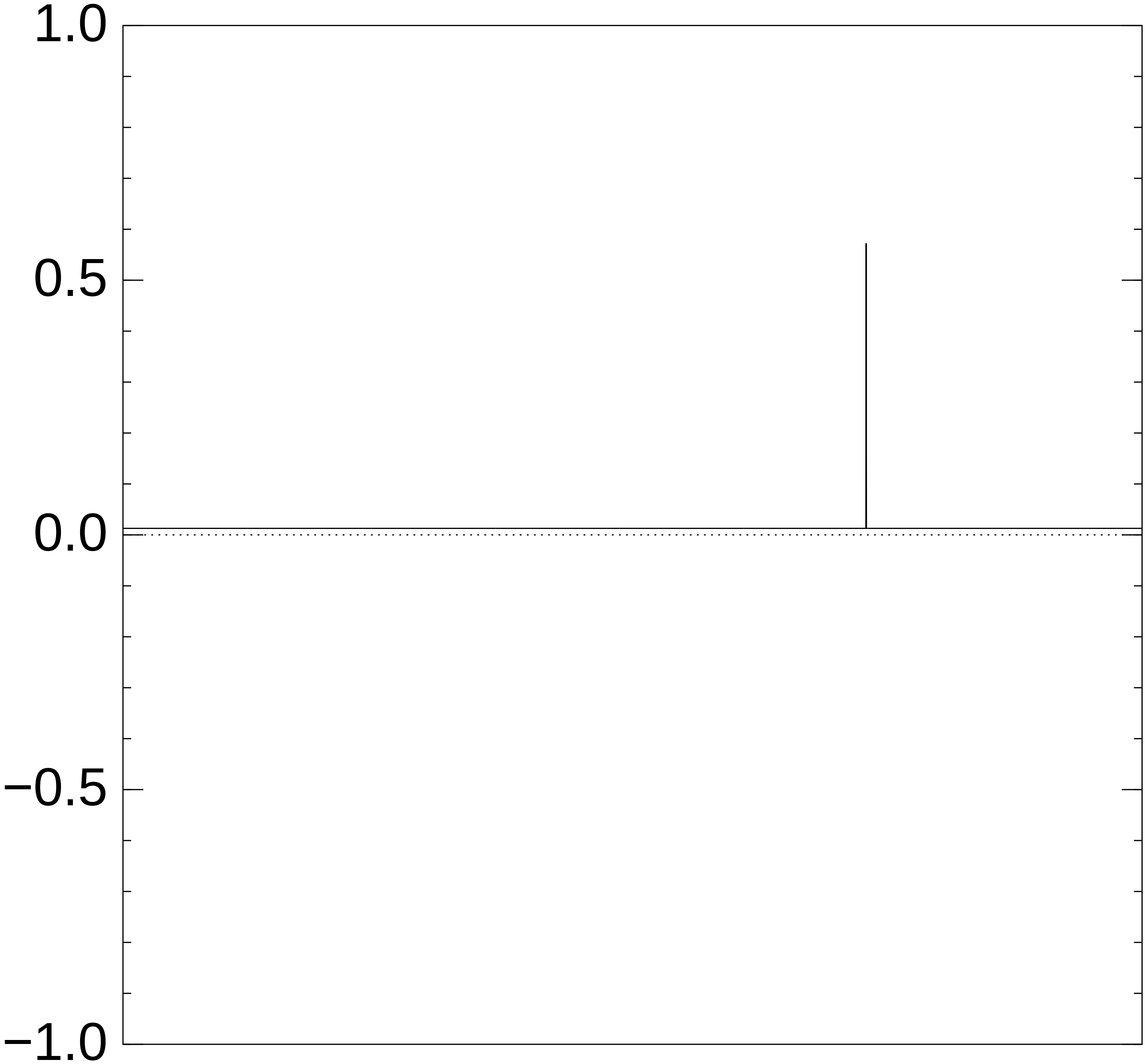
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $17 \times$ (Step $1 +$ Step 2):



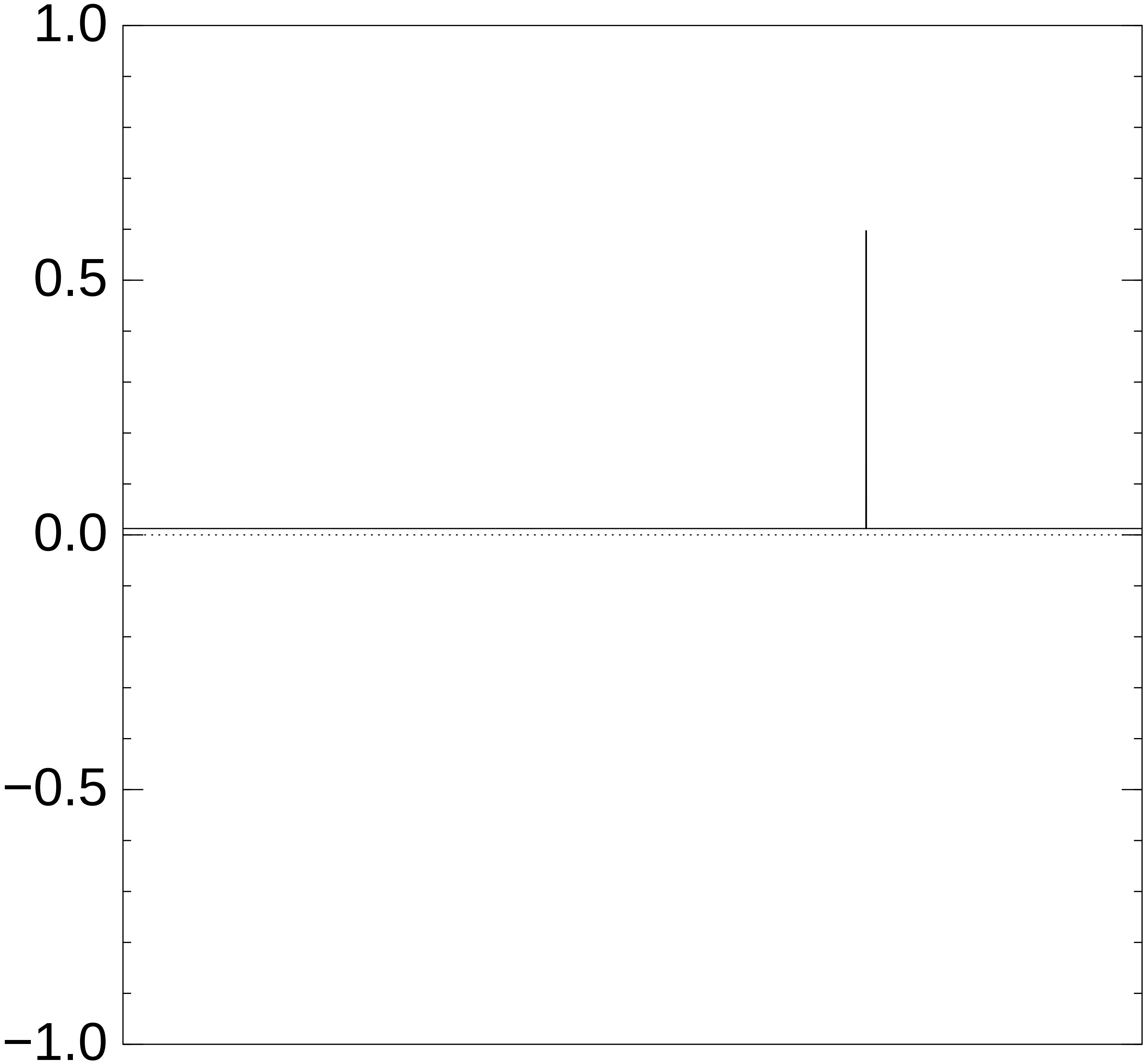
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $18 \times$ (Step 1 + Step 2):



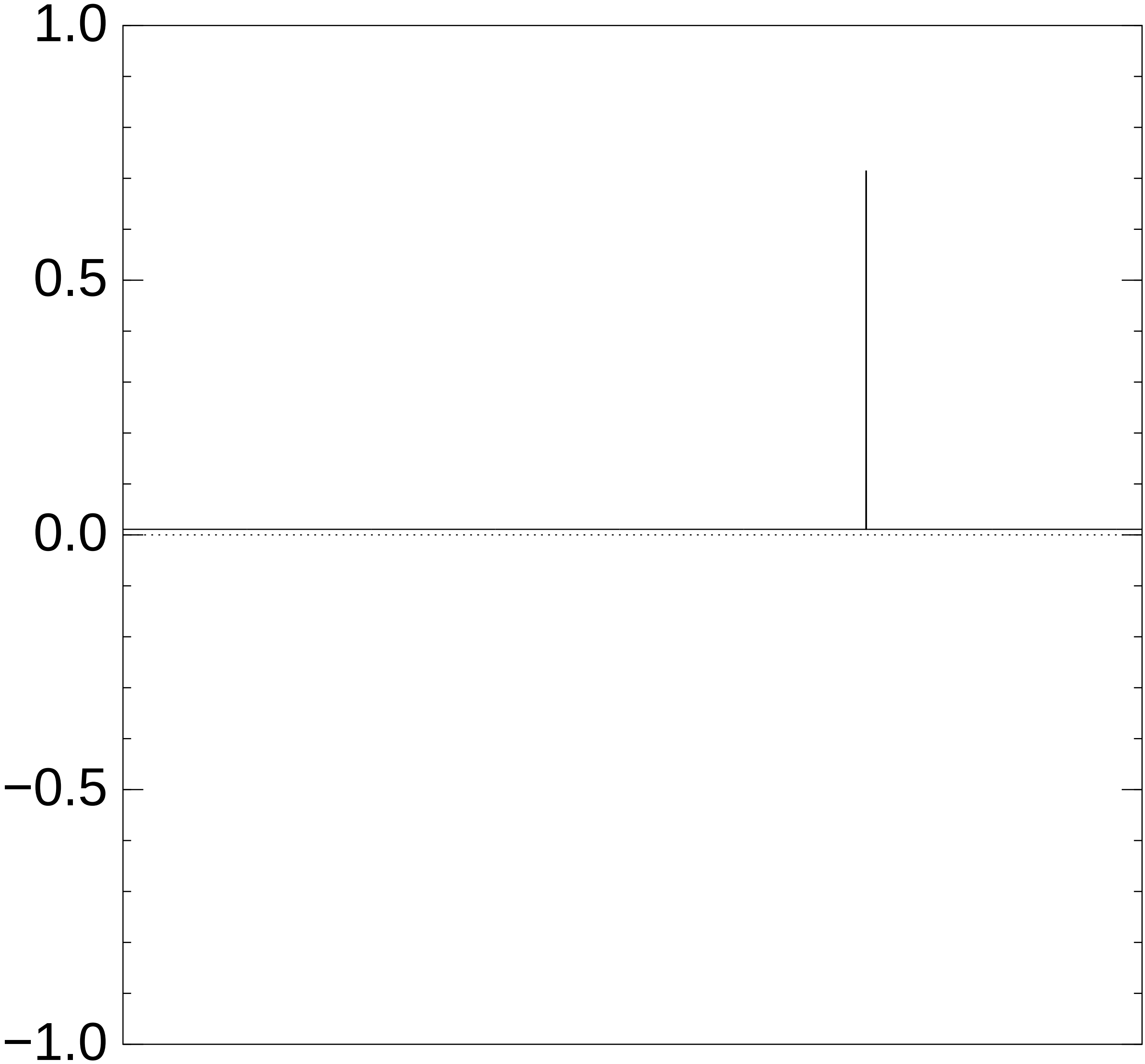
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $19 \times$ (Step $1 +$ Step 2):



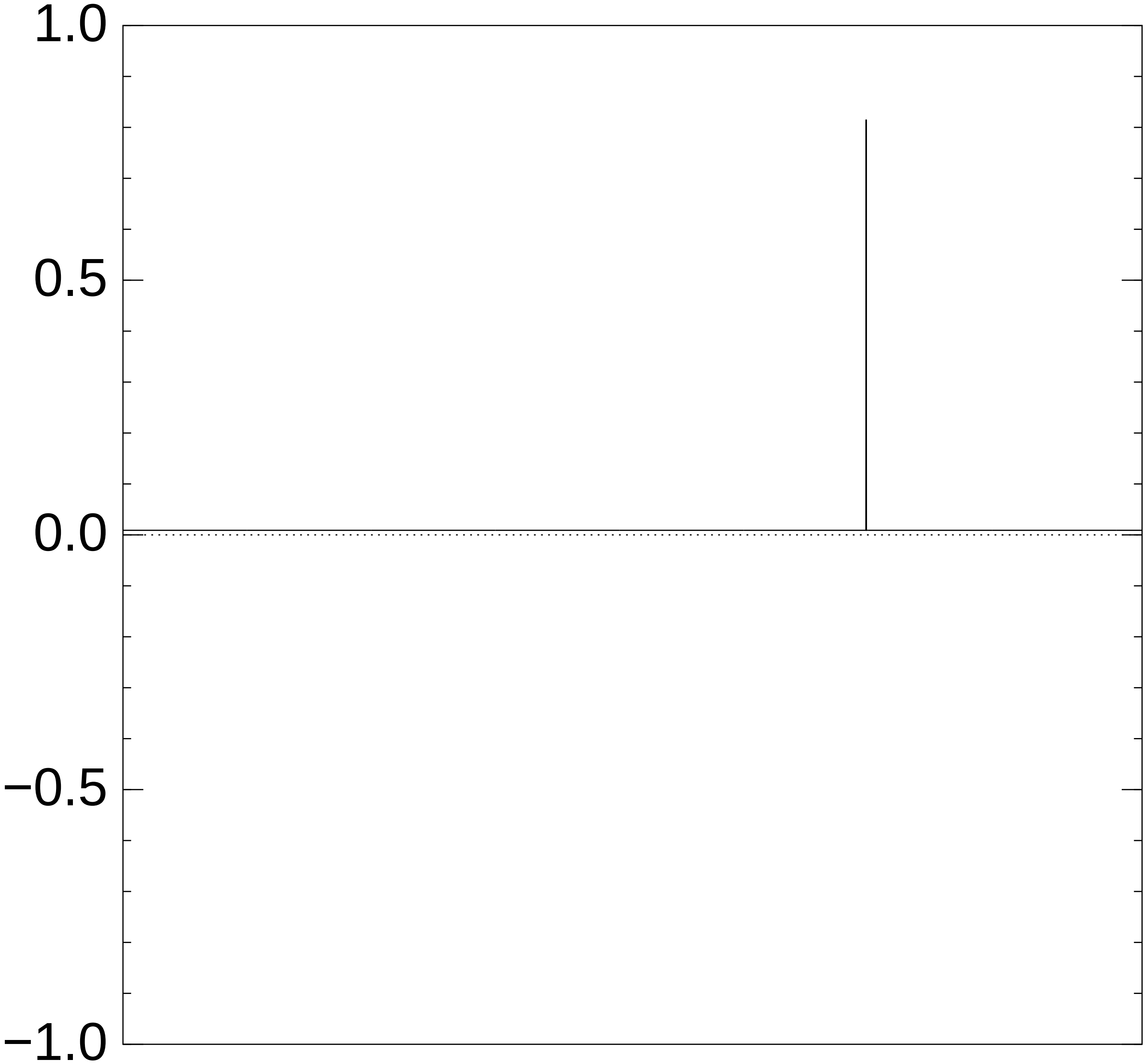
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $20 \times$ (Step $1 +$ Step 2):



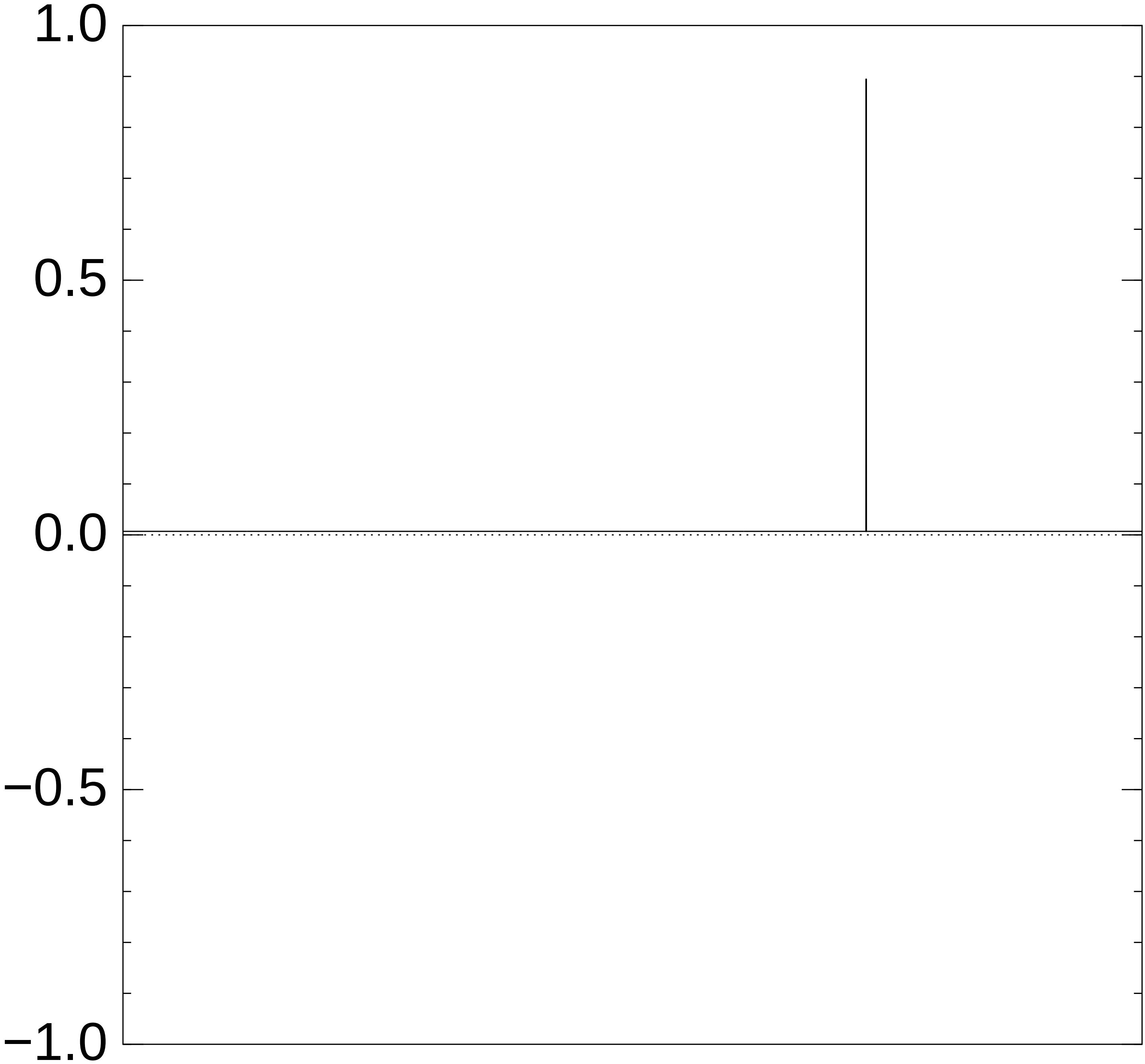
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $25 \times (\text{Step } 1 + \text{Step } 2)$:



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $30 \times (\text{Step } 1 + \text{Step } 2)$:

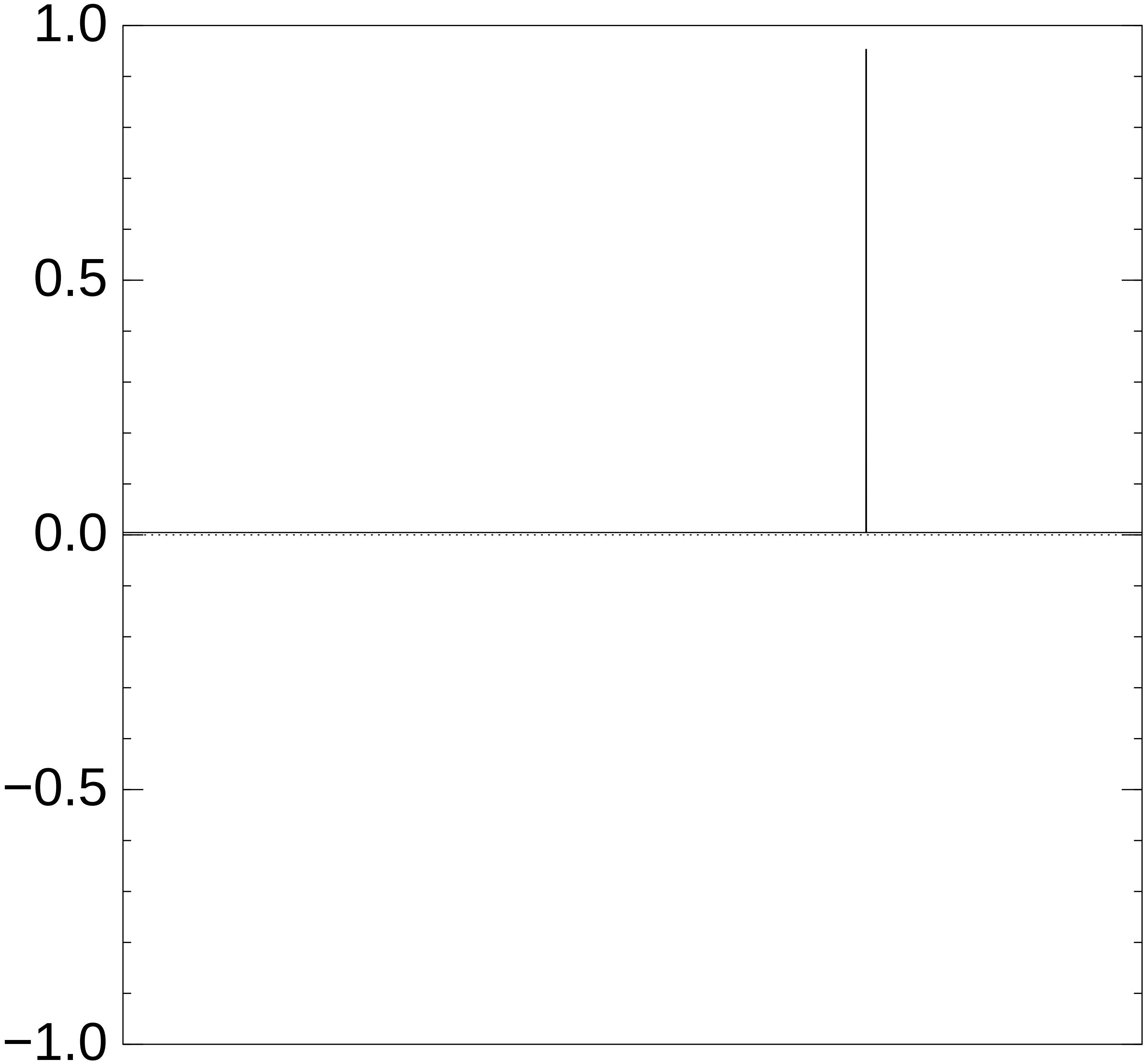


Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $35 \times$ (Step $1 +$ Step 2):

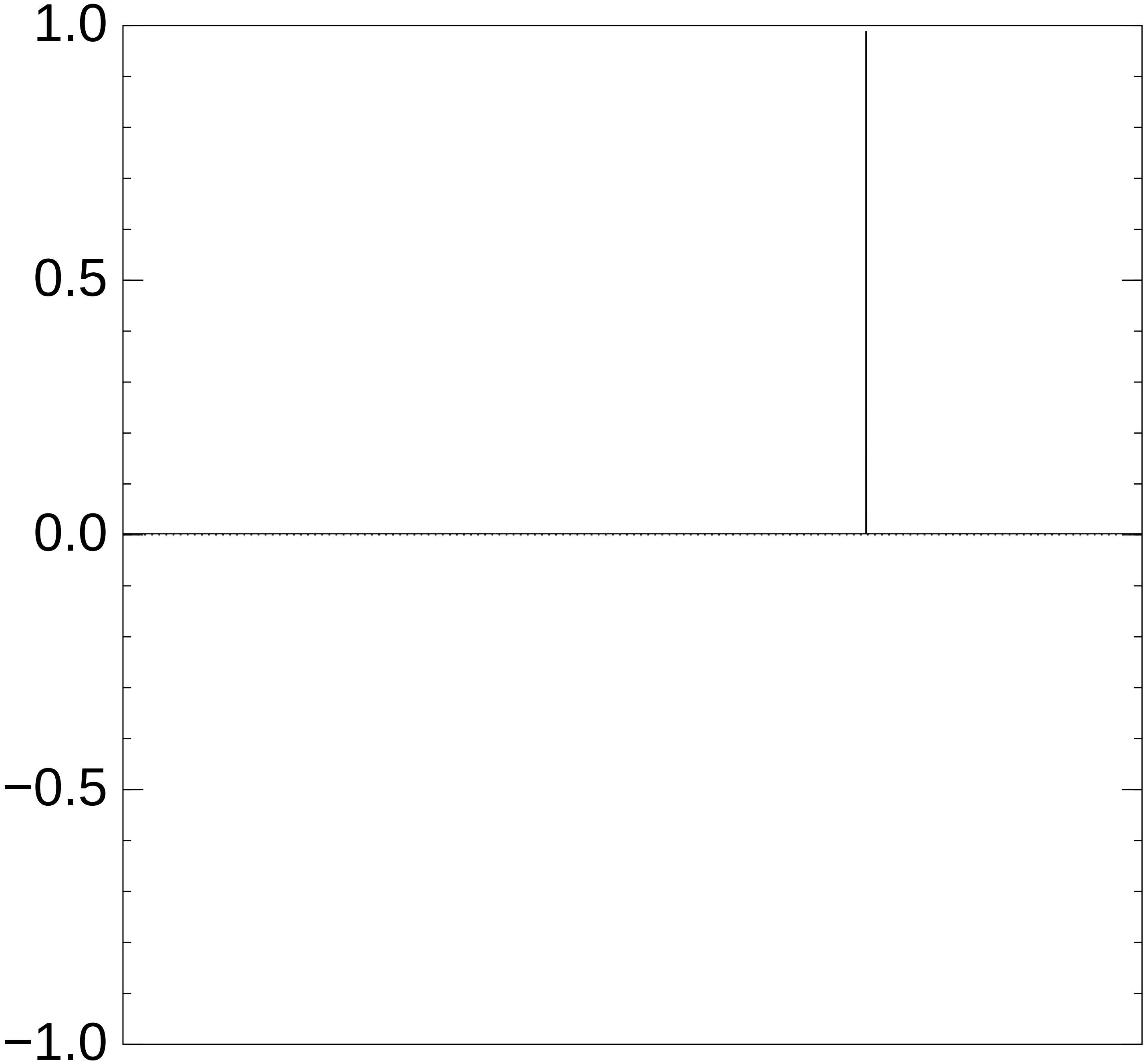


Good moment to stop, measure.

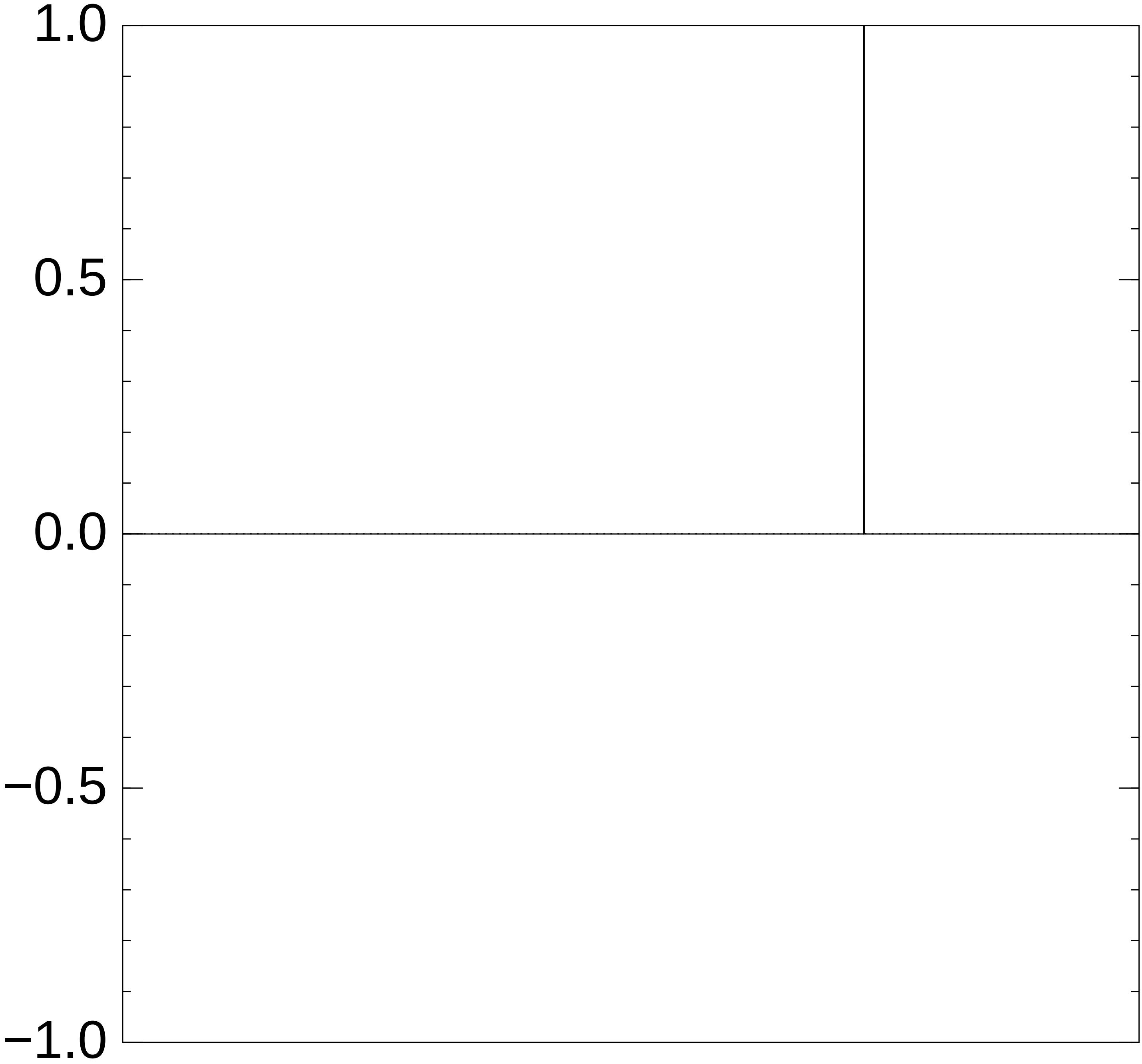
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $40 \times$ (Step $1 +$ Step 2):



Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $45 \times (\text{Step 1} + \text{Step 2})$:

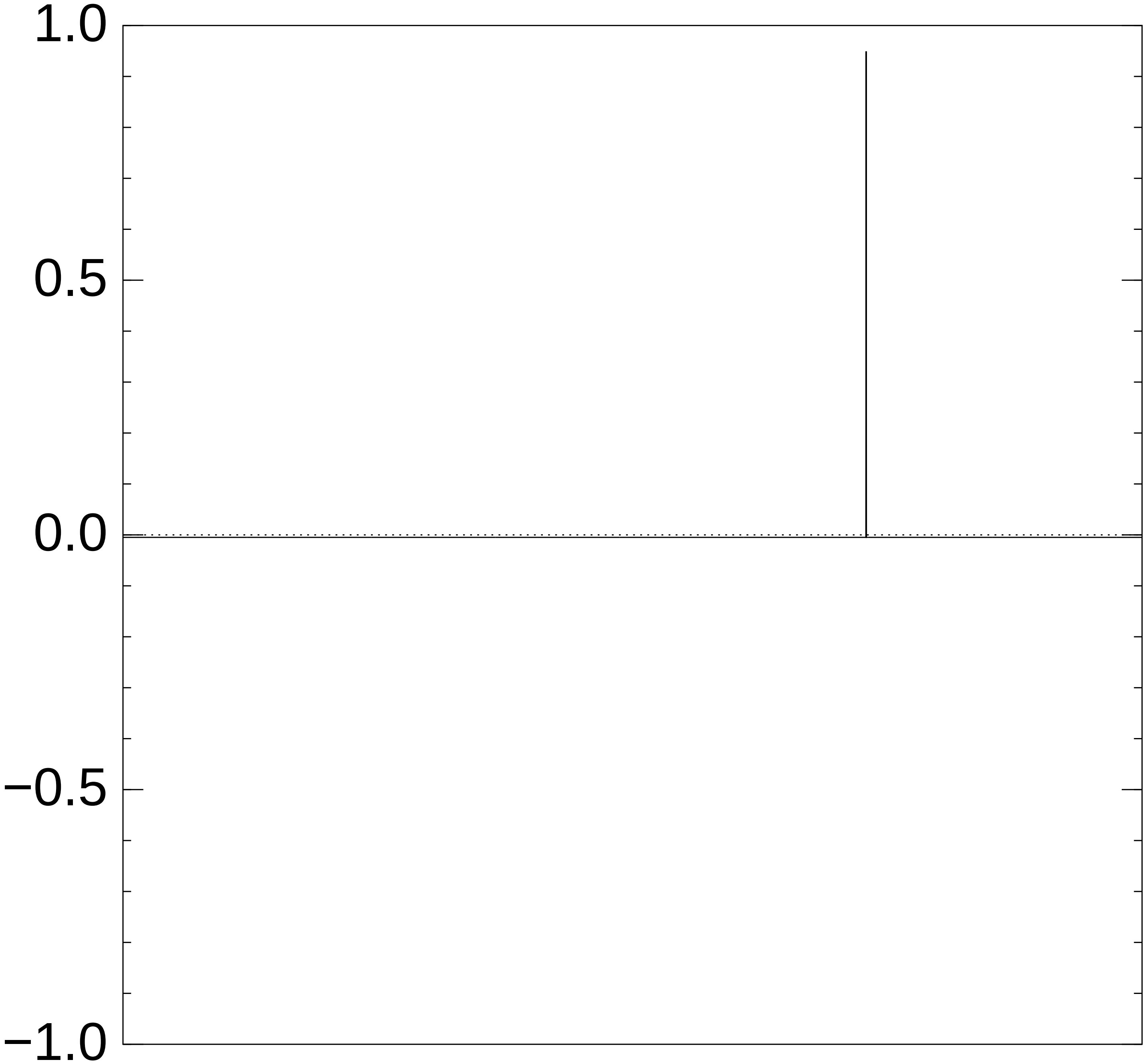


Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $50 \times$ (Step $1 +$ Step 2):

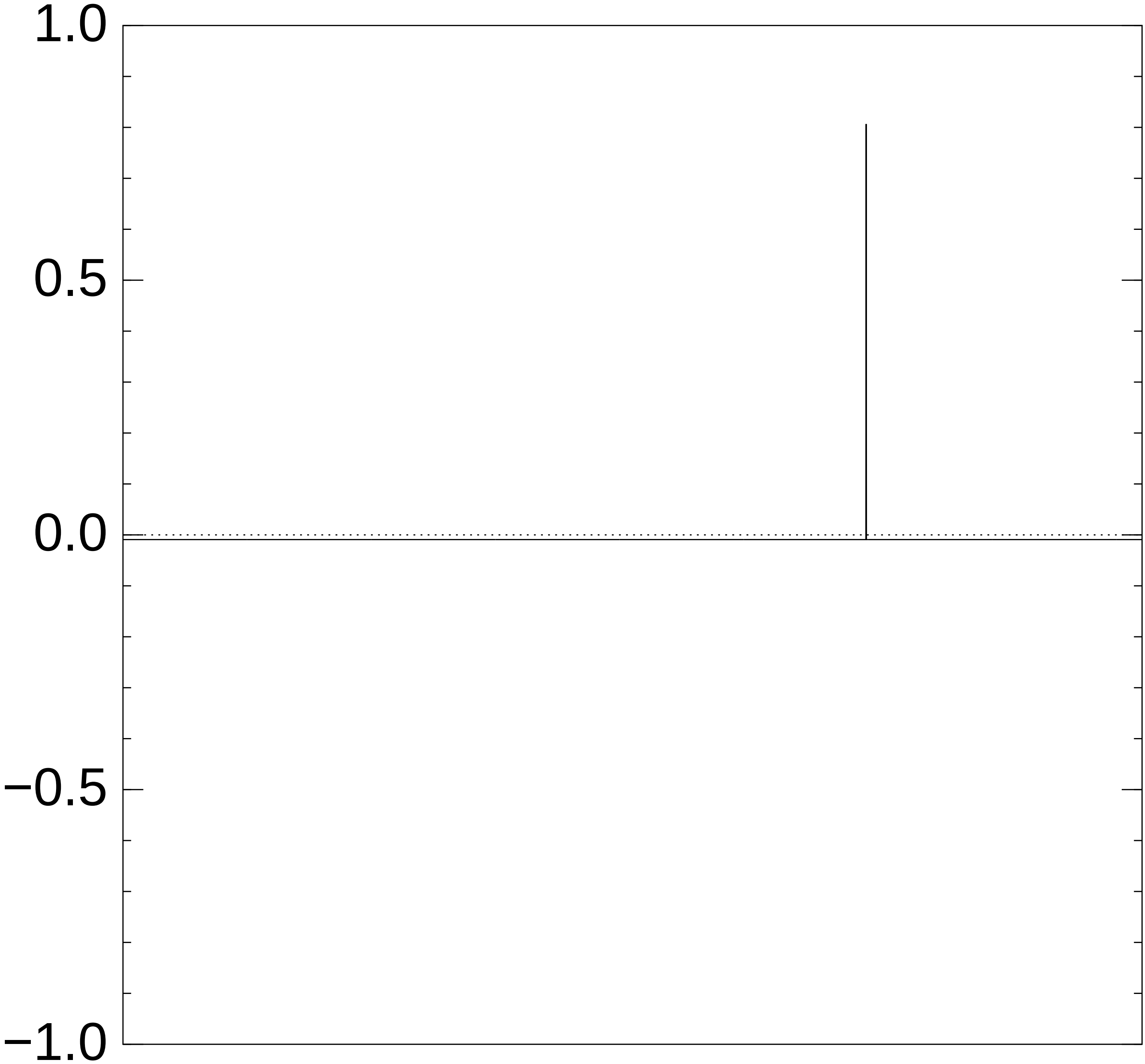


Traditional stopping point.

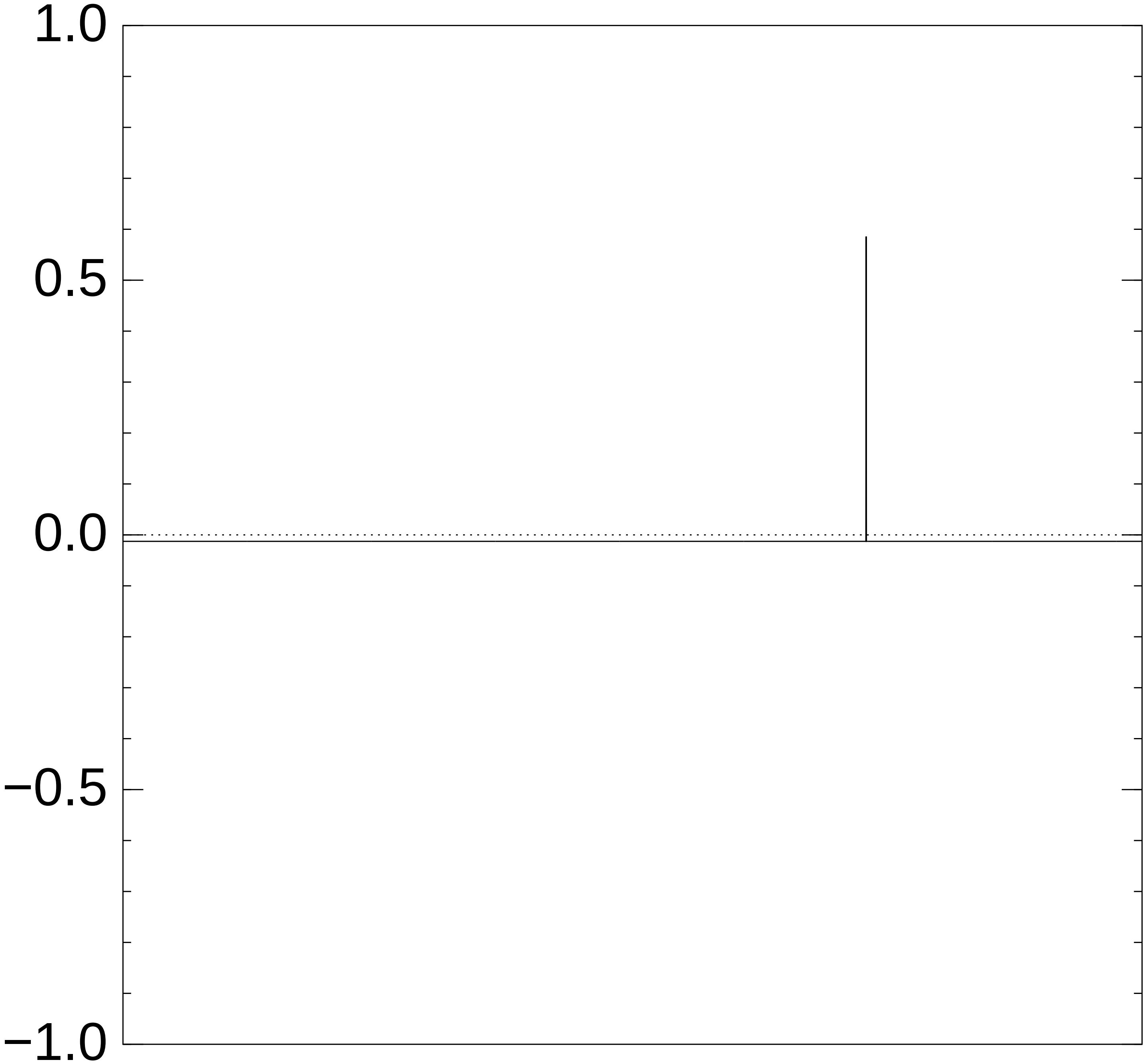
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $60 \times (\text{Step 1} + \text{Step 2})$:



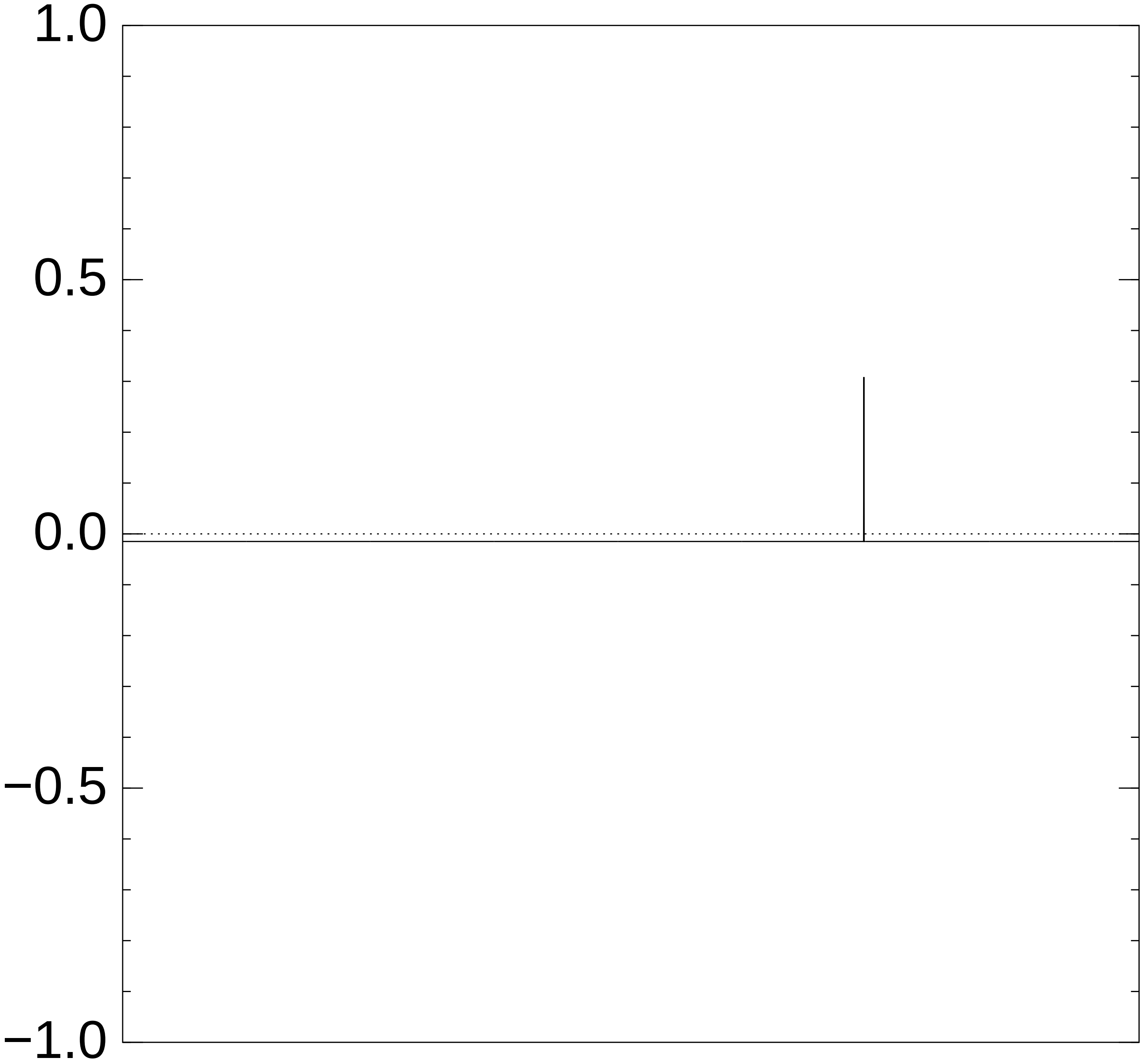
Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $70 \times (\text{Step } 1 + \text{Step } 2)$:



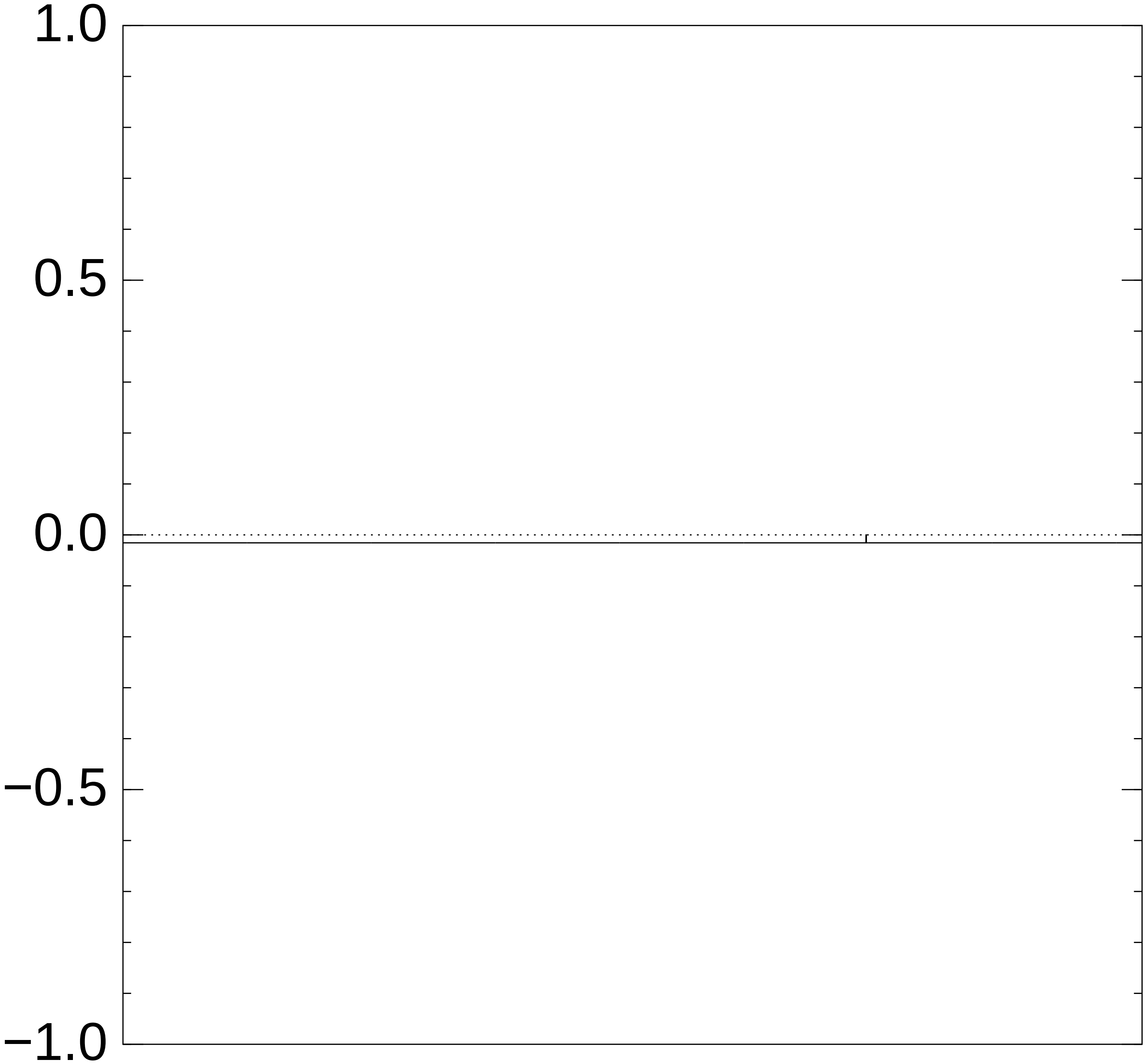
Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $80 \times$ (Step $1 +$ Step 2):



Normalized graph of $q \mapsto a_q$
for an example with $n = 12$
after $90 \times$ (Step 1 + Step 2):



Normalized graph of $q \mapsto a_q$ for an example with $n = 12$ after $100 \times$ (Step $1 +$ Step 2):



Very bad stopping point.

$q \mapsto a_q$ is completely described by a vector of two numbers (with fixed multiplicities):

(1) a_q for roots q;

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Easily compute eigenvalues
and powers of this linear map
to understand evolution
of state of Grover's algorithm.
$\Rightarrow$ Probability is $\approx$1
after $\approx(\pi/4)2^{n/2}$ iterations.

Ambainis's algorithm

Unique-collision-finding problem:
Say f has n-bit inputs,
exactly one collision $\{p, q\}$:
i.e., $p \neq q$, $f(p) = f(q)$.
Problem: find this collision.

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Compute $f(S)$, sort.

Generalize to cost r,
success probability $\approx(r/2^n)^2$:
Choose a set S of size r.
Compute $f(S)$, sort.

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2003 Ambainis, simplified 2007 Magniez–Nayak–Roland–Santha: Create superposition of states $(D(S), D(T))$ with adjacent S, T. By a quantum walk find S containing a collision.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

Negate $a_{S,T}$

if S contains collision.

Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

For each T:

Diffuse $a_{S,T}$ across all S.

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Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S,T) according to $(\#(S\cap\{p,q\}),\#(T\cap\{p,q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n=15$, $r=1024$, after 0 negations and 0 diffusions:

$\Pr[\text{class }(0,0)]\approx 0.938;+$
$\Pr[\text{class }(0,1)]\approx 0.000;+$
$\Pr[\text{class }(1,0)]\approx 0.000;+$
$\Pr[\text{class }(1,1)]\approx 0.060;+$
$\Pr[\text{class }(1,2)]\approx 0.000;+$
$\Pr[\text{class }(2,1)]\approx 0.000;+$
$\Pr[\text{class }(2,2)]\approx 0.001;+$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 1 negation and 46 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.935; +$
$\Pr[\text{class } (0, 1)] \approx 0.000; +$
$\Pr[\text{class } (1, 0)] \approx 0.000; -$
$\Pr[\text{class } (1, 1)] \approx 0.057; +$
$\Pr[\text{class } (1, 2)] \approx 0.000; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; -$
$\Pr[\text{class } (2, 2)] \approx 0.008; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 2 negations and 92 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.918; +$
$\Pr[\text{class } (0, 1)] \approx 0.001; +$
$\Pr[\text{class } (1, 0)] \approx 0.000; -$
$\Pr[\text{class } (1, 1)] \approx 0.059; +$
$\Pr[\text{class } (1, 2)] \approx 0.001; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; -$
$\Pr[\text{class } (2, 2)] \approx 0.022; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 3 negations and 138 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.897; +$
$\Pr[\text{class } (0, 1)] \approx 0.001; +$
$\Pr[\text{class } (1, 0)] \approx 0.000; -$
$\Pr[\text{class } (1, 1)] \approx 0.058; +$
$\Pr[\text{class } (1, 2)] \approx 0.002; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.042; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 4 negations and 184 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.873; +$
$\Pr[\text{class } (0, 1)] \approx 0.001; +$
$\Pr[\text{class } (1, 0)] \approx 0.000; -$
$\Pr[\text{class } (1, 1)] \approx 0.054; +$
$\Pr[\text{class } (1, 2)] \approx 0.002; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.070; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 5 negations and 230 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.838; +$
$\Pr[\text{class } (0, 1)] \approx 0.001; +$
$\Pr[\text{class } (1, 0)] \approx 0.001; -$
$\Pr[\text{class } (1, 1)] \approx 0.054; +$
$\Pr[\text{class } (1, 2)] \approx 0.003; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.104; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 6 negations and 276 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.800; +$
$\Pr[\text{class } (0, 1)] \approx 0.001; +$
$\Pr[\text{class } (1, 0)] \approx 0.001; -$
$\Pr[\text{class } (1, 1)] \approx 0.051; +$
$\Pr[\text{class } (1, 2)] \approx 0.006; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.141; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
7 negations and 322 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.758; +$
$\Pr[\text{class } (0, 1)] \approx 0.002; +$
$\Pr[\text{class } (1, 0)] \approx 0.001; -$
$\Pr[\text{class } (1, 1)] \approx 0.047; +$
$\Pr[\text{class } (1, 2)] \approx 0.007; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.184; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
8 negations and 368 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.708; +$
$\Pr[\text{class } (0, 1)] \approx 0.003; +$
$\Pr[\text{class } (1, 0)] \approx 0.001; -$
$\Pr[\text{class } (1, 1)] \approx 0.046; +$
$\Pr[\text{class } (1, 2)] \approx 0.007; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.234; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 9 negations and 414 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.658; +$
$\Pr[\text{class } (0, 1)] \approx 0.003; +$
$\Pr[\text{class } (1, 0)] \approx 0.001; -$
$\Pr[\text{class } (1, 1)] \approx 0.042; +$
$\Pr[\text{class } (1, 2)] \approx 0.009; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.287; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
10 negations and 460 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.606; +$
$\Pr[\text{class } (0, 1)] \approx 0.003; +$
$\Pr[\text{class } (1, 0)] \approx 0.002; -$
$\Pr[\text{class } (1, 1)] \approx 0.037; +$
$\Pr[\text{class } (1, 2)] \approx 0.013; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.338; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 11 negations and 506 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.547; +$
$\Pr[\text{class } (0, 1)] \approx 0.004; +$
$\Pr[\text{class } (1, 0)] \approx 0.003; -$
$\Pr[\text{class } (1, 1)] \approx 0.036; +$
$\Pr[\text{class } (1, 2)] \approx 0.015; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.394; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 12 negations and 552 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.491; +$
$\Pr[\text{class } (0, 1)] \approx 0.004; +$
$\Pr[\text{class } (1, 0)] \approx 0.003; -$
$\Pr[\text{class } (1, 1)] \approx 0.032; +$
$\Pr[\text{class } (1, 2)] \approx 0.014; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.455; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 13 negations and 598 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.436; +$
$\Pr[\text{class } (0, 1)] \approx 0.005; +$
$\Pr[\text{class } (1, 0)] \approx 0.003; -$
$\Pr[\text{class } (1, 1)] \approx 0.026; +$
$\Pr[\text{class } (1, 2)] \approx 0.017; +$
$\Pr[\text{class } (2, 1)] \approx 0.000; +$
$\Pr[\text{class } (2, 2)] \approx 0.513; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 14 negations and 644 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.377; +$
$\Pr[\text{class } (0, 1)] \approx 0.006; +$
$\Pr[\text{class } (1, 0)] \approx 0.004; -$
$\Pr[\text{class } (1, 1)] \approx 0.025; +$
$\Pr[\text{class } (1, 2)] \approx 0.022; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.566; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 15 negations and 690 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.322; +$
$\Pr[\text{class } (0, 1)] \approx 0.005; +$
$\Pr[\text{class } (1, 0)] \approx 0.004; -$
$\Pr[\text{class } (1, 1)] \approx 0.021; +$
$\Pr[\text{class } (1, 2)] \approx 0.023; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.623; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 16 negations and 736 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.270; +$
$\Pr[\text{class } (0, 1)] \approx 0.006; +$
$\Pr[\text{class } (1, 0)] \approx 0.005; -$
$\Pr[\text{class } (1, 1)] \approx 0.017; +$
$\Pr[\text{class } (1, 2)] \approx 0.022; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.680; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 17 negations and 782 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.218; +$
$\Pr[\text{class } (0, 1)] \approx 0.007; +$
$\Pr[\text{class } (1, 0)] \approx 0.005; -$
$\Pr[\text{class } (1, 1)] \approx 0.015; +$
$\Pr[\text{class } (1, 2)] \approx 0.024; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.730; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
18 negations and 828 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.172; +$
$\Pr[\text{class } (0, 1)] \approx 0.006; +$
$\Pr[\text{class } (1, 0)] \approx 0.005; -$
$\Pr[\text{class } (1, 1)] \approx 0.011; +$
$\Pr[\text{class } (1, 2)] \approx 0.029; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.775; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 19 negations and 874 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.131; +$
$\Pr[\text{class } (0, 1)] \approx 0.007; +$
$\Pr[\text{class } (1, 0)] \approx 0.006; -$
$\Pr[\text{class } (1, 1)] \approx 0.008; +$
$\Pr[\text{class } (1, 2)] \approx 0.030; +$
$\Pr[\text{class } (2, 1)] \approx 0.002; +$
$\Pr[\text{class } (2, 2)] \approx 0.816; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
20 negations and 920 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.093; +$
$\Pr[\text{class } (0, 1)] \approx 0.007; +$
$\Pr[\text{class } (1, 0)] \approx 0.007; -$
$\Pr[\text{class } (1, 1)] \approx 0.007; +$
$\Pr[\text{class } (1, 2)] \approx 0.027; +$
$\Pr[\text{class } (2, 1)] \approx 0.002; +$
$\Pr[\text{class } (2, 2)] \approx 0.857; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 21 negations and 966 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.062; +$
$\Pr[\text{class } (0, 1)] \approx 0.007; +$
$\Pr[\text{class } (1, 0)] \approx 0.006; -$
$\Pr[\text{class } (1, 1)] \approx 0.004; +$
$\Pr[\text{class } (1, 2)] \approx 0.030; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.890; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 22 negations and 1012 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.037; +$
$\Pr[\text{class } (0, 1)] \approx 0.008; +$
$\Pr[\text{class } (1, 0)] \approx 0.007; -$
$\Pr[\text{class } (1, 1)] \approx 0.002; +$
$\Pr[\text{class } (1, 2)] \approx 0.034; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.910; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 23 negations and 1058 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.017; +$
$\Pr[\text{class } (0, 1)] \approx 0.008; +$
$\Pr[\text{class } (1, 0)] \approx 0.007; -$
$\Pr[\text{class } (1, 1)] \approx 0.002; +$
$\Pr[\text{class } (1, 2)] \approx 0.034; +$
$\Pr[\text{class } (2, 1)] \approx 0.002; +$
$\Pr[\text{class } (2, 2)] \approx 0.930; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 24 negations and 1104 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.005; +$
$\Pr[\text{class } (0, 1)] \approx 0.007; +$
$\Pr[\text{class } (1, 0)] \approx 0.007; -$
$\Pr[\text{class } (1, 1)] \approx 0.000; +$
$\Pr[\text{class } (1, 2)] \approx 0.030; +$
$\Pr[\text{class } (2, 1)] \approx 0.002; +$
$\Pr[\text{class } (2, 2)] \approx 0.948; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 25 negations and 1150 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.000; +$
$\Pr[\text{class } (0, 1)] \approx 0.008; +$
$\Pr[\text{class } (1, 0)] \approx 0.008; -$
$\Pr[\text{class } (1, 1)] \approx 0.000; +$
$\Pr[\text{class } (1, 2)] \approx 0.031; +$
$\Pr[\text{class } (2, 1)] \approx 0.001; +$
$\Pr[\text{class } (2, 2)] \approx 0.952; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 26 negations and 1196 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.002; -$
$\Pr[\text{class } (0, 1)] \approx 0.008; +$
$\Pr[\text{class } (1, 0)] \approx 0.008; -$
$\Pr[\text{class } (1, 1)] \approx 0.000; -$
$\Pr[\text{class } (1, 2)] \approx 0.035; +$
$\Pr[\text{class } (2, 1)] \approx 0.002; +$
$\Pr[\text{class } (2, 2)] \approx 0.945; +$

Right column is sign of $a_{S,T}$.

Classify (S, T) according to
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Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
27 negations and 1242 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.011; -$
$\Pr[\text{class } (0, 1)] \approx 0.007; +$
$\Pr[\text{class } (1, 0)] \approx 0.007; -$
$\Pr[\text{class } (1, 1)] \approx 0.001; -$
$\Pr[\text{class } (1, 2)] \approx 0.034; +$
$\Pr[\text{class } (2, 1)] \approx 0.003; +$
$\Pr[\text{class } (2, 2)] \approx 0.938; +$

Right column is sign of $a_{S,T}$.

Data structures

Moving from $D(S)$ to $D(T)$: dominated by $O(1)$ evaluations of f if f is extremely slow.

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Minor problem: time to hash S is huge for some sets S.

Fix: randomize hash function (1979 Carter–Wegman), and specify big enough time for whole algorithm to be reliable.

Major problem: hash table depends on history, not just on S. Algorithm fails horribly.

Need history-independent $D(S)$.

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2013 Bernstein–Jeffery–Lange–Meurer: radix tree.

Simplest radix tree: Left subtree stores $\{x : (0, x) \in S\}$ if nonempty. Right subtree stores $\{x : (1, x) \in S\}$ if nonempty.

Caveats

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I recommend algorithm analysis on 2-dim mesh of tiny processors: e.g. 0.472 for MQ (vs. 0.462) from 2017 Bernstein–Yang.

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- Quantum ops are expensive.

Grover risk to cryptography is much smaller than Shor risk.

Background slides . . .

What do quantum computers do?

"Quantum algorithm" means an algorithm that a quantum computer can run.

i.e. a sequence of instructions, where each instruction is in a quantum computer's supported instruction set.

How do we know which instructions a quantum computer will support?

Quantum computer type 1 (QC1):
contains many "qubits";
can efficiently perform
"NOT gate", "Hadamard gate",
"controlled NOT gate", "T gate".

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Combine these instructions to compute "Toffoli gate"; . . . "Simon's algorithm"; . . . "Shor's algorithm"; etc.

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to compute “Toffoli gate”;
. . . “Simon’s algorithm”;
. . . “Shor’s algorithm”; etc.

General belief: Traditional CPU
isn’t QC1; e.g. can’t factor quickly.

Quantum computer type 2 (QC2): stores a simulated universe; efficiently simulates the laws of quantum physics with as much accuracy as desired.

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General belief: any QC1 is a QC2. Partial proof: see, e.g., 2011 Jordan–Lee–Preskill "Quantum algorithms for quantum field theories".

Quantum computer type 3 (QC3):
efficiently computes anything
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General belief: any QC3 is a QC1.
Argument for belief:
look, we're building a QC1.

A note on D-Wave

Apparent scientific consensus:
Current "quantum computers" from D-Wave are useless— can be more cost-effectively simulated by traditional CPUs.

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Is D-Wave a bad investment?

The state of a computer

Data ("state") stored in 3 bits:
a list of 3 elements of $\{0, 1\}$.
e.g.: $(0, 0, 0)$.

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a list of 64 elements of $\{0, 1\}$.
e.g.: $(1, 1, 1, 1, 1, 0, 0, 0, 1,$
$0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0,$
$0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 1,$
$1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1,$
$0, 0, 1, 1, 1, 0, 0, 1, 0, 0, 0,$
$1, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1)$.

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Data stored in 4 qubits: a list of 16 numbers, not all zero. e.g.:
$(3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3)$.

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Data stored in 64 qubits:
a list of 2^{64} numbers, not all zero.

The state of a quantum computer

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Data stored in 64 qubits:
a list of 2^{64} numbers, not all zero.

Data stored in 1000 qubits: a list of 2^{1000} numbers, not all zero.

Measuring a quantum computer

Can simply look at a bit.

Cannot simply look at the list of numbers stored in n qubits.

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- produces n bits and
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If n qubits have state $(a_0, a_1, \ldots, a_{2^n-1})$ then measurement produces q with probability $|a_q|^2 / \sum_r |a_r|^2$.

State is then all zeros except 1 at position q.

e.g.: Say 3 qubits have state
$(1, 1, 1, 1, 1, 1, 1, 1)$.

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Measurement produces
$000 = 0$ with probability $1/8$;
$001 = 1$ with probability $1/8$;
$010 = 2$ with probability $1/8$;
$011 = 3$ with probability $1/8$;
$100 = 4$ with probability $1/8$;
$101 = 5$ with probability $1/8$;
$110 = 6$ with probability $1/8$;
$111 = 7$ with probability $1/8$.

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"Quantum RNG."

e.g.: Say 3 qubits have state $(1,1,1,1,1,1,1,1)$.

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$010 = 2$ with probability $1/8$;
$011 = 3$ with probability $1/8$;
$100 = 4$ with probability $1/8$;
$101 = 5$ with probability $1/8$;
$110 = 6$ with probability $1/8$;
$111 = 7$ with probability $1/8$.

“Quantum RNG.”

Warning: Quantum RNGs sold today are measurably biased.

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Measurement produces
$000 = 0$ with probability $9/173$;
$001 = 1$ with probability $1/173$;
$010 = 2$ with probability $16/173$;
$011 = 3$ with probability $1/173$;
$100 = 4$ with probability $25/173$;
$101 = 5$ with probability $81/173$;
$110 = 6$ with probability $4/173$;
$111 = 7$ with probability $36/173$.

e.g.: Say 3 qubits have state $(3, 1, 4, 1, 5, 9, 2, 6)$.

Measurement produces
$000 = 0$ with probability $9/173$;
$001 = 1$ with probability $1/173$;
$010 = 2$ with probability $16/173$;
$011 = 3$ with probability $1/173$;
$100 = 4$ with probability $25/173$;
$101 = 5$ with probability $81/173$;
$110 = 6$ with probability $4/173$;
$111 = 7$ with probability $36/173$.

5 is most likely outcome.

e.g.: Say 3 qubits have state
$(0, 0, 0, 0, 0, 1, 0, 0)$.

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Measurement produces
$000 = 0$ with probability 0;
$001 = 1$ with probability 0;
$010 = 2$ with probability 0;
$011 = 3$ with probability 0;
$100 = 4$ with probability 0;
$101 = 5$ with probability 1;
$110 = 6$ with probability 0;
$111 = 7$ with probability 0.

e.g.: Say 3 qubits have state $(0, 0, 0, 0, 0, 1, 0, 0)$.

Measurement produces
$000 = 0$ with probability 0;
$001 = 1$ with probability 0;
$010 = 2$ with probability 0;
$011 = 3$ with probability 0;
$100 = 4$ with probability 0;
$101 = 5$ with probability 1;
$110 = 6$ with probability 0;
$111 = 7$ with probability 0.

5 is guaranteed outcome.

NOT gates

NOT_0 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(1, 3, 1, 4, 9, 5, 6, 2)$.

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NOT_0 gate on 4 qubits:
$(3,1,4,1,5,9,2,6,5,3,5,8,9,7,9,3) \mapsto$
$(1,3,1,4,9,5,6,2,3,5,8,5,7,9,3,9)$.

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$(3,1,4,1,5,9,2,6,5,3,5,8,9,7,9,3) \mapsto$
$(1,3,1,4,9,5,6,2,3,5,8,5,7,9,3,9)$.

NOT_1 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(4, 1, 3, 1, 2, 6, 5, 9)$.

NOT gates

NOT_0 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(1, 3, 1, 4, 9, 5, 6, 2)$.

NOT_0 gate on 4 qubits:
$(3,1,4,1,5,9,2,6,5,3,5,8,9,7,9,3) \mapsto$
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NOT_1 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(4, 1, 3, 1, 2, 6, 5, 9)$.

NOT_2 gate on 3 qubits:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(5, 9, 2, 6, 3, 1, 4, 1)$.

state	measurement
$(1,0,0,0,0,0,0,0)$	000
$(0,1,0,0,0,0,0,0)$	001
$(0,0,1,0,0,0,0,0)$	010
$(0,0,0,1,0,0,0,0)$	011
$(0,0,0,0,1,0,0,0)$	100
$(0,0,0,0,0,1,0,0)$	101
$(0,0,0,0,0,0,1,0)$	110
$(0,0,0,0,0,0,0,1)$	111

Operation on quantum state:
NOT_0, swapping pairs.

Operation after measurement:
flipping bit 0 of result.

Flip: output is not input.

Controlled-NOT gates

e.g. $\text{CNOT}_{1,0}$:

$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$

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flipping bit 0 *if* bit 1 is set; i.e.,
$(q_2, q_1, q_0) \mapsto (q_2, q_1, q_0 \oplus q_1)$.

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e.g. $\text{CNOT}_{2,0}$:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(3, 1, 4, 1, 9, 5, 6, 2)$.

Controlled-NOT gates

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$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
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e.g. $\text{CNOT}_{0,2}$:
$(3, 1, 4, 1, 5, 9, 2, 6) \mapsto$
$(3, 9, 4, 6, 5, 1, 2, 1)$.

Toffoli gates

Also known as controlled-controlled-NOT gates.

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e.g. $\text{CCNOT}_{0,1,2}$:
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More shuffling

Combine NOT, CNOT, Toffoli to build other permutations.

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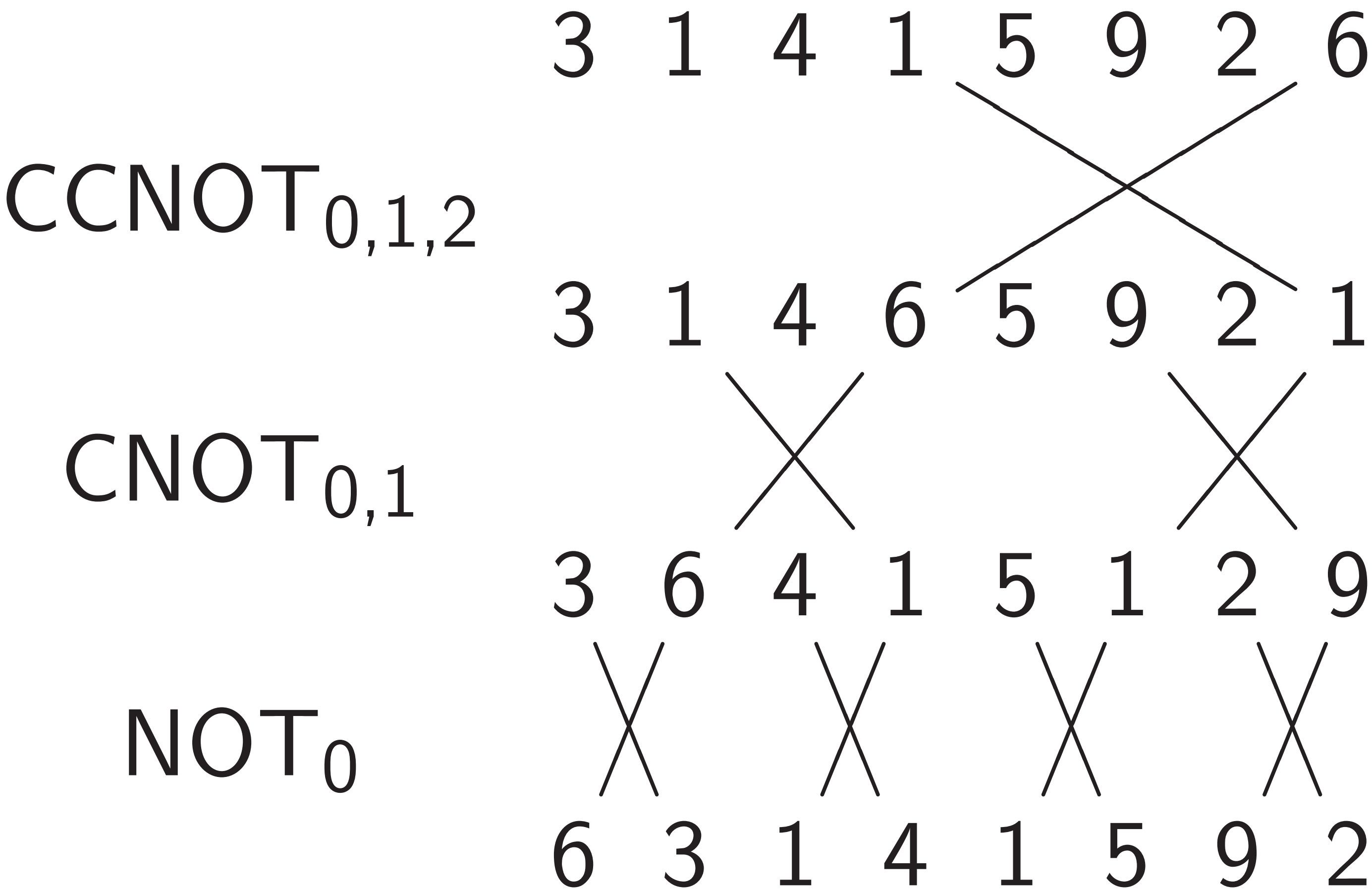
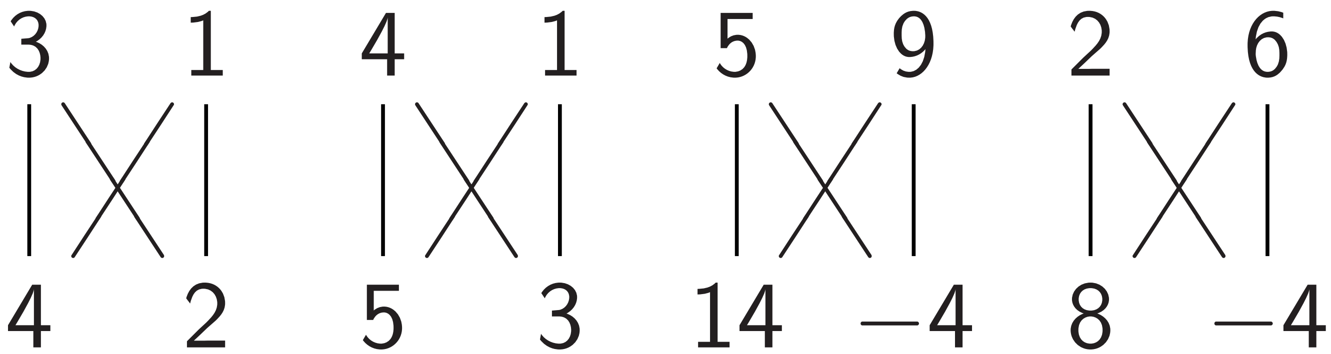
e.g. series of gates to rotate 8 positions by distance 1:



Hadamard gates

Hadamard$_0$:

$(a, b) \mapsto (a + b, a - b)$.



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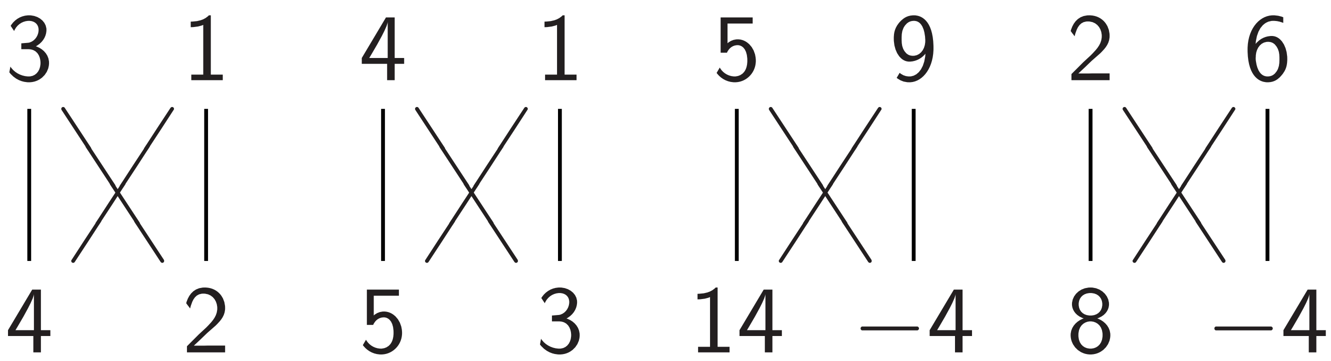
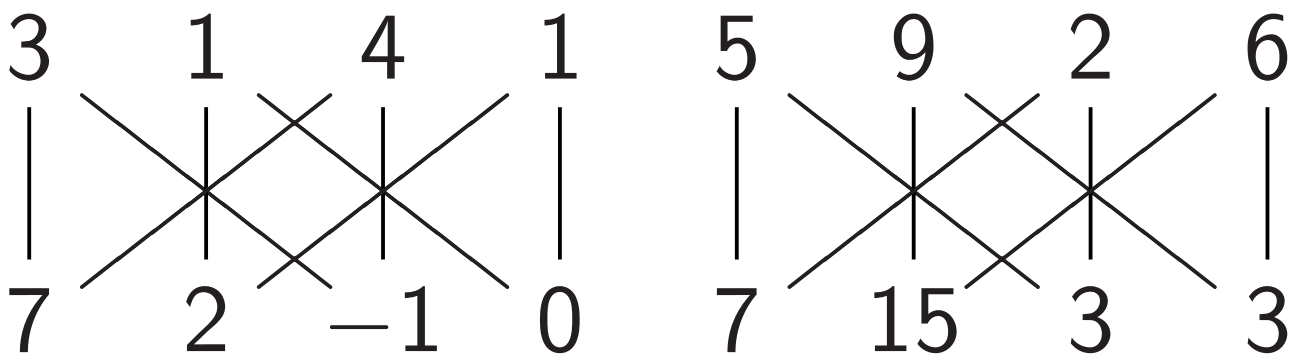


Hadamard$_1$:

$(a, b, c, d) \mapsto$
$(a + c, b + d, a - c, b - d)$.



Simon's algorithm

Step 1. Set up pure zero state:

1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Simon's algorithm

Step 2. Hadamard$_0$:

1, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Simon's algorithm

Step 3. Hadamard$_1$:

1, 1, 1, 1, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Simon's algorithm

Step 4. Hadamard_2:

1, 1, 1, 1, 1, 1, 1, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe.

Simon's algorithm

Step 5. $\text{CNOT}_{0,3}$:

1, 0, 1, 0, 1, 0, 1, 0,

0, 1, 0, 1, 0, 1, 0, 1,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5b. More shuffling:

1, 0, 0, 0, 1, 0, 0, 0,

0, 1, 0, 0, 0, 1, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 1, 0,

0, 0, 0, 1, 0, 0, 0, 1,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5c. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

0, 0, 1, 0, 0, 0, 0, 0,

0, 0, 0, 1, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 1, 0,

0, 0, 0, 0, 0, 0, 0, 1.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5d. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 1,

0, 0, 0, 0, 0, 0, 1, 0,

0, 0, 0, 1, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5e. More shuffling:

1, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 1, 0, 0,
0, 0, 0, 0, 1, 0, 0, 0,
0, 1, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5f. More shuffling:

0, 0, 0, 0, 0, 1, 0, 0,

1, 0, 0, 0, 0, 0, 0, 0,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 1,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 1, 0, 0, 1, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5g. More shuffling:

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

1, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 1, 0, 0, 1, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 1.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5h. More shuffling:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 1, 0, 0, 1, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 1,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

1, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5i. More shuffling:

0, 0, 0, 0, 0, 0, 1, 0,

0, 0, 0, 1, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 1,

0, 0, 1, 0, 0, 0, 0, 0,

0, 1, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 1, 0, 0,

1, 0, 0, 0, 0, 0, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5j. Final shuffling:

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 1, 0, 0, 1, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 1, 0, 0, 0, 0, 1,

0, 1, 0, 0, 1, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

1, 0, 0, 0, 0, 1, 0, 0.

Each column is a parallel universe performing its own computations.

Simon's algorithm

Step 5j. Final shuffling:

0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 1, 0, 0, 1, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 1, 0, 0, 0, 0, 1,
0, 1, 0, 0, 1, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0,
1, 0, 0, 0, 0, 1, 0, 0.

Each column is a parallel universe performing its own computations.

Surprise: u and $u \oplus 101$ match.

Simon's algorithm

Step 6. Hadamard$_0$:

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 1, \overline{1}, 0, 0, 1, 1,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 1, 1, 0, 0, 1, \overline{1},$

$1, \overline{1}, 0, 0, 1, 1, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, 1, 0, 0, 1, \overline{1}, 0, 0.$

Simon's algorithm

Step 7. Hadamard$_1$:

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, \overline{1}, \overline{1}, 1, 1, 1, \overline{1}, \overline{1},$

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, 1, \overline{1}, \overline{1}, 1, \overline{1}, \overline{1}, 1,$

$1, \overline{1}, 1, \overline{1}, 1, 1, 1, 1,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$0, 0, 0, 0, 0, 0, 0, 0,$

$1, 1, 1, 1, 1, \overline{1}, 1, \overline{1}.$

Simon's algorithm

Step 8. Hadamard$_2$:

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, $\overline{2}$, 0, 2,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, 2, 0, $\overline{2}$,

2, 0, 2, 0, 0, $\overline{2}$, 0, $\overline{2}$,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, 2, 0, 0, 2, 0, 2.

Simon's algorithm

Step 8. Hadamard$_2$:

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, $\overline{2}$, 0, 2,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, $\overline{2}$, 0, 0, 2, 0, $\overline{2}$,

2, 0, 2, 0, 0, $\overline{2}$, 0, $\overline{2}$,

0, 0, 0, 0, 0, 0, 0, 0,

0, 0, 0, 0, 0, 0, 0, 0,

2, 0, 2, 0, 0, 2, 0, 2.

Step 9: Measure. Obtain some information about the surprise: a random vector orthogonal to 101.