Algorithms for multiquadratic number fields

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Paper and software:
https://multiquad.cr.yp.to
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Can other fields be attacked? Are there non-quantum attacks? What about other cryptosystems?
Compare to 2013 Lyubashevsky–Peikert–Regev: “All of the algebraic and algorithmic tools (including quantum computation) that we employ . . . can also be brought to bear against SVP and other problems on ideal lattices. Yet despite considerable effort, no significant progress in attacking these problems has been made. The best known algorithms for ideal lattices perform essentially no better than their generic counterparts, both in theory and in practice.”
Secret key in Gentry’s system:
short element $g$ of $R$.

$R$: e.g., ring of integers $\mathcal{O}_K$
of a cyclotomic field $K$.

Public key: ideal $gR$. 
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Attack stage 1, quantum:
SODA 2016 Biasse–Song
finds some generator of $gR$.
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Standard algebraic-number-theory view of all generators of $gR$, i.e., all $ug$ where $u \in R^*$:

\[ \log u \text{ ranges over } \] 

Dirichlet’s log-unit lattice;

\[ \log ug = \log u + \log g. \]
Standard algebraic-number-theory view of all generators of $gR$, i.e., all $ug$ where $u \in R^*$: Log $u$ ranges over Dirichlet’s log-unit lattice; Log $ug = \text{Log } u + \text{Log } g$.

Given any generator $ug$, try to find short Log $g$ by finding lattice vector Log $u$ close to Log $ug$. 
Standard algebraic-number-theory view of all generators of \( gR \), i.e., all \( ug \) where \( u \in R^* \):

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Given any generator \( ug \), try to find short \( \text{Log } g \) by finding lattice vector \( \text{Log } u \) close to \( \text{Log } ug \).

Apply, e.g., embedding or Babai, starting from basis for \( \text{Log } R^* \)? Hard to find short enough basis, unless \( g \) is extremely short.
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Et cetera. Obtain short basis.

Now embedding easily finds $g$. 
Are you a lattice salesman? Try to dismiss lattice attacks.
Ask: Do attacks against
• the $gR \mapsto g$ problem,
• Gentry’s original FHE system,
• the original Garg–Gentry–Halevi multilinear maps, . . .
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My response to the salesman: Maybe not—but this problem is a natural starting point for studying other lattice problems that we certainly care about.

“Canary in the coal mine.”
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$I \mapsto$ shortest nonzero vector in $I$.

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“ Exact Ideal-SVP”:
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Attack is against ideal $I$ with a short generator.

2015 Peikert says idea is “useless” for more general principal ideals:
“We simply hadn’t realized that the added guarantee of a short generator would transform the technique from useless to devastatingly effective.”
2015 Peikert also says idea is limited to principal ideals: “Although cyclotomics have a lot of structure, nobody has yet found a way to exploit it in attacking Ideal-SVP/BDD . . . For commonly used rings, principal ideals are an extremely small fraction of all ideals. . . . The weakness here is not so much due to the structure of cyclotomics, but rather to the extra structure of principal ideals that have short generators.”
Actually, the idea produces attacks far beyond this case.

2016 Cramer–Ducas–Wesolowski: Ideal-SVP attack for approx factor $2^{N^{1/2+o(1)}}$ in deg-$N$ cyclotomics, under plausible assumptions about class-group generators etc. Start from Biasse–Song, use more features of cyclotomic fields.
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Can techniques be pushed to smaller approx factors? Can techniques be adapted to break, e.g., Ring-LWE?
NIST post-quantum competition

69 submissions (5 withdrawn), including 20 lattice-based enc.
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Most lattice-based enc systems use power-of-2 cyclotomics. Some non-power-of-2 cyclotomics:
LIMA has \( \Phi_{1019} \) option, “more conservative choice of field”;
NTRU-HRSS-KEM uses \( \Phi_{701} \); NTRUEncrypt uses \( \Phi_{743} \) etc.
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Can cyclotomic attacks on Gentry be extended to these systems?
Some systems avoid cyclotomics.

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Streamlined NTRU Prime $4591^{761}$, 1218-byte key: see Tanja’s talk later today.
Two theories of lattice safety

Theory 1: Best choices of field $F$ are choices where we know proofs “attack against cryptosystem $C_F$ $\Rightarrow$ attack against problem $L_F$”, where $L_F$ is a “lattice problem”.
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Theory 2: Safety of field $F$ is damaged by extra automorphisms, extra subfields, etc. Similar situation to discrete-log crypto.
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What’s a good test case for $F$?
Multiquadratic fields

Assumptions: $n \in \{0, 1, 2, \ldots\}$; squarefree $d_1, \ldots, d_n \in \mathbb{Z}$; $\prod_{j \in J} d_j$ non-square for each nonempty subset $J \subseteq \{1, \ldots, n\}$.

$K = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_n})$: smallest subfield of $\mathbb{C}$ containing $\sqrt{d_1}, \ldots, \sqrt{d_n}$.

$K$ is a degree-$2^n$ number field.

Basis: $\prod_{j \in J} d_j$ for each subset $J \subseteq \{1, \ldots, n\}$.

e.g. $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q} \oplus \mathbb{Q}\sqrt{2} \oplus \mathbb{Q}\sqrt{3} \oplus \mathbb{Q}\sqrt{6}$. 
This field is Galois:
has $2^n$ automorphisms.

e.g. automorphisms of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$
map $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ to
$a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6};$
$a - b\sqrt{2} + c\sqrt{3} - d\sqrt{6};$
$a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6};$
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$a - b\sqrt{2} + c\sqrt{3} - d\sqrt{6};$
$a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6};$
$a - b\sqrt{2} - c\sqrt{3} + d\sqrt{6}.$

About $2^{n^2}/4$ subfields.

e.g. subfields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$: $\mathbb{Q}(\sqrt{2}, \sqrt{3}),$
$\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{6}), \mathbb{Q}.$
Gentry for multiquadratics

Use optimizations from
PKC 2010 Smart–Vercauteren,
Eurocrypt 2011 Gentry–Halevi.
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$F$: monic irreducible polynomial. Ring $R = \mathbb{Z}[x]/F$; not required to be ring of integers of $\mathbb{Q}[x]/F$. 
Gentry for multiquadratics

Use optimizations from PKC 2010 Smart–Vercauteren, Eurocrypt 2011 Gentry–Halevi.

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Multiquadratics: take, e.g.,

$F = (x - \sqrt{2} - \sqrt{3}) \cdot (x + \sqrt{2} - \sqrt{3}) \cdot (x - \sqrt{2} + \sqrt{3}) \cdot (x + \sqrt{2} + \sqrt{3})$.

Note $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. 
Smart–Vercauteren keygen:
Take short random $g \in R$.
Compute $q$, absolute norm of $g$.
Start over if $q$ is not prime.
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Compute root $r$ of $g$ in $\mathbb{Z}/q$.
Public key $gR = qR + (x - r)R$
is represented as $(q, r)$.
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is represented as \((q, r)\).

(We implemented multiquadratic adaptation of Gentry–Halevi cyclotomic keygen speedup:
instead of requiring prime \( q \),
require \( \gcd\{b, q\} > 1 \) for each relative norm \( a + b\sqrt{d} \) of \( g \).
Any squarefree \( q \) will work.)
Smart–Vercauteren encryption:
Take short $m \in \mathbb{Z}[x]/F$.
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Homomorphic operations:
add/multiply ciphertexts $m(r)$
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Decryption:
given $c \in \{0, 1, \ldots, q - 1\}$,
compute $c/g \in \mathbb{Q}[x]/F$,
round to element of $\mathbb{Z}[x]/F$,
multiply by $g$, subtract from $c$. 
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Decryption works if
each coefficient of $m/g \in \mathbb{Q}[x]/F$
is in $(-1/2, 1/2)$. 
Gentry says “computational complexity of all of these algorithms must be polynomial in security parameter”.

Flaw in Smart–Vercauteren: for some choices of $F$, keygen time is not polynomial in security parameter.
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Flaw in Smart–Vercauteren: for some choices of $F$, keygen time is not polynomial in security parameter.

For multiquadratic $F$, keygen is disastrously slow: far too many tries to find prime $q$. (Adaptation of Gentry–Halevi speedup gives only a polynomial improvement.)
Why this happens: Fix prime $p$. Take field $k$ of size $p^2$. 
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d_1, \ldots, d_n$ are squares in $k$, so $F$ splits completely in $k[x]$. 
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Heuristic: for most $p \leq 2^n$, have $\Theta(p)$ distinct linear factors $h$.

For each linear factor $h$: with probability $\approx 1/p$, 
h divides $g$ in $F_p[x]$, forcing $p^2$ to divide norm of $g$ if any $d_i$ is non-square in $F_p$. 
Our multiquadratic tweaks to Smart–Vercauteren (including adaptation of Gentry–Halevi):

1. Generalize cryptosystem to support \( n \) polynomial variables. Use \( R = \mathbb{Z}[\sqrt{d_1}, \ldots, \sqrt{d_n}] \).
Our multiquadratic tweaks to Smart–Vercauteren (including adaptation of Gentry–Halevi):

1. Generalize cryptosystem to support \( n \) polynomial variables. Use \( R = \mathbb{Z}[^{\sqrt{d_1}, \ldots, \sqrt{d_n}}] \).

2. Subroutine: Construct uniform random invertible element of \( R/p \).
Our multiquadratic tweaks to Smart–Vercauteren (including adaptation of Gentry–Halevi):

1. Generalize cryptosystem to support $n$ polynomial variables. Use $R = \mathbb{Z}[\sqrt{d_1}, \ldots, \sqrt{d_n}]$.


3. Choose $y \in \Theta(2^n/n)$. Force $g$ to be invertible mod all primes $p \leq y$. Heuristically, good chance of squarefree norm.
Computing units

Fix positive non-square $d \in \mathbb{Z}$. Assume $d$ quasipoly in $2^n$; i.e., $\log d \in n^{O(1)}$. 
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\[
\{ \ldots, \pm \varepsilon^{-2}, \pm \varepsilon^{-1}, \pm 1, \pm \varepsilon, \pm \varepsilon^2, \ldots \}
\]
is unit group of ring of integers of $\mathbb{Q}(\sqrt{d})$ for a unique $\varepsilon > 1$, the normalized fundamental unit. $\log \varepsilon < \sqrt{d}(2 + \log 4d)$; quasipoly.
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is unit group of ring of integers of \( \mathbb{Q}(\sqrt{d}) \) for a unique \( \varepsilon > 1 \), the
normalized fundamental unit.
\( \log \varepsilon < \sqrt{d}(2 + \log 4d) \); quasipoly.

Standard algorithms compute \( a, b \in \mathbb{Q} \) with \( \varepsilon = a + b\sqrt{d} \)
in time \( (\log \varepsilon)^{1+o(1)} \); quasipoly.
(Can save time by instead representing \( \varepsilon \) as product.)
Take a multiquadratic field $K = \mathbb{Q}(\sqrt{d_1}, \ldots, \sqrt{d_n})$. Assume $n > 0$ and all $d_i > 0$.

The set of **multiquadratic units** is the group generated by units of all $2^n - 1$ quadratic subfields. Analogous to cyclotomic units.

Compute this group by computing all normalized fundamental units.
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We go beyond this: compute \( \mathcal{O}_K^\times \). Could use Eisenträger–Hallgren–Kitaev–Song, but we don’t want to wait for quantum computers.
1966 Wada: exponential-time $\mathcal{O}_K^*$ algorithm for multiquadratics.

First step: Recursively compute unit groups for three proper subfields $K_\sigma, K_\tau, K_{\sigma\tau}$ of $K$.

Base cases: $\mathbb{Q}; \mathbb{Q}(\sqrt{d})$.

$\sigma, \tau$: distinct non-identity automorphisms of $K$.

$K_\sigma = \{x \in K : \sigma(x) = x\}$. 

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$\sigma$, $\tau$: distinct non-identity automorphisms of $K$.

$K_{\sigma} = \{x \in K : \sigma(x) = x\}$.

e.g. $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$,

appropriate $\sigma$, $\tau$: have

$K_{\sigma} = \mathbb{Q}(\sqrt{2}, \sqrt{3})$;

$K_{\tau} = \mathbb{Q}(\sqrt{2}, \sqrt{5})$;

$K_{\sigma\tau} = \mathbb{Q}(\sqrt{2}, \sqrt{15})$. 
Second step:
Compute $U = \mathcal{O}_{K_\sigma}^* \mathcal{O}_{K_T}^* \sigma(\mathcal{O}_{K_{\sigma T}}^*)$. 
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Compute \( U = O_{K_\sigma}^* O_{K_\tau}^* \sigma(O_{K_{\sigma\tau}}^*). \)

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Fact: \( (O_K^*)^2 \leq U \).

Proof:
If \( u \in O_K^* \) then

\( u\sigma(u) \in O_{K_{\sigma}}^* \);
\( u\tau(u) \in O_{K_T}^* \);
\( u\sigma(\tau(u)) \in O_{K_{\sigma T}}^* \); so
\( u\sigma(u)u\tau(u)/\sigma(u\sigma(\tau(u))) \in U \).
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Proof:

If \( u \in \mathcal{O}_K^* \) then
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\begin{align*}
u\sigma(u) & \in \mathcal{O}_{K_\sigma}^*; \\
u\tau(u) & \in \mathcal{O}_{K_\tau}^*; \\
u\sigma(\tau(u)) & \in \mathcal{O}_{K_{\sigma\tau}}^*; \text{ so} \\
u\sigma(u)\nu\tau(u)/\sigma(u\sigma(\tau(u))) & \in U.
\end{align*}
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In other words, \( u^2 \in U \).
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$$\alpha_1^{e_1} \cdots \alpha_k^{e_k} \text{ square } \Rightarrow \chi(\alpha_1)^{e_1} \cdots \chi(\alpha_k)^{e_k} = 1$$

for any quadratic character $\chi$ with $\chi(\alpha_1), \ldots, \chi(\alpha_k) \in \{-1, 1\}$. 
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$\alpha_1^{e_1} \cdots \alpha_k^{e_k}$ square $\Rightarrow$
$\chi(\alpha_1)^{e_1} \cdots \chi(\alpha_k)^{e_k} = 1$
for any quadratic character $\chi$
with $\chi(\alpha_1), \ldots, \chi(\alpha_k) \in \{-1, 1\}$.

Linear equation, usually reducing $\dim \{e\}$ by 1. Use many such $\chi$. 
Computing generators

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Strategy: Reuse the equation

$$g^2 = g\sigma(g)g\tau(g)/\sigma(g\sigma(\tau(g)))$$

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How to compute \( g\sigma(g) \)?
Computing generators

Main goal: Find $g$ given $gR$, where $R = \mathbb{Z}[\sqrt{d_1}, \ldots, \sqrt{d_n}]$.

Strategy: Reuse the equation $g^2 = g\sigma(g)g\tau(g)/\sigma(g\sigma(\tau(g)))$. Square root of $g^2$ is $\pm g$.

How to compute $g\sigma(g)$?

First compute relative norm of ideal $gR$ from $K$ to $K_\sigma$. Obtain ideal generated by $g\sigma(g)$. 
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How to compute $g\sigma(g)$?

First compute relative norm of ideal $gR$ from $K$ to $K_\sigma$. Obtain ideal generated by $g\sigma(g)$.

Recursively compute a generator of this ideal: probably not $g\sigma(g)$. Some $ug\sigma(g)$ with $u \in \mathcal{O}_{K_\sigma}^*$.
Unit multiple of $g\sigma(g)$, unit multiple of $g\tau(g)$, unit multiple of $g\sigma(\tau(g))$ \implies \text{some } ug^2 \text{ with } u \in \mathcal{O}_K^*.$
Unit multiple of $g\sigma(g)$, unit multiple of $g\tau(g)$, unit multiple of $g\sigma(\tau(g))$ \( \Rightarrow \) some $ug^2$ with $u \in \mathcal{O}_K^*$. 

Use quadratic characters (with values $\pm 1$ on $g$) to identify $\nu \in \mathcal{O}_K^*$ such that $\nu ug^2$ is a square.
Unit multiple of $g\sigma(g)$,
unit multiple of $g\tau(g)$,
unit multiple of $g\sigma(\tau(g))$
$\Rightarrow$ some $ug^2$ with $u \in \mathcal{O}_K^*$.

Use quadratic characters (with values $\pm 1$ on $g$) to identify $v \in \mathcal{O}_K^*$ such that $vug^2$ is a square.

Now compute square root: some unit multiple of $g$,
i.e., some $g'$ with $g'O_K = gO_K$. 
Unit multiple of $g\sigma(g)$,
unit multiple of $g\tau(g)$,
unit multiple of $g\sigma(\tau(g))$
⇒ some $ug^2$ with $u \in \mathcal{O}_K^*$.

Use quadratic characters
(with values ±1 on $g$)
to identify $v \in \mathcal{O}_K^*$
such that $v u g^2$ is a square.

Now compute square root:
some unit multiple of $g$,
i.e., some $g'$ with $g'\mathcal{O}_K = g\mathcal{O}_K$.

All of this takes quasipoly time.
Computing short generators

Assume $d_1, \ldots, d_n \geq 2^{1.03n}$.

(More work seems to push bound to $<n^2$; see paper and software.)
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Find multiquadratic (MQ) units.
Find all units.
Find some generator \( ug \).
Computing short generators

Assume $d_1, \ldots, d_n \geq 2^{1.03n}$. (More work seems to push bound to $<n^2$; see paper and software.)

Find multiquadratic (MQ) units.
Find all units.
Find some generator $ug$.

Heuristic: For most $d_1, \ldots, d_n$, all regulators $\log \epsilon$ are larger than $2^{0.51n}$; so coefficients of $2^n \log g$ on MQ unit basis are almost certainly in $(-0.1, 0.1)$. 
$u^{2^n}$ is an MQ unit.

$\log u^{2^n} = 2^n \log u$ is closest vector to $2^n \log u g$. 
$u^{2^n}$ is an MQ unit.
\[
\log u^{2^n} = 2^n \log u
\]
is closest vector to $2^n \log ug$.

MQ unit lattice is orthogonal.
Round $2^n \log ug$ to find $2^n \log u$ and $2^n \log g$. Deduce $\pm g^{2^n}$. 
\( u^{2^n} \) is an MQ unit.

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MQ unit lattice is orthogonal.

Round \( 2^n \log ug \) to find \( 2^n \log u \) and \( 2^n \log g \). Deduce \( \pm g^{2^n} \).

Use quadratic character: \( g^{2^n} \).
$u^{2^n}$ is an MQ unit.
Log $u^{2^n} = 2^n \log u$ is closest vector to $2^n \log ug$.

MQ unit lattice is orthogonal.
Round $2^n \log ug$ to find $2^n \log u$ and $2^n \log g$. Deduce $\pm g^{2^n}$.

Use quadratic character: $g^{2^n}$.
Square root: $\pm g^{2^{n-1}}$. 
$u^{2^n}$ is an MQ unit.

Log $u^{2^n} = 2^n \log u$ is closest vector to $2^n \log ug$.

MQ unit lattice is orthogonal. 

Round $2^n \log ug$ to find $2^n \log u$ and $2^n \log g$. Deduce $\pm g^{2^n}$.

Use quadratic character: $g^{2^n}$.

Square root: $\pm g^{2^n-1}$.

Use quadratic character: $g^{2^n-1}$.

Square root: $\pm g^{2^n-2}$. 
$u^{2^n}$ is an MQ unit.

Log $u^{2^n} = 2^n \log u$ is closest vector to $2^n \log ug$.

MQ unit lattice is orthogonal.
Round $2^n \log ug$ to find $2^n \log u$ and $2^n \log g$. Deduce $\pm g^{2^n}$.

Use quadratic character: $g^{2^n}$.
Square root: $\pm g^{2^n-1}$.

Use quadratic character: $g^{2^n-1}$.
Square root: $\pm g^{2^n-2}$.

\vdots

Square root: $\pm g$. Done!

MQ cryptosystem is broken for all of these fields.
Slightly simpler:

Find MQ units,
but skip finding all units.
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Take logs: $\log ug^{2^{n-1}}$.

Round: $\log u$. 
Slightly simpler:
Find MQ units, 
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Recursively find $ug^{2^{n-1}}$
where $u$ is an MQ unit; i.e., 
skip square-root computations.

Take logs: $\log u g^{2^{n-1}}$.
Round: $\log u$.

Deduce $\pm g^{2^{n-1}}$.

Use quadratic character: $g^{2^{n-1}}$.

Square root: $\pm g^{2^{n-2}}$.

Square root: $\pm g$. 