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Paper and software:

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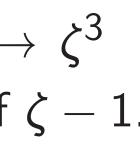
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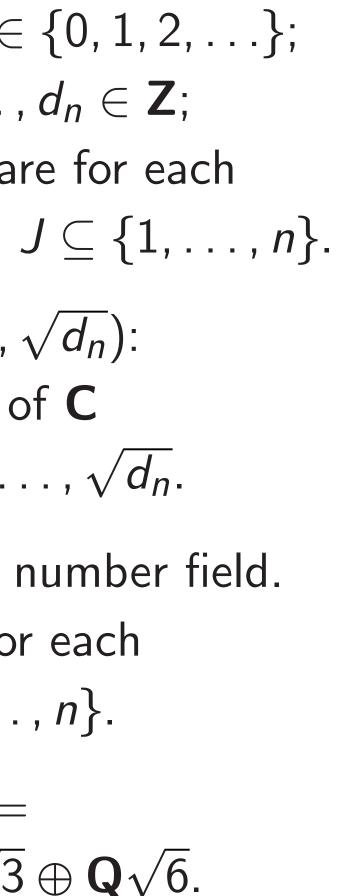
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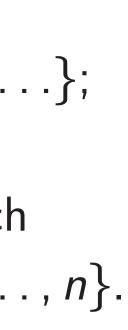
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This field is Galois has 2^n automorphism map $a + b\sqrt{2} + c$ $a + b\sqrt{2} + c\sqrt{3} + c$ $a - b\sqrt{2} + c\sqrt{3} + c$ $a + b\sqrt{2} - c\sqrt{3} - c$ $a + b\sqrt{2} - c\sqrt{3} - c$ $a - b\sqrt{2} - c\sqrt{3} + c$

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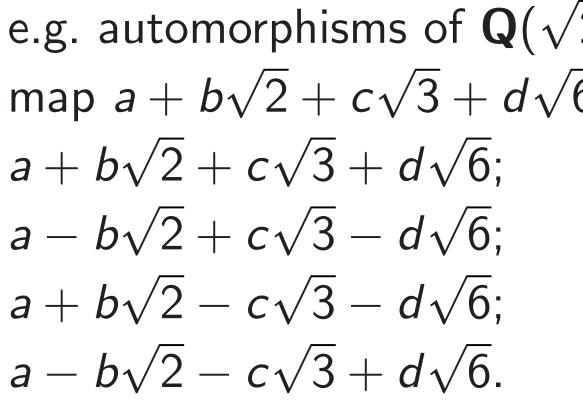
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K is a degree- 2^n number field. Basis: $\prod_{i \in J} d_i$ for each subset $J \subseteq \{1, \ldots, n\}$.

e.g. $\mathbf{Q}(\sqrt{2}, \sqrt{3}) =$ $\mathbf{Q} \oplus \mathbf{Q}\sqrt{2} \oplus \mathbf{Q}\sqrt{3} \oplus \mathbf{Q}\sqrt{6}.$

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This field is Galois: has 2^n automorphisms.



Multiquadratic fields

Assumptions: $n \in \{0, 1, 2, ...\}$; squarefree $d_1, \ldots, d_n \in \mathbf{Z}$; $\prod_{i \in J} d_i$ non-square for each nonempty subset $J \subseteq \{1, \ldots, n\}$.

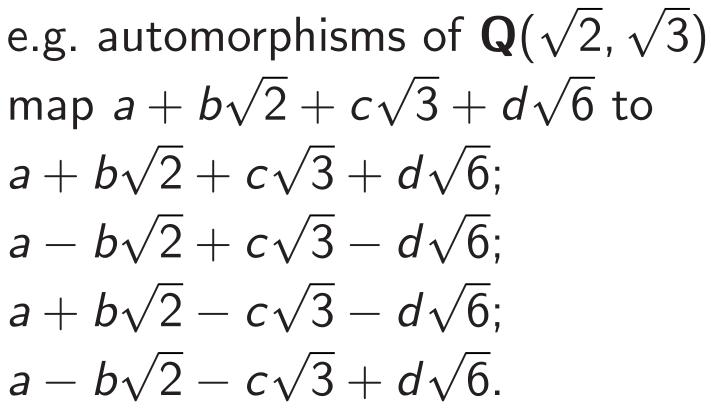
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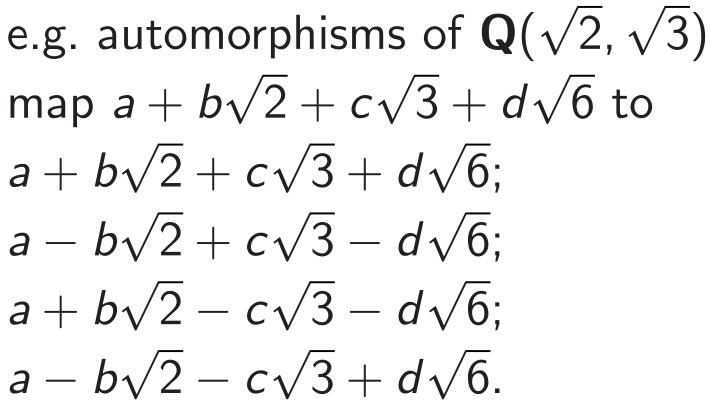
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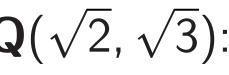
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Gentry for multiquadratics

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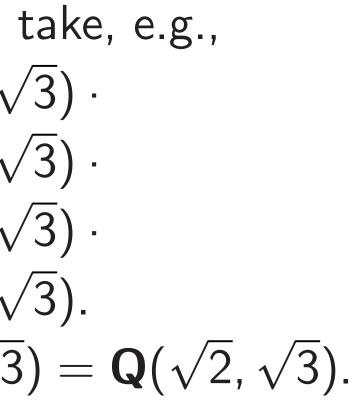
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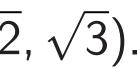
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Gentry says "computational complexity of all of these algorithms must be polynomial in security parameter". Flaw in Smart–Vercauteren: for some choices of F, keygen time is not polynomial in security parameter. For multiquadratic F, keygen is disastrously slow: far too many tries to find prime q. (Adaptation

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Gentry says "computational complexity of all of these algorithms must be polynomial in security parameter".

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Flaw in Smart–Vercauteren: for some choices of F, keygen time is not polynomial in security parameter.

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e.g. $K = \mathbf{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}),$ appropriate σ, τ : have $K_{\sigma} = \mathbf{Q}(\sqrt{2}, \sqrt{3});$

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Now compute square root: some unit multiple of g, i.e., some g' with $g'\mathcal{O}_K = g\mathcal{O}_K$.

All of this takes quasipoly time.

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