Engineering
cryptographic software
Daniel J. Bernstein
University of Illinois at Chicago \& Technische Universiteit Eindhoven

This is easy, right?

1. Take general principles of software engineering.
2. Apply principles to crypto.

Let's try some examples ...

1972 Parnas "On the criteria
to be used in decomposing
systems into modules":
"We propose instead that
one begins with a list of difficult design decisions or design decisions which are likely to change. Each module is then designed to hide such a decision from the others."
e.g. If number of cipher rounds is properly modularized as \#define ROUNDS 20
then it is easy to change.

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Do not design APIs like this:
"The sample code used in this manual omits the checking of status values for clarity, but when using cryptlib you should check return values, particularly for critical functions ..."

Not so easy: Timing attacks
1970s: TENEX operating system compares user-supplied string against secret password one character at a time, stopping at first difference:

- AAAAAA vs. SECRET: stop at 1. - SAAAAA vs. SECRET: stop at 2. - SEAAAA vs. SECRET: stop at 3.

Attacker sees comparison time, deduces position of difference. A few hundred tries reveal secret password.

How typical software checks
16-byte authenticator:
for (i = 0;i < 16;++i)
if (x[i] != y[i]) return 0;
return 1;
Fix, eliminating information flow from secrets to timings:

$$
\begin{aligned}
& \operatorname{diff}=0 ; \\
& \text { for }(i=0 ; i<16 ;++i) \\
& \operatorname{diff} \text { |=x[i] y[i]; }
\end{aligned}
$$

return 1 \& ((di ff-1) >> 8);
Notice that the language makes the wrong thing simple and the right thing complex.

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One of many examples,
part of the reference software for CAESAR candidate CLOC:
/* compare the tag */
int i;
for $(\mathrm{i}=0 ; \mathrm{i}$ < CRYPTO_ABYTES;i++)
if (ta gui] != c[(*mlen) + i])\{ return RETURN_TAG_NO_MATCH; \}
return RETURN_SUCCESS;

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Answer \#2: Attacker uses statistics to eliminate noise.

Answer \#3, what the 1970s attackers actually did:
Cross page boundary, inducing page faults, to amplify timing signal.

## Defenders don't learn

Some of the literature:
1996 Kocher pointed out timing attacks on cryptographic key bits.

Briefly mentioned by
Kocher and by 1998 Kelsey-
Schneier-Wagner-Hall:
secret array indices can affect timing via cache misses.

2002 Page, 2003 Tsunoo-Saito-
Suzaki-Shigeri-Miyauchi:
timing attacks on DES.

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What is safe: kill all data flow from secrets to array indices.

2005 Tromer-Osvik-Shamir:
65ms to steal Linux AES key
used for hard-disk encryption.

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OpenSSL integrates, cheaper
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Same issues described in 2004.
2016 Yarom-Genkin-Heninger "CacheBleed" steals RSA secret key via timings of OpenSSL.

2008 RFC 5246 "The Transport Layer Security (TLS) Protocol,
Version 1.2": "This leaves a small timing channel, since MAC performance depends to some extent on the size of the data fragment, but it is not believed to be large enough to be exploitable, due to the large block size of existing MACs and the small size of the timing signal."

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2013 AIFardan-Paterson "Lucky Thirteen: breaking the TLS and DTLS record protocols" : exploit these timings; steal plaintext.

## How to write constant-time code

If possible, write code in asm to control instruction selection.

Look for documentation identifying variability: e.g., "Division operations terminate when the divide operation completes, with the number of cycles required dependent on the values of the input operands."

Measure cycles rather than trusting CPU documentation.

Cut off all data flow from secrets to branch conditions.

Cut off all data flow from secrets to array indices.

Cut off all data flow from secrets to shift/rotate distances.

Prefer logic instructions.
Prefer vector instructions.
Watch out for PUs with variable-time multipliers: e.g.,
Cortex-M3 and most PowerPCs.

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But crypto software should be applied to all communication.

Crypts that's too slow $\Rightarrow$
fewer users $\Rightarrow$ fewer cryptanalysts
$\Rightarrow$ less attractive for everybody.

Typical situation:
You want (constant-time)
software that computes cipher $X$ as efficiently as possible.

Starting point:
You have written a reference implementation of $X$.

You have chosen a target CPU.
(Can repeat for other PUs.)
You measure performance of the implementation. Now what?

## A simplified example

Target CPU: TI LM4F120H5QR microcontroller containing one ARM Cortex-M4F core.

Reference implementation:
int sum(int *x)
\{
int result $=0$;
int i;
for ( $i=0 ; i<1000 ;++i$ )
result $+=x[i]$;
return result;
\}

## Counting cycles:

static volatile unsigned int *const DWT_CYCCNT
$=$ (void *) 0xE0001004;
int beforesum = *DWT_CYCCNT; int result $=$ sum (x);
int aftersum = *DWT_CYCCNT;
UARTprintf("sum \%d \%d\n", result, aftersum-beforesum) ;

Output shows 8012 cycles. Change 1000 to 500: 4012.
"Okay, 8 cycles per addition. Um, are microcontrollers really this slow at addition?"
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Apply random "optimizations"
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until you get bored.
Keep the fastest results.
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Good practice:
Figure out lower bound for cycles spent on arithmetic etc. Understand gap between lower bound and observed time.

Find "ARM Cortex-M4 Processor Technical Reference Manual".
Rely on Wikipedia comment that $\mathrm{M} 4 \mathrm{~F}=\mathrm{M} 4+$ floating-point unit.

Manual says that Cortex-M4 "implements the ARMv7E-M architecture profile".

Points to the "ARMv7-M Architecture Reference Manual", which defines instructions:
e.g., "ADD" for 32-bit addition.

First manual says that
ADD takes just 1 cycle.

Inputs and output of ADD are "integer registers". ARMv7-M has 16 integer registers, including special-purpose "stack pointer" and "program counter".

Each element of x array needs to be "loaded" into a register.

Basic load instruction: LDR.
Manual says 2 cycles but adds a note about "pipelining". Then more explanation: if next instruction is also LDR (with address not based on first LDR) then it saves 1 cycle.
$n$ consecutive LDRs
takes only $n+1$ cycles
("more multiple LDRs can be pipeline together").

Can achieve this speed in other ways (LDRD, LDM) but nothing seems faster.

Lower bound for $n$ LDR $+n$ ADD:
$2 n+1$ cycles,
including $n$ cycles of arithmetic.
Why observed time is higher: non-consecutive LDRs;
costs of manipulating i.
int sum(int *x)
\{
int result $=0$;
int $* y=x+1000 ;$
int $x 0, x 1, x 2, x 3, x 4$,
$\mathrm{x} 5, \mathrm{x} 6, \mathrm{x} 7, \mathrm{x} 8, \mathrm{x} 9$;
while (x ! = y) \{
$\mathrm{x} 0=0[($ volatile int $*) \mathrm{x}]$;
$\mathrm{x} 1=1[(\operatorname{volatile}$ int $*) \mathrm{x}]$;
$\mathrm{x} 2=2[(\operatorname{volatile}$ int $*) \mathrm{x}]$;
xU $=3[($ volatile int $*) x]$; $\mathrm{x} 4=4[($ volatile int $*) \mathrm{x}]$; $x 5=5[($ volatile int $*) x]$; $\mathrm{x} 6=6[(\operatorname{volatile}$ int $*) \mathrm{x}]$;
$x 7=7[($ volatile int $*) x]$;
$x 8=8[($ volatile int $*) x] ;$
x9 = 9[(volatile int *)x];
result $+=x 0$;
result += xi;
result $+=\mathrm{x} 2$;
result $+=$ xS;
result $+=x 4$;
result += xS;
result $+=\mathrm{x} 6$;
result += xT;
result += x8;
result $+=\mathrm{x} 9$;
$x 0=10[(v o l a t i l e ~ i n t ~ *) x] ;$
$\mathrm{x} 1=11[($ volatile int $*) \mathrm{x}]$;
$x 2=12[(v o l a t i l e ~ i n t ~ *) ~ x] ;$
x3 $=13[($ volatile int $*) x]$; $\mathrm{x} 4=14[($ volatile int $*) \mathrm{x}]$; $\mathrm{x} 5=15[(\mathrm{volatile}$ int $*) \mathrm{x}]$; $\mathrm{x} 6=16[(\mathrm{volatile}$ int $*) \mathrm{x}]$; $\mathrm{x} 7=17[($ volatile int $*) \mathrm{x}]$; $\mathrm{x} 8=18[($ volatile int $*) \mathrm{x}]$; x9 = 19[(volatile int *) x] ; $x+=20 ;$
result $+=\mathrm{x} 0$;
result $+=\mathrm{x} 1$;
result $+=$ x2;
result $+=$ xS;
result $+=\mathrm{x} 4$;
result $+=$ xS;
result += x6;
result $+=\mathrm{x} 7$;
result += x8;
result $+=\mathrm{x} 9$;
\}
return result;
\}
result += x6;
result $+=\mathrm{x} 7$;
result $+=$ x8;
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2526 cycles. Even better in asm.
result += x6;
result $+=$ xT;
result $+=x 8$;
result $+=\mathrm{x} 9$;
\}

## return result;

\}

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Wikipedia: "By the late 1990s for even performance sensitive code, optimizing compilers exceeded the performance of human experts."
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A real example
Salsa20 reference software:
30.25 cycles/byte on this CPU.

Lower bound for arithmetic:
64 bytes require
21 - 16 1-cycle ADDs,
20-16 1-cycle KORs,
so at least 10.25 cycles/byte.
ARMv7-M instruction set includes free rotation as part of XOR instruction.
(Compiler knows this.)

Detailed benchmarks show several cycles/byte spent on load_littleendian and store_littleendian.

Can replace with LDR and STR.
(Compiler doesn't see this.)
Then observe 23 cycles/ byte:
18 cycles/byte for rounds,
plus 5 cycles/byte overhead.
Still far above 10.25 cycles/byte.

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Can replace with LDR and STR.
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plus 5 cycles/byte overhead.
Still far above 10.25 cycles/ byte.
Gap is mostly loads, stores.
Minimize load/store cost by
choosing "spills" carefully.

Which of the 16 Salsa 20 words should be in registers?
Don't trust compiler to optimize register allocation.

Make loads consecutive?
Don't trust compiler to optimize instruction scheduling.

Spill to FPU instead of stack?
Don't trust compiler to
optimize instruction selection.
On bigger PUs, selecting vector instructions is critical for performance.

## The big picture

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Minor optimization challenges:

- Pipelining.
- Superscalar processing.

Major optimization challenges:

- Vectorization.
- Many threads; many cores.
- The memory hierarchy; the ring; the mesh.
- Larger-scale parallelism.
- Larger-scale networking.


## CPU design in a nutshell



Gates $\pi: a, b \mapsto 1-a b$ computing product $h_{0}+2 h_{1}+4 h_{2}+8 h_{3}$ of integers $f_{0}+2 f_{1}, g_{0}+2 g_{1}$.

## Electricity takes time to

 percolate through wires and gates. If $f_{0}, f_{1}, g_{0}, g_{1}$ are stable then $h_{0}, h_{1}, h_{2}, h_{3}$ are stable a few moments later.Electricity takes time to
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Build circuit with more gates to multiply (e.g.) 32-bit integers:

(Details omitted.)

Build circuit to compute
32-bit integer $r_{i}$
given 4-bit integer $i$
and 32 -bit integers $r_{0}, r_{1}, \ldots, r_{15}$ :


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Build circuit for "register write":
$r_{0}, \ldots, r_{15}, s, i \mapsto r_{0}^{\prime}, \ldots, r_{15}^{\prime}$
where $r_{j}^{\prime}=r_{j}$ except $r_{i}^{\prime}=s$.

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where $r_{j}^{\prime}=r_{j}$ except $r_{i}^{\prime}=s$. Build circuit for addition. Etc.

$r_{0}, \ldots, r_{15}, i, j, k \mapsto r_{0}^{\prime}, \ldots, r_{15}^{\prime}$ where $r_{\ell}^{\prime}=r_{\ell}$ except $r_{i}^{\prime}=r_{j} r_{k}$ : | register | register |
| :---: | :---: |
| read | read |

## Add more flexibility.

More arithmetic:
replace $(i, j, k)$ with
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More (but slower) storage: "load" from and "store" to larger "RAM" arrays.

Build "flip-flops"
storing $\left(p, r_{0}, \ldots, r_{15}\right)$.
$\operatorname{Hook}\left(p, r_{0}, \ldots, r_{15}\right)$
flip-flops into circuit inputs.
Hook outputs $\left(p^{\prime}, r_{0}^{\prime}, \ldots, r_{15}^{\prime}\right)$
into the same flip-flops.
At each "clock tick",
flip-flops are overwritten with the outputs.

Clock needs to be slow enough for electricity to percolate all the way through the circuit, from flip-flops to flip-flops.

Now have semi-flexible CPU:


## Further flexibility is useful:

 e.g., rotation instructions."Pipelining" allows faster clock:


Goal: Stage $n$ handles instruction one tick after stage $n-1$.

Instruction fetch reads next instruction,
feeds $p^{\prime}$ back, sends instruction.
After next clock tick, instruction decode uncompresses this instruction, while instruction fetch reads another instruction.

Some extra flip-flop area.
Also extra area to
preserve instruction semantics:
e.g., stall on read-after-write.

# "Superscalar" processing: 



## "Vector" processing:

Expand each 32-bit integer into $n$-vector of 32-bit integers.
ARM "NEON" has $n=4$; Intel "AVX2" has $n=8$;
Intel "AVX-512" has $n=16$;
GPU have larger $n$.

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Huge effect on higher-level algorithms and data structures.

Network on chip: the mesh
How expensive is sorting?
Input: array of $n$ numbers. Each number in $\left\{1,2, \ldots, n^{2}\right\}$, represented in binary.

Output: array of $n$ numbers, in increasing order, represented in binary; same multiset as input.

Network on chip: the mesh

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Output: array of $n$ numbers,
in increasing order,
represented in binary; same multiset as input.

Metric: seconds used by circuit of area $n^{1+o(1)}$.

For simplicity assume $n=4^{k}$.

Spread array across
square mesh of $n$ small cells, each of area $n^{o(1)}$,
with near-neighbor wiring:


Sort row of $n^{0.5}$ cells
in $n^{0.5+o(1)}$ seconds:

- Sort each pair in parallel.

$$
\frac{31}{13} \frac{41}{14} \frac{59}{59} \frac{26}{26} \mapsto
$$

- Sort alternate pairs in parallel.

$$
\begin{aligned}
& 13145 \\
& 11345 \\
& 296
\end{aligned}
$$

- Repeat until number of steps equals row length.

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$$
\begin{aligned}
& 13145 \\
& 11345 \\
& 296 \\
& 6
\end{aligned}
$$

- Repeat until number of steps equals row length.

Sort each row, in parallel, in a total of $n^{0.5+o(1)}$ seconds.

Sort all $n$ cells
in $n^{0.5+o(1)}$ seconds:

- Recursively sort quadrants in parallel, if $n>1$.
- Sort each column in parallel.
- Sort each row in parallel.
- Sort each column in parallel.
- Sort each row in parallel.

With proper choice of left-to-right/right-to-left
for each row, can prove that this sorts whole array.

For example, assume that this $8 \times 8$ array is in cells:
$\begin{array}{llllllll}3 & 1 & 4 & 1 & 5 & 9 & 2 & 6\end{array}$
$\begin{array}{llllllll}5 & 3 & 5 & 8 & 9 & 7 & 9 & 3\end{array}$
$\begin{array}{llllllll}2 & 3 & 8 & 4 & 6 & 2 & 6 & 4\end{array}$
$\begin{array}{llllllll}3 & 3 & 8 & 3 & 2 & 7 & 9 & 5\end{array}$
$\begin{array}{llllllll}0 & 2 & 8 & 8 & 4 & 1 & 9 & 7\end{array}$
$\begin{array}{llllllll}1 & 6 & 9 & 3 & 9 & 9 & 3 & 7\end{array}$
$\begin{array}{llllllll}5 & 1 & 0 & 5 & 8 & 2 & 0 & 9\end{array}$
$\begin{array}{llllllll}7 & 4 & 9 & 4 & 4 & 5 & 9 & 2\end{array}$

Recursively sort quadrants,
top $\rightarrow$, bottom $\leftarrow$ :

| 1 | 1 | 2 | 3 | 2 | 2 | 2 | 3 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |

$\begin{array}{llllllll}3 & 3 & 3 & 3 & 4 & 5 & 5 & 6\end{array}$
$\begin{array}{llllllll}3 & 4 & 4 & 5 & 6 & 6 & 7 & 7\end{array}$
$\begin{array}{llllllll}5 & 8 & 8 & 8 & 9 & 9 & 9 & 9\end{array}$

| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |

$\begin{array}{llllllll}4 & 4 & 3 & 2 & 5 & 4 & 4 & 3\end{array}$
$\begin{array}{llllllll}7 & 6 & 5 & 5 & 9 & 8 & 7 & 7\end{array}$
$\begin{array}{llllllll}9 & 9 & 8 & 8 & 9 & 9 & 9 & 9\end{array}$

Sort each column
in parallel:

| 1 | 1 | 0 | 0 | 2 | 2 | 1 | 0 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 2 | 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 4 | 4 | 4 | 3 |
| 3 | 4 | 3 | 3 | 5 | 5 | 5 | 6 |
| 4 | 4 | 4 | 5 | 6 | 6 | 7 | 7 |
| 5 | 6 | 5 | 5 | 9 | 8 | 7 | 7 |
| 7 | 8 | 8 | 8 | 9 | 9 | 9 | 9 |
| 9 | 9 | 8 | 8 | 9 | 9 | 9 | 9 |

Sort each row in parallel, alternately $\leftarrow, \rightarrow$ :
$\begin{array}{llllllll}0 & 0 & 0 & 1 & 1 & 1 & 2 & 2\end{array}$
$\begin{array}{llllllll}3 & 2 & 2 & 2 & 2 & 2 & 1 & 1\end{array}$
$\begin{array}{llllllll}3 & 3 & 3 & 3 & 3 & 4 & 4 & 4\end{array}$
$\begin{array}{llllllll}6 & 5 & 5 & 5 & 4 & 3 & 3 & 3\end{array}$
$\begin{array}{llllllll}4 & 4 & 4 & 5 & 6 & 6 & 7 & 7\end{array}$
$\begin{array}{llllllll}9 & 8 & 7 & 7 & 6 & 5 & 5 & 5\end{array}$
$\begin{array}{llllllll}7 & 8 & 8 & 8 & 9 & 9 & 9 & 9\end{array}$
$\begin{array}{llllllll}9 & 9 & 9 & 9 & 9 & 9 & 8 & 8\end{array}$

Sort each column
in parallel:

| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 4 | 4 | 4 | 5 | 4 | 4 | 4 | 4 |
| 6 | 5 | 5 | 5 | 6 | 5 | 5 | 5 |
| 7 | 8 | 7 | 7 | 6 | 6 | 7 | 7 |
| 9 | 8 | 8 | 8 | 9 | 9 | 8 | 8 |
| 9 | 9 | 9 | 9 | 9 | 9 | 9 | 9 |

Sort each row in parallel,
$\leftarrow$ or $\rightarrow$ as desired:
$\begin{array}{llllllll}0 & 0 & 0 & 1 & 1 & 1 & 1 & 1\end{array}$
22222223
$\begin{array}{llllllll}3 & 3 & 3 & 3 & 3 & 3 & 3 & 3\end{array}$
$\begin{array}{llllllll}4 & 4 & 4 & 4 & 4 & 4 & 4 & 5\end{array}$
$\begin{array}{llllllll}5 & 5 & 5 & 5 & 5 & 5 & 6 & 6\end{array}$
$\begin{array}{llllllll}6 & 6 & 7 & 7 & 7 & 7 & 7 & 8\end{array}$
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GPUs: parallel + global RAM.
Old Xeon Phi: parallel + ring. New Xeon Phi: parallel + mesh.

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Shock waves from subroutines into high-level algorithm design.

