Hyper-and-elliptic-curve cryptography

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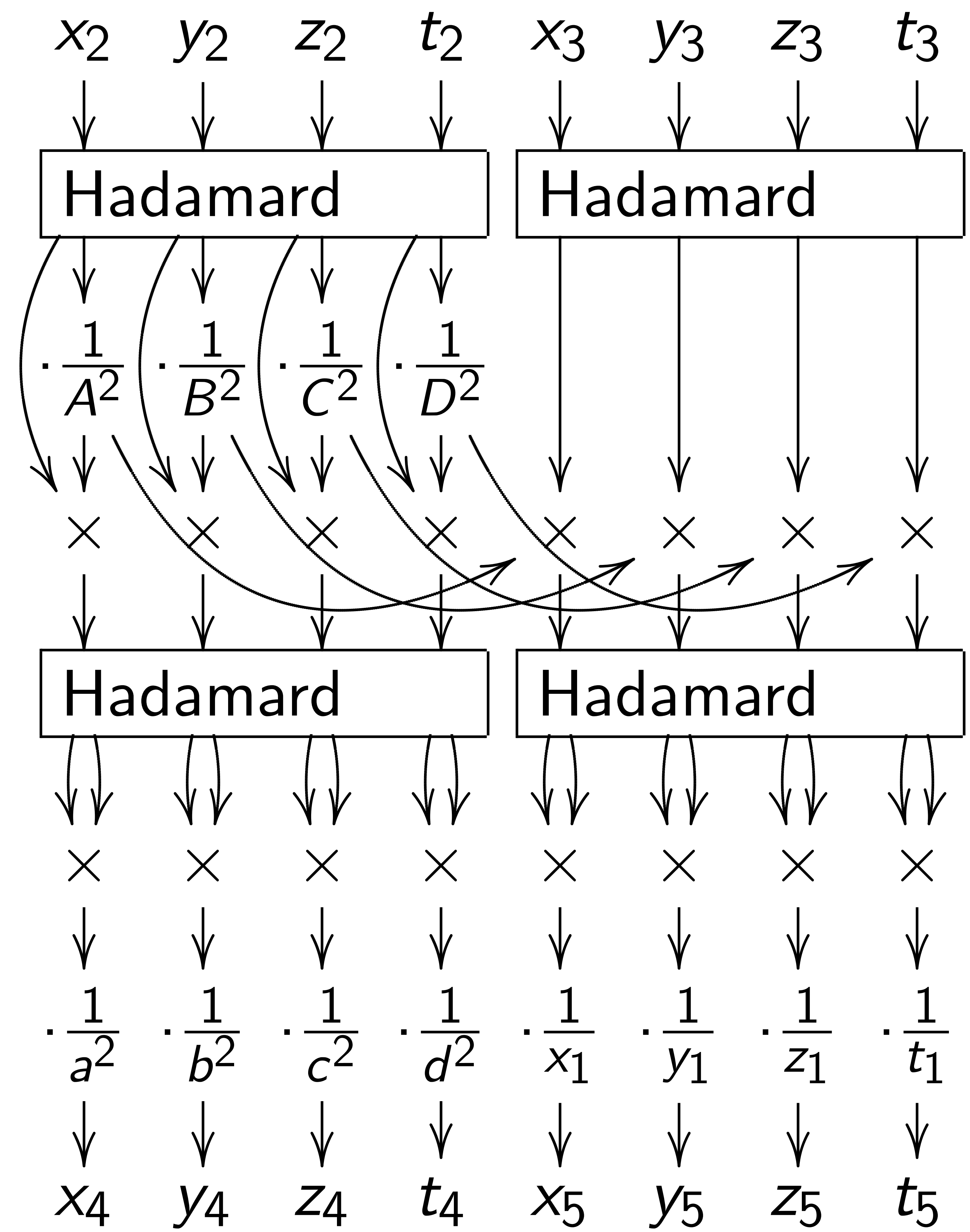
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Joint work with: Tanja Lange
Technische Universiteit Eindhoven

cr.yp.to/papers.html#hyperand
(2014) + new examples (2015)

Rewind to 2012 Gaudry–Schost:
"the computation took
more than 1,000,000 CPU hours".

The Gaudry–Schost motivation:



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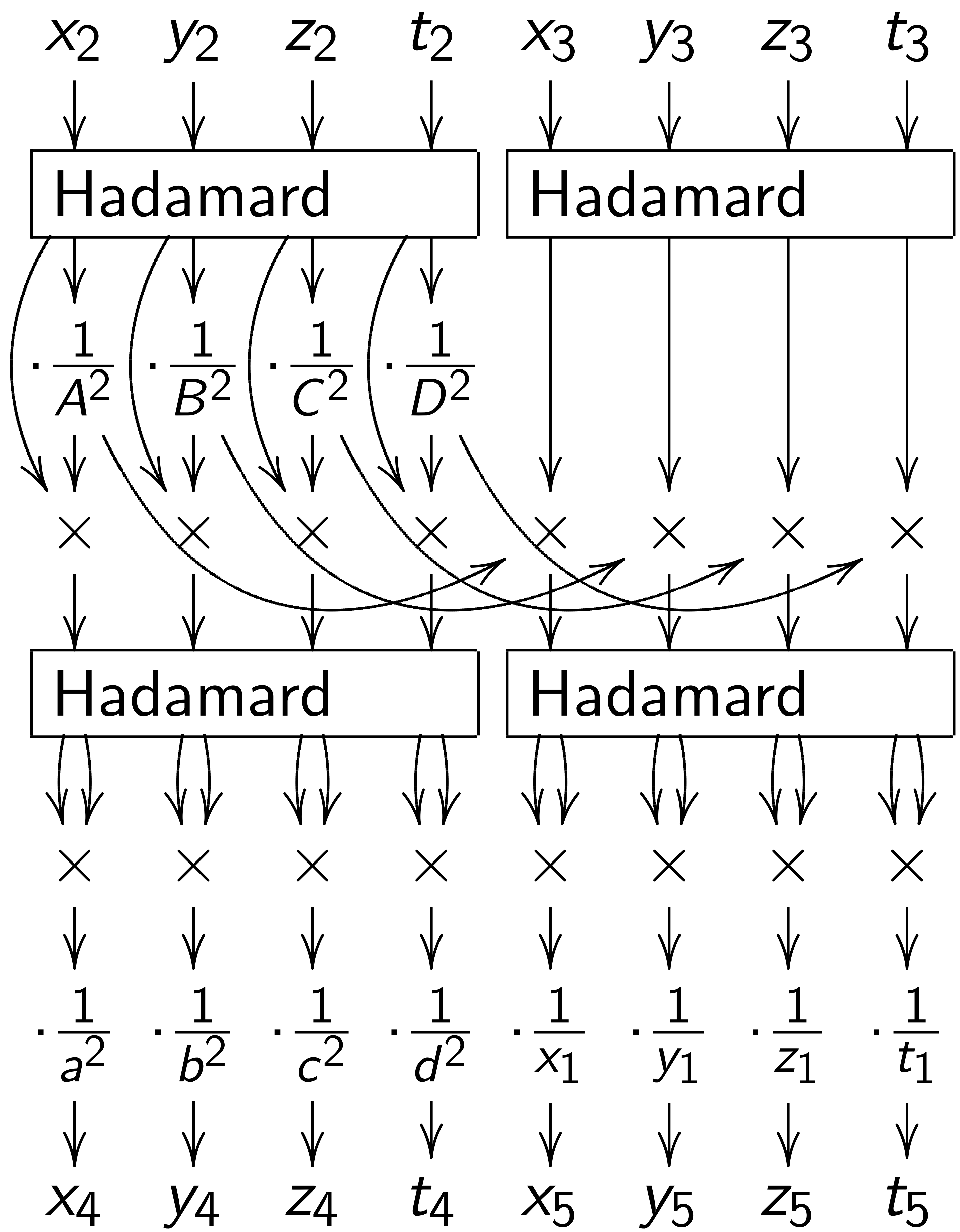

Inputs:

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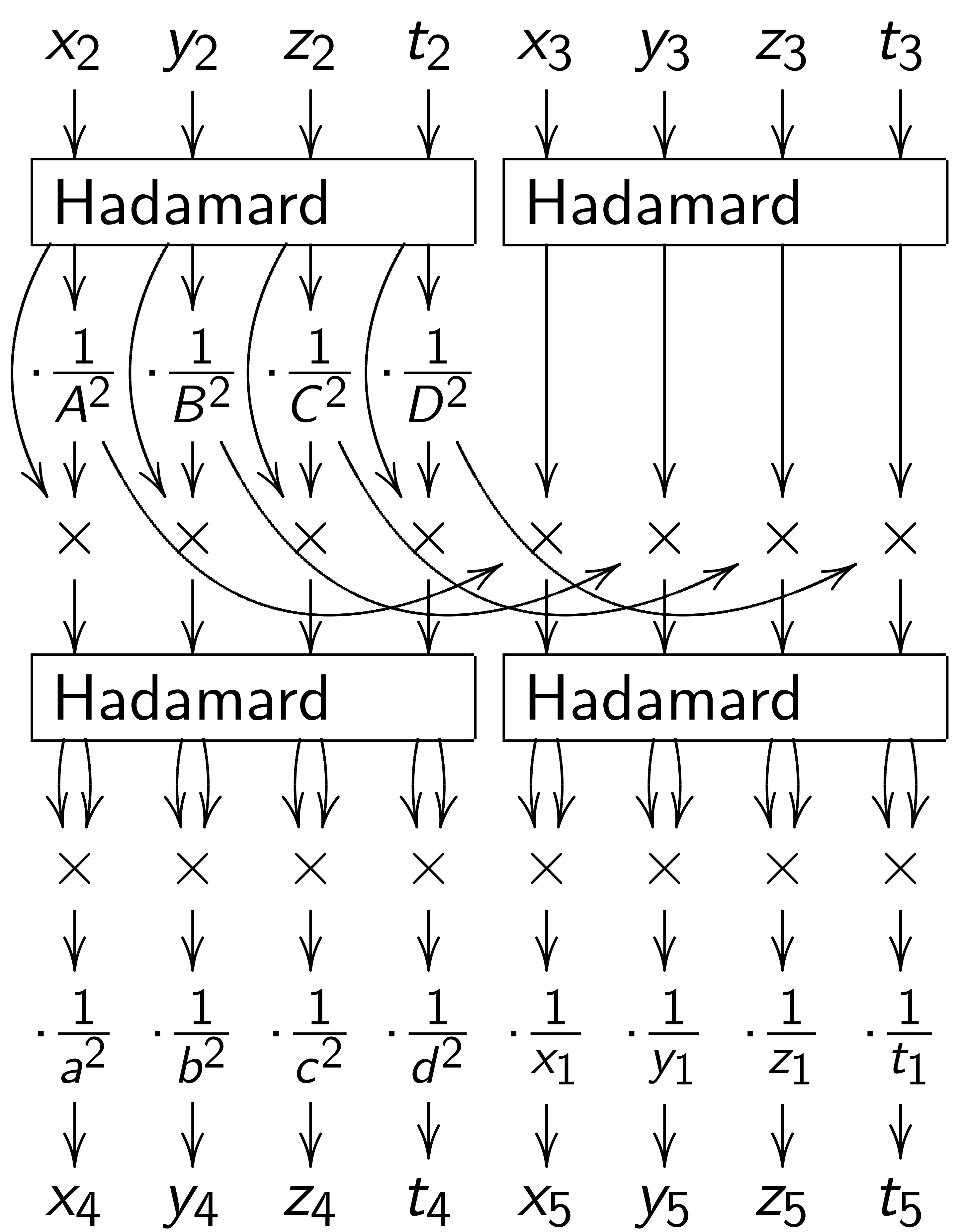

Inputs: “squared 6

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$(x_3 : y_3 : z_3 : t_3)$ for

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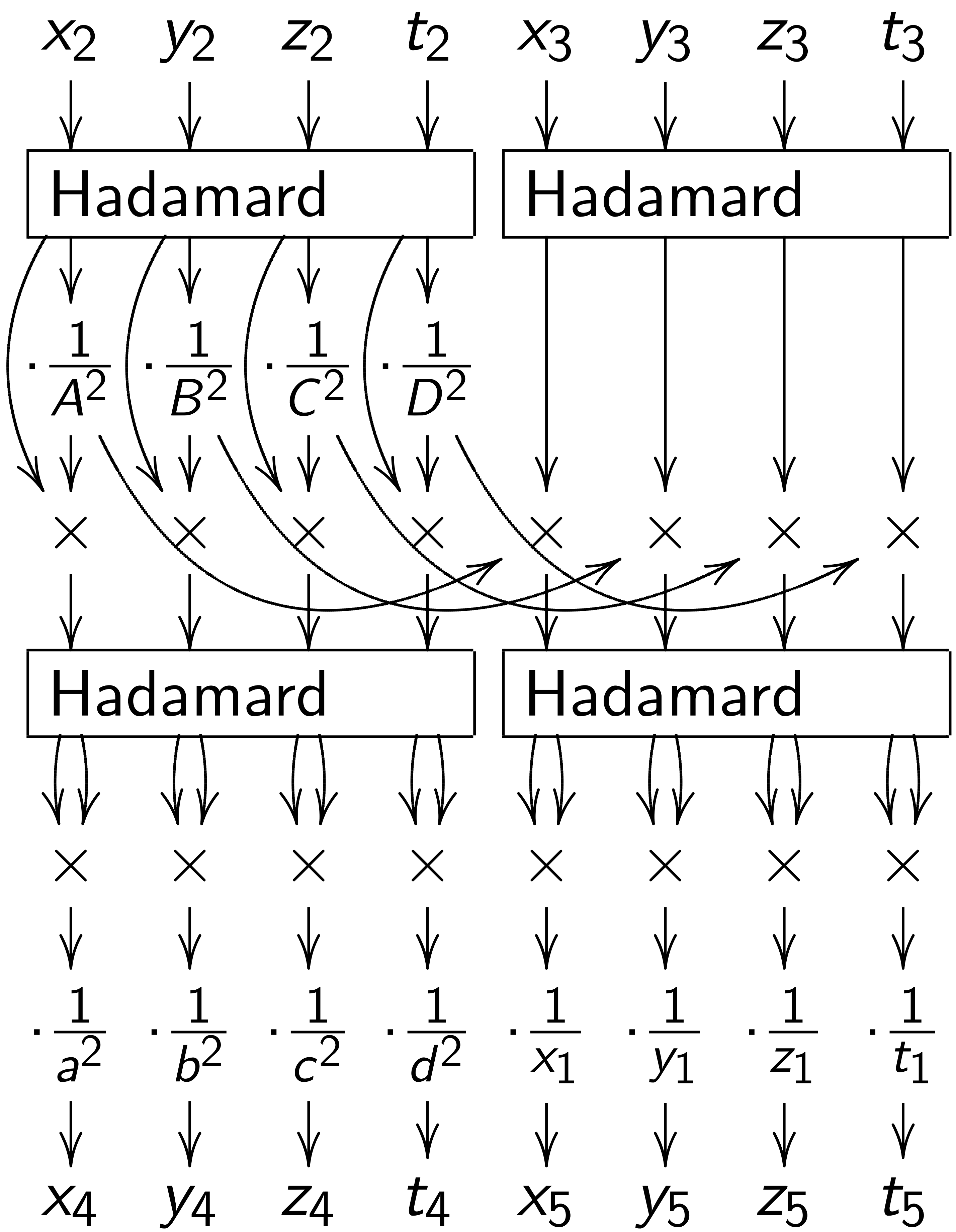

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$(x_2 : y_2 : z_2 : t_2)$ for Q_2,
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This diagram computes
$(x_4 : y_4 : z_4 : t_4)$ for $Q_4 = 2Q$
$(x_5 : y_5 : z_5 : t_5)$ for $Q_5 = Q_3$

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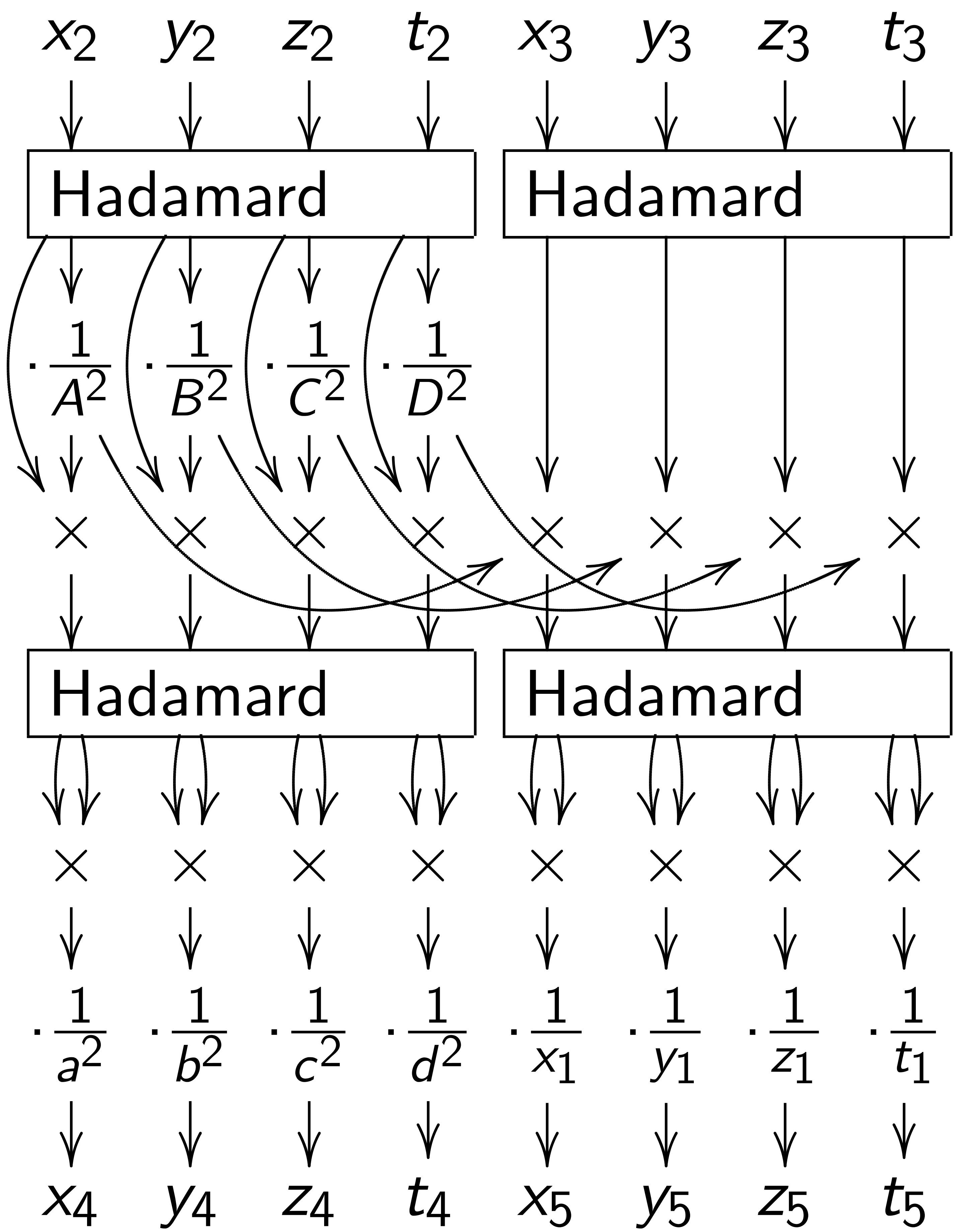

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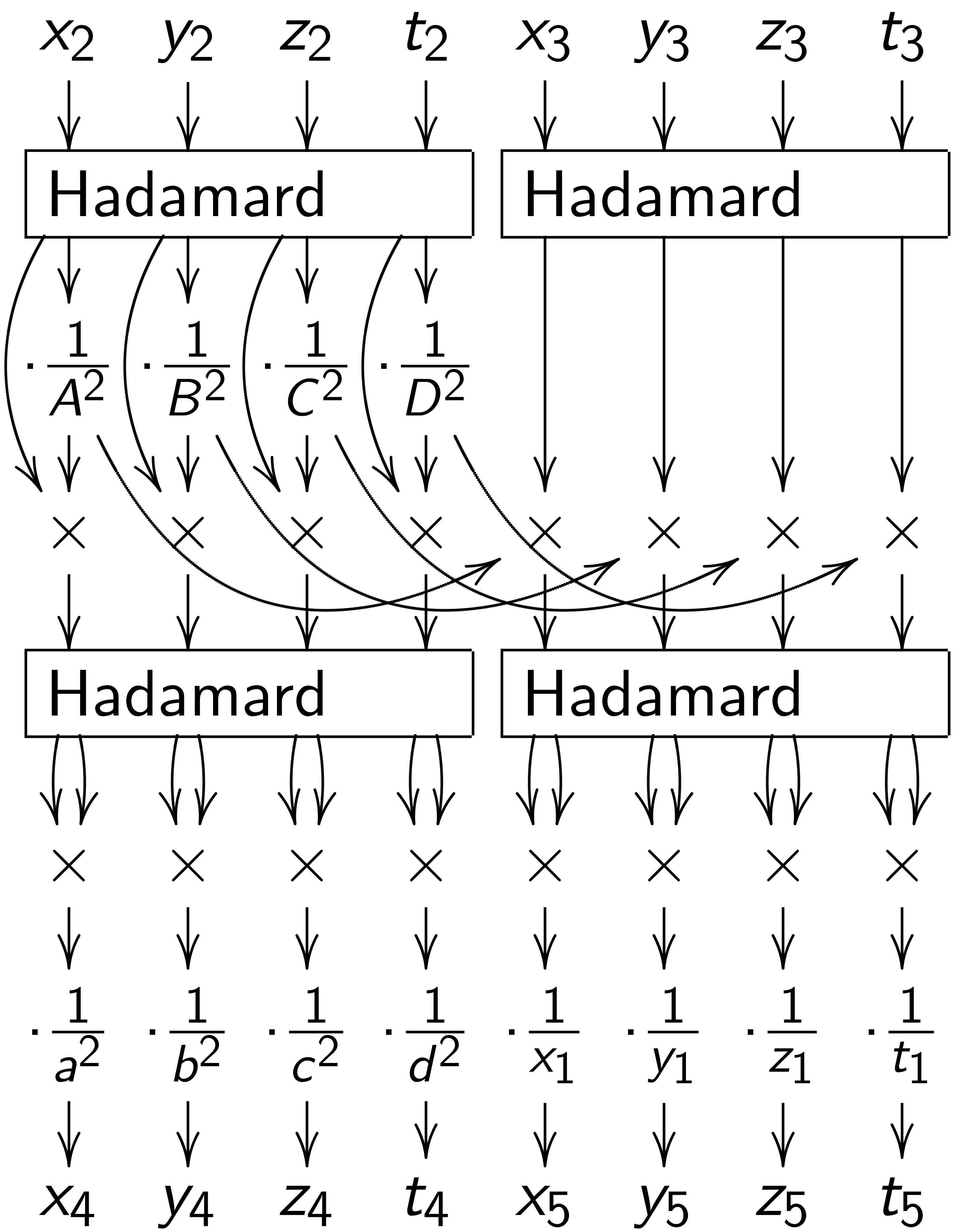

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Just 14 mults for Q_4
(1986 Chudnovsky–Chudnovsky).
Huge speedup if constants
$\left(\frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2} : \frac{1}{d^2}\right)$ etc. are small.

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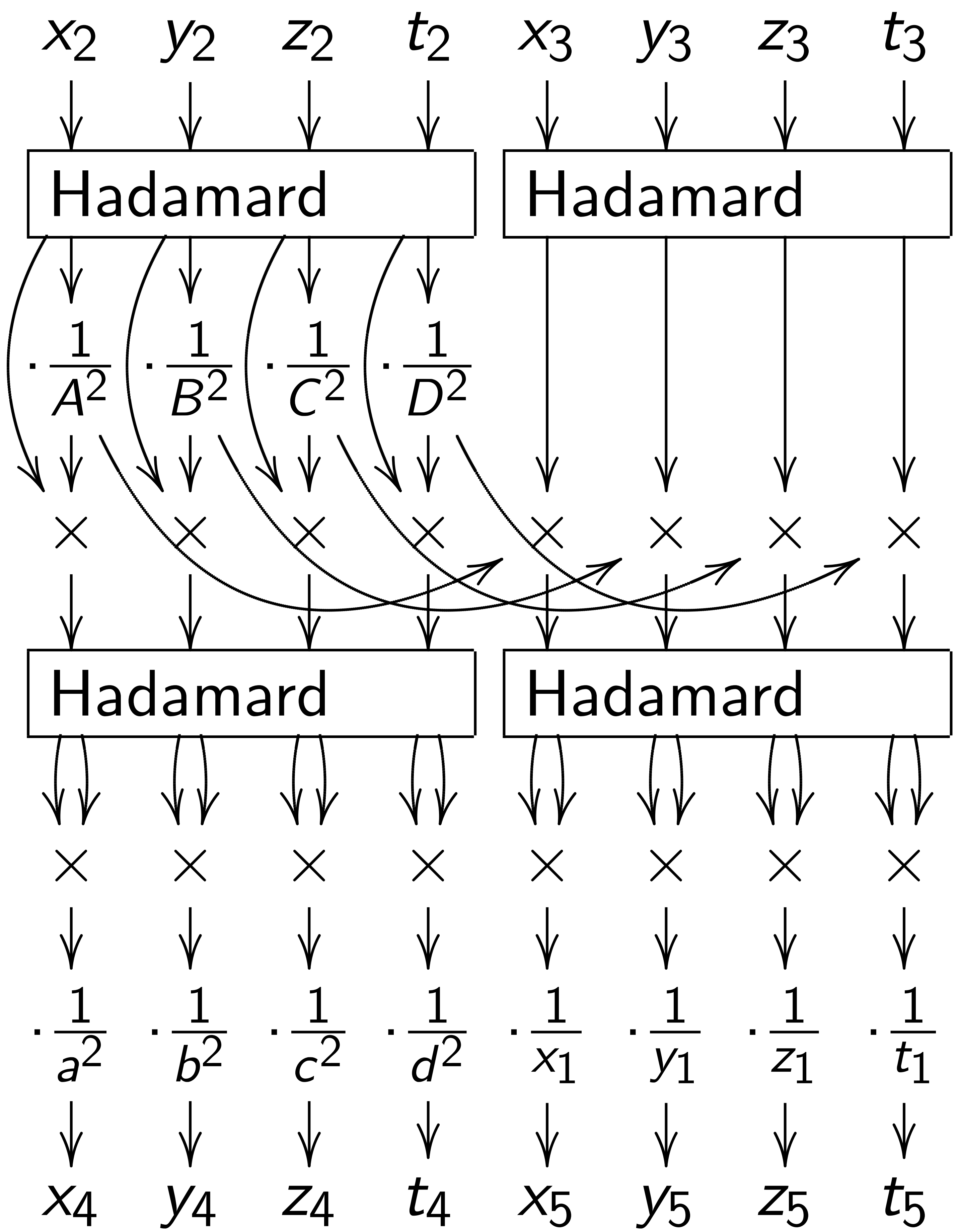

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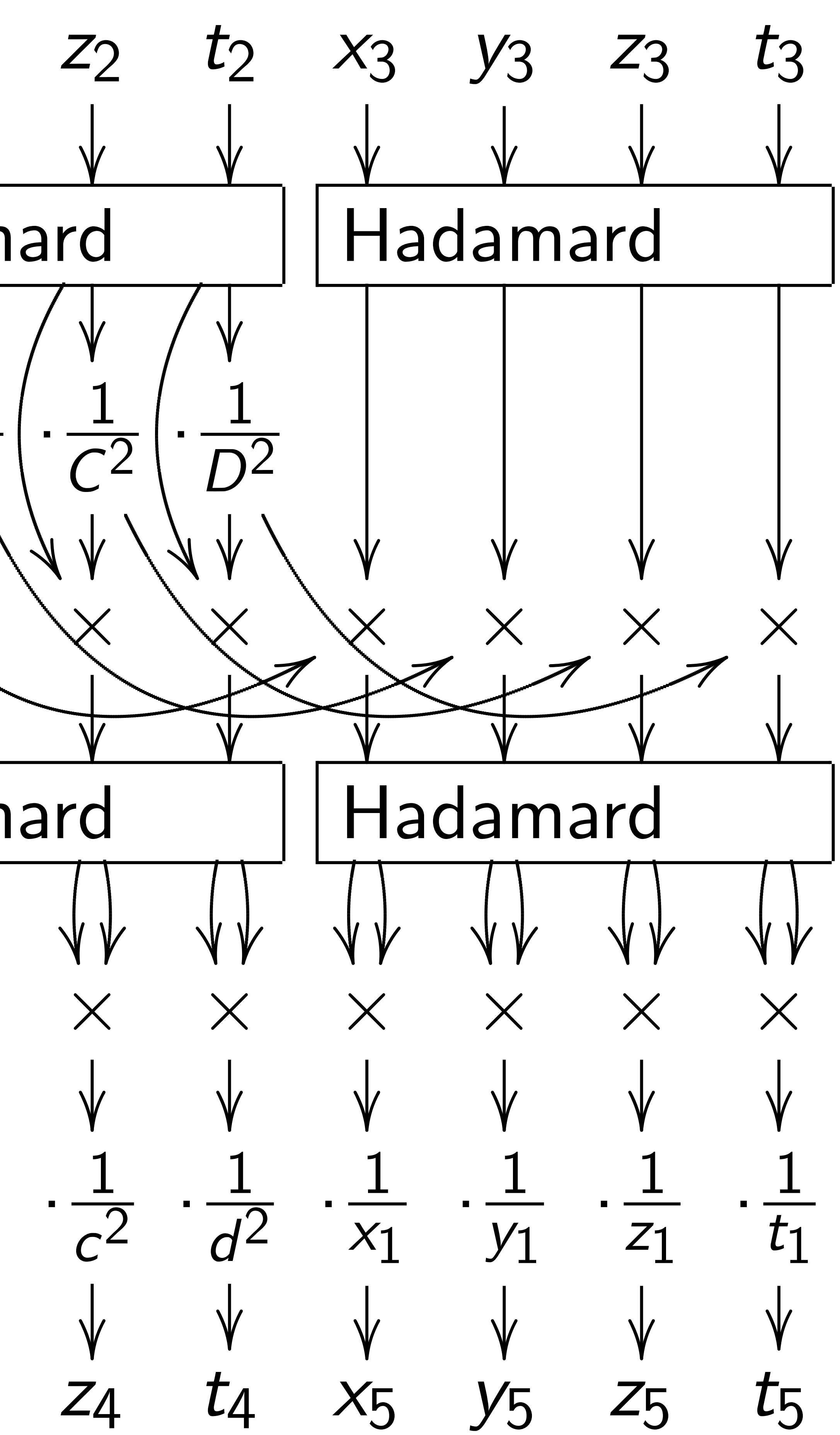

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$-H$
where
$A^2 = a^2$
$B^2 = a^2$
$C^2 = a^2$
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$F = (a^4$
$G = (a^4$
$H = (a^4$
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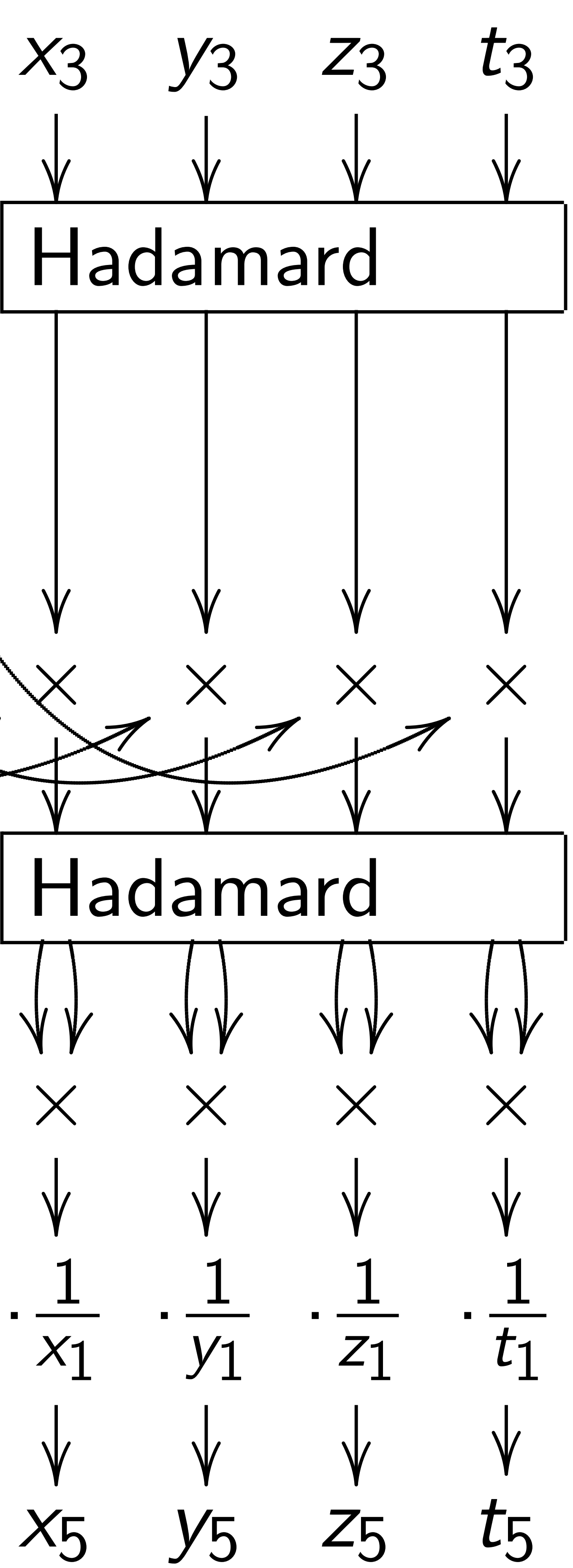

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$B^2 = a^2 + b^2 - c$
$C^2 = a^2 - b^2 + c$
$D^2 = a^2 - b^2 - c$
$F = (a^4 - b^4 - c^4 +$
$G = (a^4 - b^4 + c^4 -$
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$E^2 = F^2 + G^2 +$

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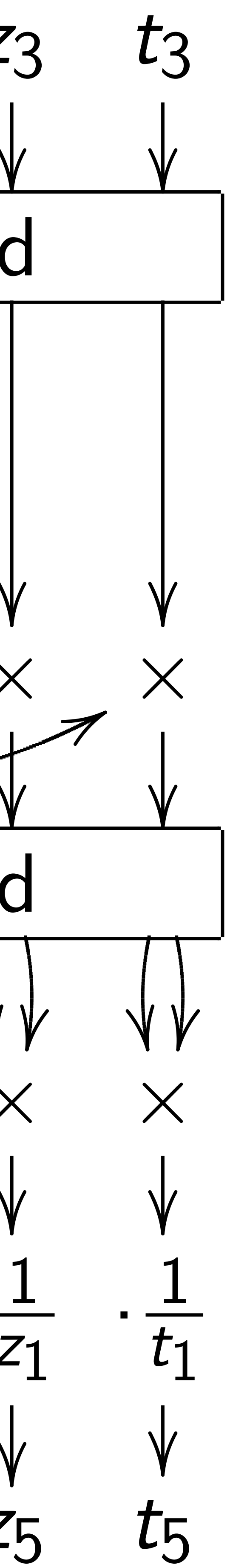

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$(x_i : y_i : z_i : t_i)$ are points on
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$4E^2xyzt = ((x^2 + y^2 + z^2$
$\quad -F(xt + yz) - G(xz +$
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where
$A^2 = a^2 + b^2 + c^2 + d^2$,
$B^2 = a^2 + b^2 - c^2 - d^2$,
$C^2 = a^2 - b^2 + c^2 - d^2$,
$D^2 = a^2 - b^2 - c^2 + d^2$,
$F = (a^4 - b^4 - c^4 + d^4)/(a^2 d^2$
$G = (a^4 - b^4 + c^4 - d^4)/(a^2 c^2$
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$C^2 = a^2 - b^2 + c^2 - d^2$,
$D^2 = a^2 - b^2 - c^2 + d^2$,
$F = (a^4 - b^4 - c^4 + d^4)/(a^2d^2 - b^2c^2)$,
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Surface is from 18

Über die Flächen

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Endlich möge hier noch
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singulären Tangentialebenen

$p = 0, \; q = 0,$

als die Fundamentalebenen, also
genen Coordinaten, und bezeic
teren durch r und s, so erhält
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10., $\phi^2 =$

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$\phi = p^2 + q^2 + r^2 + s^2 + 2a(q$

$K = a^2 + b^2 + c^2 - 2a$

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Surface is from 1864 Kumm

Über die Flächen vierten Gr

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men kann. Wählt man die vier in der Form (4
singulären Tangentialebenen

$$p = 0,\ q = 0,\ p' = 0,\ q' = 0$$

als die Fundamentalebenen, also p, q, p', q', als d
genen Coordinaten, und bezeichnet demgemäſs di
teren durch r und s, so erhält man folgende Fo
chung:

10., $$\phi^2 = 16\,Kpqrs,$$

wo

$$\phi = p^2 + q^2 + r^2 + s^2 + 2a(qr + ps) + 2b(rp + qs)$$
$$K = a^2 + b^2 + c^2 - 2abc - 1.$$

in welcher die sieben Constanten a, b, c, d, e, f,
auf die richtige Anzahl von drei Constanten a, b, c
ist. Wählt man in dieser Form die Coefficienten
Ausdrücke p, q, r, s real, und die drei Constanten

$(x_i : y_i : z_i : t_i)$ are points on original Kummer surface K :

$$4E^2xyzt = ((x^2 + y^2 + z^2 + t^2) \\ -F(xt + yz) - G(xz + yt) \\ -H(xy + zt))^2$$

where

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Surface is from 1864 Kummer, *Über die Flächen vierten Grades mit sechzehn singulären Punkten*:

Endlich möge hier noch eine Formveränderung erwähnt werden, welche man mit der Gleichung dieser Flächen vornehmen kann. Wählt man die vier in der Form (4.) enthaltenen singulären Tangentialebenen

$$p = 0,\ q = 0,\ p' = 0,\ q' = 0$$

als die Fundamentalebenen, also p, q, p', q', als die vier homogenen Coordinaten, und bezeichnet demgemäſs die beiden letzteren durch r und s, so erhält man folgende Form der Gleichung:

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$$K = a^2 + b^2 + c^2 - 2abc - 1.$$

in welcher die sieben Constanten a, b, c, d, e, f, k jener Form auf die richtige Anzahl von drei Constanten a, b, c eingeschränkt ist. Wählt man in dieser Form die Coefficienten der linearen Ausdrücke p, q, r, s real, und die drei Constanten a, b, c eben-

$z_i : t_i)$ are points on
Kummer surface K :
$t = ((x^2 + y^2 + z^2 + t^2)$
$(xt + yz) - G(xz + yt)$
$(xy + zt))^2$

$+ b^2 + c^2 + d^2,$
$+ b^2 - c^2 - d^2,$
$- b^2 + c^2 - d^2,$
$- b^2 - c^2 + d^2,$
$-b^4-c^4+d^4)/(a^2d^2-b^2c^2),$
$-b^4+c^4-d^4)/(a^2c^2-b^2d^2),$
$+b^4-c^4-d^4)/(a^2b^2-c^2d^2),$
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points on
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$- y^2 + z^2 + t^2)$
$- G(xz + yt)$
$)^2$

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$^2 - d^2,$
$^2 + d^2,$
$d^4)/(a^2d^2 - b^2c^2),$
$d^4)/(a^2c^2 - b^2d^2),$
$d^4)/(a^2b^2 - c^2d^2),$
$H^2 + FGH - 4.$

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vom 18. *April* 1864. 253

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$$p = 0,\ q = 0,\ p' = 0,\ q' = 0$$

als die Fundamentalebenen, also p, q, p', q', als die vier homogenen Coordinaten, und bezeichnet demgemäfs die beiden letzteren durch r und s, so erhält man folgende Form der Gleichung:

10., $$\phi^2 = 16\,Kpqrs,$$

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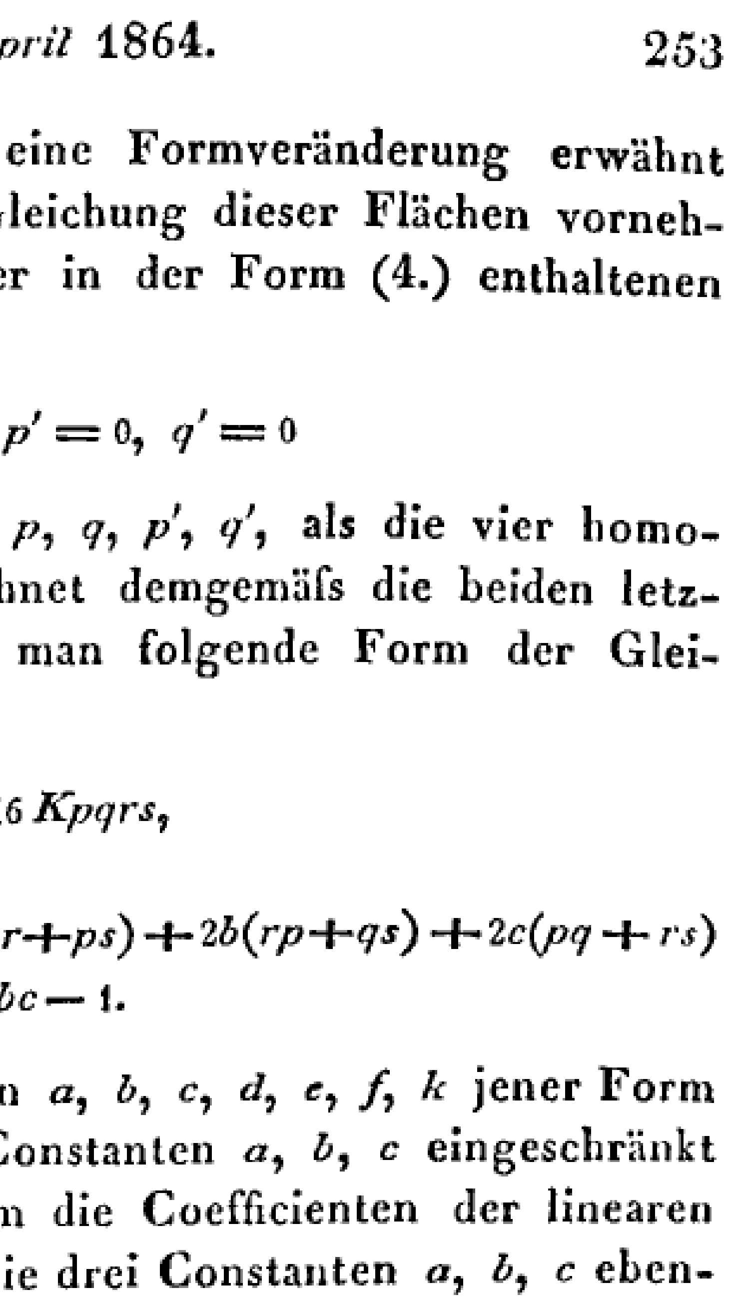

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Counting ops suggests: Yes, especially with small a^2 etc.

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$(x-1)(x-\lambda)(x-\mu)(x-\nu)$.

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$\frac{\lambda\mu}{\nu}$, $a^2 = \frac{b^2c^2\nu}{\mu}$, $d^2 = 1$.

Take $s_1, s_2, s_3 \in \mathbf{F}_{p^2}$, norm 1,
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$-s_1^2 s_2^2 s_3^2$ has norm 1.
Write it as $\overline{r}/r$ with $r \in \mathbf{F}_{p^2}^*$.

Define $s = -r(s_1 + s_2 + s_3)$.

Take any $\beta \in \mathbf{F}_{p^2} - \mathbf{F}_p$
with $(\overline{\beta}/\beta)^2 \notin \{s_1^2, s_2^2, s_3^2\}$.
Then g has 6 distinct roots
$(1 \pm s_j)/(\overline{\beta} \pm \beta s_j) \in \mathbf{F}_p$.

Hope that $a^2, b^2, c^2, d^2 \in \mathbf{F}_p$;
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$$y^2 = (x - 16/3)(x + 3/1072)(x - 1/16)(x + 16/67)(x + 1/20)(x - 20/67)$$

with Kummer surface
$a^2 = 194769$, $b^2 = 126939$,
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A good example for crypto:

$$y^2 = (z + 3)(z + 1/9)(z - 1/7)(z - 7/3)(z - 8/7)(z - 7/24).$$

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$= 32\ell$ for a prime $\ell \approx 2^{249}$.

$\#E'(\mathbf{F}_{p^2}) = 12 \cdot \text{prime}$.

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Particularly nice arithmetic:
$(a^2 : b^2 : c^2 : d^2) = (20:12:12:5)$;
$(A^2 : \ldots) = (49:15:15:1)$;
$(\frac{1}{a^2} : \cdots) = (3:5:5:12)$;
$(\frac{1}{A^2} : \cdots) = (15:49:49:735)$.