Advanced code-based cryptography

Daniel J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Lattice-basis reduction

Define $L = (0, 24)\mathbf{Z} + (1, 17)\mathbf{Z}$
$= \{(b, 24a + 17b) : a, b \in \mathbf{Z}\}$.

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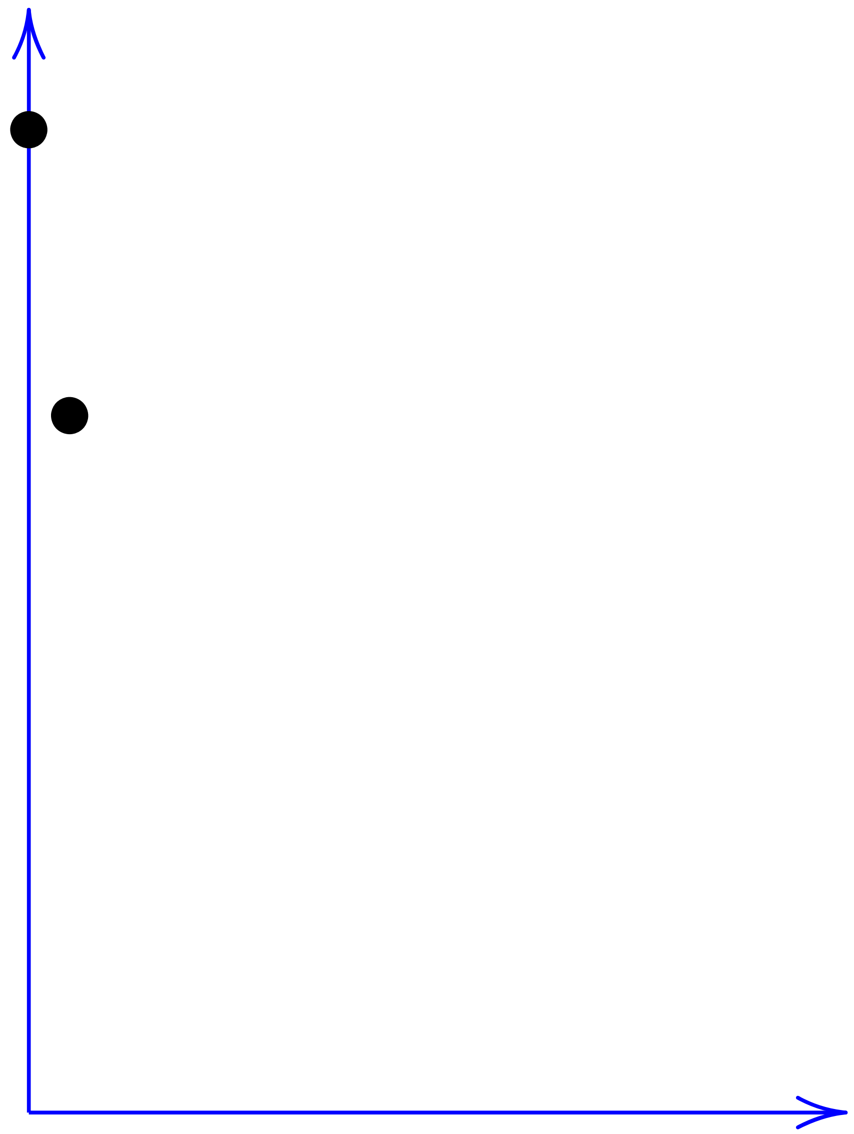
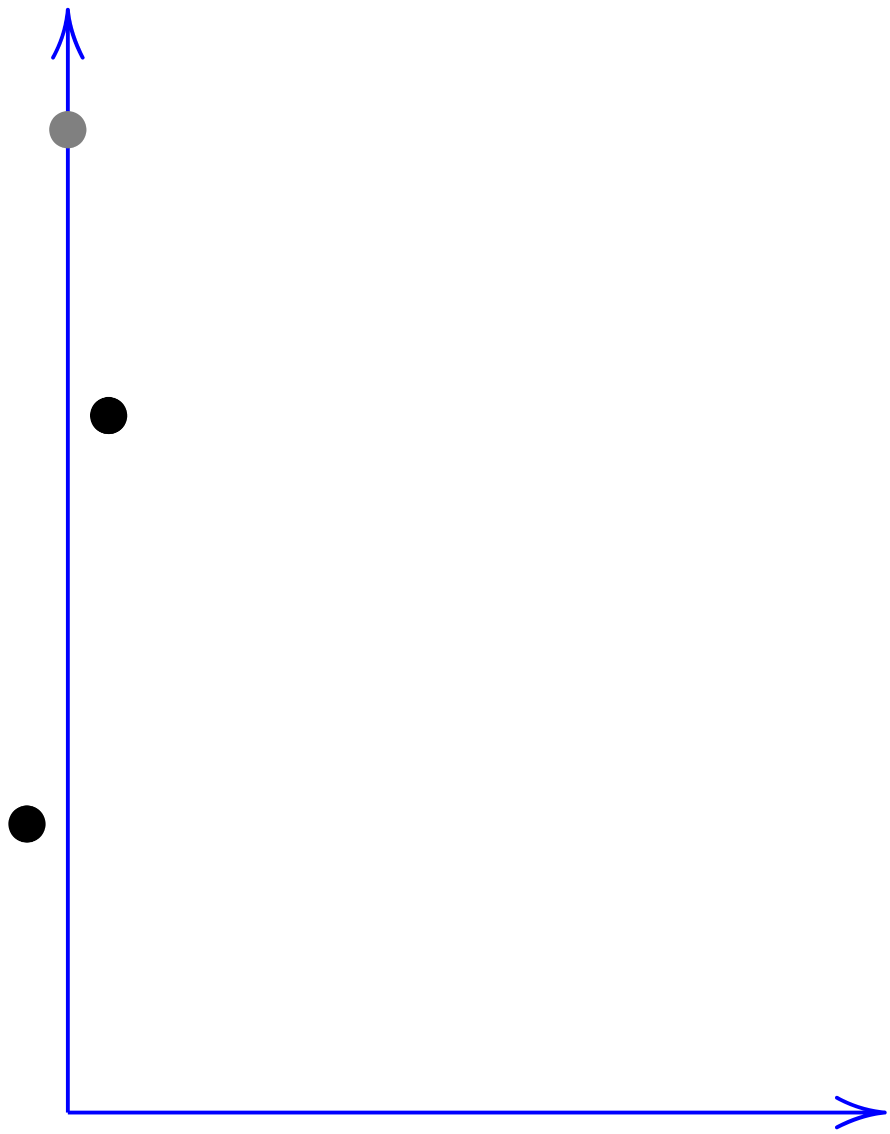
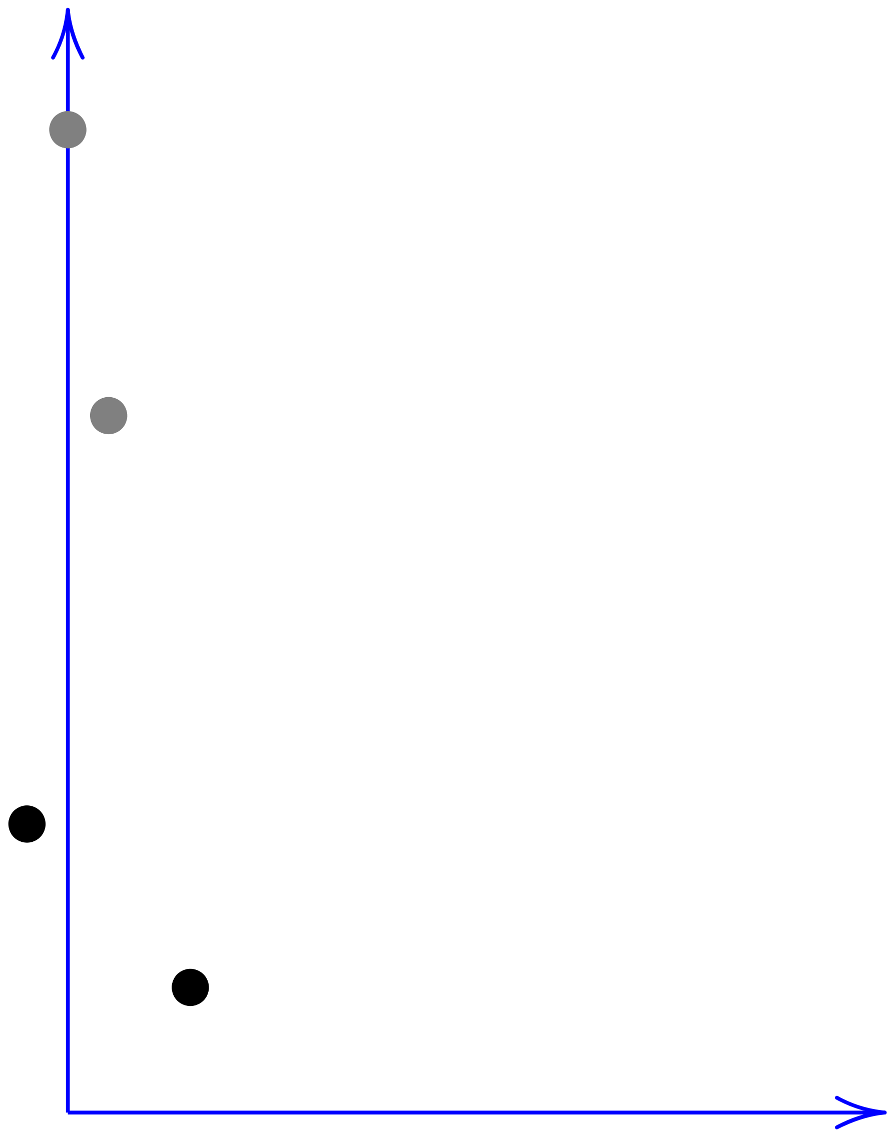
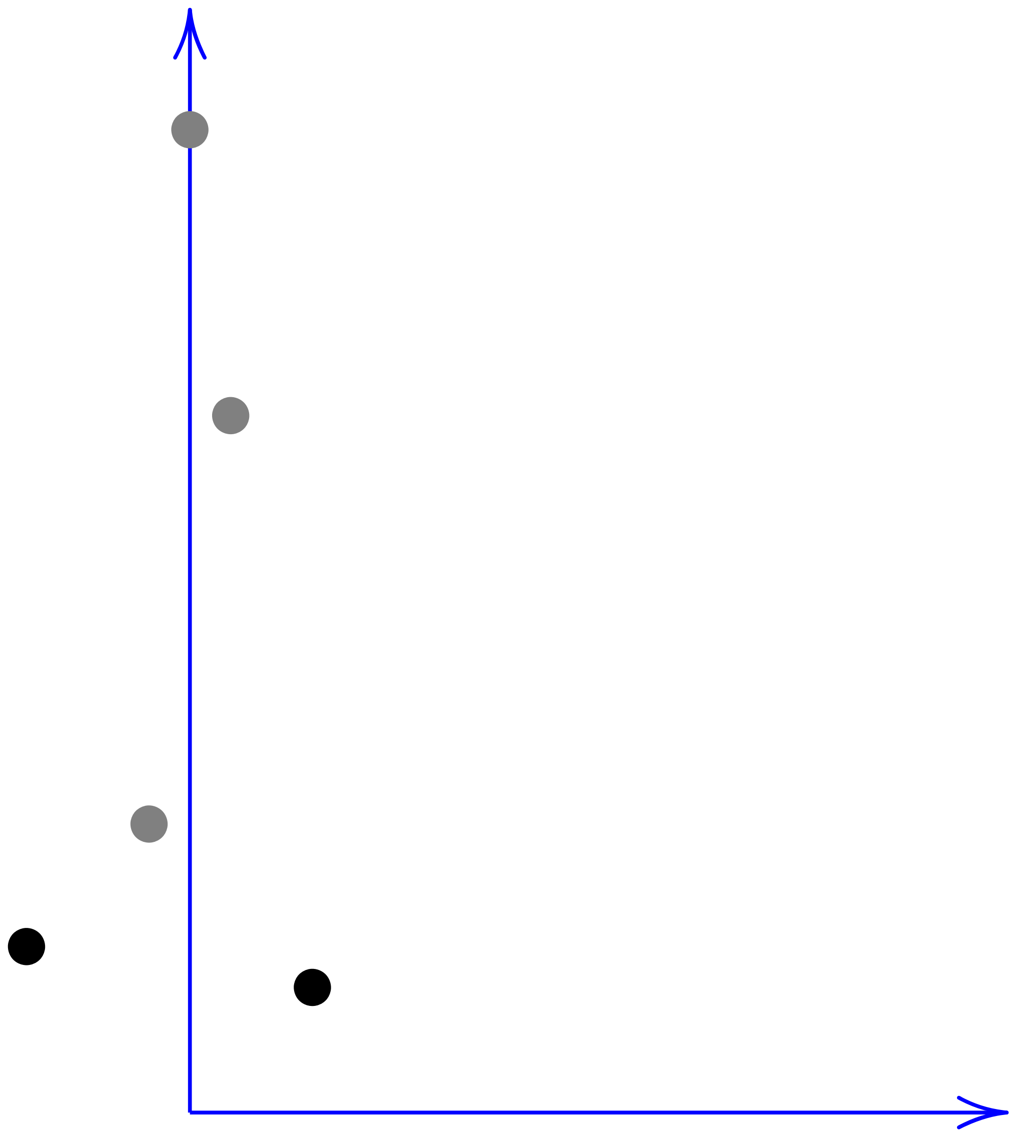
$$= (-4, 4)\mathbf{Z} + (3, 3)\mathbf{Z}.$$

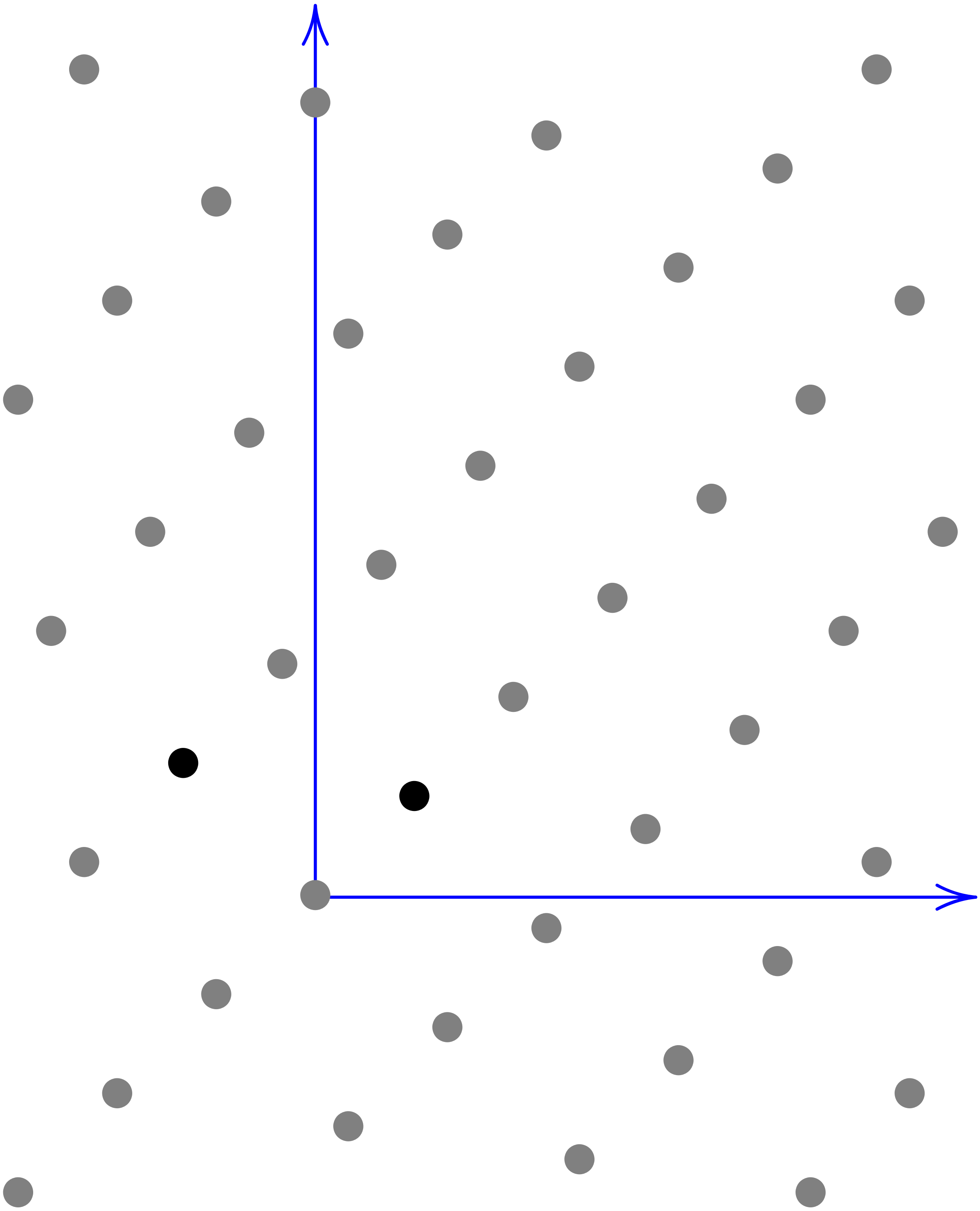
$(-4, 4), (3, 3)$ are orthogonal.
Shortest vectors in L are
$(0, 0)$, $(3, 3)$, $(-3, -3)$.











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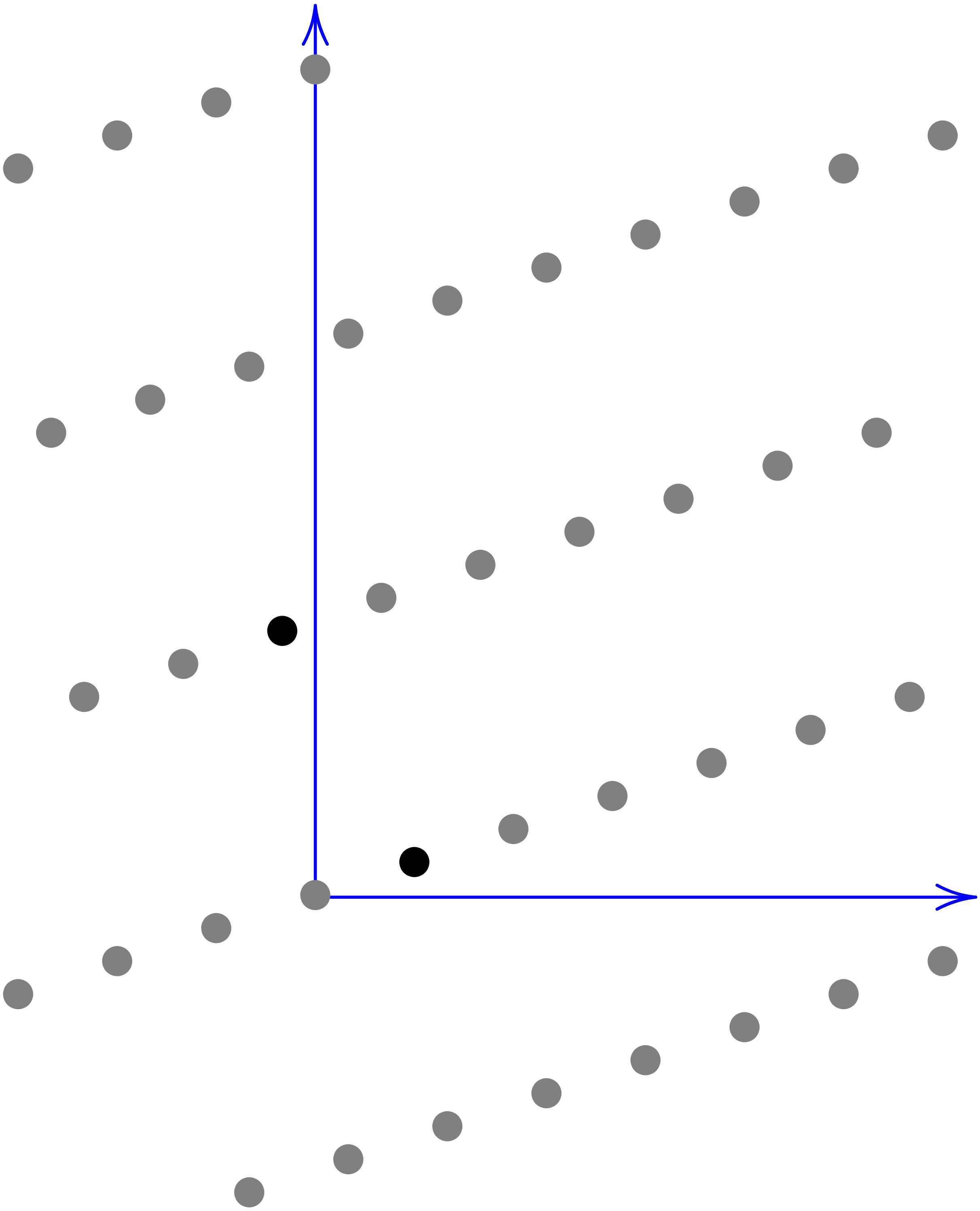
Nearly orthogonal.

Shortest vectors in L are $(0,0)$, $(3,1)$, $(-3,-1)$.



Polynomial lattices

Define $P = \mathbf{F}_2[x]$,
$r_0 = (101000)_x = x^5 + x^3 \in P$,
$r_1 = (10011)_x = x^4 + x + 1 \in P$,
$L = (0, r_0)P + (1, r_1)P$.

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$(111, 1)$: shortest nonzero vector.
$(10, 1110)$: shortest independent vector.

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Can use other metrics, or equivalently rescale L.

e.g. Define $L \subseteq \mathbf{F}_2[\sqrt{x}] \times \mathbf{F}_2[\sqrt{x}]$ as $(0, r_0\sqrt{x})P + (1, r_1\sqrt{x})P$.

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Successive generators for L:
$(0, 101000\sqrt{x})$, degree 5.5.
$(1, 10011\sqrt{x})$, degree 4.5.
$(10, 1110\sqrt{x})$, degree 3.5.
$(111, 1\sqrt{x})$, degree 2.

Warning: Sometimes
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e.g. Define
$r_0 = 101000$, $r_1 = 10111$,
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Successive generators for L:
$(0, 101000\sqrt{x})$, degree 5.5.
$(1, 10111\sqrt{x})$, degree 4.5.
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$(1101, 11\sqrt{x})$, degree 3.

For any field k, any r_0, r_1
in $P = k[x]$ with $\deg r_0 > \deg r_1$:

Euclid/Stevin computation:
Define $r_2 = r_0 \bmod r_1$,
$r_3 = r_1 \bmod r_2$, etc.

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Extended: $q_0 = 0$; $q_1 = 1$;
$q_{i+2} = q_i - \lfloor r_i / r_{i+1} \rfloor q_{i+1}$.
Then $q_i r_1 \equiv r_i \pmod{r_0}$.

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Lattice view: Have $(0, r_0\sqrt{x})P + (1, r_1\sqrt{x})P = (q_i, r_i\sqrt{x})P + (q_{i+1}, r_{i+1}\sqrt{x})P$.

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Can continue until $r_{i+1} = 0$. $\gcd\{r_0, r_1\} = r_i / \operatorname{leadcoeff} r_i$.

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Say j is minimal with $\deg r_j\sqrt{x} \le (\deg r_0)/2$.
Then $\deg q_j \le (\deg r_0)/2$ so $\deg(q_j, r_j\sqrt{x}) \le (\deg r_0)/2$.
Shortest nonzero vector.

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Shortest nonzero vector.

$(q_{j+\epsilon}, r_{j+\epsilon} \sqrt{x})$ has degree
$\deg r_0 \sqrt{x} - \deg(q_j, r_j \sqrt{x})$
for some $\epsilon \in \{-1, 1\}$.
Shortest independent vector.

Proof of “shortest”:

Take any $(q, r\sqrt{x})$ in lattice.

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(q, r\sqrt{x}) = {} & u(q_j, r_j\sqrt{x}) \\
& + v(q_{j+\epsilon}, r_{j+\epsilon}\sqrt{x})
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for some $u, v \in P$.

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for some $u, v \in P$.

$q_j r_{j+\epsilon} - q_{j+\epsilon} r_j = \pm r_0$

so $v = \pm(rq_j - qr_j)/r_0$

and $u = \pm(qr_{j+\epsilon} - rq_{j+\epsilon})/r_0$.

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$q_j r_{j+\epsilon} - q_{j+\epsilon} r_j = \pm r_0$
so $v = \pm(r q_j - q r_j)/r_0$
and $u = \pm(q r_{j+\epsilon} - r q_{j+\epsilon})/r_0$.

If $\deg(q, r\sqrt{x})$
$< \deg(q_{j+\epsilon}, r_{j+\epsilon}\sqrt{x})$
then $\deg v < 0$ so $v = 0$;
i.e., any vector in lattice
shorter than $(q_{j+\epsilon}, r_{j+\epsilon}\sqrt{x})$
is a multiple of $(q_j, r_j\sqrt{x})$.

Classical binary Goppa codes

Fix integer $n \geq 0$;
integer $m \geq 1$ with $2^m \geq n$;
integer $t \geq 0$;
distinct $a_1, \ldots, a_n \in \mathbf{F}_{2^m}$;
monic $g \in \mathbf{F}_{2^m}[x]$ of degree t
with $g(a_1) \cdots g(a_n) \neq 0$.

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Define linear subspace $\Gamma \subseteq \mathbf{F}_2^n$
as set of $(c_1, \ldots, c_n)$ with
$\sum_i c_i/(x - a_i) = 0$ in $\mathbf{F}_{2^m}[x]/g$.

Then $\#\Gamma \geq 2^{n-mt}$.

Goal: Find $c \in \Gamma$ given $v = c + e$, assuming $|e| \leq t/2$.

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Lift $\sum_i v_i/(x - a_i)$ from $\mathbf{F}_{2^m}[x]/g$ to $s \in \mathbf{F}_{2^m}[x]$ with $\deg s < t$.
Find shortest nonzero $(q_j, r_j\sqrt{x})$ in the lattice $L = (0, g\sqrt{x})\mathbf{F}_{2^m}[x] + (1, s\sqrt{x})\mathbf{F}_{2^m}[x]$.

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Define $E, F \in \mathbf{F}_{2^m}[x]$ by $F = \prod_{i:e_i \neq 0}(x - a_i)$ and $E = \sum_i Fe_i/(x - a_i)$.
Fact: $E/F = r_j/q_j$ so F is monic denominator of r_j/q_j.

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$e_i = 0$ if $F(a_i) \neq 0$.
$e_i = E(a_i)/F'(a_i)$ if $F(a_i) = 0$.

This decoder
"corrects $\lfloor t/2 \rfloor$ errors for Γ".

Why does this work?

$\sum_i e_i/(x - a_i) = E/F$ and
$\sum_i c_i/(x - a_i) = 0$ in $\mathbf{F}_{2^m}[x]/g$
so $s = E/F$ in $\mathbf{F}_{2^m}[x]/g$
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$(F, E\sqrt{x})$ is a short vector:
$\deg(F, E\sqrt{x}) \leq |e| \leq t/2$
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Recall proof of "shortest":
$(F, E\sqrt{x}) \in (q_j, r_j\sqrt{x})\mathbf{F}_{2^m}[x]$,
so $E/F = r_j/q_j$. Done!

The squarefree case

$\Gamma(g)$ contains $\Gamma(g^2)$:
$\sum_i c_i/(x - a_i) = 0$ in $\mathbf{F}_{2^m}[x]/g$ if
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(Not covered in this talk:
correcting $\approx t + t^2/n$ errors.
See, e.g., "jet list decoding".)

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F' is a square:

if $F = \sum_j F_j x^j$ then

$$F' = \sum_j jF_j x^{j-1}$$
$$= \sum_{j \in 1+2\mathbf{Z}} jF_j x^{j-1}$$
$$= (\sum_{j \in 1+2\mathbf{Z}} \sqrt{jF_j} x^{(j-1)/2})^2.$$

The McEliece cryptosystem

Standardize integers $n \geq 0$; $t \geq 2$; $m \geq 1$ with $2^m \geq n$.

1978 McEliece example: $n = 1024$, $m = 10$, $t = 50$.
This is too small:
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$n = 3408$, $m = 12$, $t = 67$:
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$n = 6960$, $m = 13$, $t = 119$:
$\approx 2^{263}$ pre-quantum security.

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Alice receives Ke,
finds $v \in \mathbf{F}_2^n$ with $Kv = Ke$,
decodes v to find $v - e$.

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Bob chooses random $c \in \Gamma$
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1986 Niederreiter improvements:

Send Ke instead of $c + e$.

K is smaller than G
whenever $mt < n - mt$.
Compress K to $mt(n - mt)$ bits
by requiring systematic form.

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Our constant-time software for batches of 256 decodings:

26544 Ivy Bridge cycles for $(n, t) = (2048, 32)$; $\approx 2^{87}$.

79715 Ivy Bridge cycles for $(n, t) = (3408, 67)$; $\approx 2^{146}$.

306102 Ivy Bridge cycles for $(n, t) = (6960, 119)$; $\approx 2^{263}$.

The additive FFT

Fix $n = 4096 = 2^{12}$, $t = 41$.

Big final decoding step
is to find all roots in $\mathbf{F}_{2^{12}}$
of $F = F_{41}x^{41} + \cdots + F_0 x^0$.

For each $\alpha \in \mathbf{F}_{2^{12}}$,
compute $F(\alpha)$ by Horner's rule:
41 adds, 41 mults.

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Our cost: **6.01** adds, **2.09** mults.

Asymptotics:
normally $t \in \Theta(n / \lg n)$,
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Wait a minute.
Didn't we learn in school
that FFT evaluates
an n-coeff polynomial
at n points
using $n^{1+o(1)}$ operations?
Isn't this better than $n^2/\lg n$?

Standard radix-2 FFT:

Want to evaluate
$F = F_0 + F_1 x + \cdots + F_{n-1} x^{n-1}$
at all the nth roots of 1.

Write F as $F_0(x^2) + x F_1(x^2)$.
Observe big overlap between
$F(\alpha) = F_0(\alpha^2) + \alpha F_1(\alpha^2)$,
$F(-\alpha) = F_0(\alpha^2) - \alpha F_1(\alpha^2)$.

F_0 has $n/2$ coeffs;
evaluate at $(n/2)$nd roots of 1
by same idea recursively.
Similarly F_1.

Useless in char 2: $\alpha = -\alpha$.
Standard workarounds are painful.
FFT considered impractical.

1988 Wang–Zhu,
independently 1989 Cantor:
"additive FFT" in char 2.
Still quite expensive.

1996 von zur Gathen–Gerhard:
some improvements.

2010 Gao–Mateer:
much better additive FFT.

We use Gao–Mateer,
plus some new improvements.

Gao and Mateer evaluate $F = F_0 + F_1 x + \cdots + F_{n-1} x^{n-1}$ on a size-n $\mathbf{F}_2$-linear space.

Main idea: Write F as $F_0(x^2 + x) + x F_1(x^2 + x)$.

Big overlap between $F(\alpha) = F_0(\alpha^2 + \alpha) + \alpha F_1(\alpha^2 + \alpha)$ and $F(\alpha + 1) = F_0(\alpha^2 + \alpha) + (\alpha + 1) F_1(\alpha^2 + \alpha)$.

"Twist" to ensure $1 \in$ space. Then $\{\alpha^2 + \alpha\}$ is a size-$(n/2)$ $\mathbf{F}_2$-linear space. Apply same idea recursively.

We generalize to
$F = F_0 + F_1 x + \cdots + F_t x^t$
for any $t < n$.

$\Rightarrow$ several optimizations,
not all of which are automated
by simply tracking zeros.

For $t = 0$: copy F_0.

For $t \in \{1, 2\}$:
F_1 is a constant.
Instead of multiplying
this constant by each α,
multiply only by generators
and compute subset sums.

Syndrome computation

Initial decoding step: compute

$s_0 = r_1 + r_2 + \cdots + r_n,$
$s_1 = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_n\alpha_n,$
$s_2 = r_1\alpha_1^2 + r_2\alpha_2^2 + \cdots + r_n\alpha_n^2,$
$\vdots,$
$s_t = r_1\alpha_1^t + r_2\alpha_2^t + \cdots + r_n\alpha_n^t.$

$r_1, r_2, \ldots, r_n$ are received bits scaled by Goppa constants.

Typically precompute matrix mapping bits to syndrome.

Not as slow as Chien search but still $n^{2+o(1)}$ and huge secret key.

Compare to multipoint evaluation:

$$F(\alpha_1) = F_0 + F_1\alpha_1 + \cdots + F_t\alpha_1^t,$$
$$F(\alpha_2) = F_0 + F_1\alpha_2 + \cdots + F_t\alpha_2^t,$$
$$\vdots,$$
$$F(\alpha_n) = F_0 + F_1\alpha_n + \cdots + F_t\alpha_n^t.$$

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Matrix for syndrome computation
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matrix for multipoint evaluation.

Amazing consequence:
syndrome computation is as few
ops as multipoint evaluation.
Eliminate precomputed matrix.

Transposition principle:
If a linear algorithm computes a matrix M then reversing edges and exchanging inputs/outputs computes the transpose of M.

1956 Bordewijk; independently 1957 Lupanov for Boolean matrices.

1973 Fiduccia analysis: preserves number of mults; preserves number of adds plus number of nontrivial outputs.

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Still excessive code size.

Built new interpreter, allowing some code compression.
Still big; still some overhead.

Better solution:
stared at additive FFT,
wrote down transposition
with same loops etc.

Small code, no overhead.

Speedups of additive FFT
translate easily
to transposed algorithm.

Further savings:
merged first stage with
scaling by Goppa constants.

Results

60493 Ivy Bridge cycles:

8622 for permutation.
20846 for syndrome.
7714 for BM.
14794 for roots.
8520 for permutation.

Code will be public domain.
We're still speeding it up.

More information:
`cr.yp.to/papers.html#mcbits`