Hyper-and-elliptic-curve cryptography

Daniel J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Includes recent joint work with:
Tanja Lange
Technische Universiteit Eindhoven

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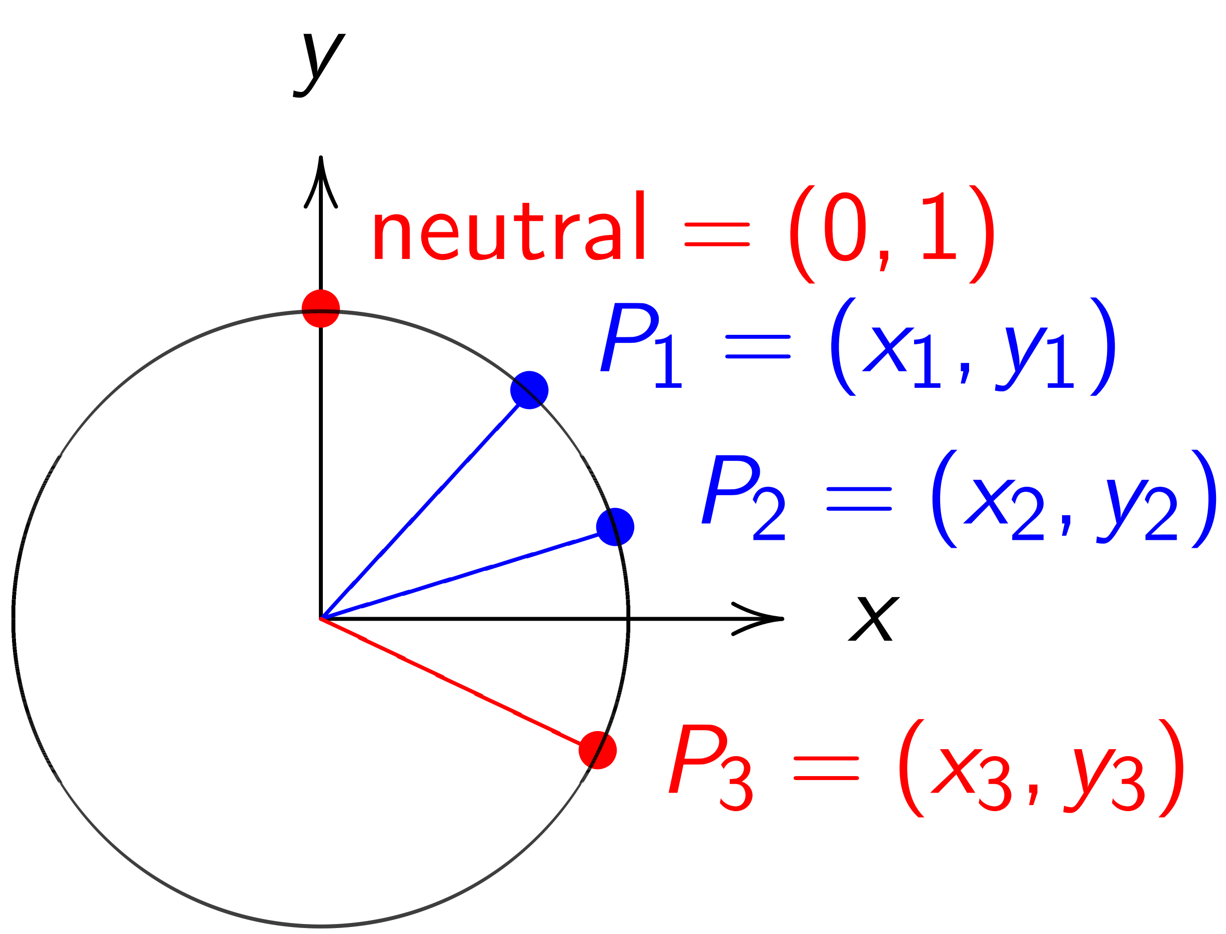
Clock($\mathbf{R}$): the commutative group $\{(x,y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 = 1\}$ under the operations

"0": $() \mapsto (0,1)$;

"−": $(x,y) \mapsto (-x,y)$;

"+": $(x_1,y_1),(x_2,y_2) \mapsto (x_1y_2 + y_1x_2, y_1y_2 - x_1x_2)$.



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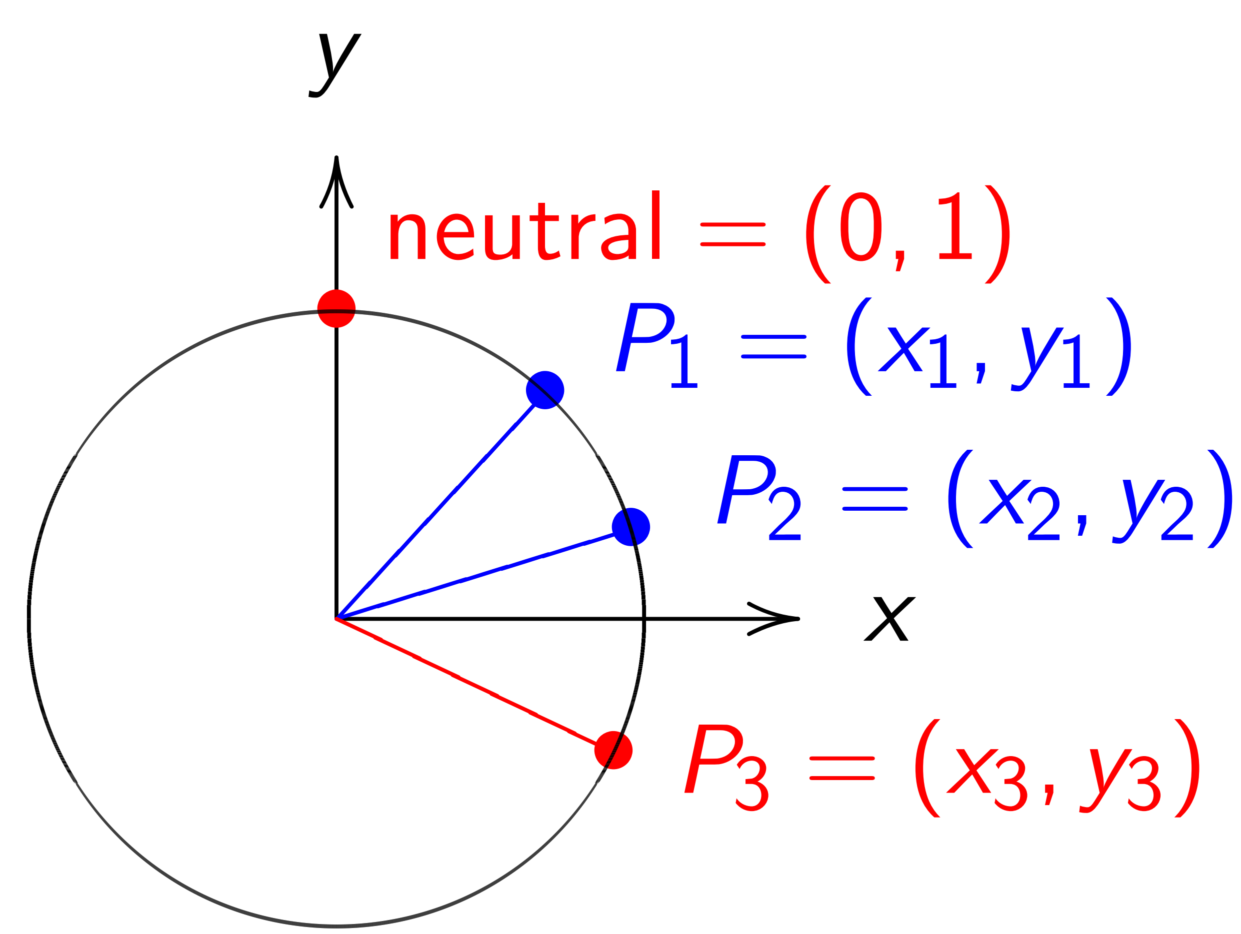

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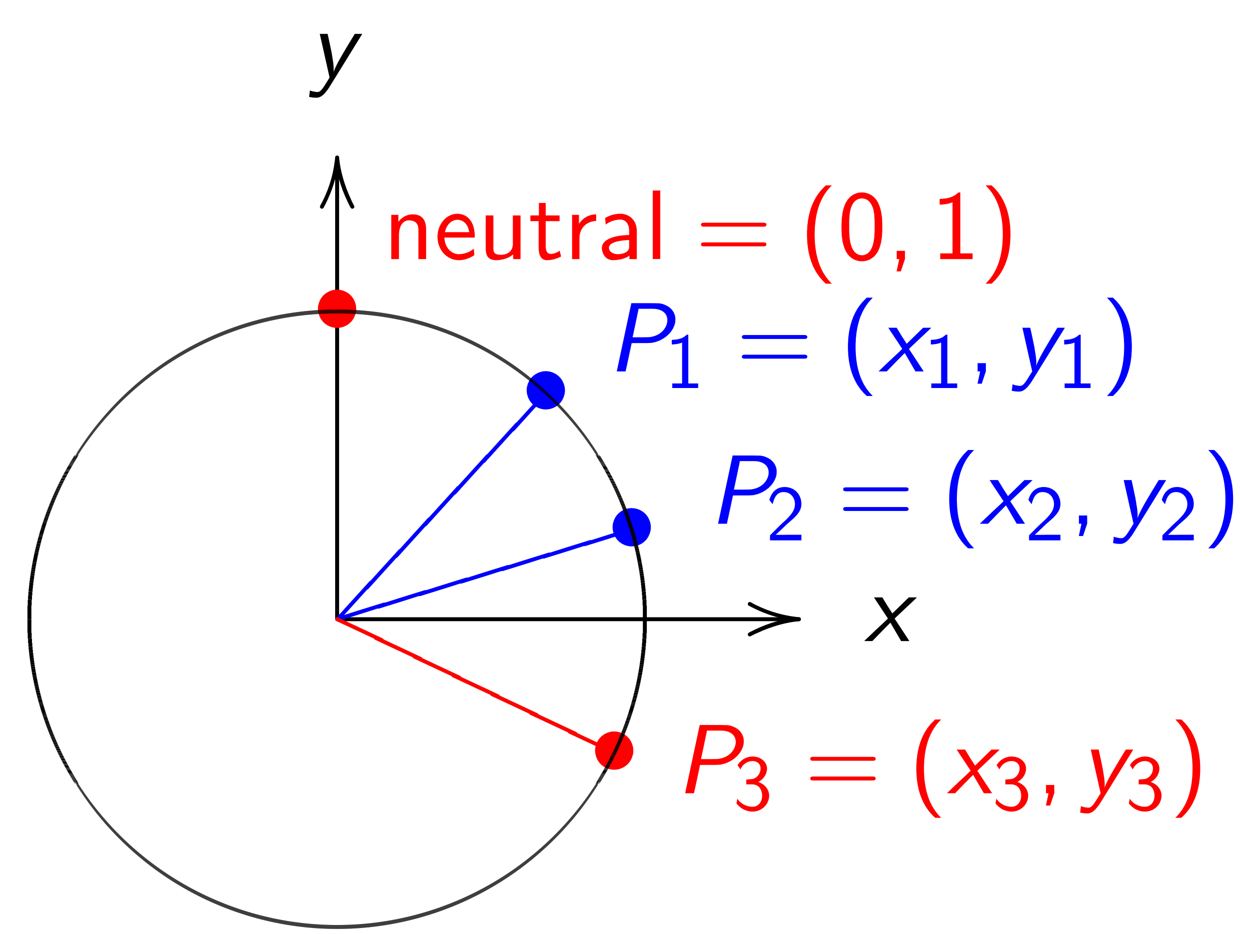

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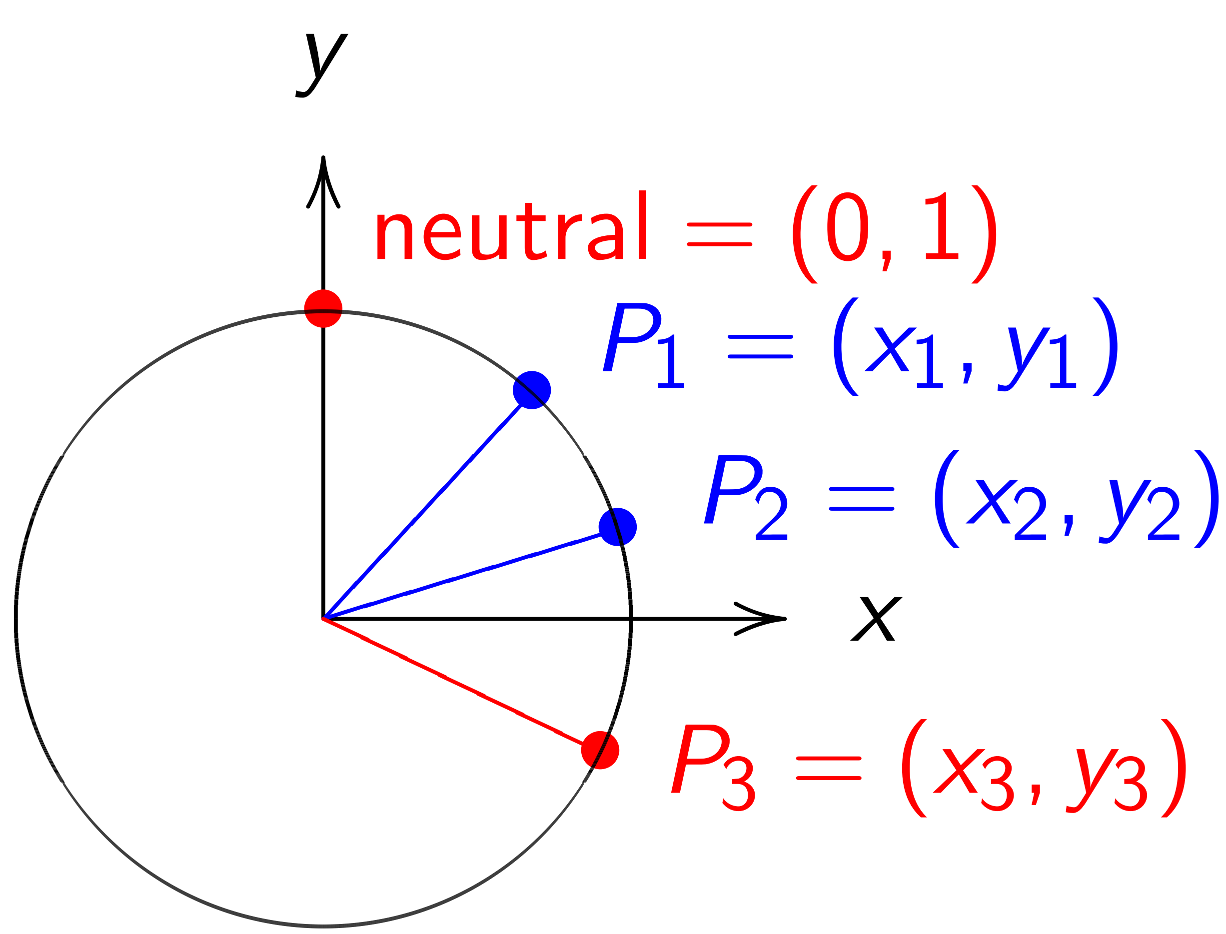

More clock perspectives:

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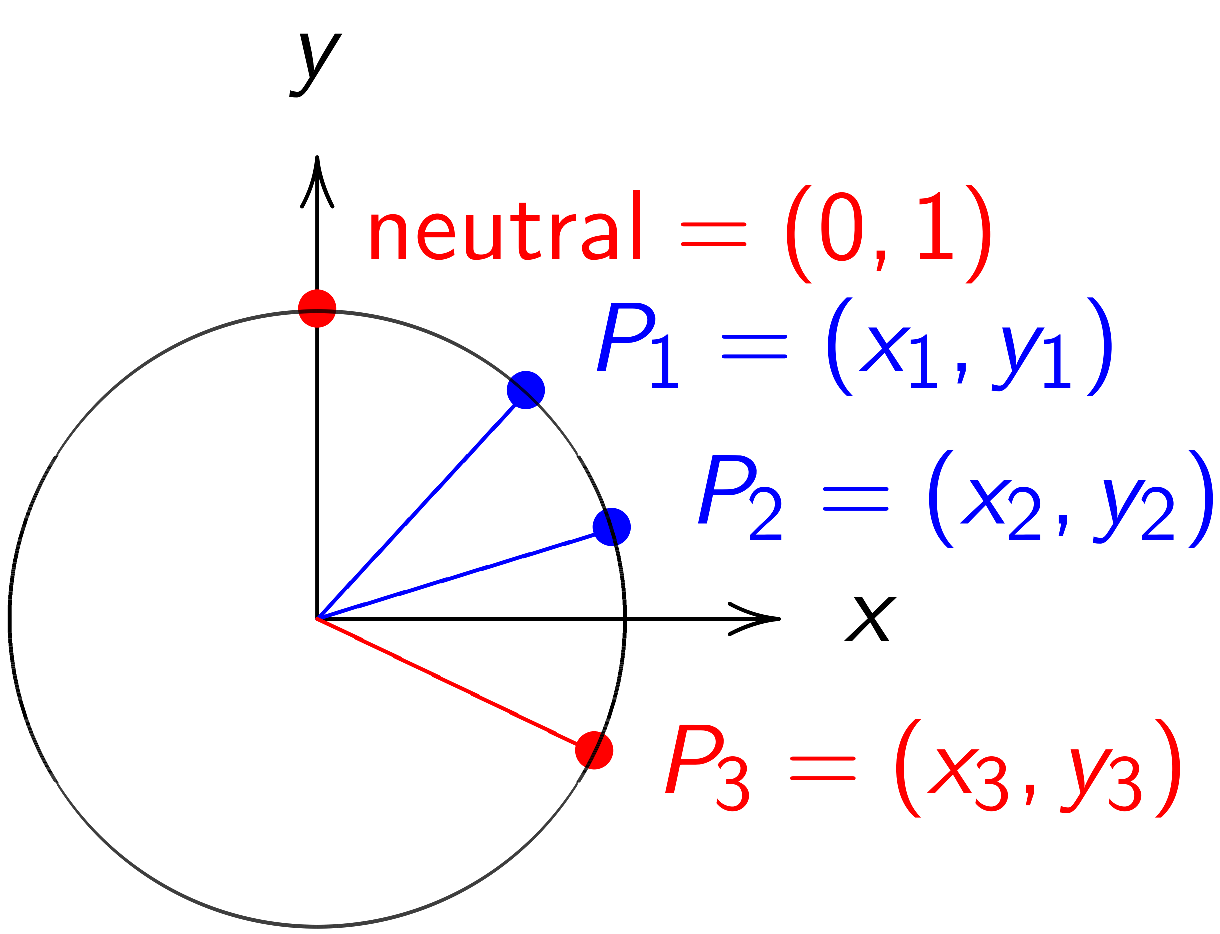

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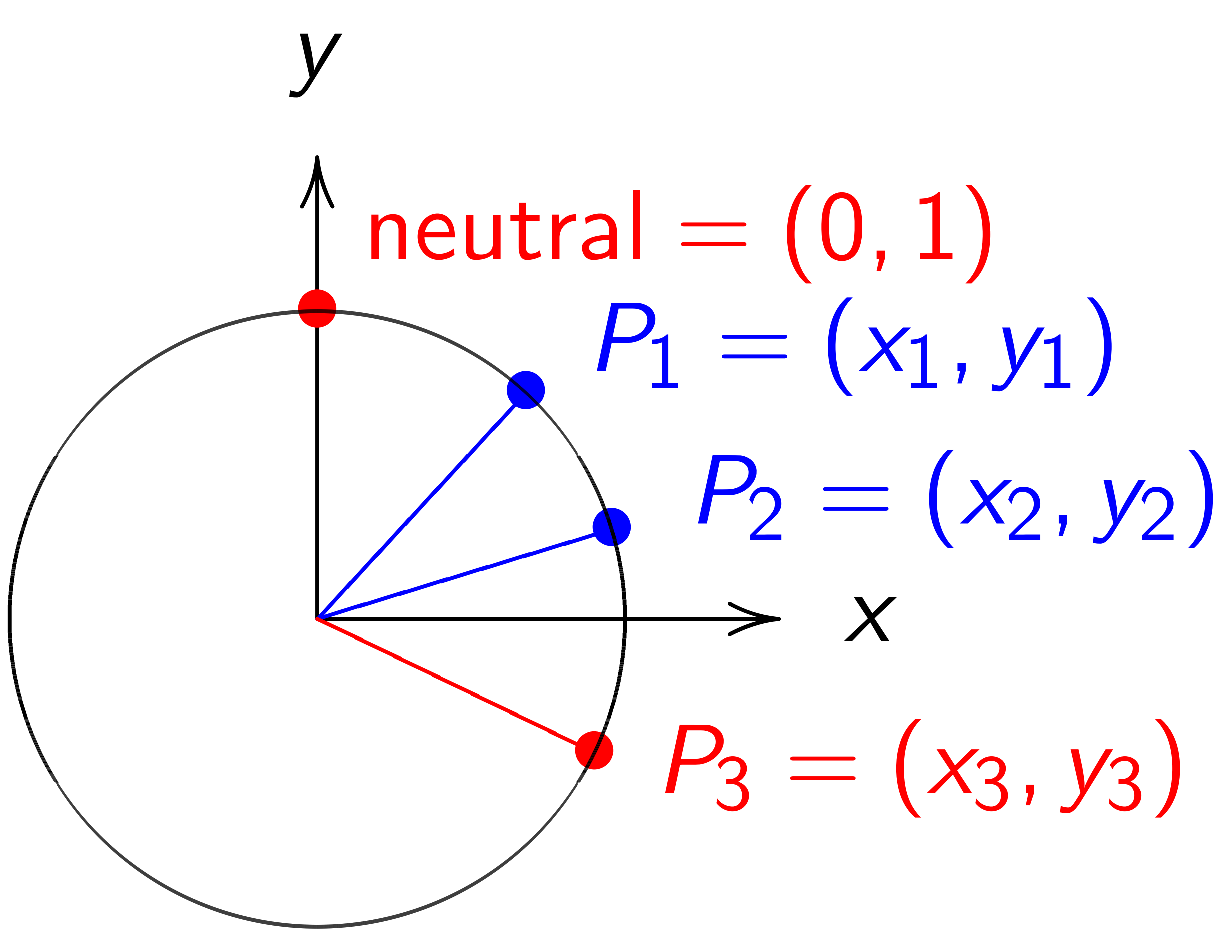

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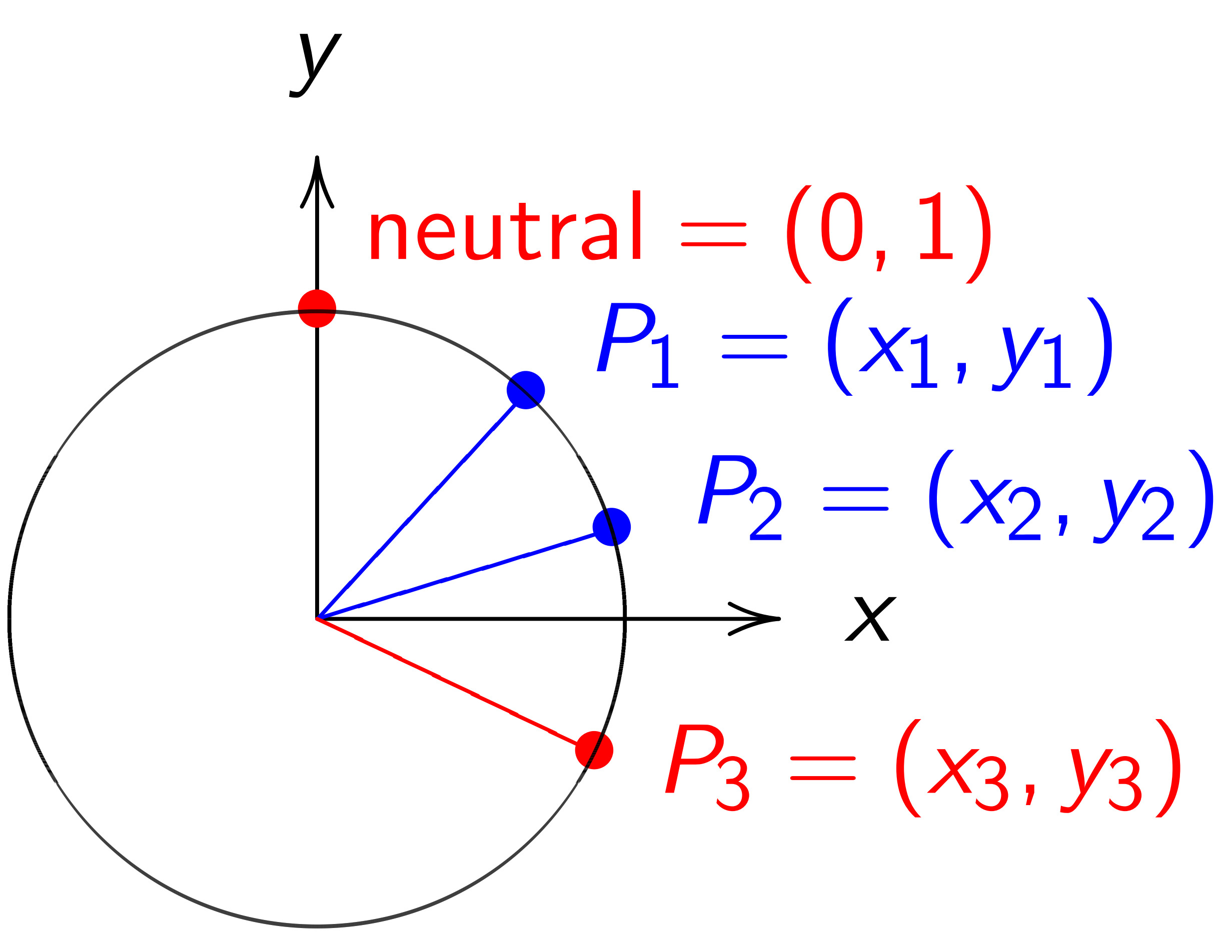

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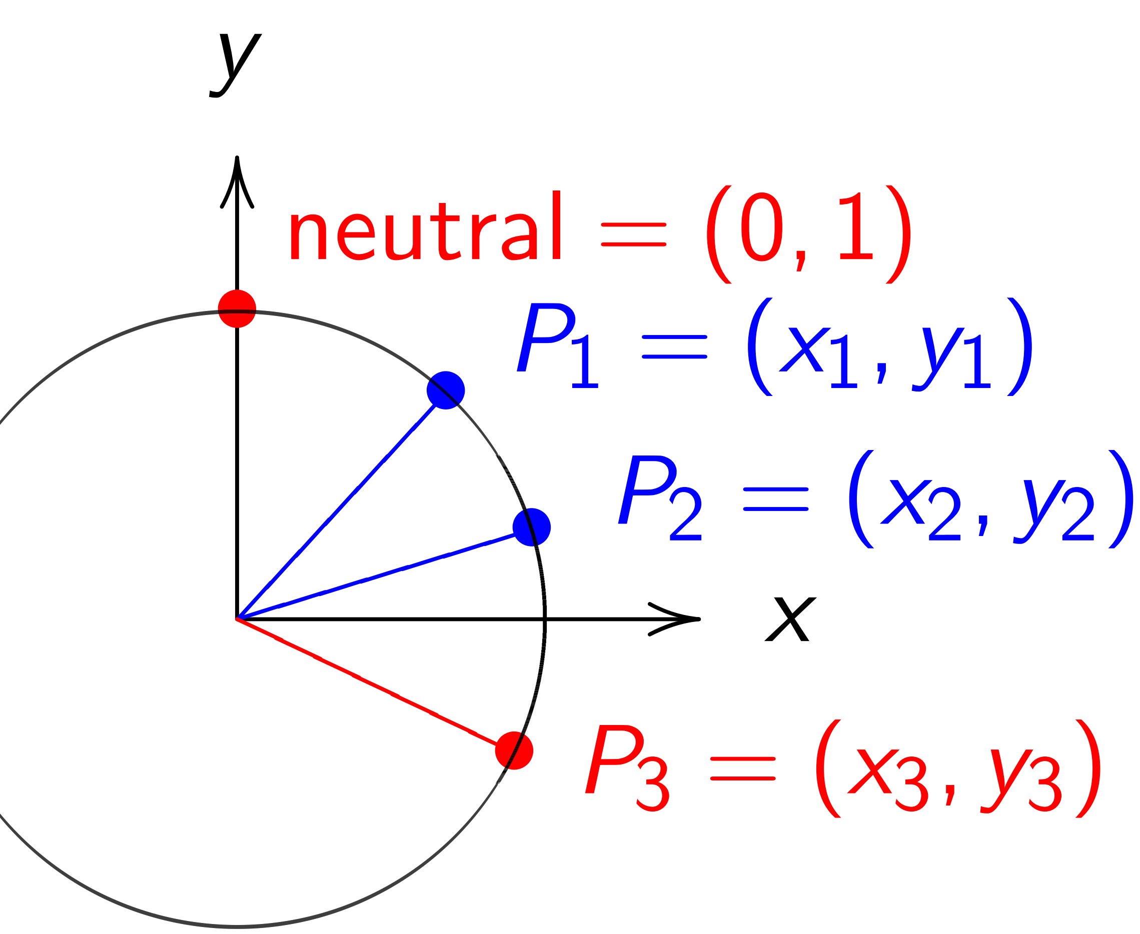

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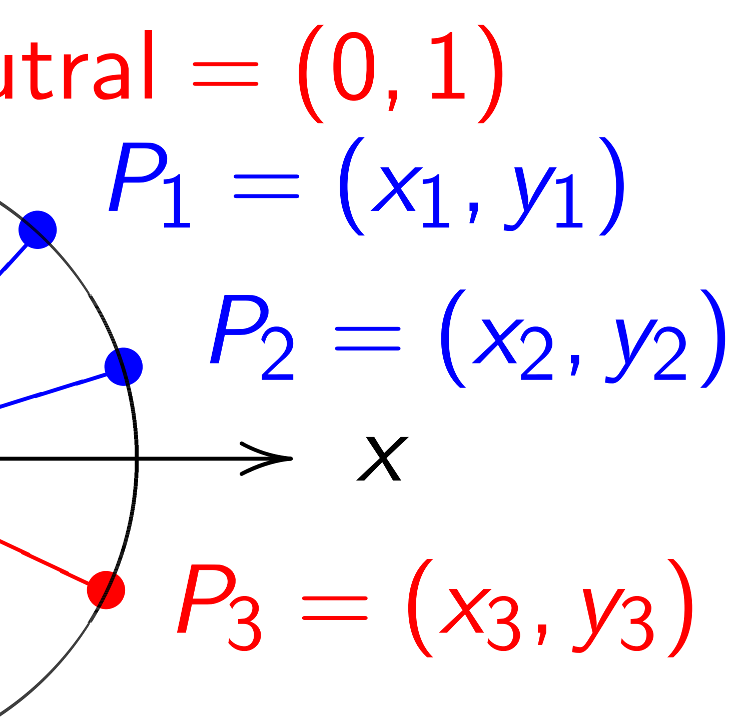

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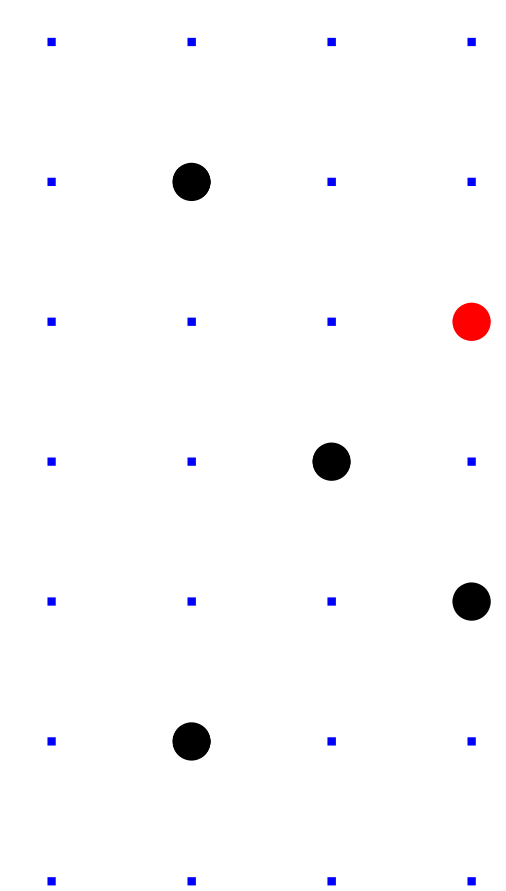

Diagram plots $\mathbf{F}_7$
$-3, -2, -1, 0, 1, 2$

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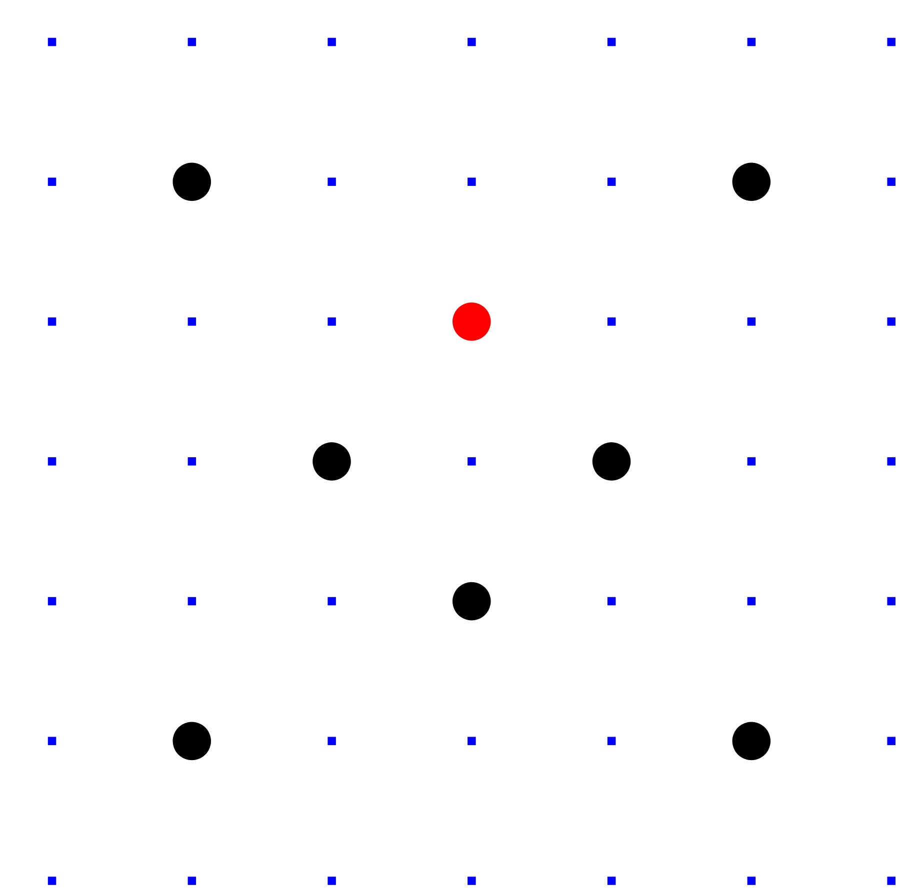

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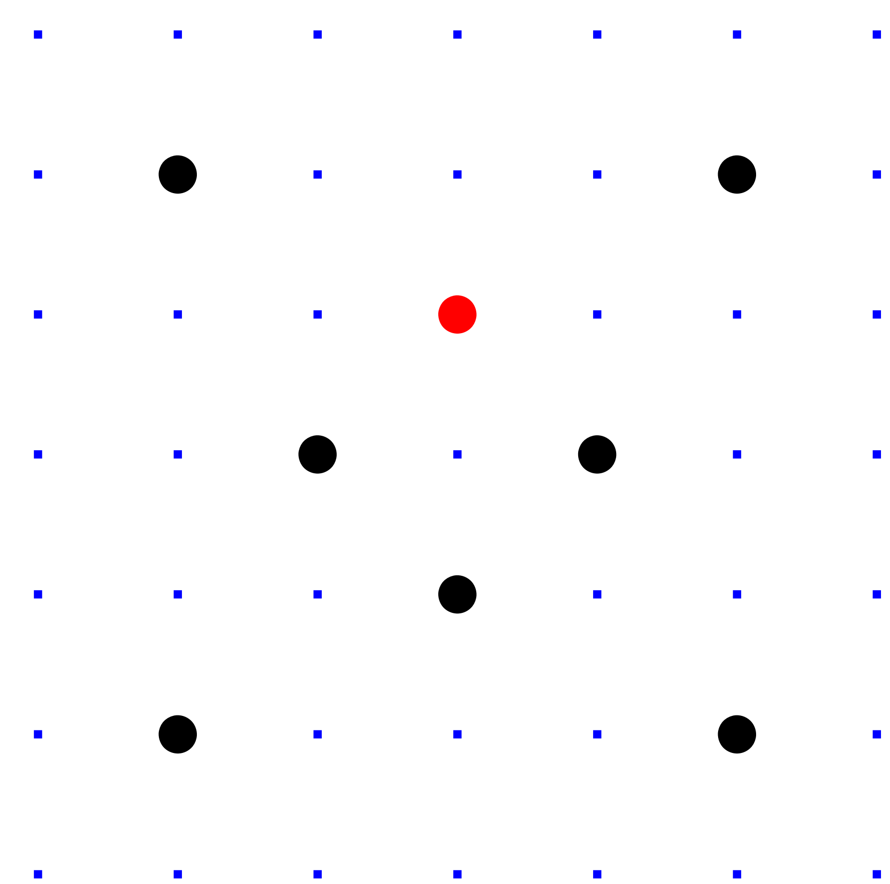

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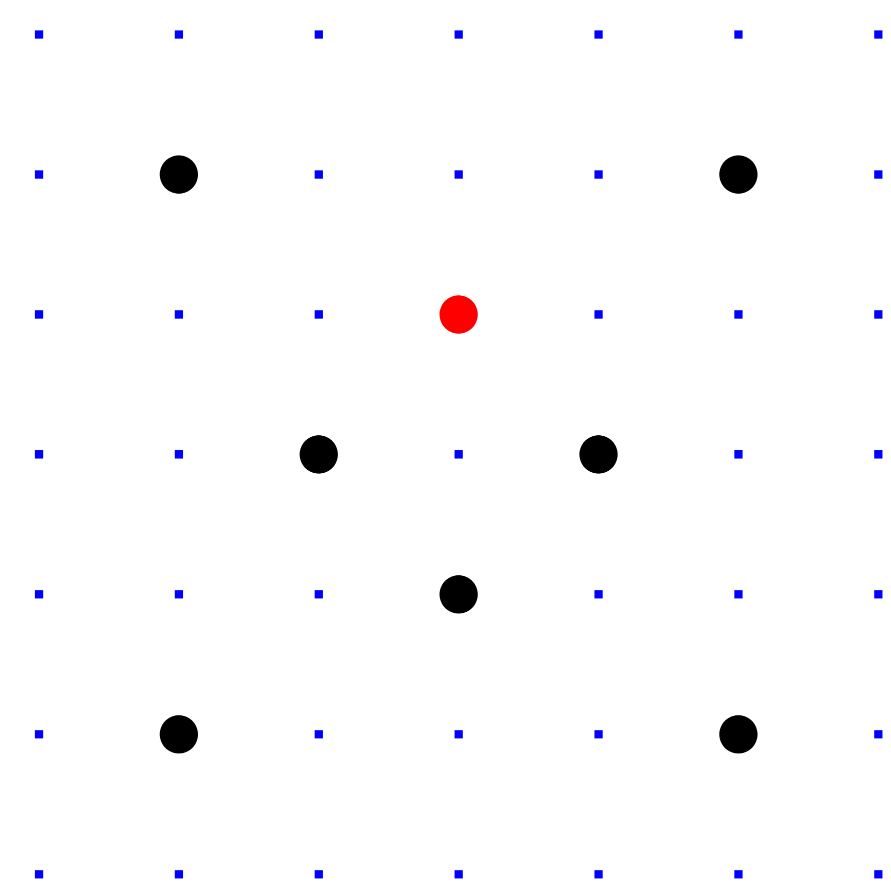

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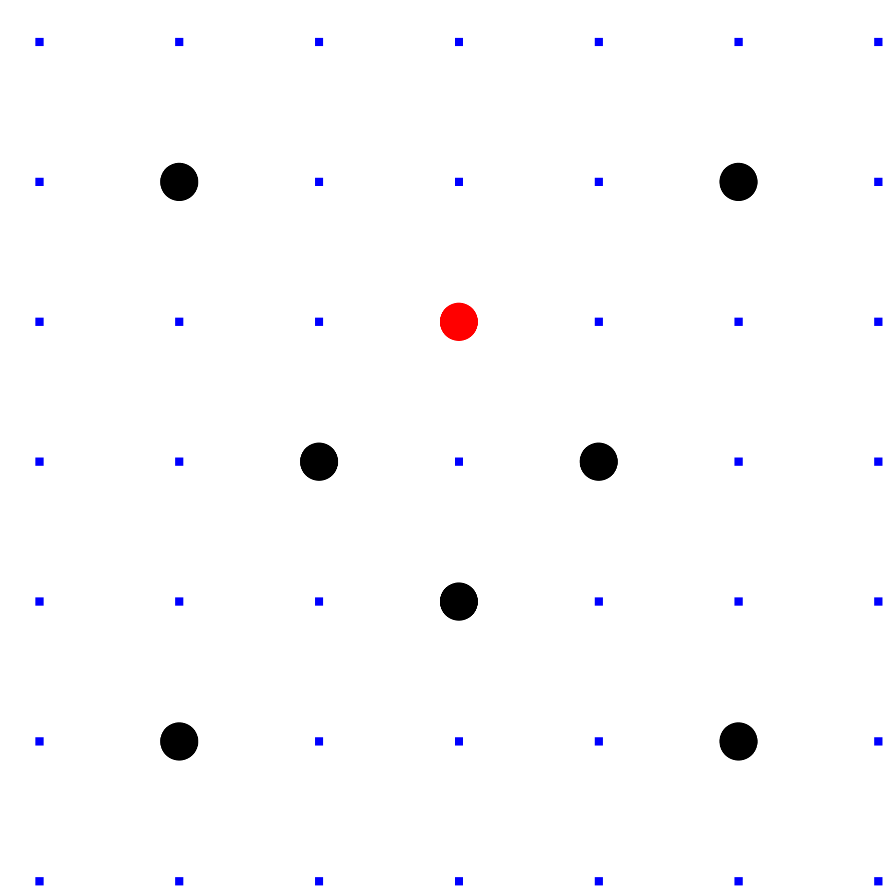

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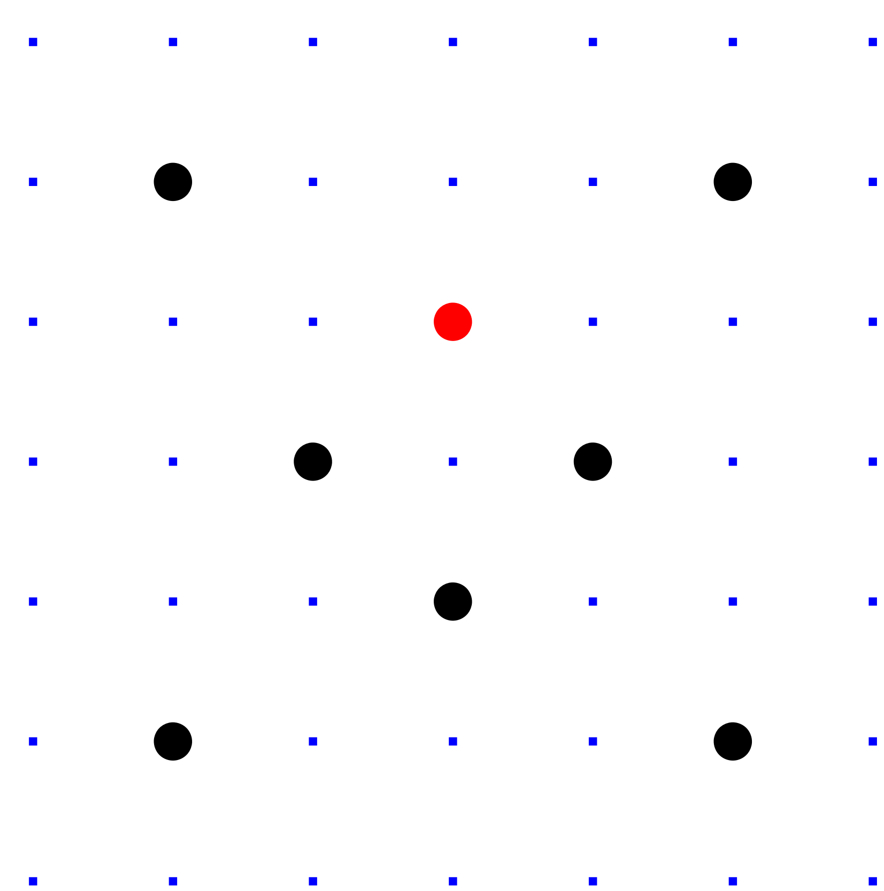

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Larger example: Clock($\mathbf{F}_{100}$

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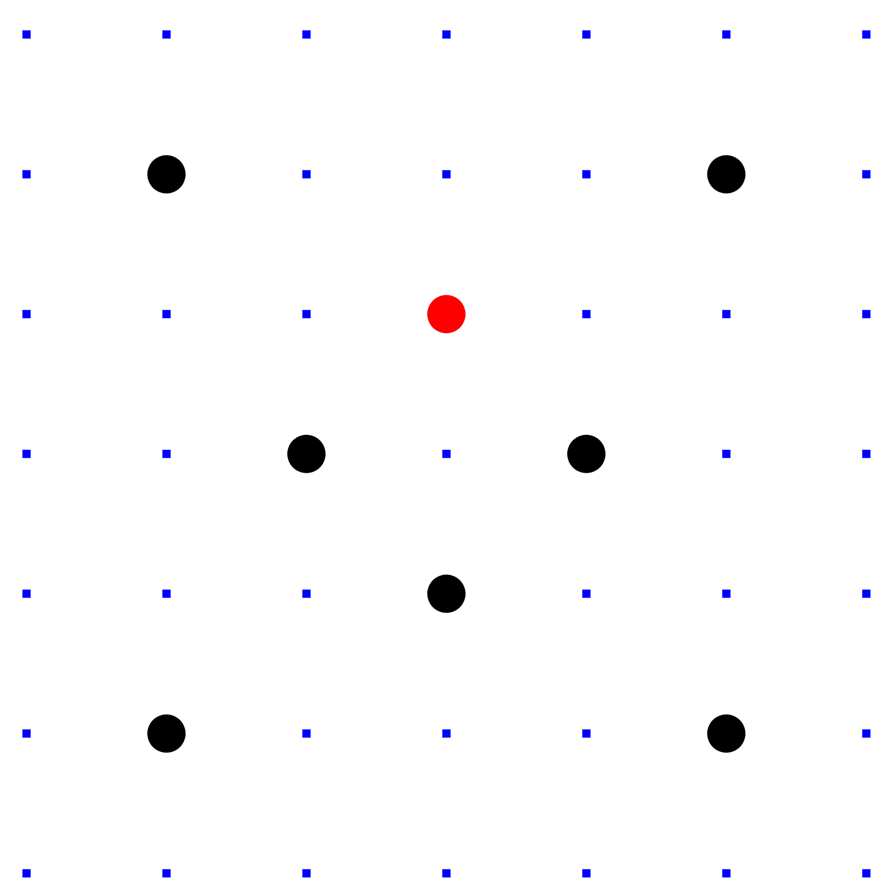

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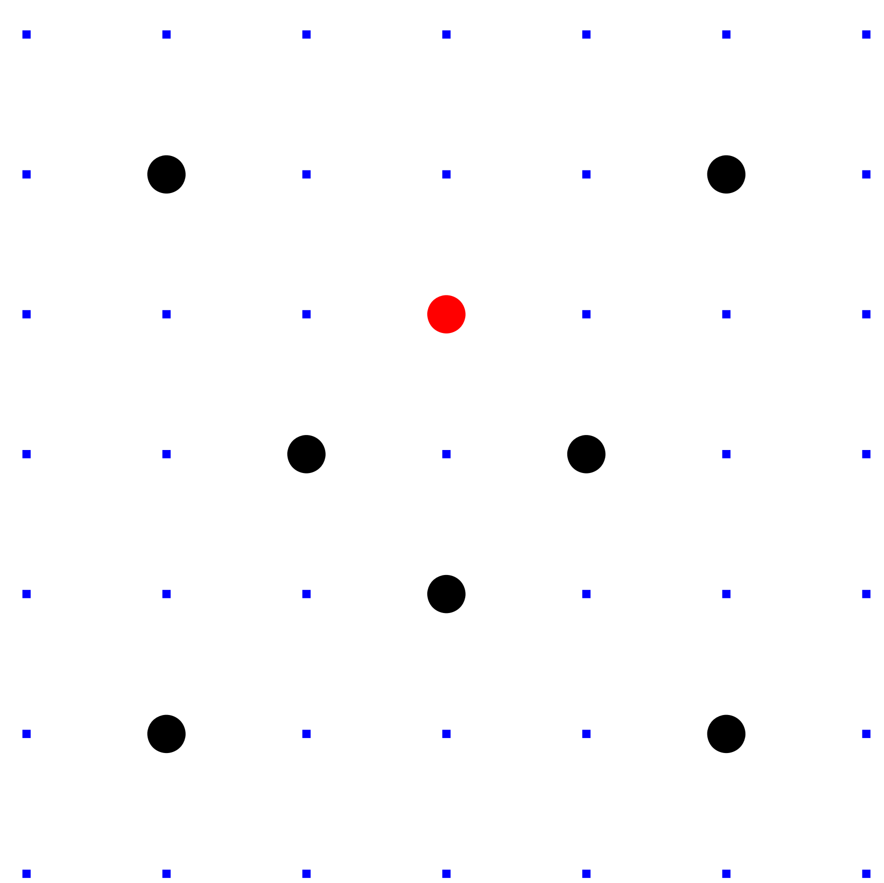

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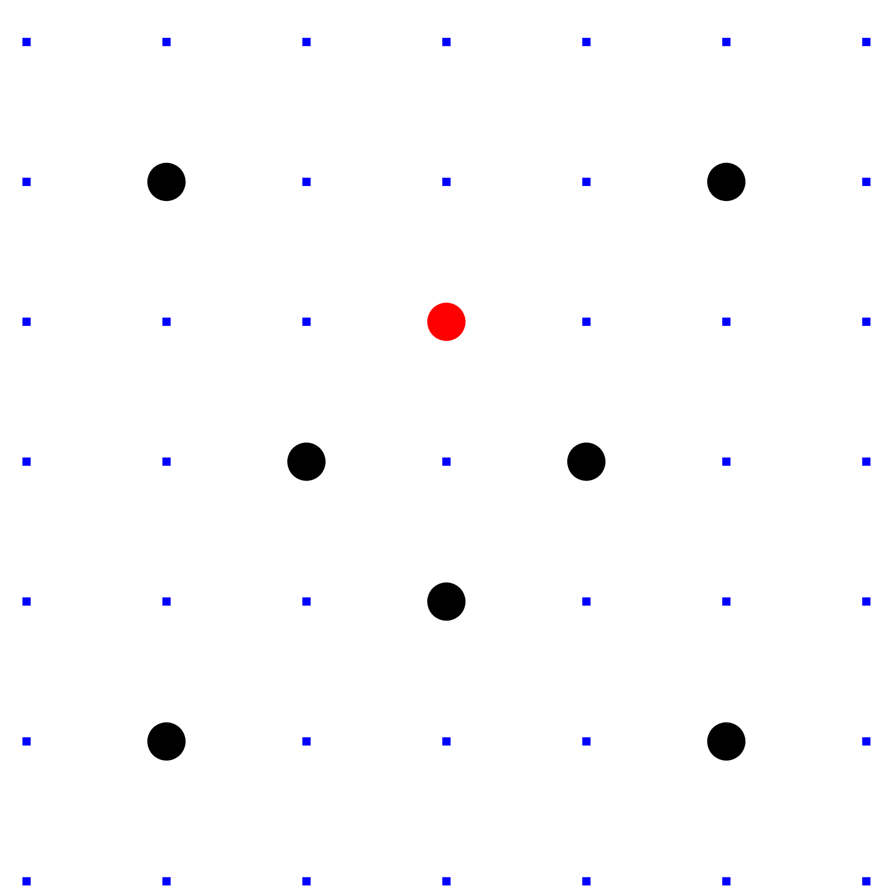

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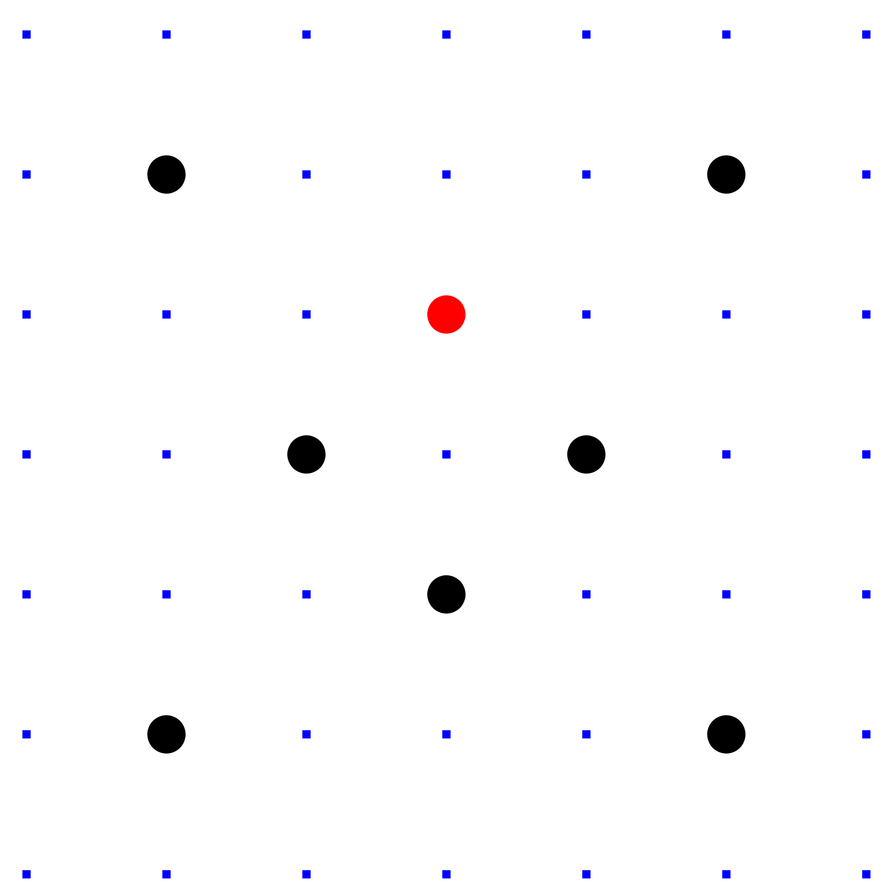

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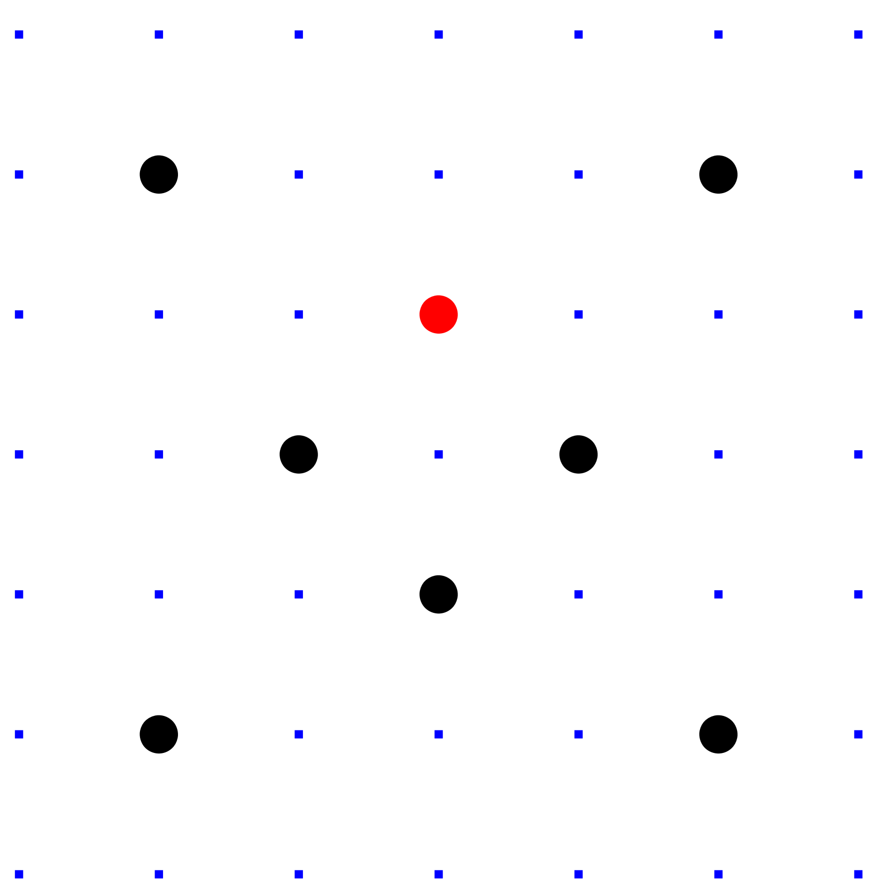

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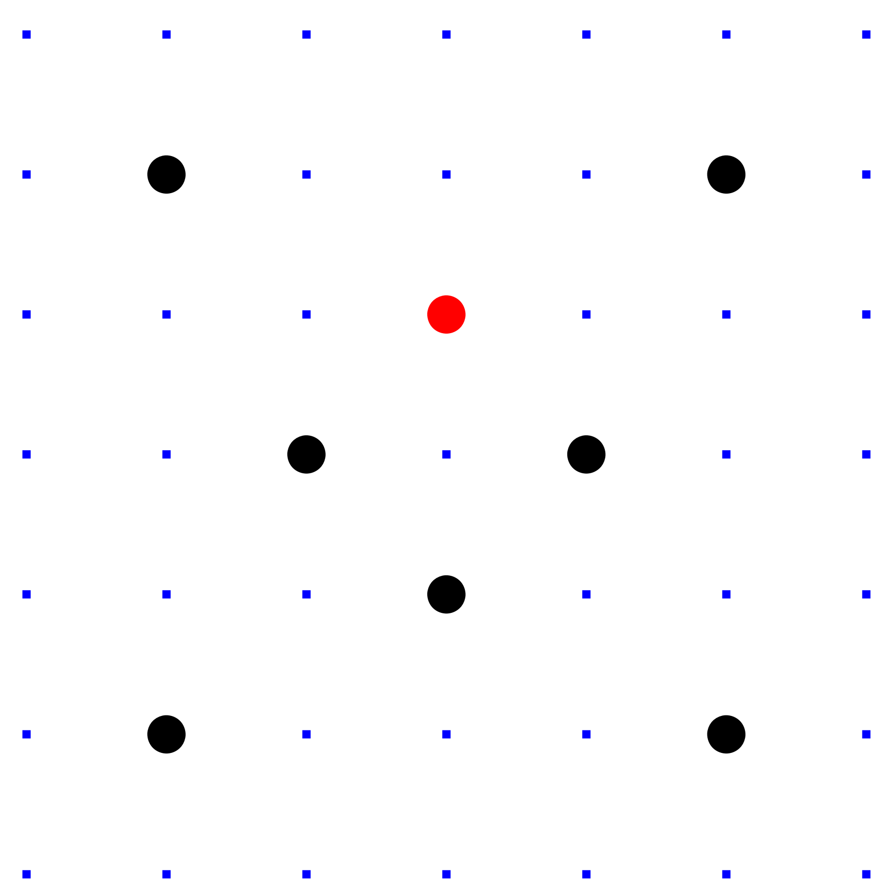

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We'll build cryptography
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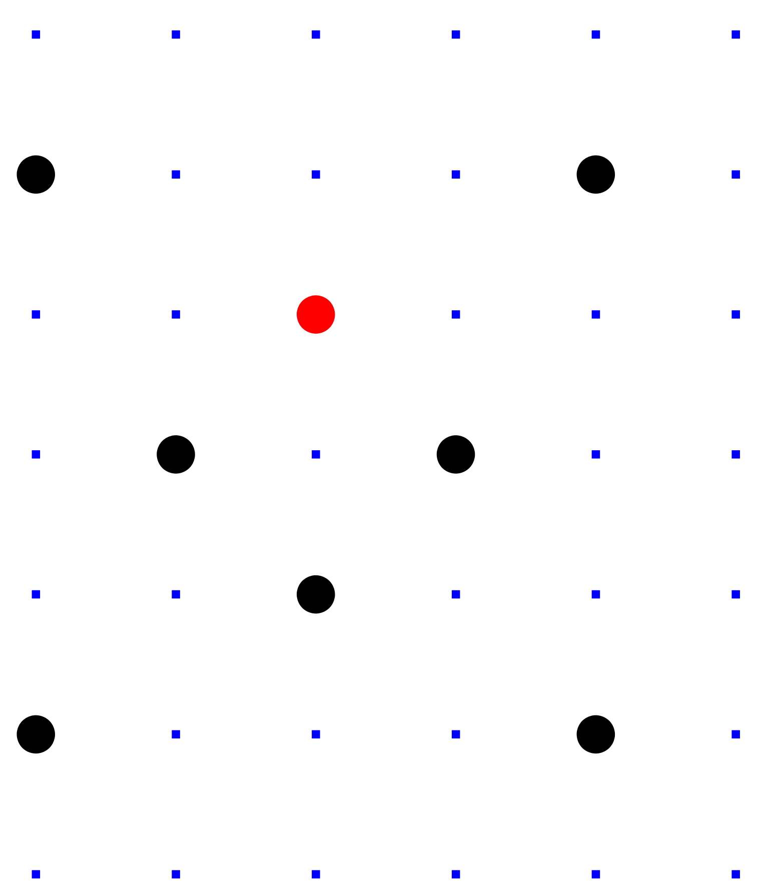

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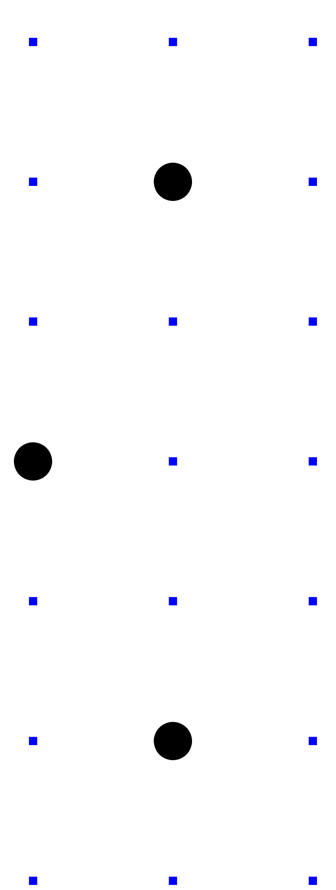

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"Scalar multiplication" maps
$\mathbf{Z} \times \text{Clock}(\mathbf{F}_q) \to \text{Clock}(\mathbf{F}_q)$
by $n, P \mapsto nP$.

We'll build cryptography
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negate $(-n)P$ if n
double $(n/2)P$ if n
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Larger example: Clock($\mathbf{F}_{1000003}$).

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30 clock additions produce
$n(1000, 2) = (947472, 736284)$
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Can you figure out n?

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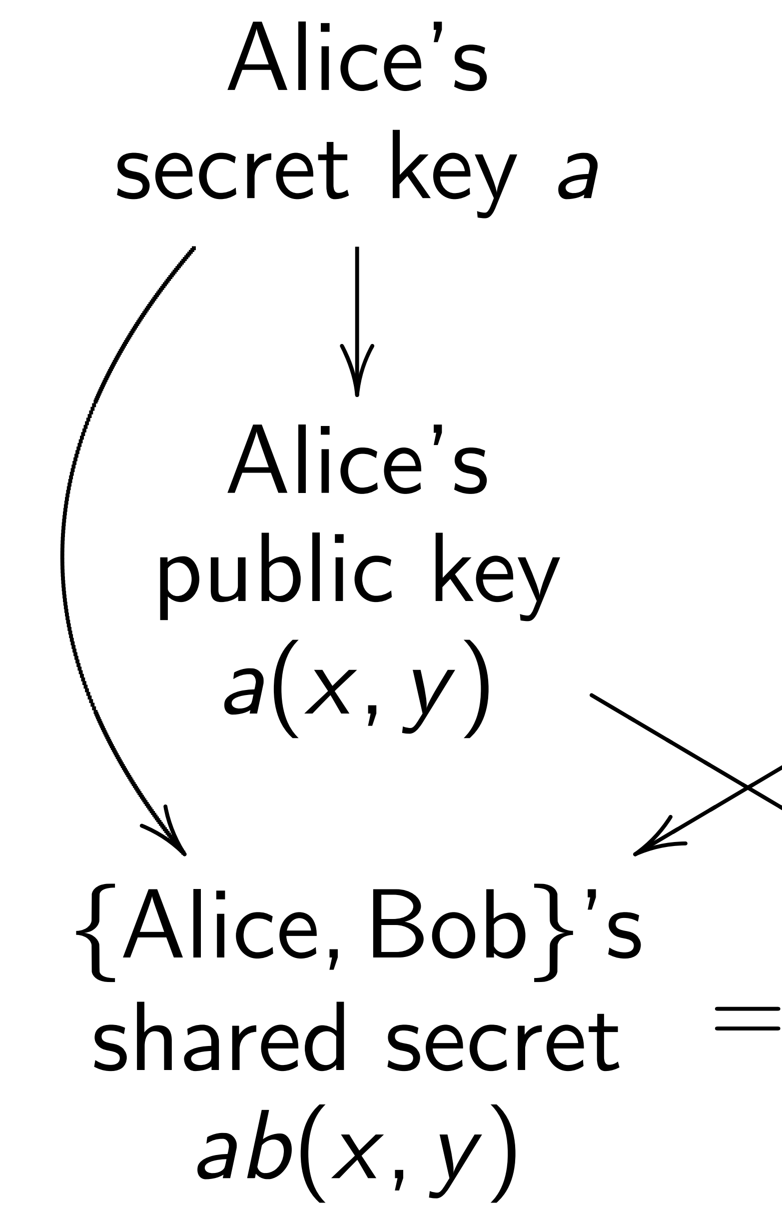

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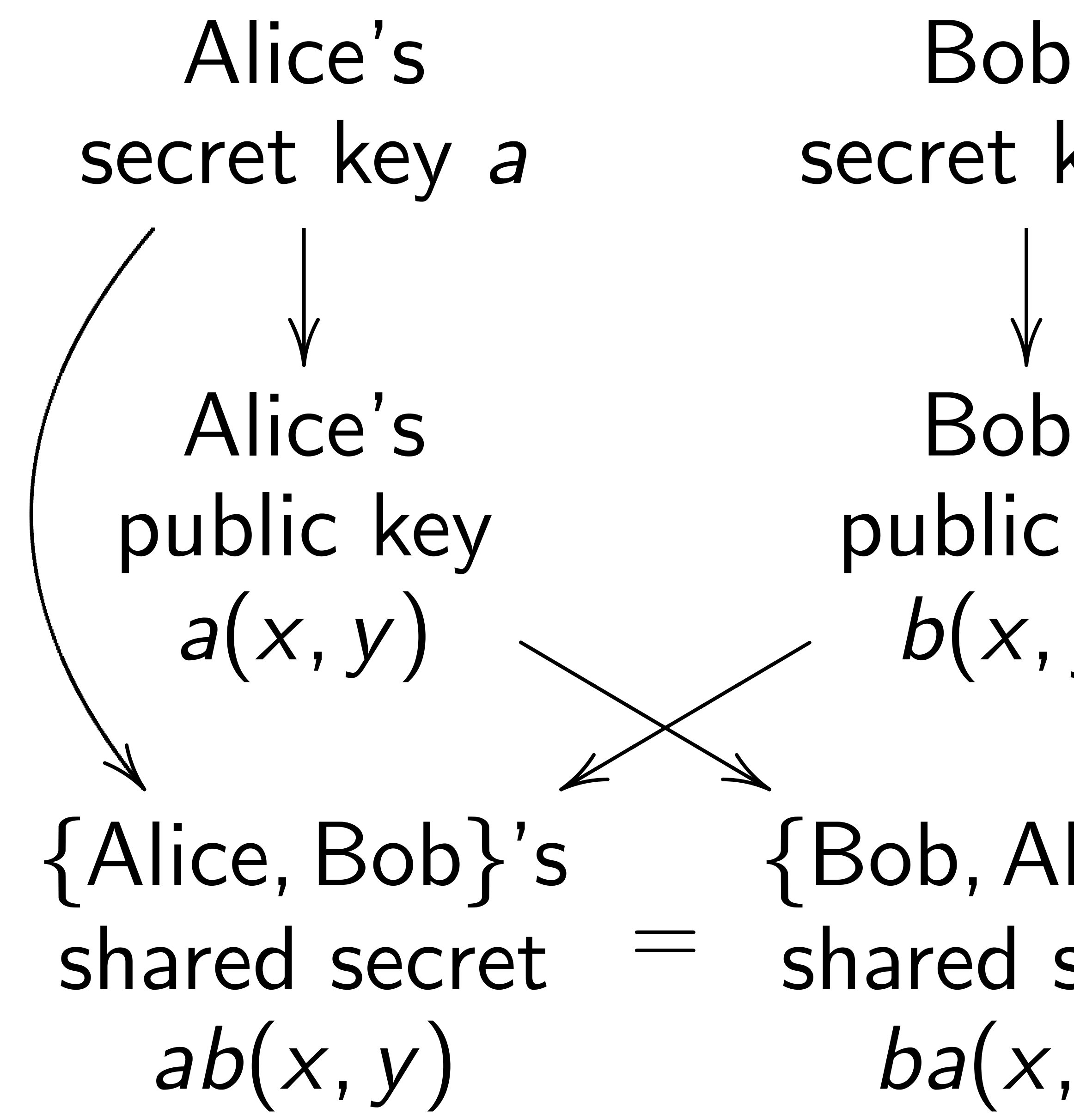

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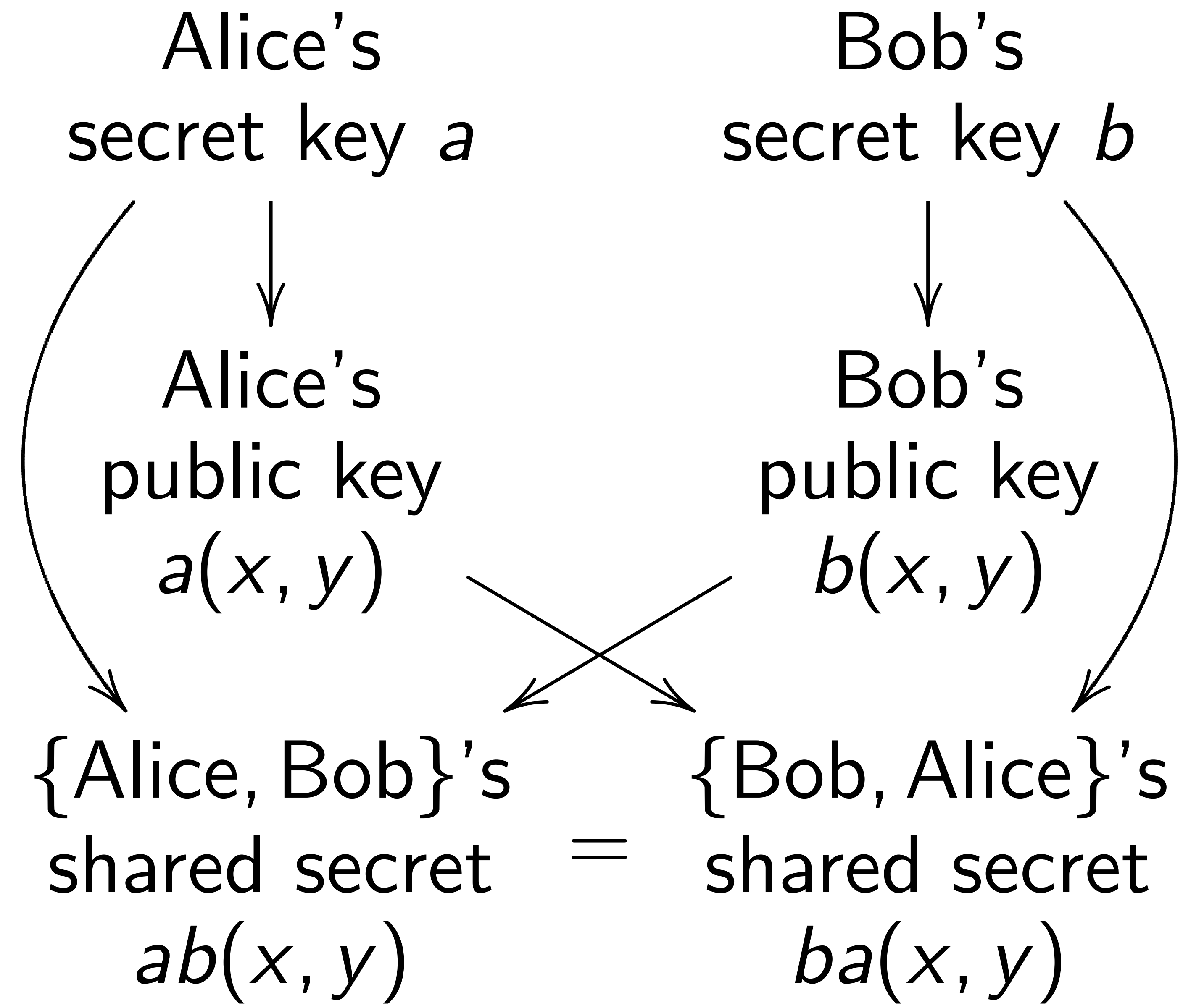

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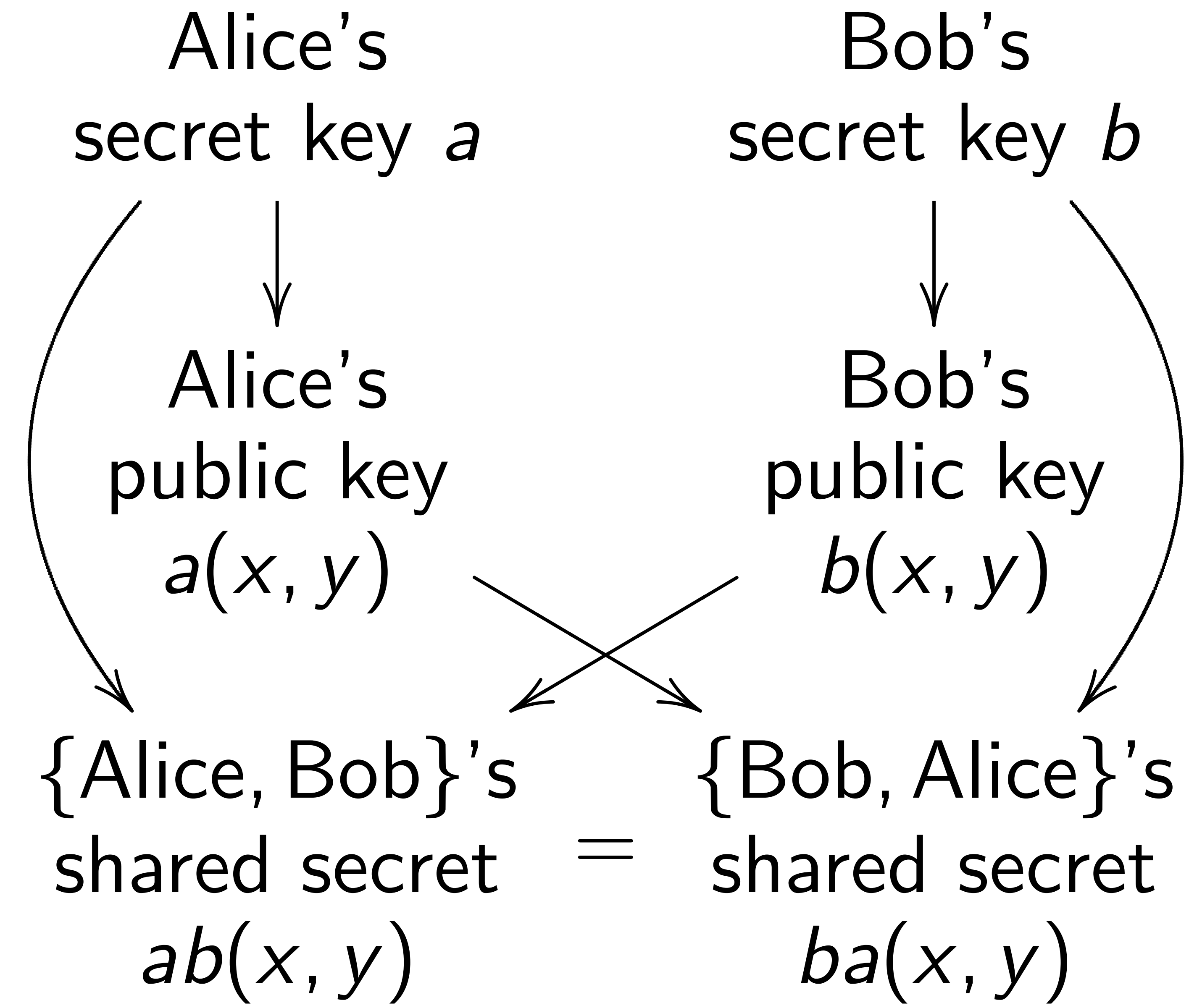

Need surprisingly large q to avoid state-of-the-art attacks. Recommendation: $q > 2^{1500}$. Better: Switch to elliptic curves.

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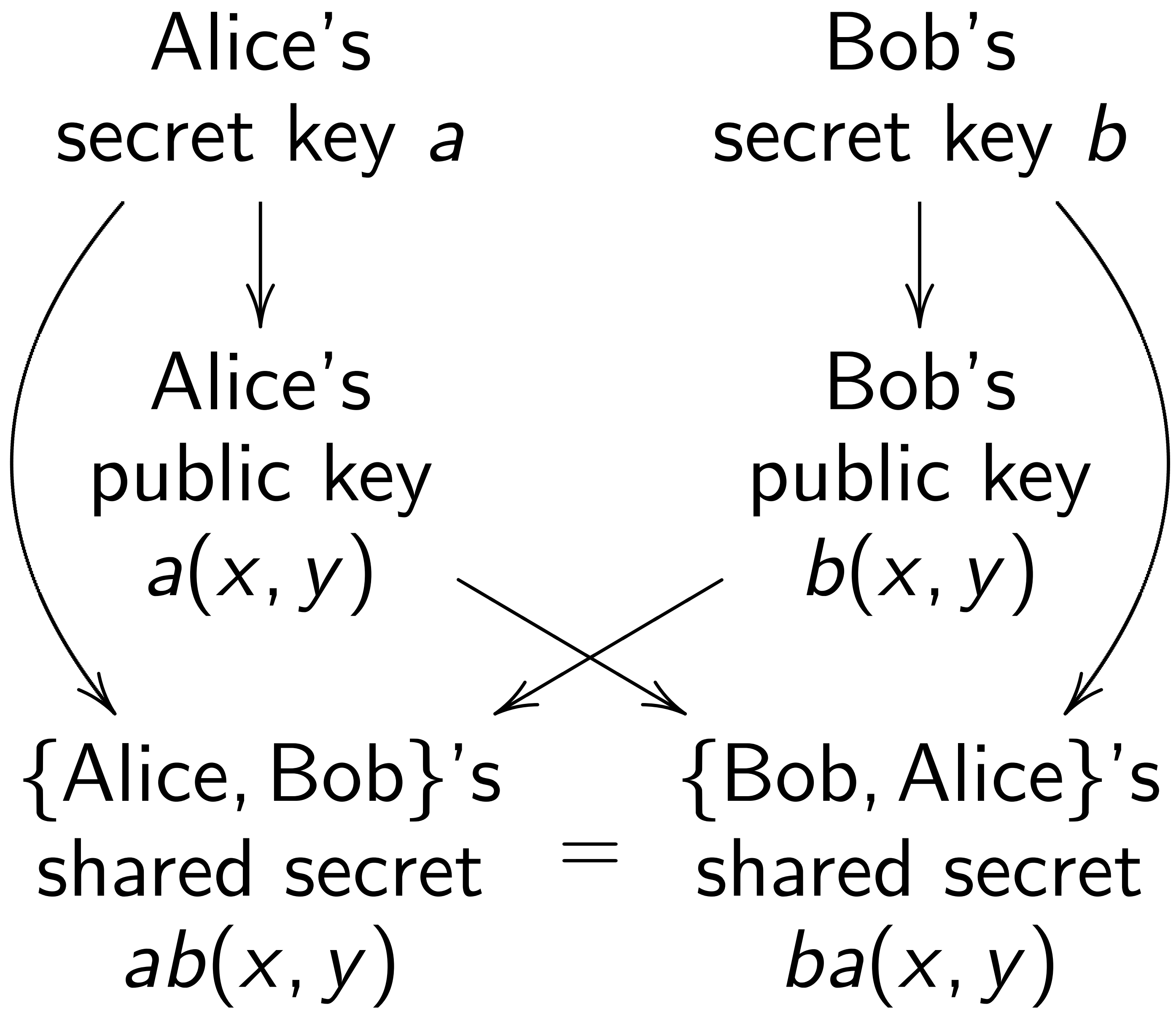

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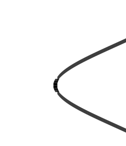

$x^2 + y^2$

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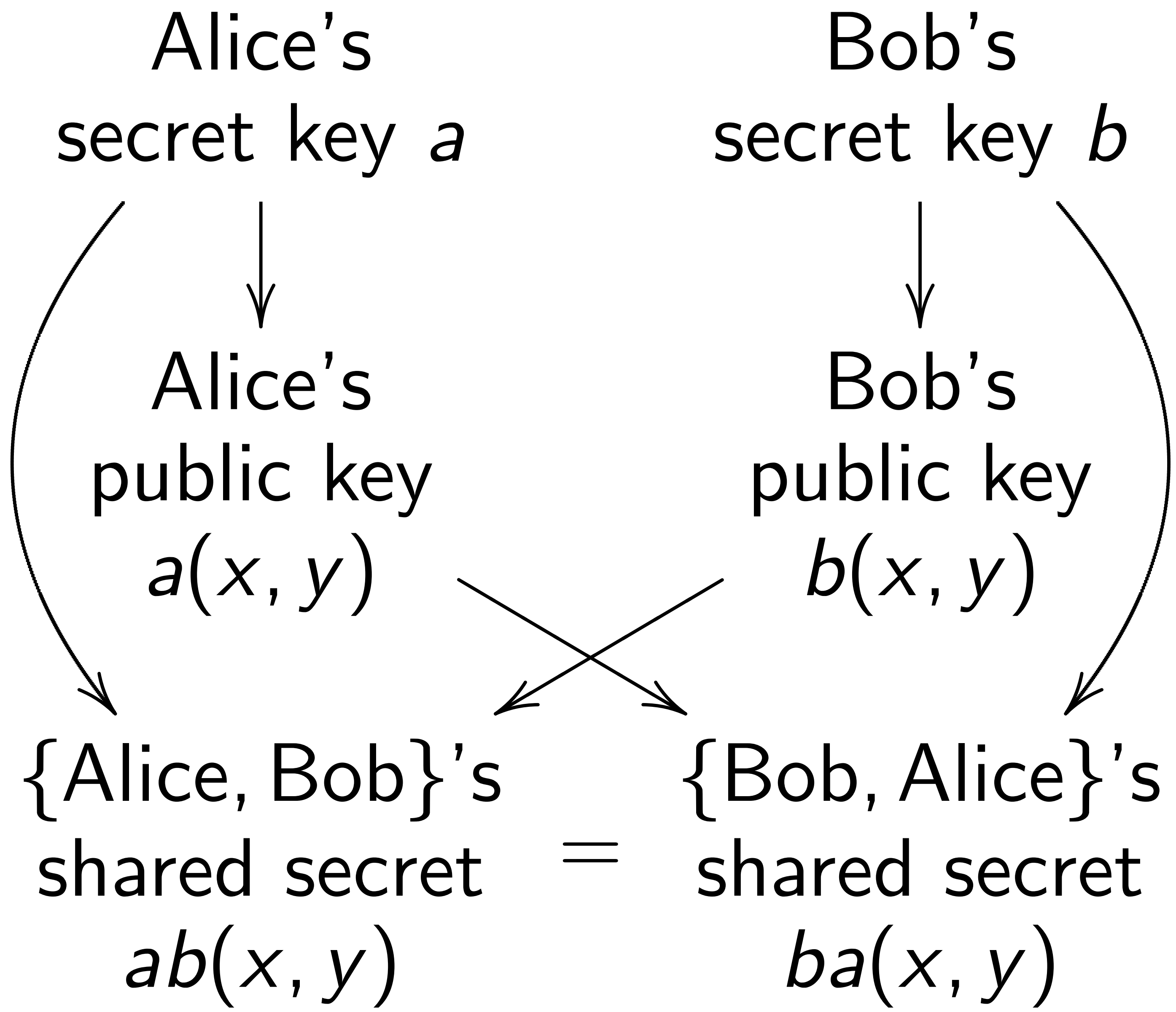

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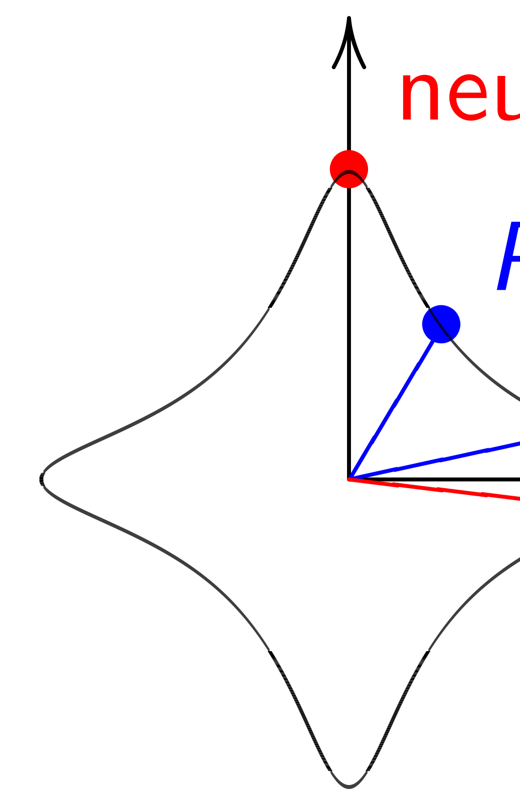

$x^2 + y^2 = 1 - 30x$

Sum of (x_1, y_1) an

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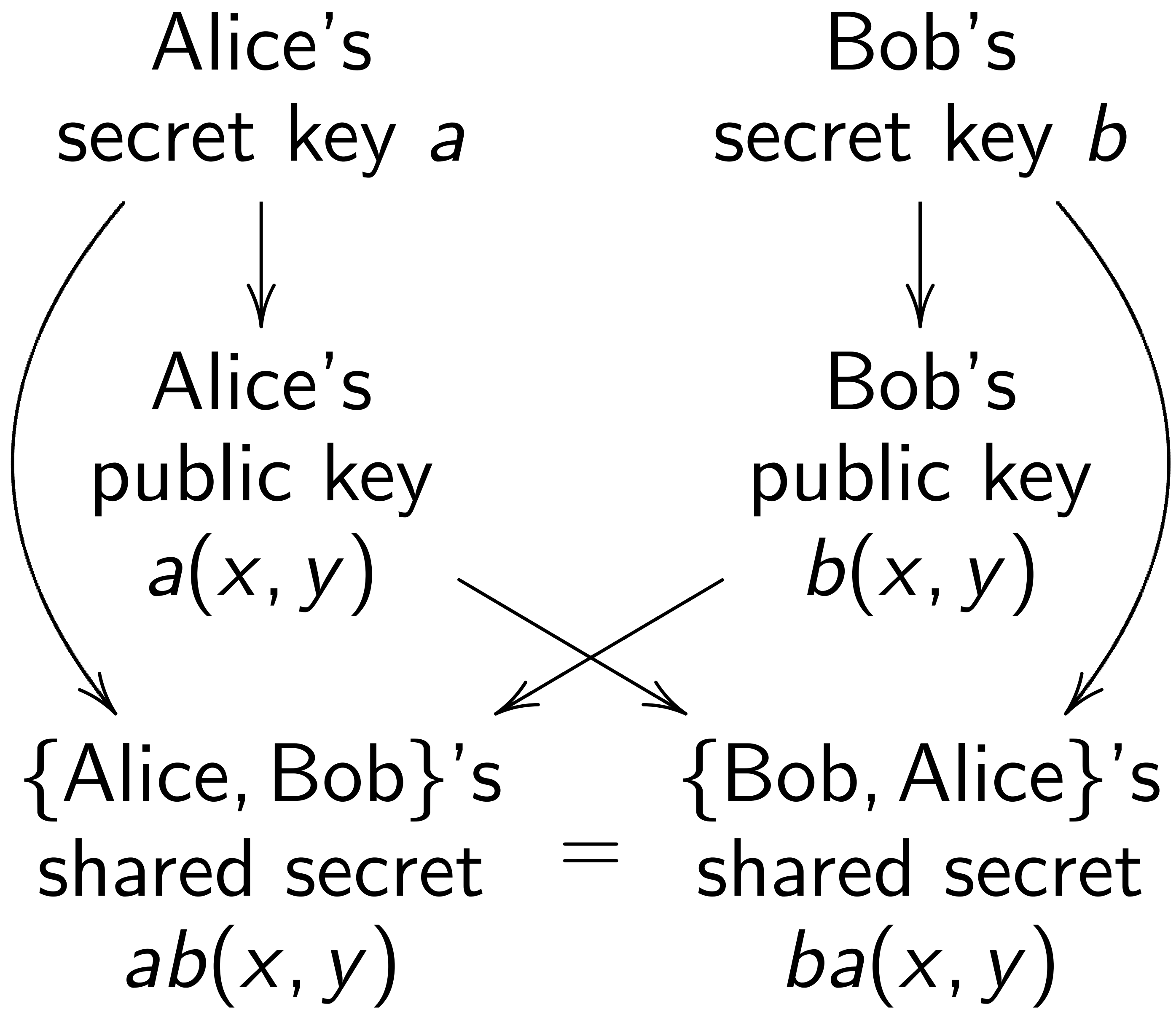

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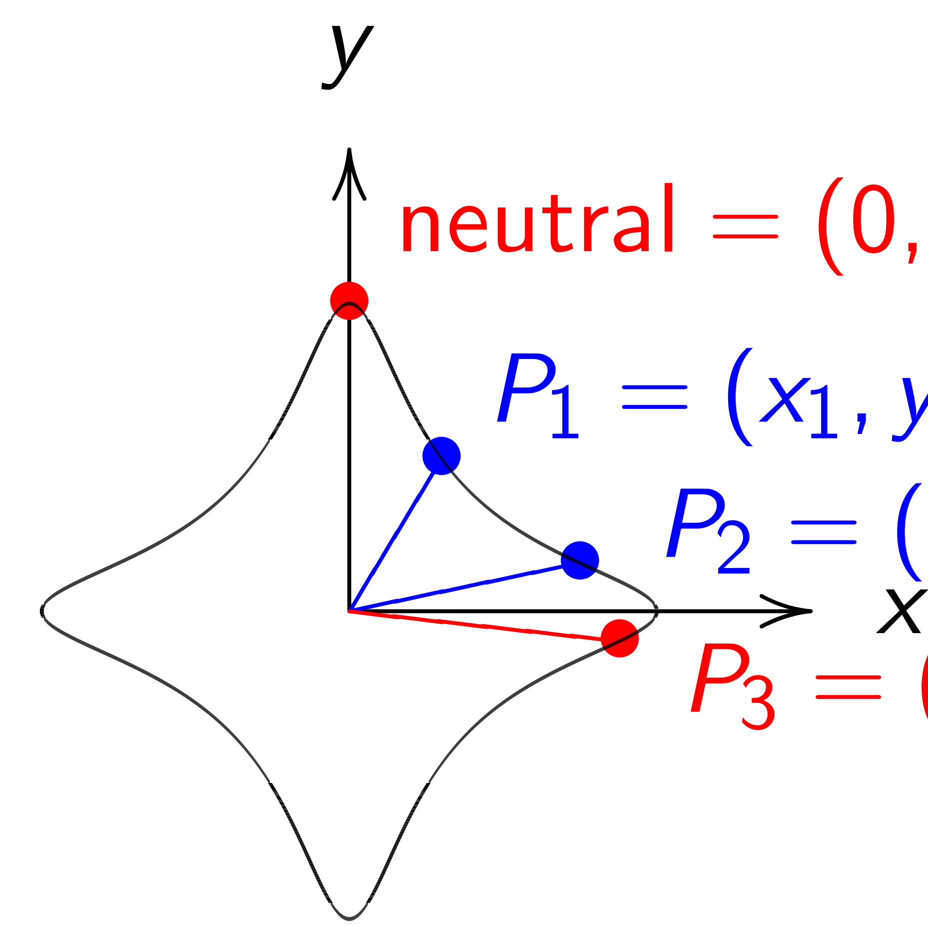

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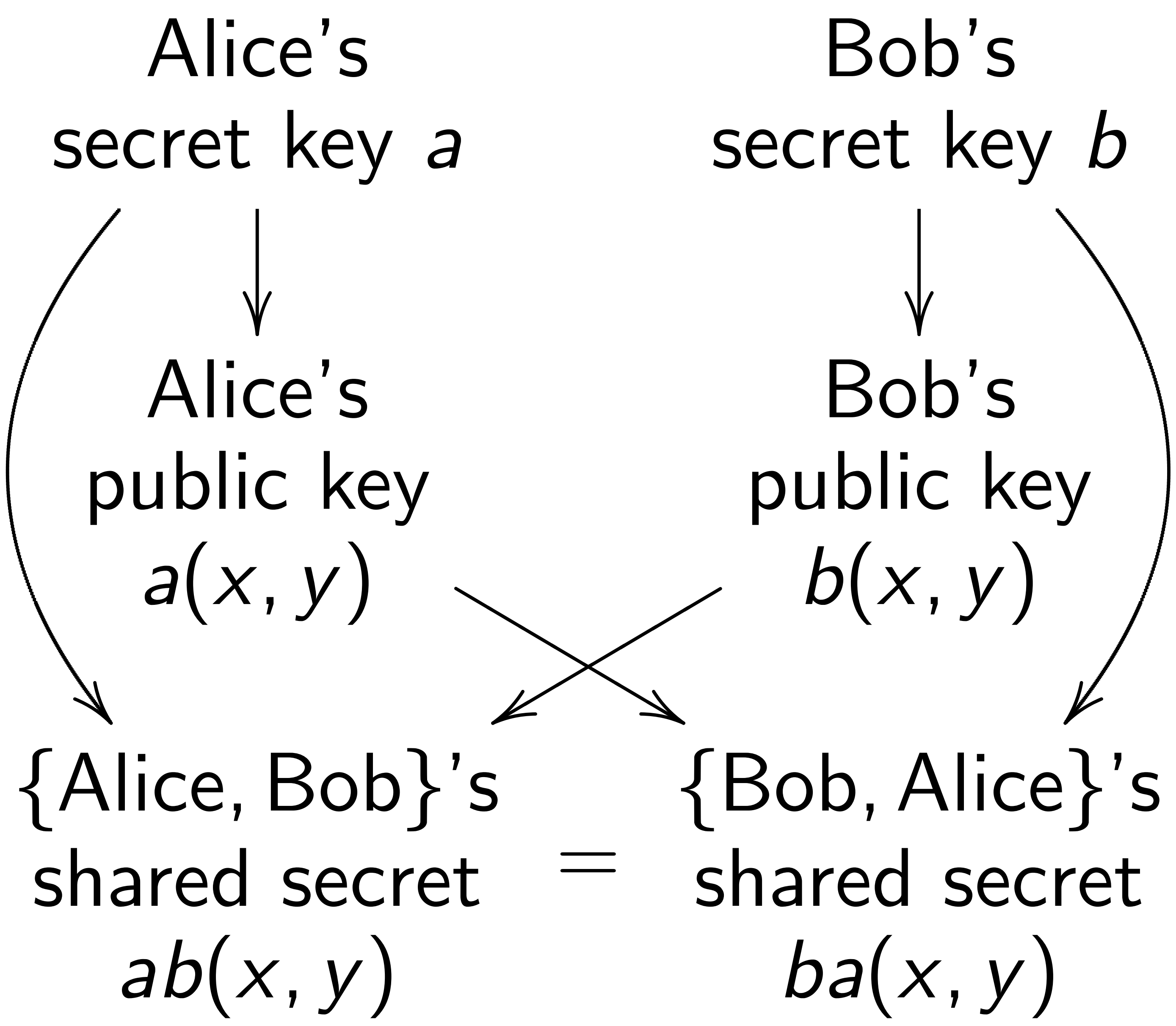

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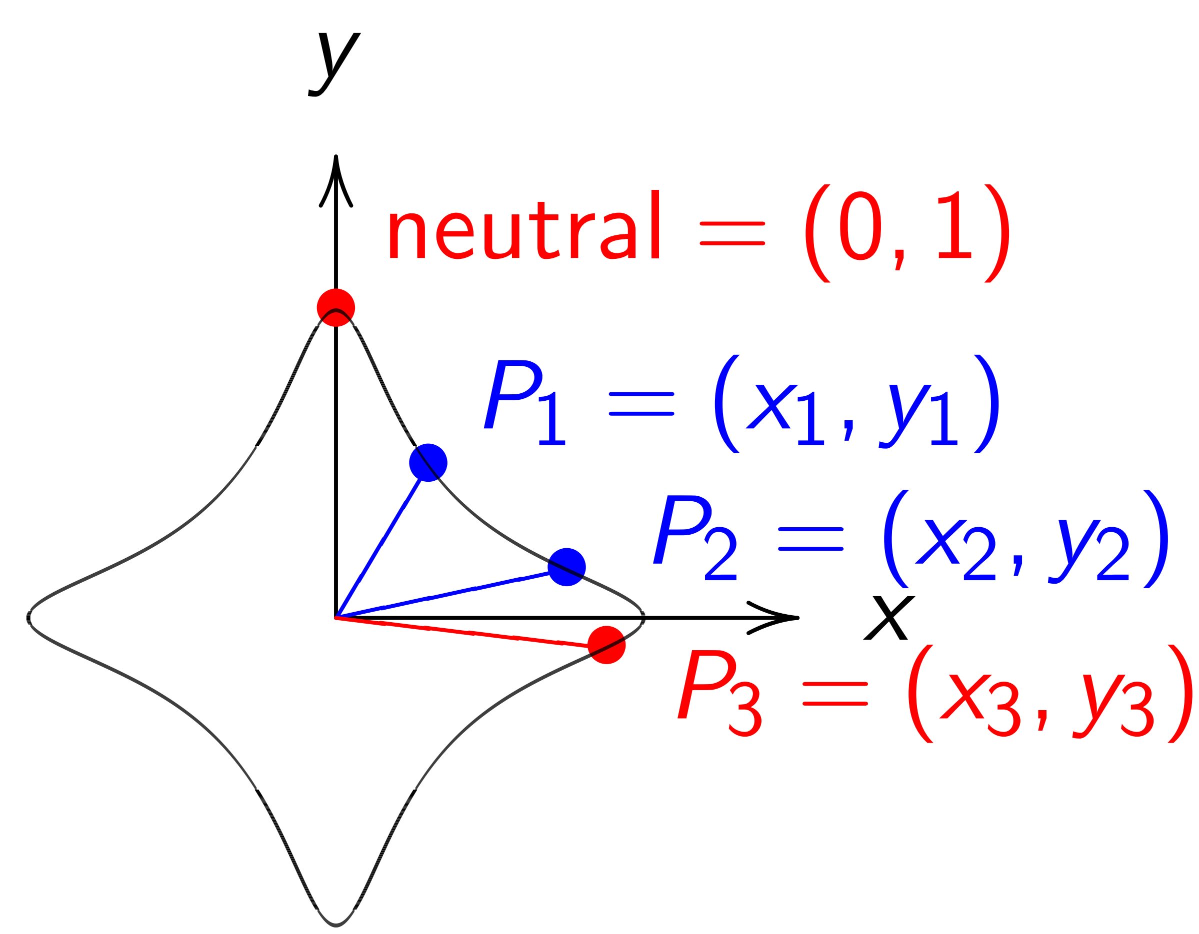

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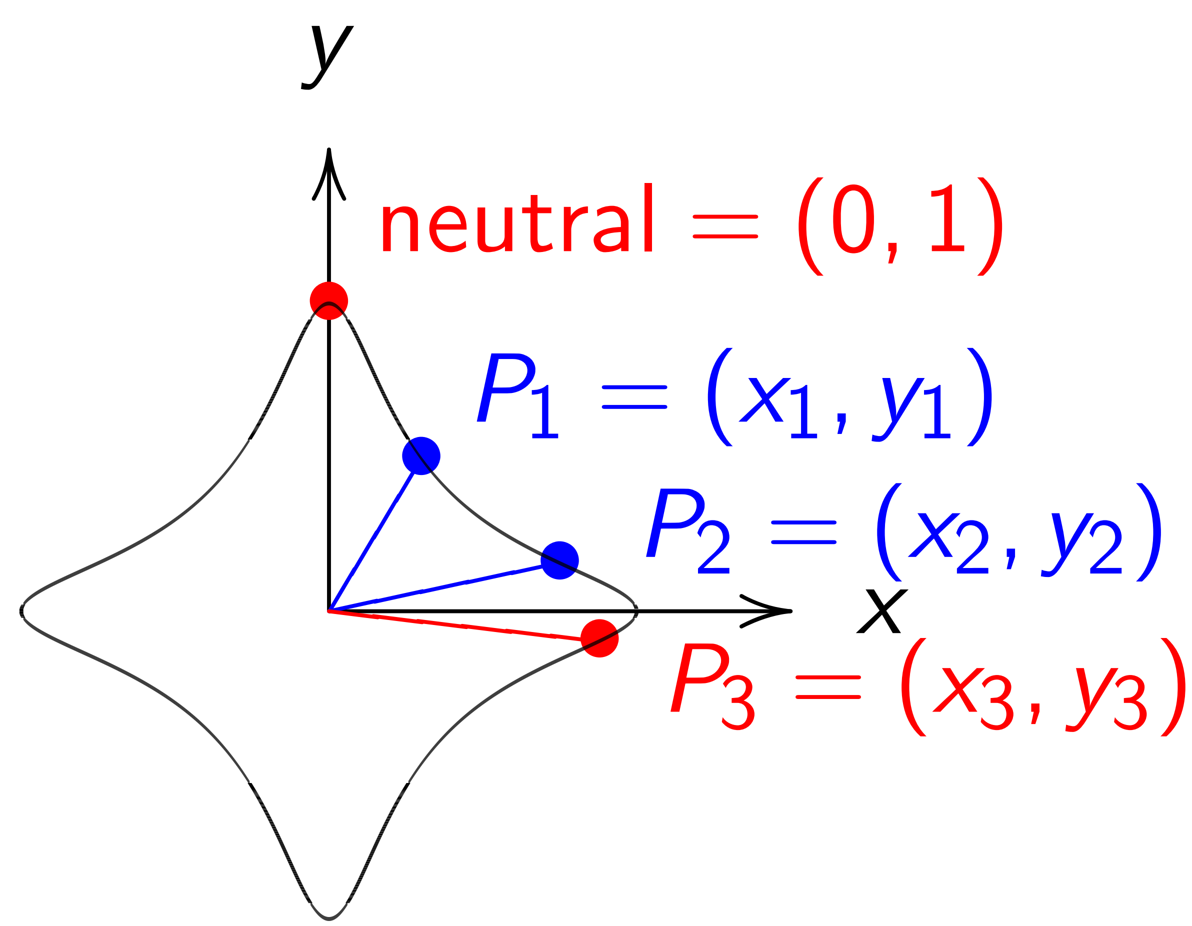

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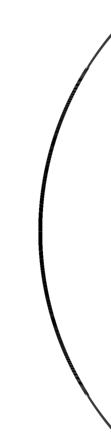

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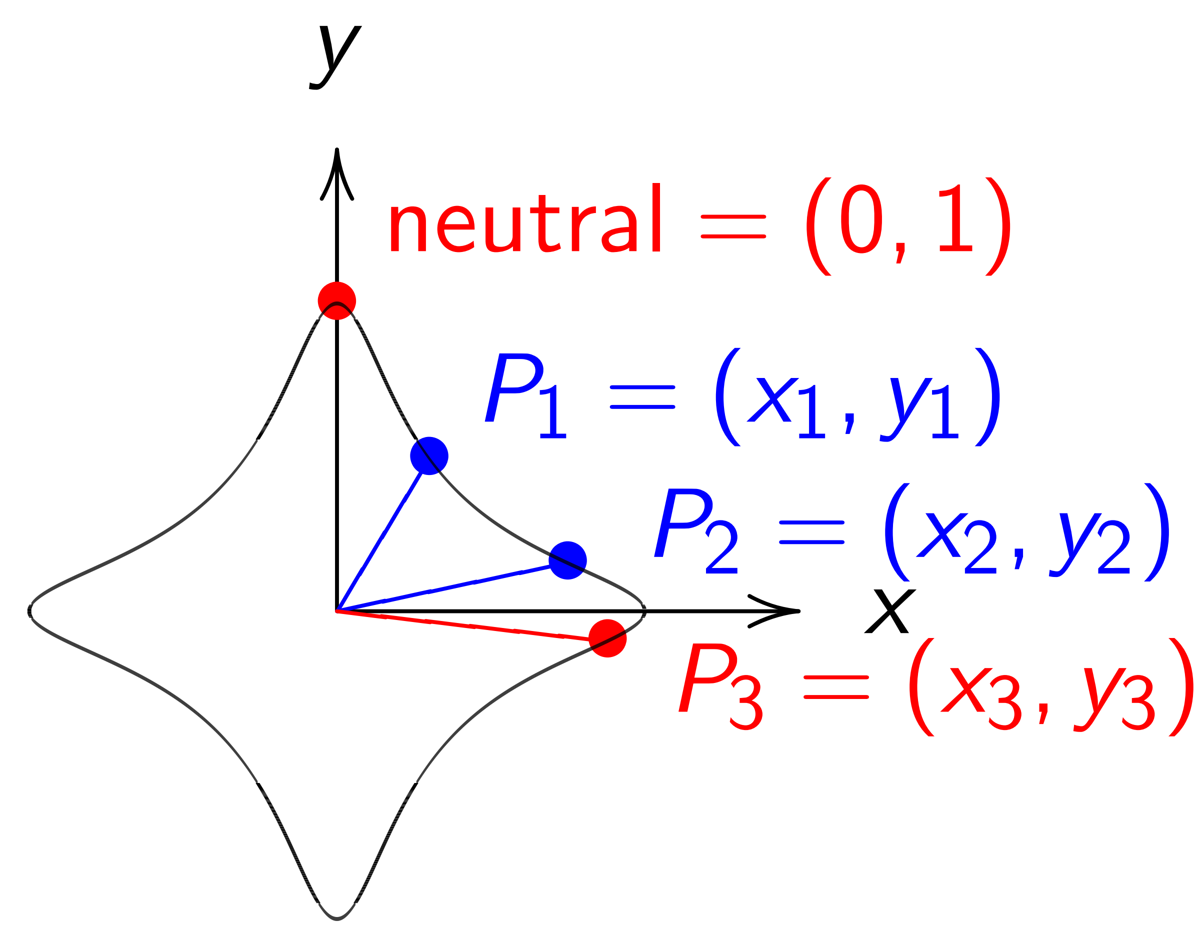

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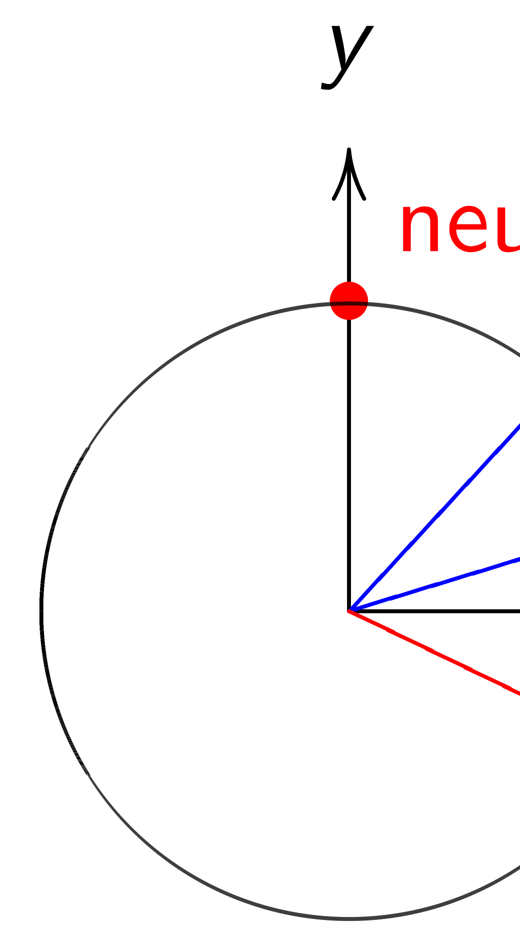

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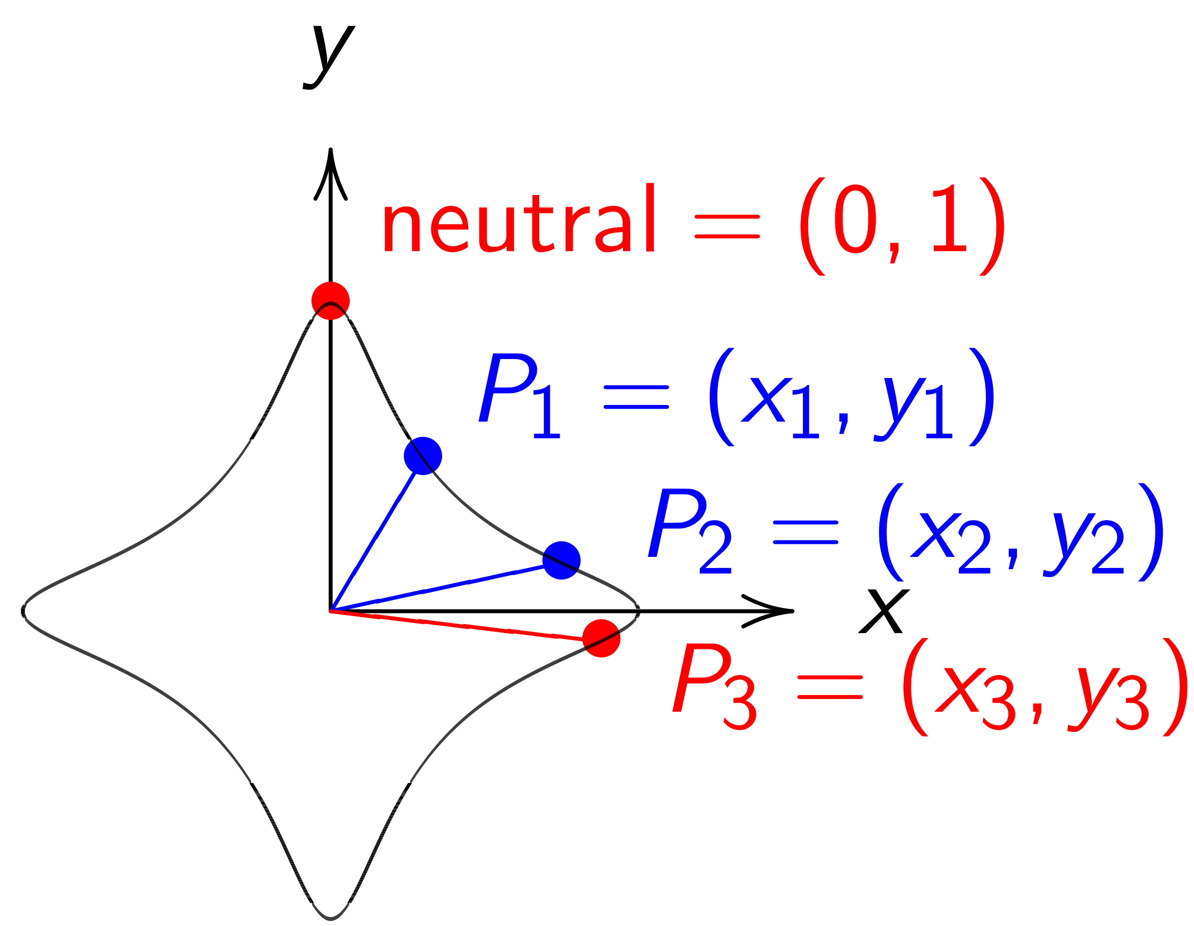
Addition on an elliptic curve



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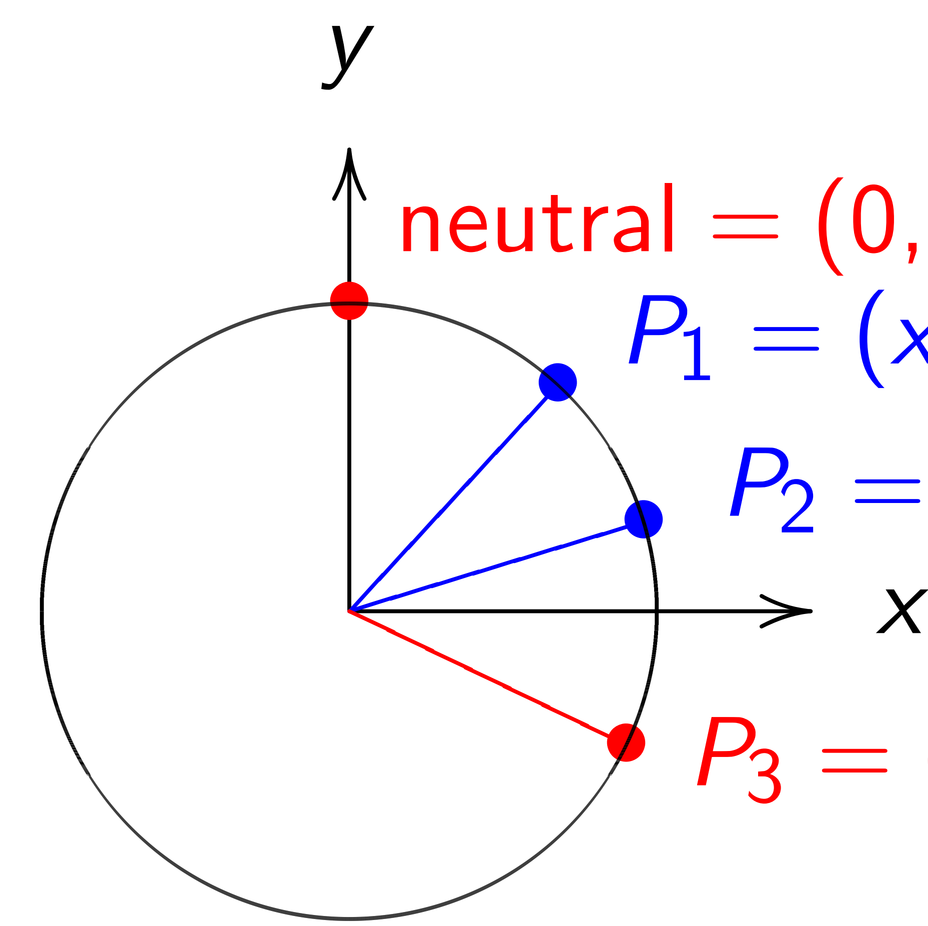

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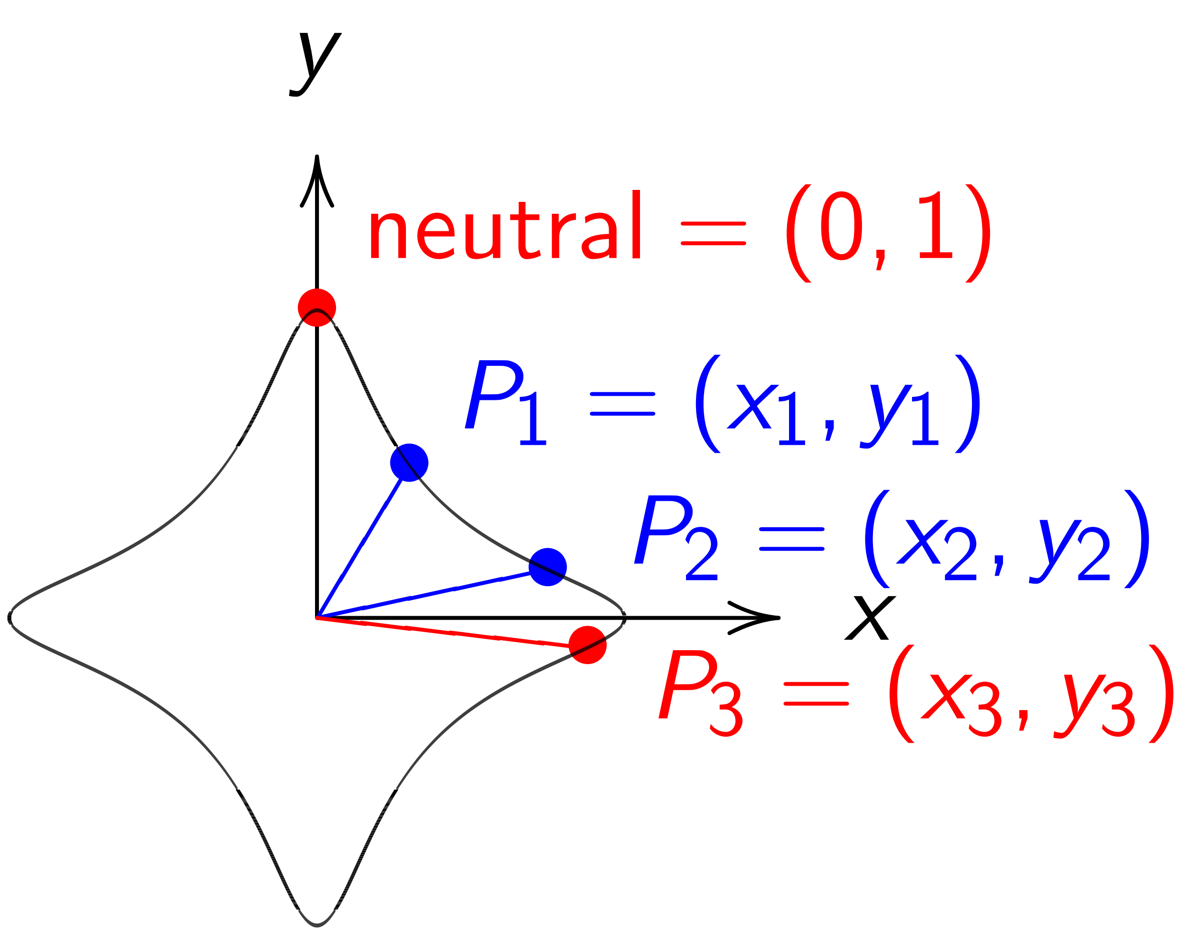

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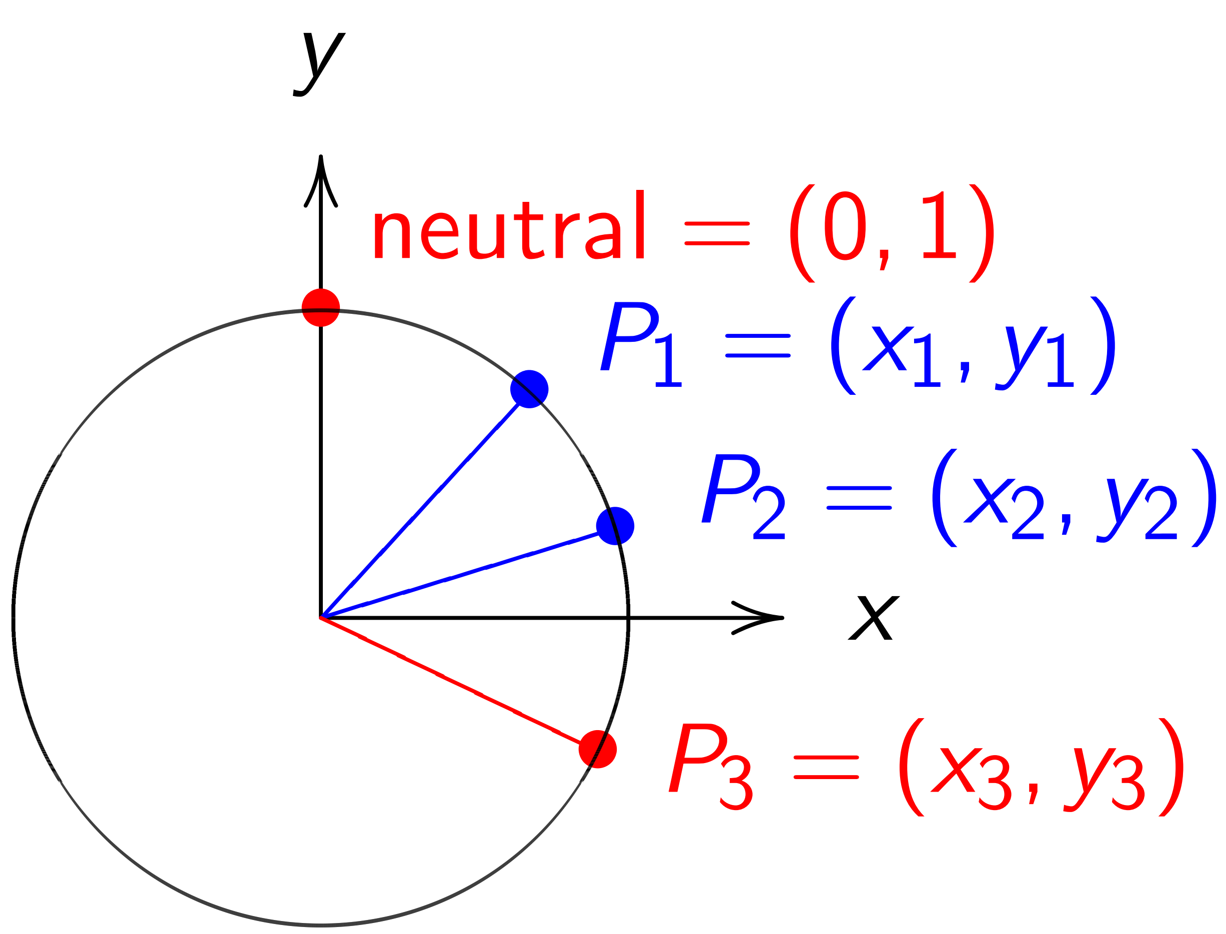

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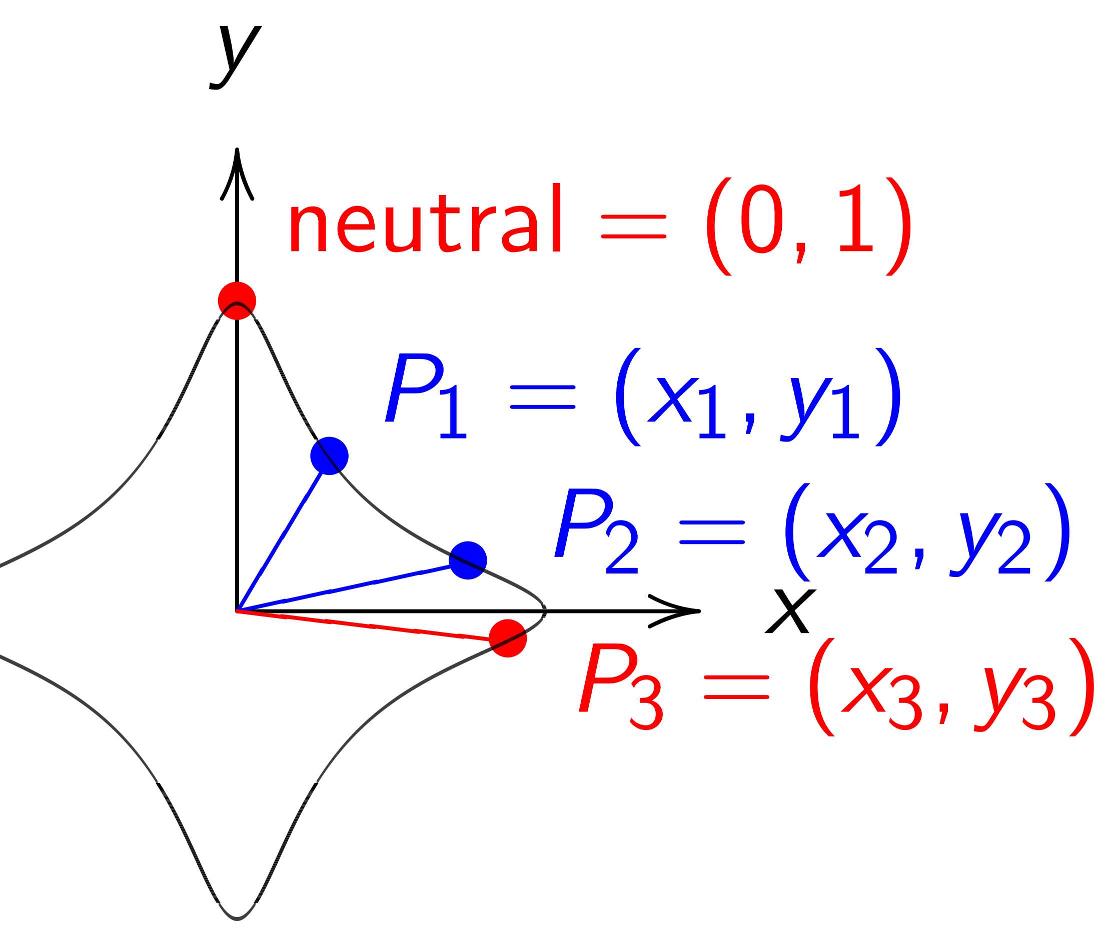

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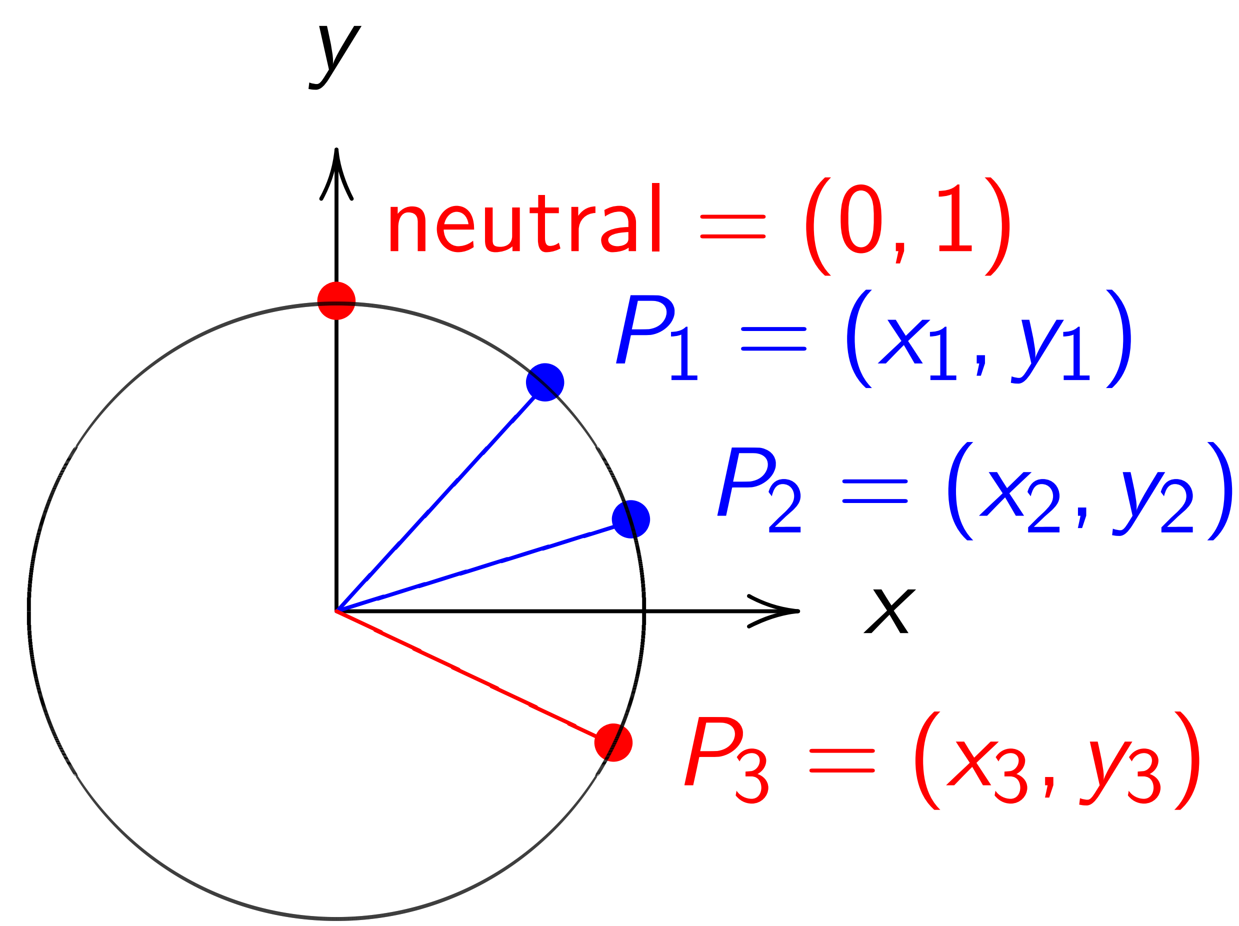

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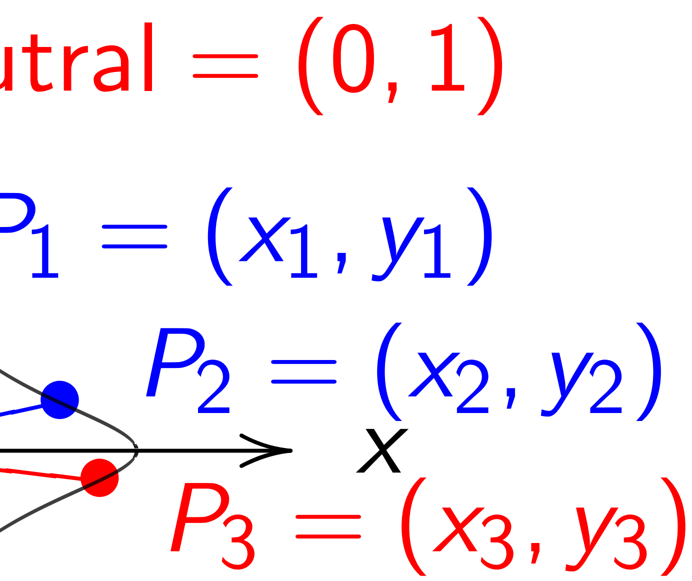

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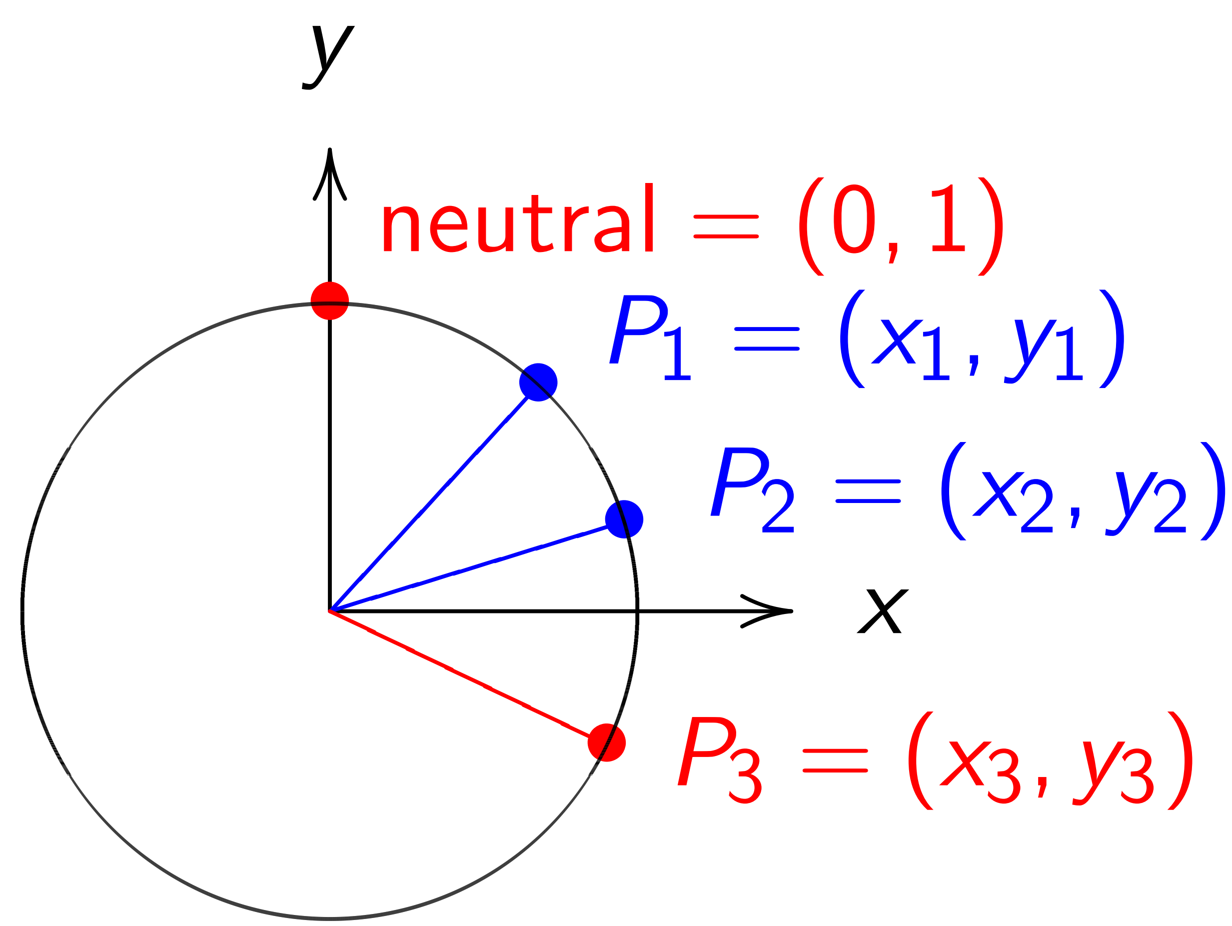

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More elliptic curve

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$x^2 + y^2 = 1 -$

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where

$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1}$

$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1}$

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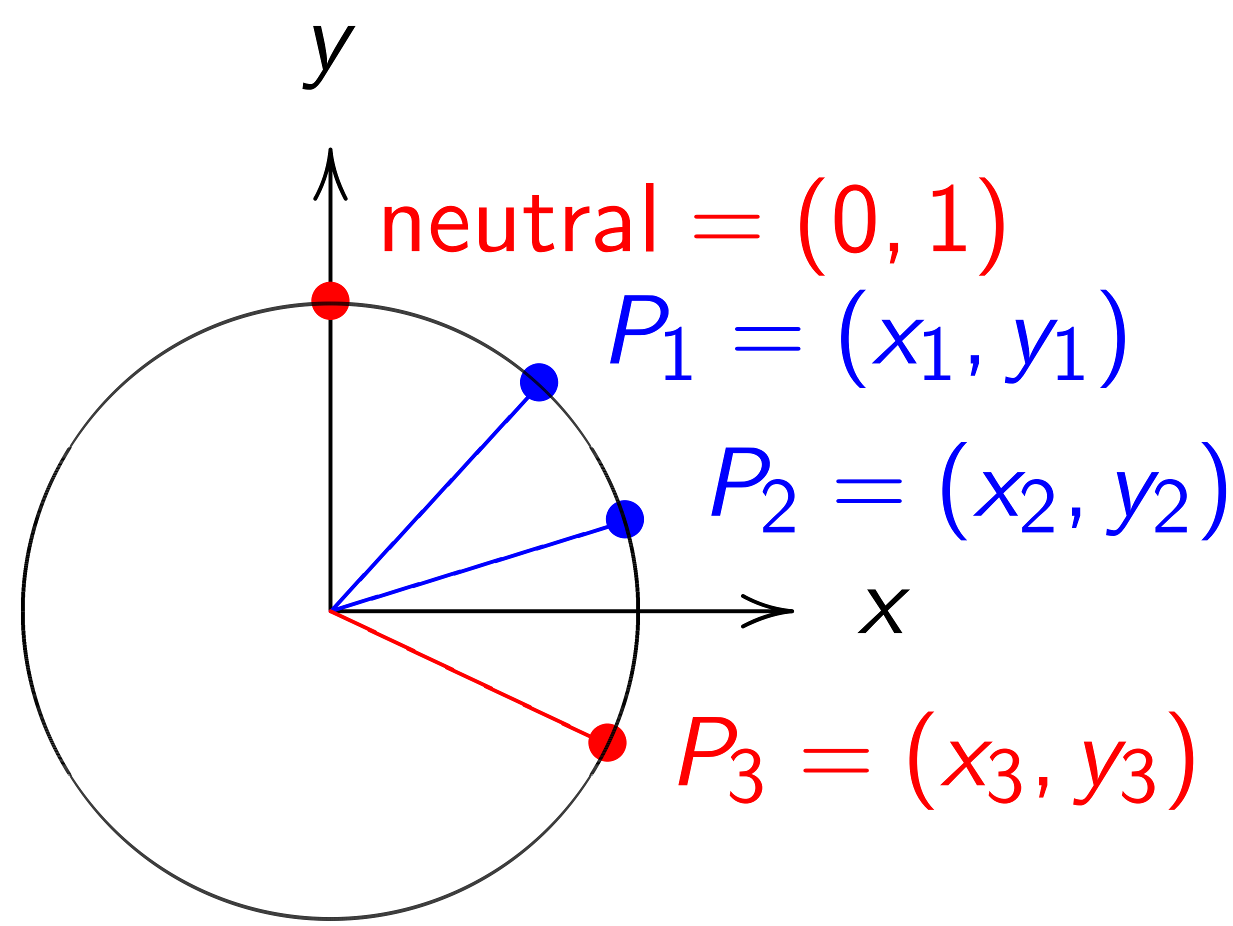
The clock again, for comparison:



$x^2 + y^2 = 1.$

Sum of (x_1, y_1) and (x_2, y_2) is

$(x_1 y_2 + y_1 x_2,$

$y_1 y_2 - x_1 x_2).$

More elliptic curves

Choose an odd prime power

Choose a *non-square* $d \in \mathbf{F}_q$

$\{(x, y) \in \mathbf{F}_q \times \mathbf{F}_q :$
$x^2 + y^2 = 1 + dx^2y^2\}$

is a "complete Edwards curv

"The Edwards addition law"

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where

$$x_3 = \frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2},$$

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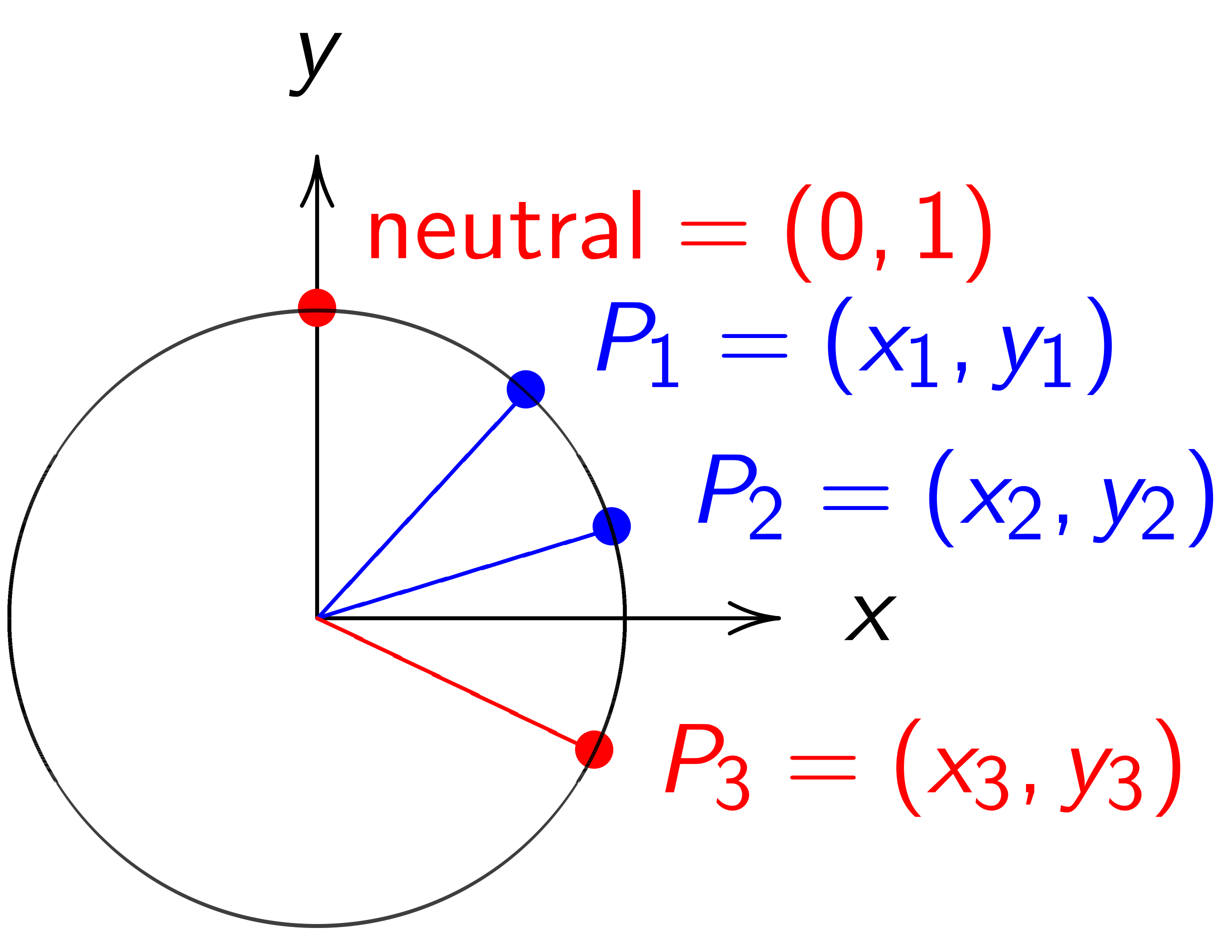

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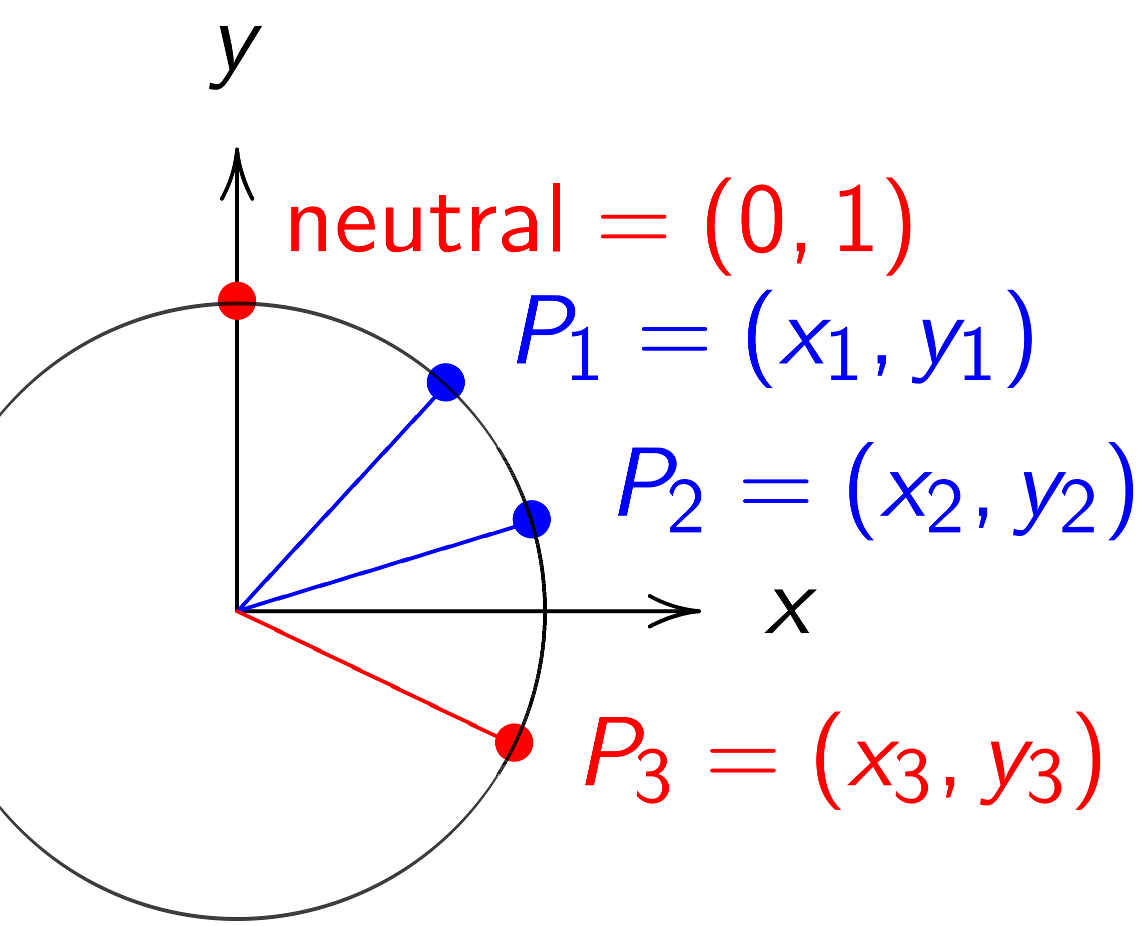

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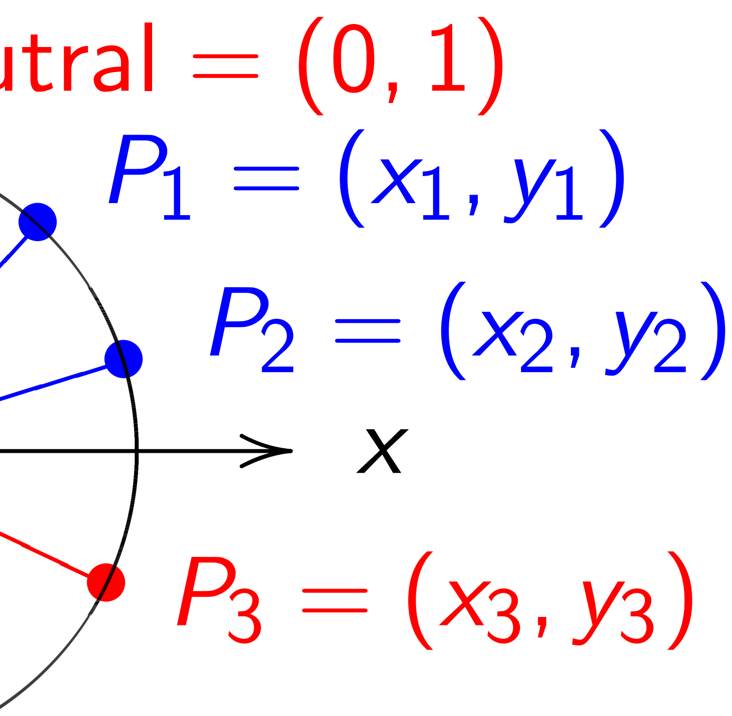

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$(x_1 + dx_1x_2y_1y_2y_1)^2 =$
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Conclude that d is a square.
But d is not a square! Q.E.D.

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<u>A class group of a quadratic</u>

Fix prime $p \in 3 + 4\mathbf{Z}$ with p

e.g. $p = 2^{127} - 309$.

Define C as the curve $y^2 =$

$\delta t(t-1)(t-10)(t-5/8)(t$

over $\mathbf{F}_p$ where $\delta = -2/3^5 5^4$

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View J projectively,
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Define rational operations $0, -, +$ making J a group.
J is an "Abelian variety".

Rationally map C to J,
taking ∞ to 0.
J is a "C-Abelian variety".

J is initial:
maps uniquely to
any C-Abelian variety.

A class group of a quadratic field

Fix prime $p \in 3 + 4\mathbf{Z}$ with $p \geq 19$.
e.g. $p = 2^{127} - 309$.

Define C as the curve $y^2 = \delta t(t-1)(t-10)(t-5/8)(t-25)$ over $\mathbf{F}_p$ where $\delta = -2/3^5 5^4$, with specified point ∞.

Define J as "Jac C": surface defined by equation

$$\delta t(t-1)(t-10)(t-5/8)(t-25) - (v_1 t + v_0)^2$$

mod $t^2 + u_1 t + u_0 = 0$
in variables (u_0, u_1, v_0, v_1).

View J projectively, handling ∞ carefully.
Define rational operations $0, -, +$ making J a group.
J is an "Abelian variety".

Rationally map C to J, taking ∞ to 0.
J is a "C-Abelian variety".

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2006 Gaudry)

Linear combinatio
$1, u_0, u_1, u_0^2, u_0 u_1,$
$x = 16 u_0 u_1^2 - 8 u_0^2$
$5 u_1^2 - 1215000 v_0 v$
$175 u_1 - 1250$, etc.
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given P_3 and P_2 and $P_1 = F$
(1986 Chudnovsky–Chudnov
2006 Gaudry)

Linear combinations of
$1, u_0, u_1, u_0^2, u_0 u_1, u_1^2, u_0 u_1^2, v$
$x = 16 u_0 u_1^2 - 8 u_0^2 + 573 u_0$
$5 u_1^2 - 1215000 v_0 v_1 + 2460 u$
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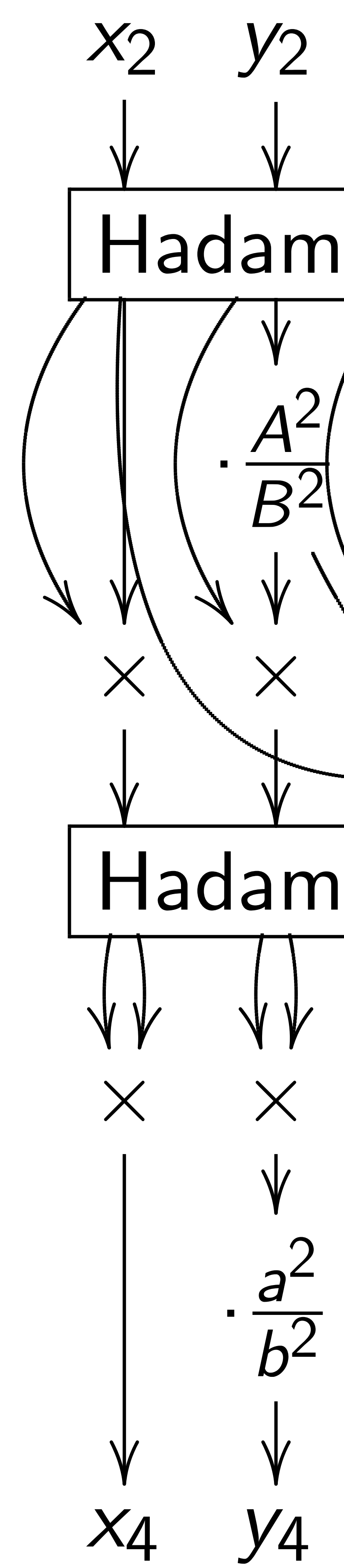

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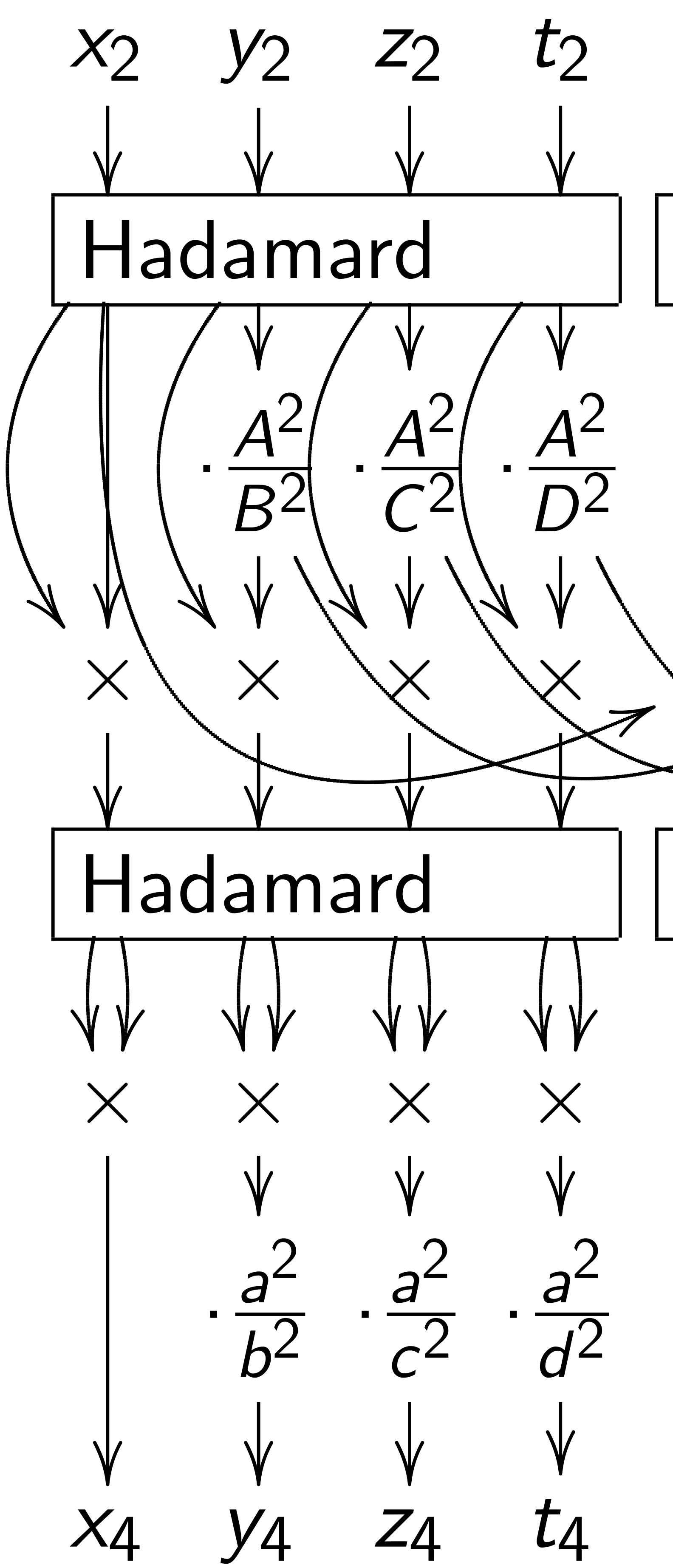

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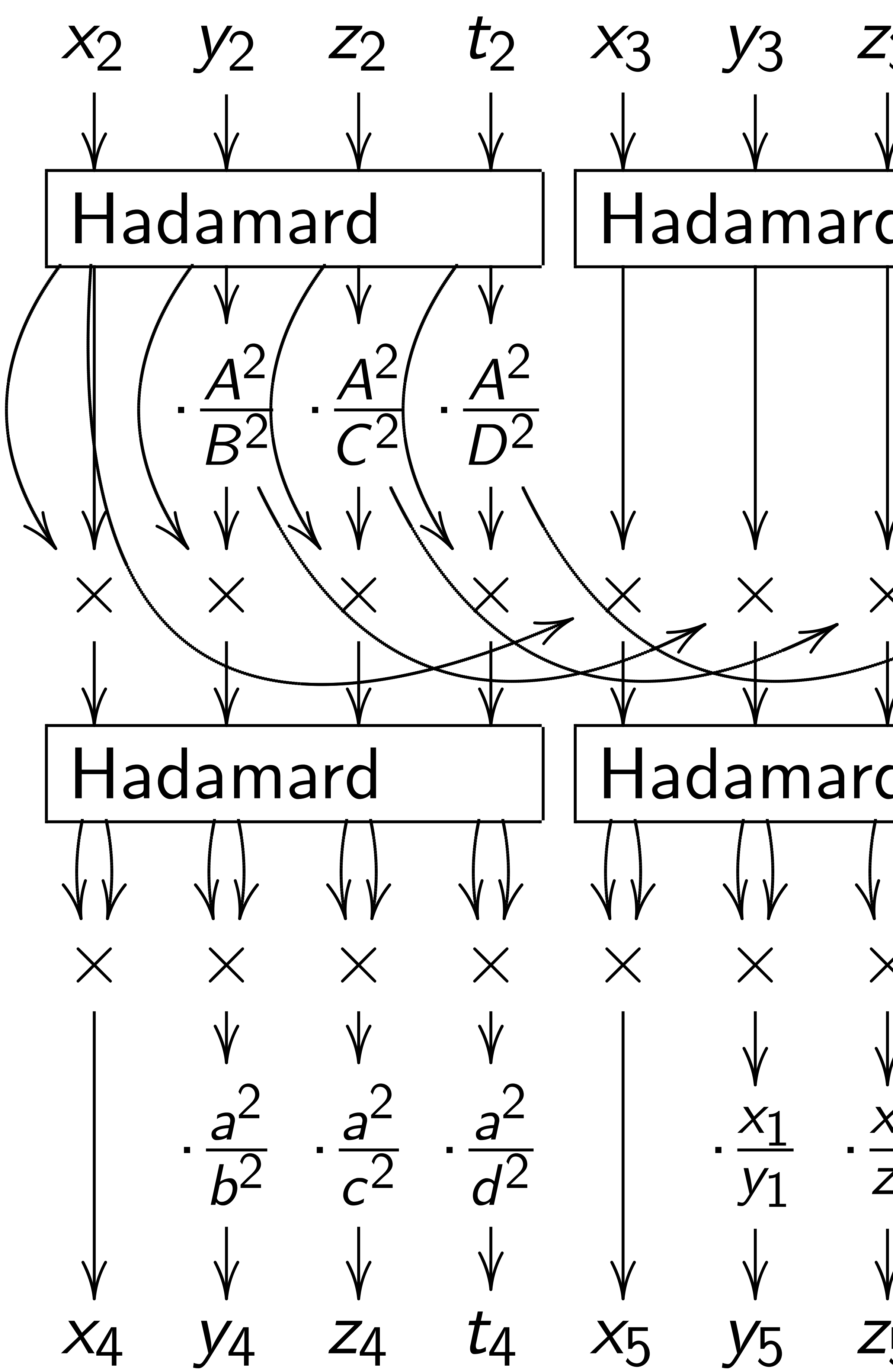

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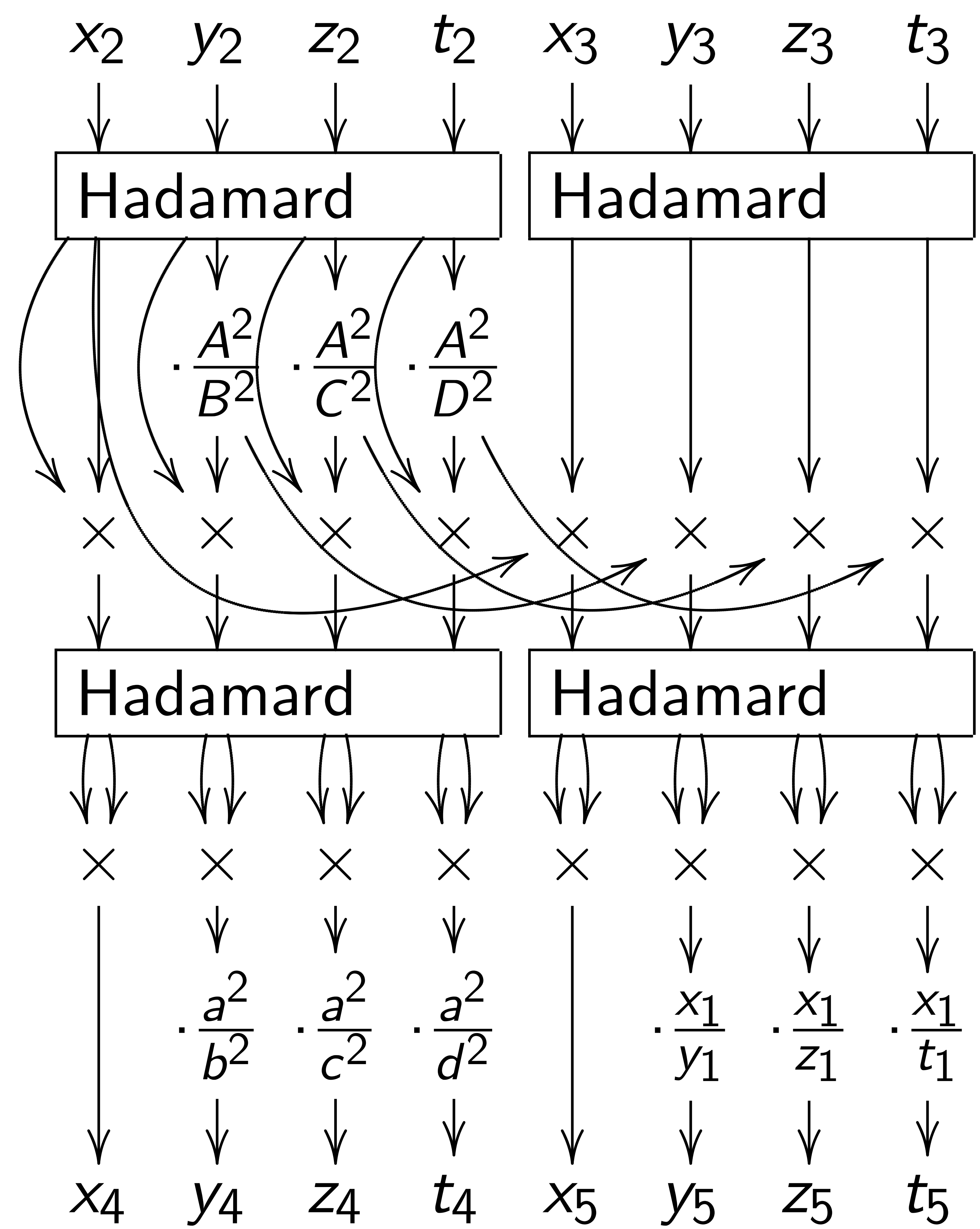

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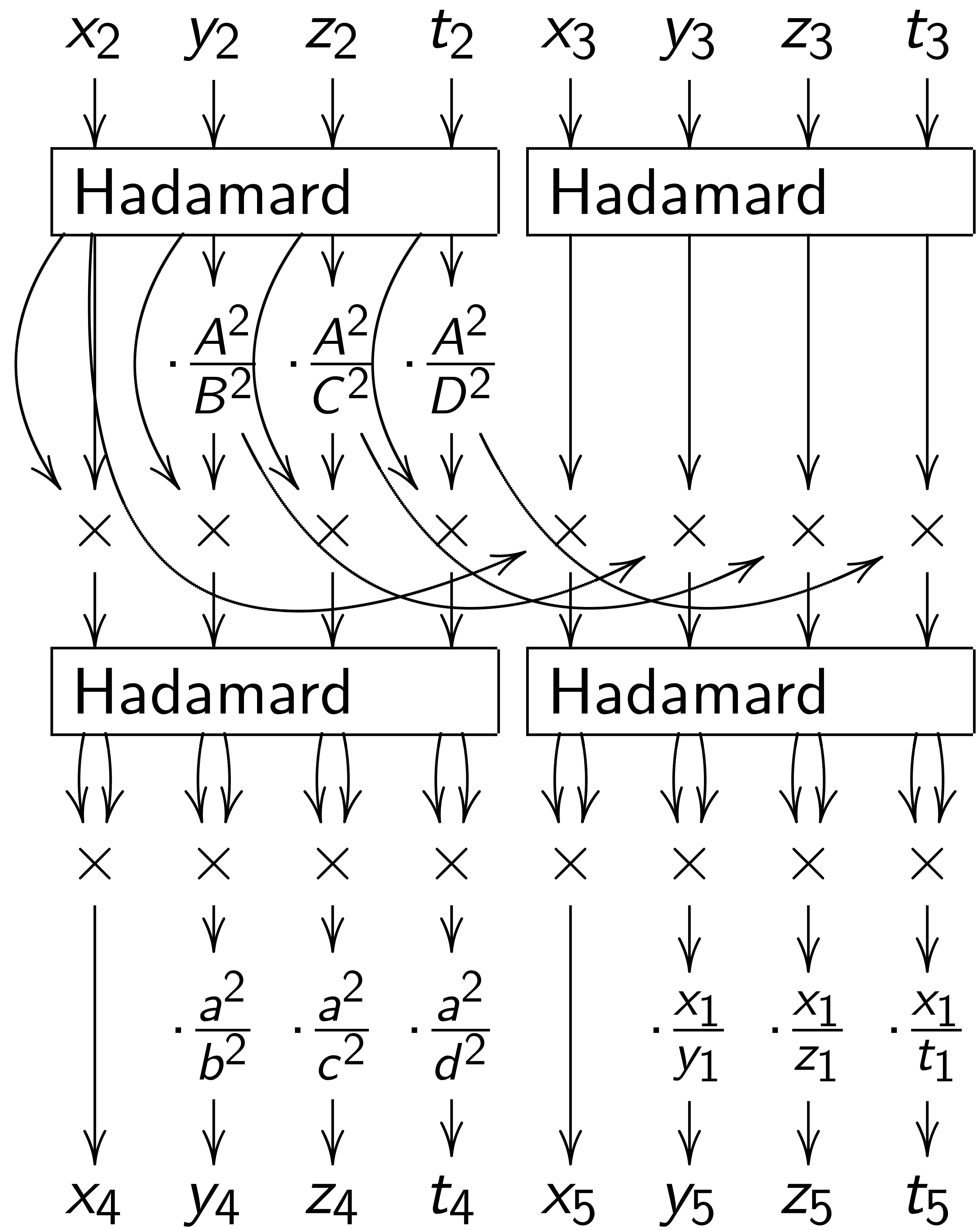

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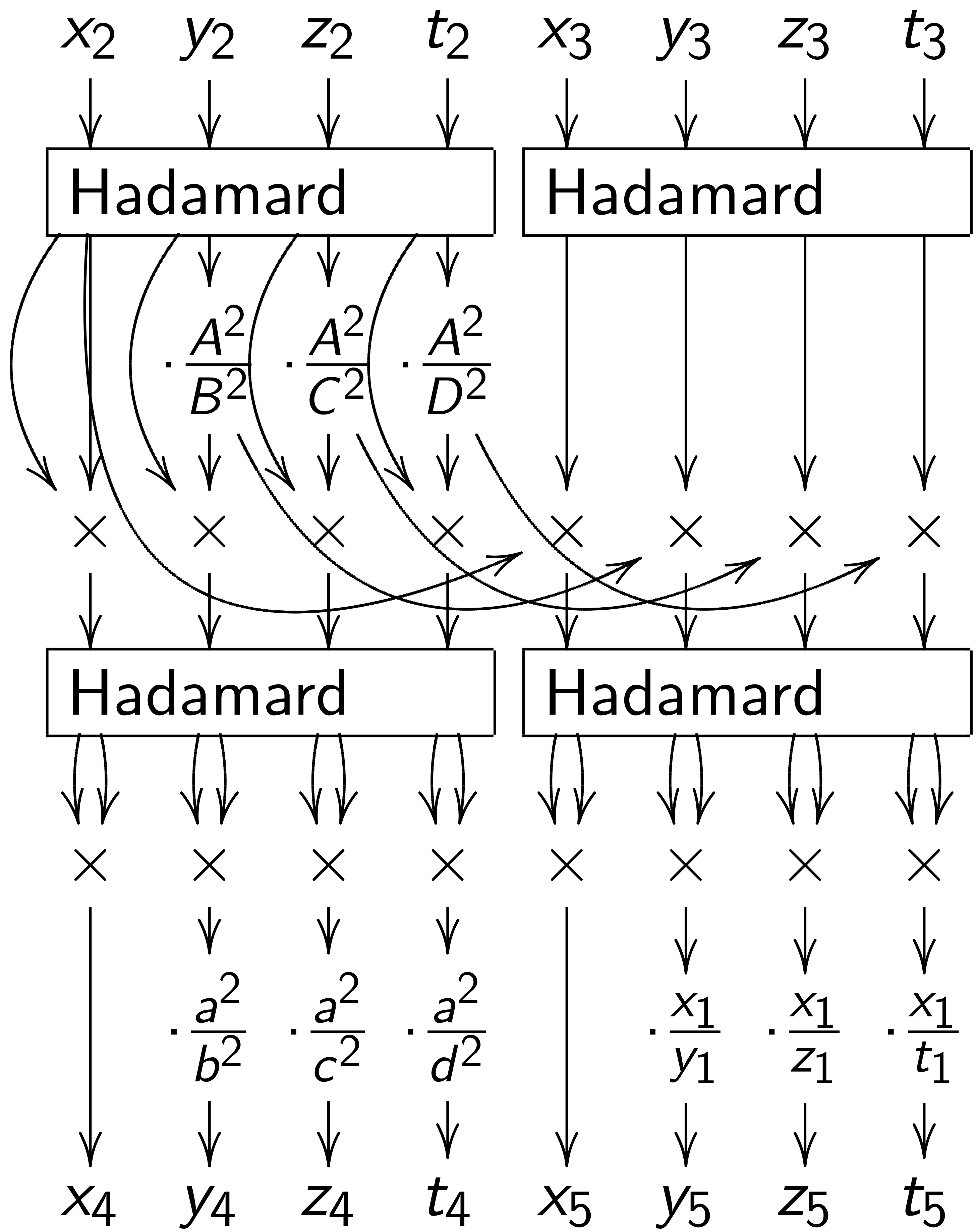

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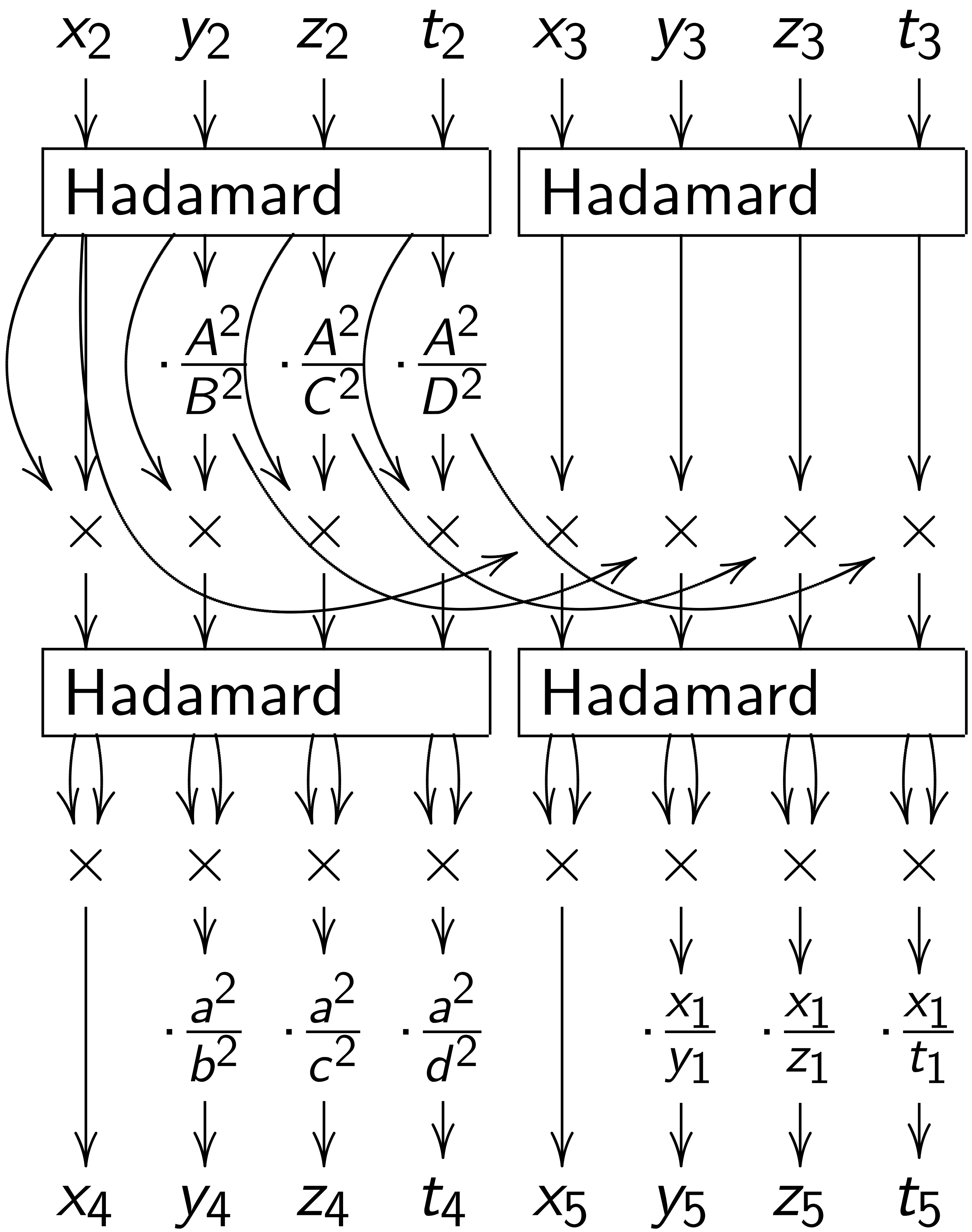

These coordinates
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so they don't support
rational group operations,
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Coefficients in computation
are all small, saving time:
$(a^2 : b^2 : c^2 : d^2)$
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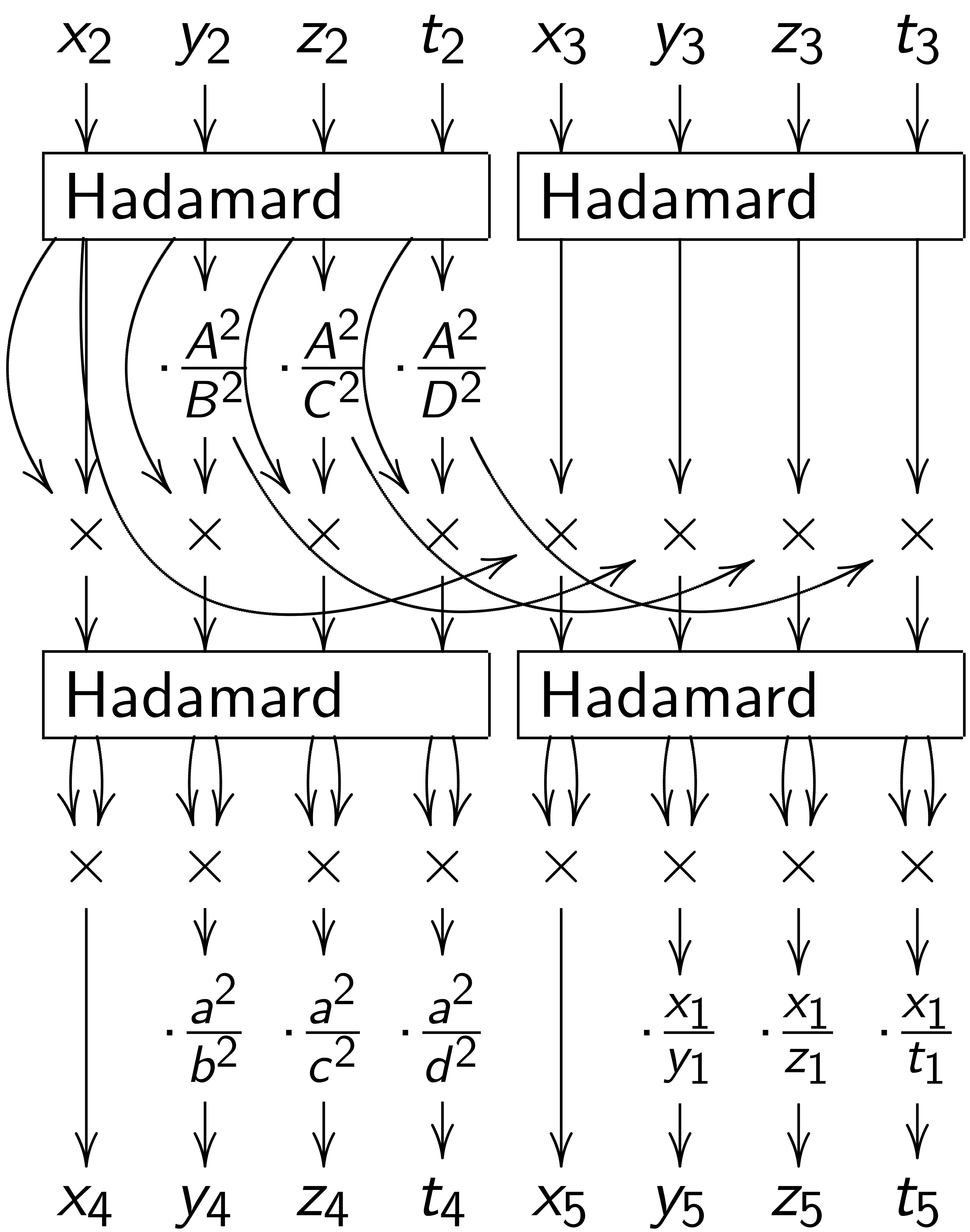

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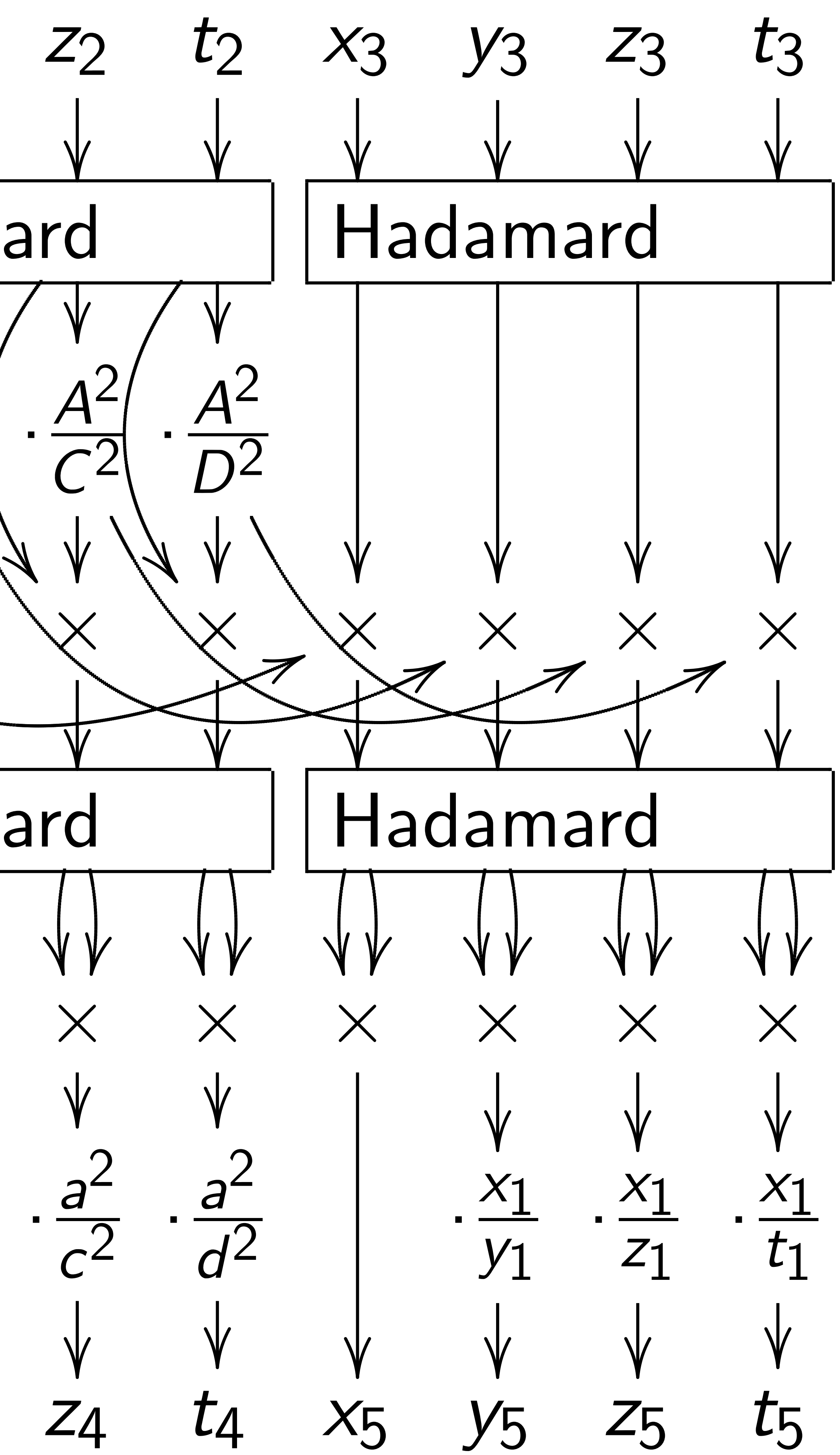

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If $y^2 =$
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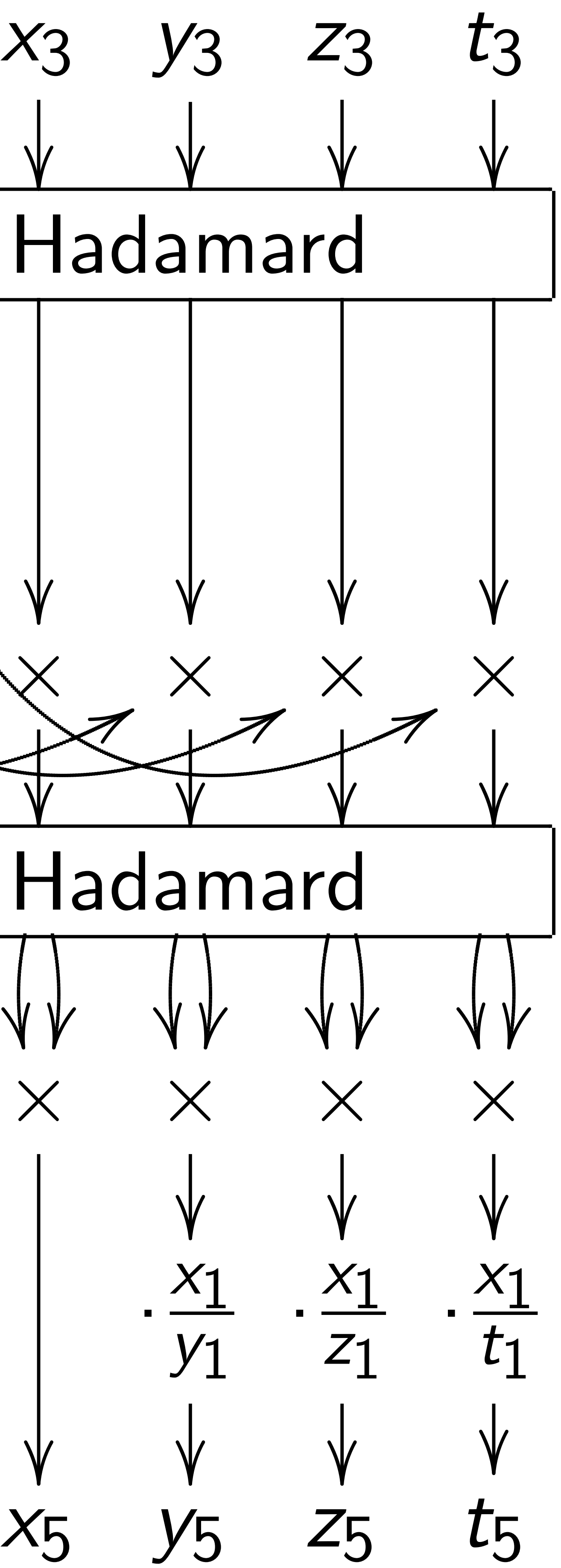

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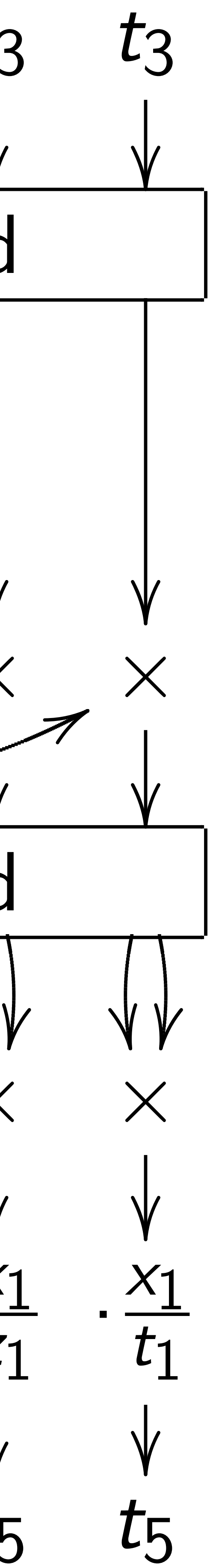

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A Kummer-friendly Scholten

If $y^2 =$
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then
$(y(z+2)^3)^2 = (z-1)(z+1)$
$\quad (z-1/2)(z+3/2)(z-$
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A Kummer-friendly Scholten curve

If $y^2 = \delta t(t-1)(t-10)(t-5/8)(t-25)$ then

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Define $\mathbf{F}_{p^2} = \mathbf{F}_p[i]/(i^2+1)$; $r = (7+4i)^2 = 33+56i$; $s = 159+56i$; $\omega = \sqrt{-384}$.

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Then $(\omega y(z+2)^3/(1-iz)^3)^2 = rx^3 + sx^2 + \bar{s}x + \bar{r}$ where $x = (1+iz)^2/(1-iz)^2$.

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Map $(x, \omega y(z+2)$
to an Edwards cur
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where $x = (1+iz)^2/(1-iz)^2$.

Map $(x, \omega y(z+2)^3/(1-iz$
to an Edwards curve E over
by chain of "2-isogenies".

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Map $(x, \omega y(z+2)^3/(1-iz)^3)$
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Map $(x, \omega y(z+2)^3/(1-iz)^3)$
to an Edwards curve E over $\mathbf{F}_{p^2}$
by chain of "2-isogenies".

View two coordinates over $\mathbf{F}_{p^2}$
as four coordinates over $\mathbf{F}_p$;
view curve E as surface W.
Have now mapped C rationally
to this Abelian variety W.

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Compute formulas for
the unique map $J \to W$
of C-Abelian varieties
and a "dual isogeny" $W \to J$.
Composition has small kernel.

ner-friendly Scholten curve

)$(t-10)(t-5/8)(t-25)$

$)^3)^2 = (z-1)(z+1)(z+2)$
$-1/2)(z+3/2)(z-2/3)$
$= (5-2t)/(5+t)$.

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$= (1+iz)^2/(1-iz)^2$.

Map $(x, \omega y(z+2)^3/(1-iz)^3)$ to an Edwards curve E over $\mathbf{F}_{p^2}$ by chain of "2-isogenies".

View two coordinates over $\mathbf{F}_{p^2}$ as four coordinates over $\mathbf{F}_p$; view curve E as surface W. Have now mapped C rationally to this Abelian variety W.

Compute formulas for the unique map $J \to W$ of C-Abelian varieties and a "dual isogeny" $W \to J$. Composition has small kernel.

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$-1)(z+1)(z+2)$

$-3/2)(z - 2/3)$

$)/(5 + t)$.

$]/(i^2 + 1)$;

$3 + 56i$;

$= \sqrt{-384}$.

$/(1 - iz)^3)^2$

$+ \bar{r}$

$)^2/(1 - iz)^2$.

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"Hyper-and-elliptic-curve" groups support Edwards coo *and* support Kummer coords with small coefficients. 3 independent constraints on 2 degrees of freedom, but everything lifts to $\mathbf{Q}$.

Map $(x, \omega y(z+2)^3/(1-iz)^3)$ to an Edwards curve E over $\mathbf{F}_{p^2}$ by chain of "2-isogenies".

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Cryptographic consequences

Speed records for high-security $a \mapsto aP$ use Edwards coords.

Speed records for high-security $a, P \mapsto aP$ use Kummer coords for Jacobians of genus-2 curves with small Kummer coefficients.

"Hyper-and-elliptic-curve" groups support Edwards coords *and* support Kummer coords with small coefficients. 3 independent constraints on 2 degrees of freedom, but everything lifts to $\mathbf{Q}$.