Quantum algorithms for the subset-sum problem

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cr.yp.to/qsubsetsum.html

Joint work with:

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University of Waterloo

Tanja Lange
Technische Universiteit Eindhoven

Alexander Meurer
Ruhr-Universität Bochum

Subset-sum example:
Is there a subsequence of
(499, 852, 1927, 2535, 3596, 3608, 4688, 5989, 6385, 7353, 7650, 9413)
having sum 36634?

Many variations: e.g.,
find such a subsequence *if* one exists;
find such a subsequence *knowing that* one exists;
allow range of sums;
coefficients outside $\{0, 1\}$; etc.

"Subset-sum problem";
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Define $u \in \mathbf{Z}^{12}$ as
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and $\sum_{i \in J} x_i = 36$
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If $J \subseteq \{1, 2, \ldots, 12\}$
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Define $L \subseteq \mathbf{Z}^{12}$ as $\{v : v_1 x_1 + \cdots + v_{12} x_{12} = 0\}$.

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Is there a subsequence of
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having length w and xor 1060?

This is the central algorithmic
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Quantum search (0.5)

Assume that function f has n-bit input, unique root

Generic brute-force search finds this root using $\approx 2^n$ evaluations of f.

1996 Grover method finds this root using $\approx 2^{0.5n}$ quantum evaluations on superpositions of inputs.

Cost of quantum evaluation $\approx$ cost of evaluation of f if cost counts qubit "operati

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Apply to the function
$J \mapsto \Sigma(J) - t$ where
$\Sigma(J) = \sum_{i \in J} x_i$.

Cost $2^{0.5n}$ to find root (i.e.,
to find indices of subsequence
of $x_1, \ldots, x_n$ with sum t)
or to decide that no root exists.
We suppress poly factors in cost.

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Algorithm details f

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Easily adapt to handle
different # of roots,
and # not known in advance.
Faster if # is large,
but typically # is not very large.
Most interesting: $\# \in \{0, 1\}$.

Apply to the function
$J \mapsto \Sigma(J) - t$ where
$\Sigma(J) = \sum_{i \in J} x_i$.

Cost $2^{0.5n}$ to find root (i.e.,
to find indices of subsequence
of $x_1, \ldots, x_n$ with sum t)
or to decide that no root exists.
We suppress poly factors in cost.

Algorithm details for unique

Represent $J \subseteq \{1, \ldots, n\}$ as
integer between 0 and $2^n -$

n bits are enough space
to store one such integer.

n qubits store much more,
a superposition over sets J:
2^n complex amplitudes
$a_0, \ldots, a_{2^n-1}$ with
$|a_0|^2 + \cdots + |a_{2^n-1}|^2 = 1$.
Measuring these n qubits
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Start from uniform superpos
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Repeat steps 1 and 2 about $0.58 \cdot 2^{0.5n}$ times.

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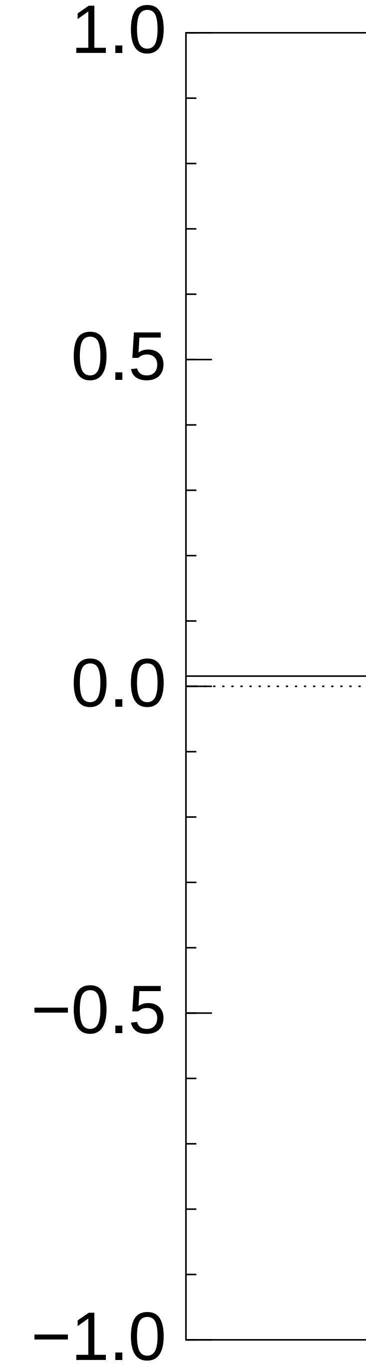

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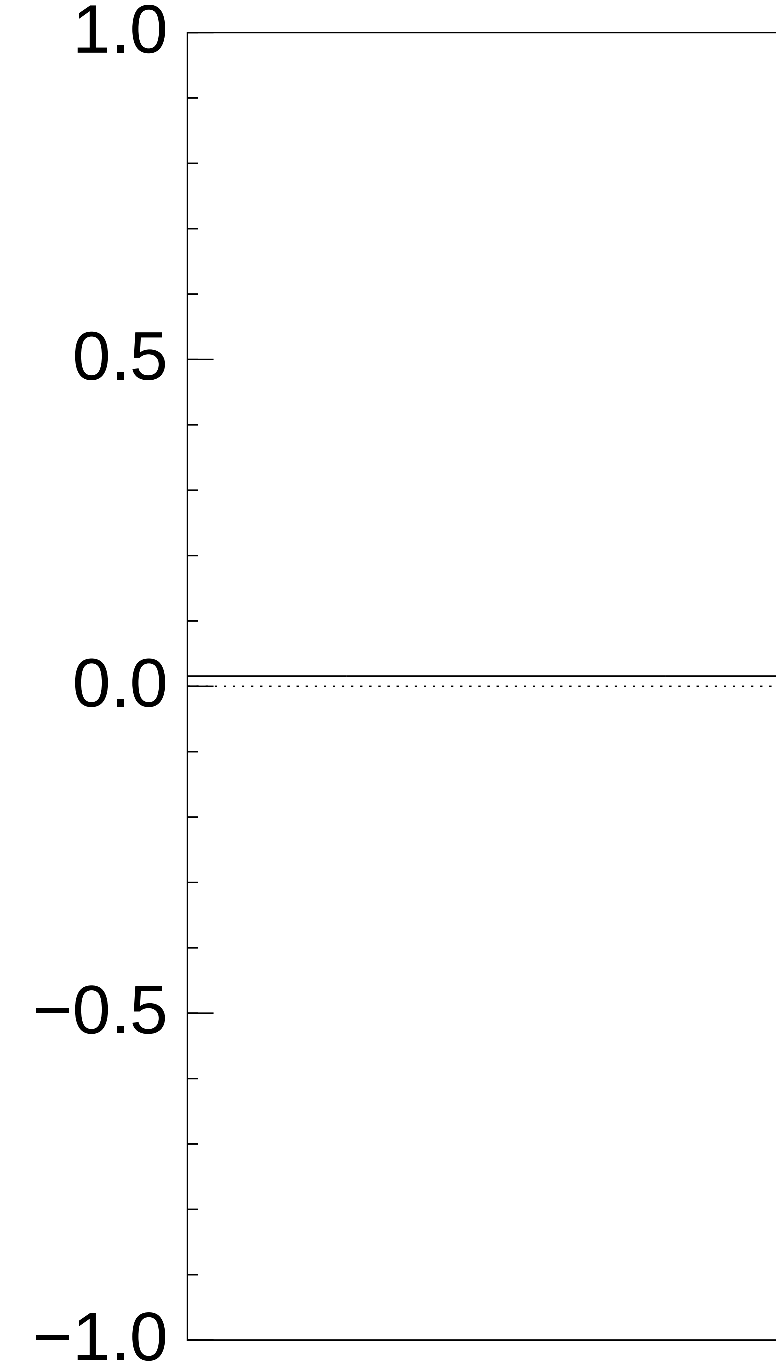
Graph of $J \mapsto a_J$
for 36634 example
after 0 steps:



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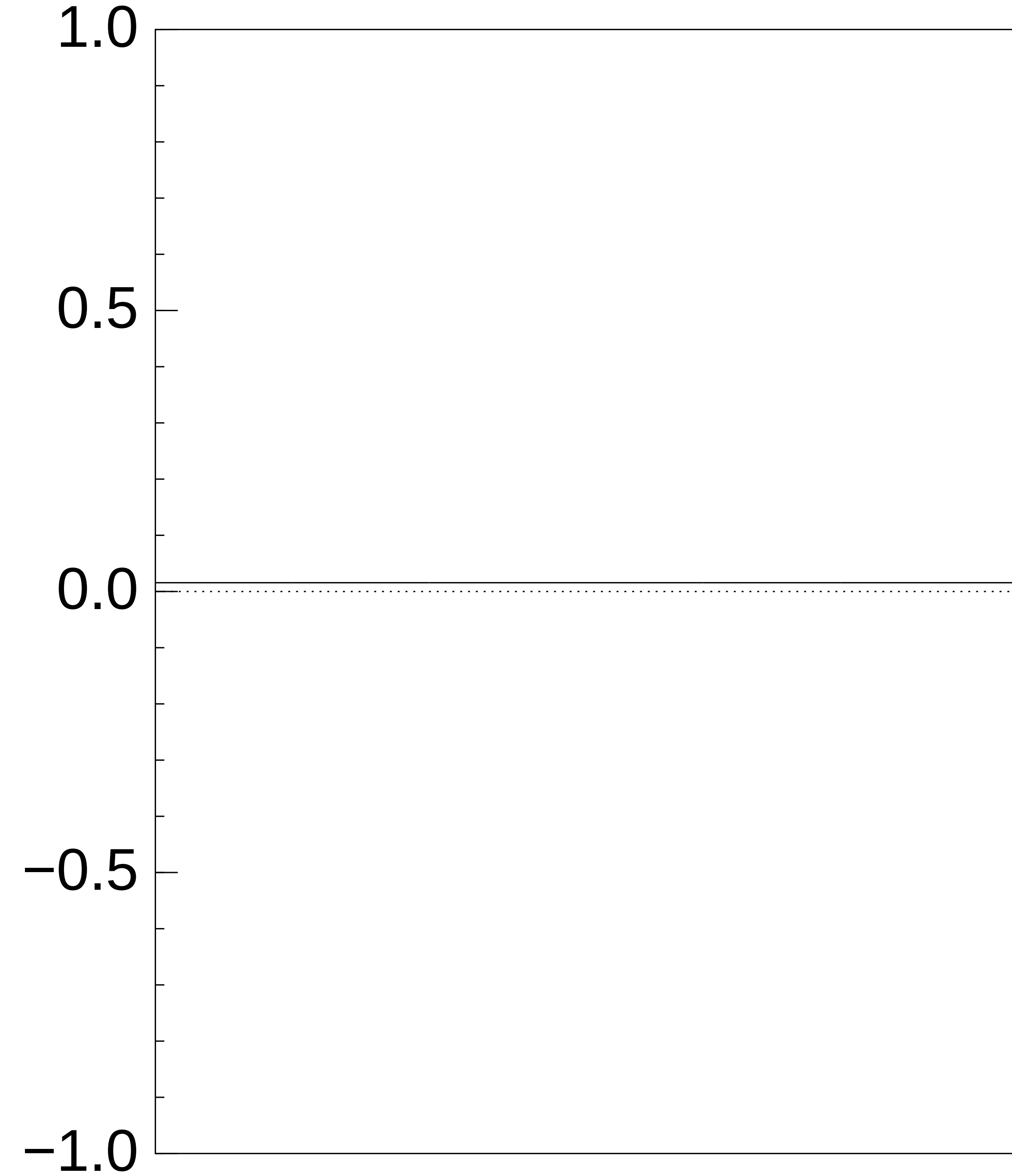
Graph of $J \mapsto a_J$
for 36634 example with $n =$
after 0 steps:



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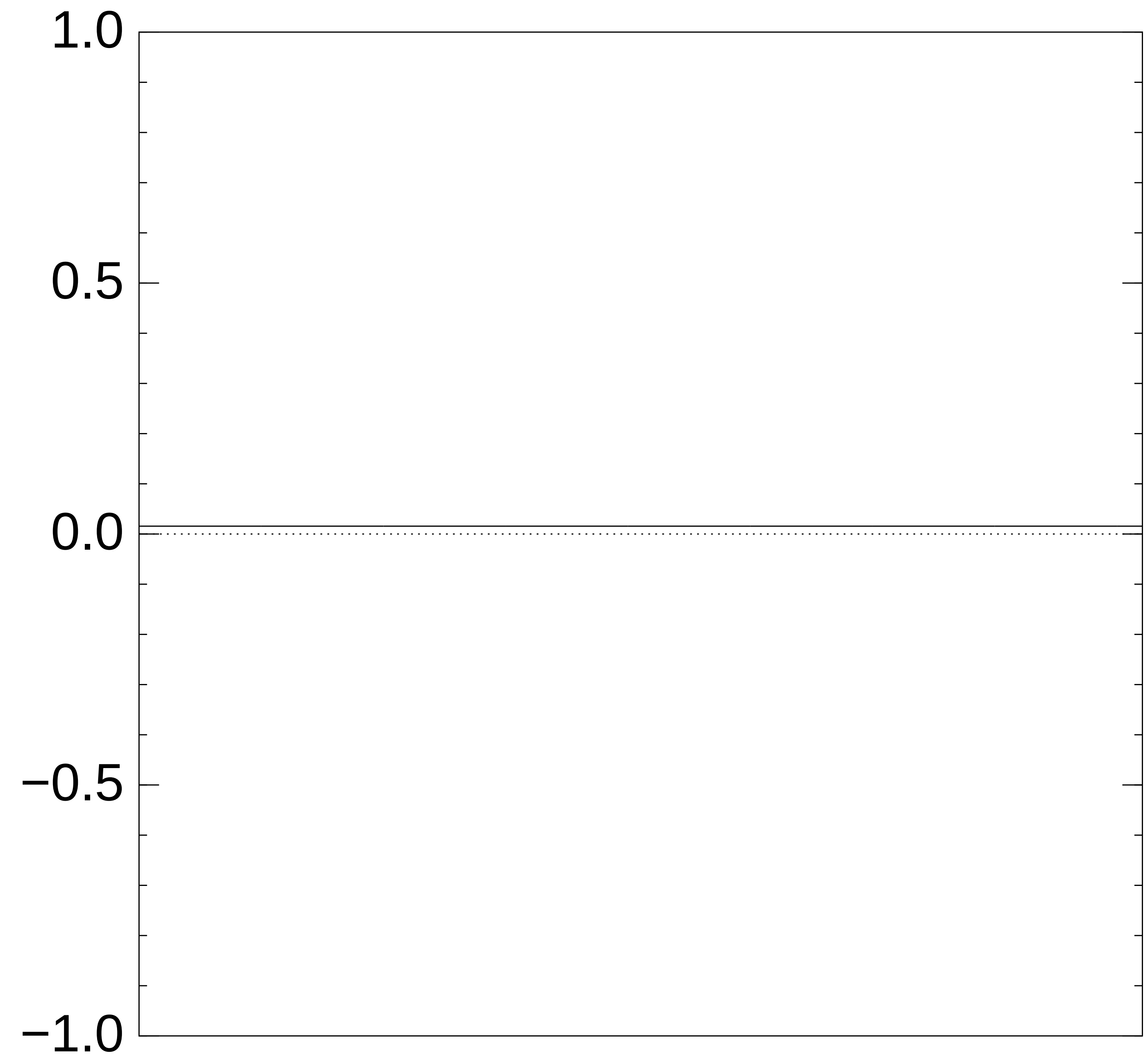

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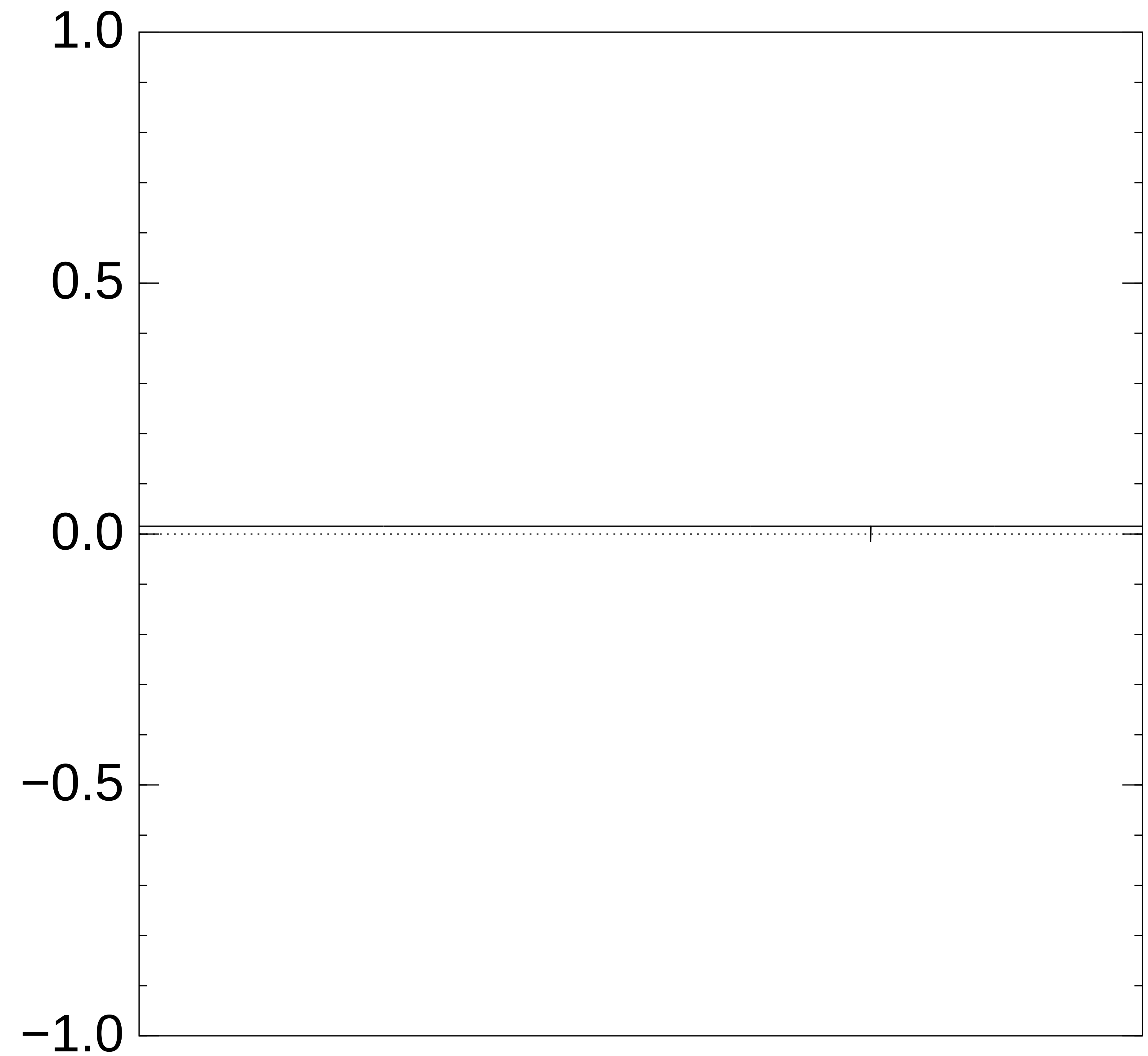
Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after Step 1:



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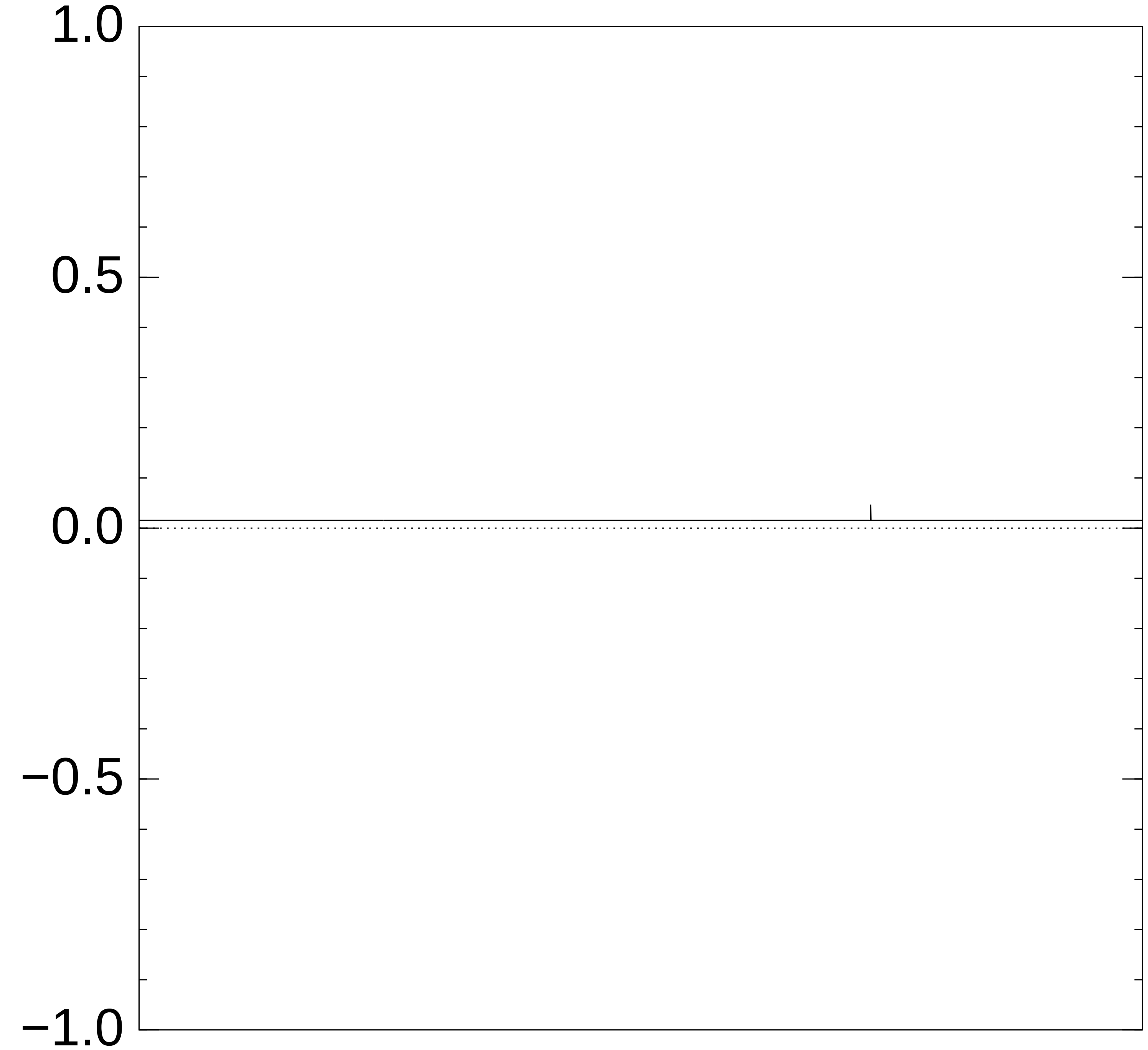
Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after Step 1 + Step 2:



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Measure the n qubits.
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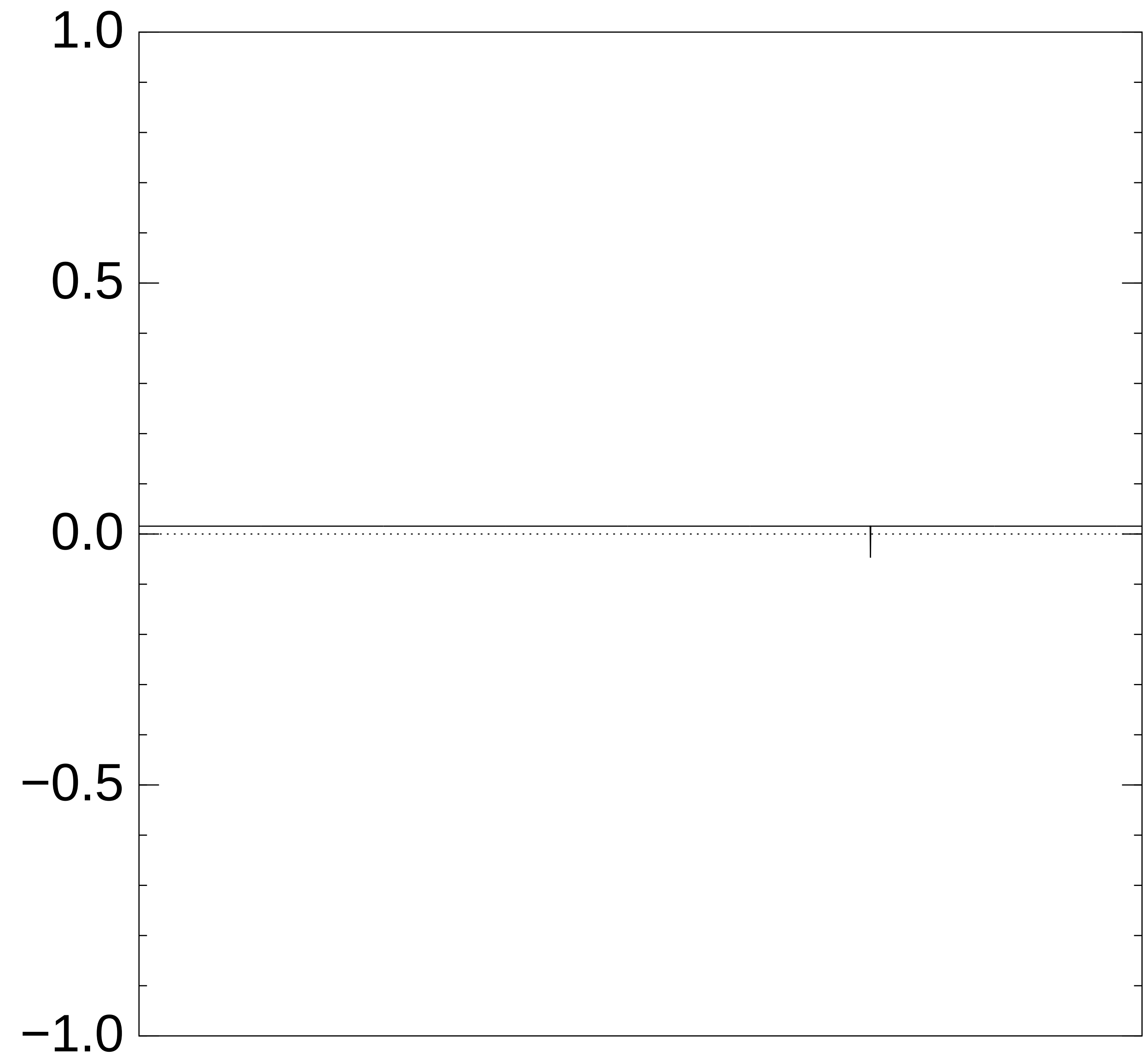
Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after Step 1 + Step 2 + Step 1:



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Measure the n qubits.
With high probability this finds
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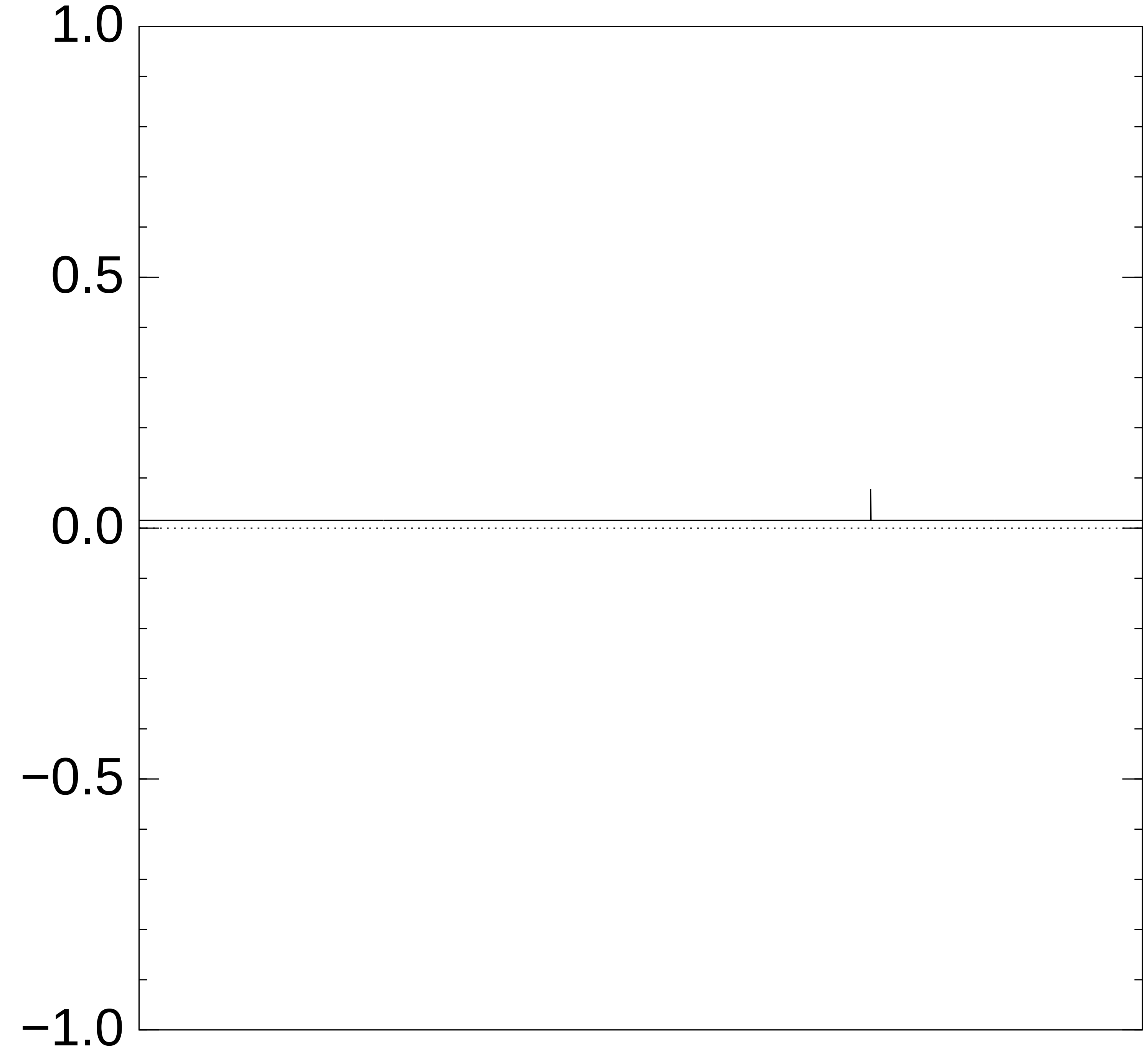
Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $2 \times$ (Step 1 + Step 2):



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Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
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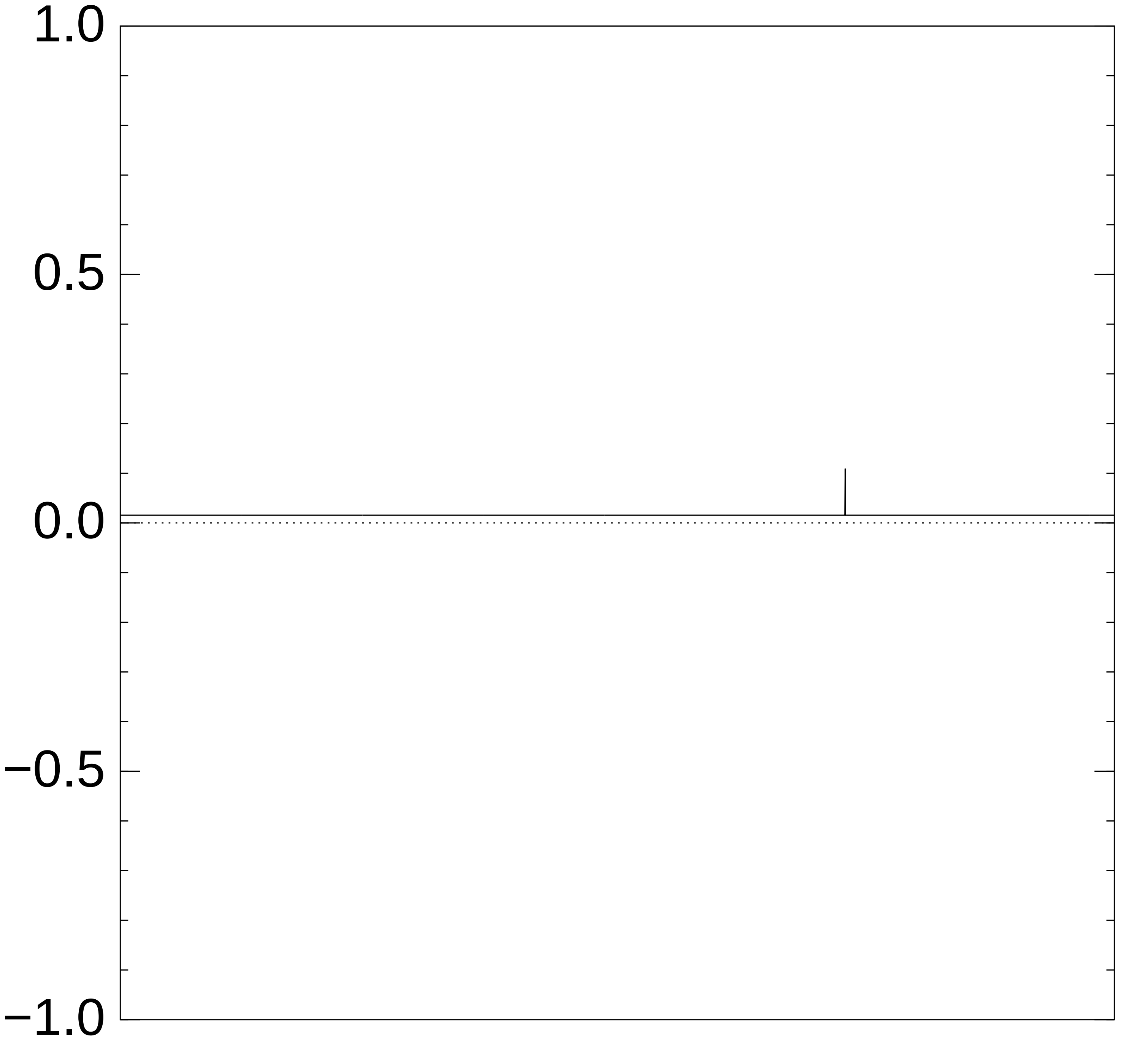
Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $3 \times$ (Step 1 + Step 2):



Step 1: Set $a \leftarrow b$ where
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$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
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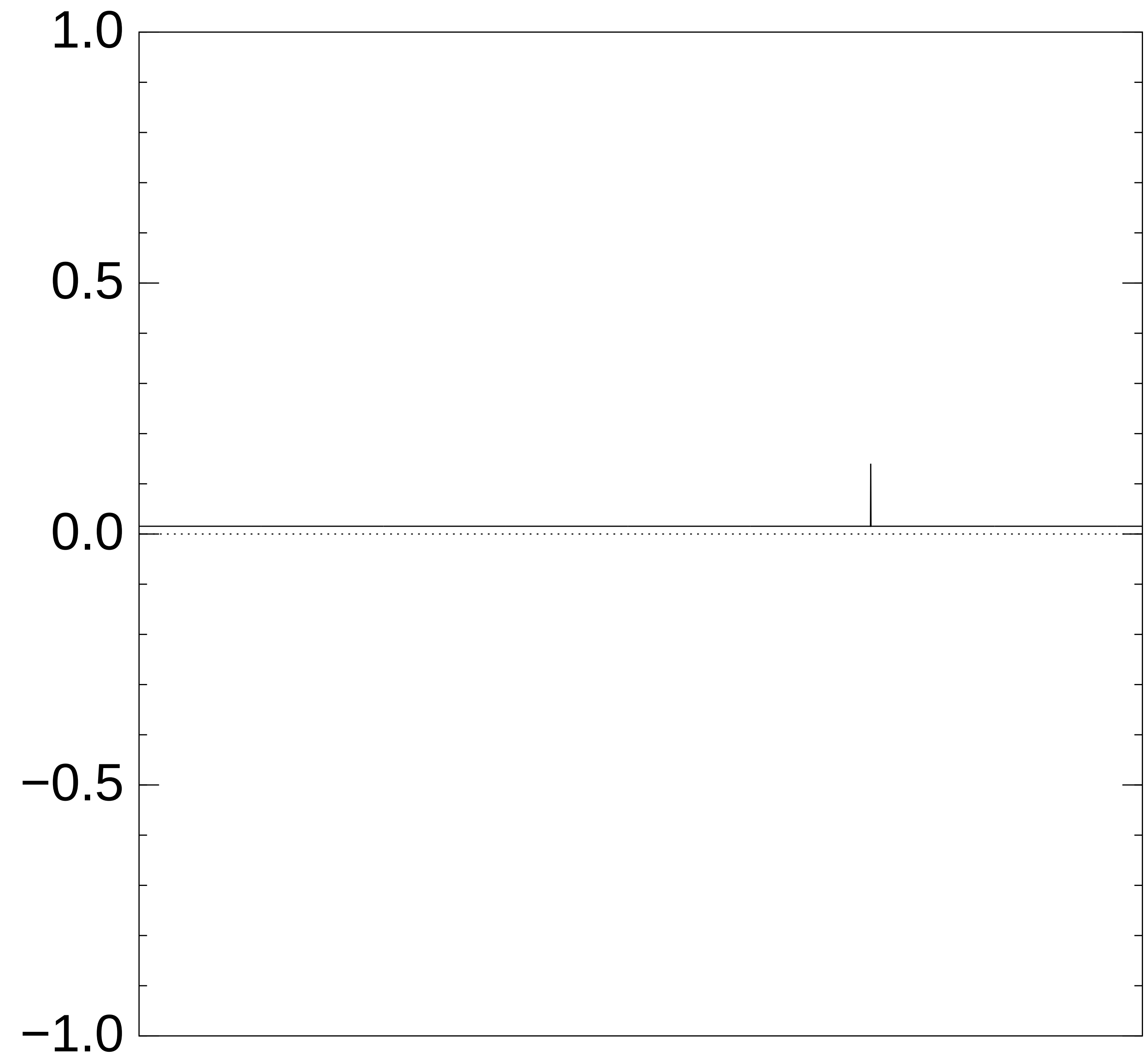
Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $4 \times$ (Step 1 + Step 2):



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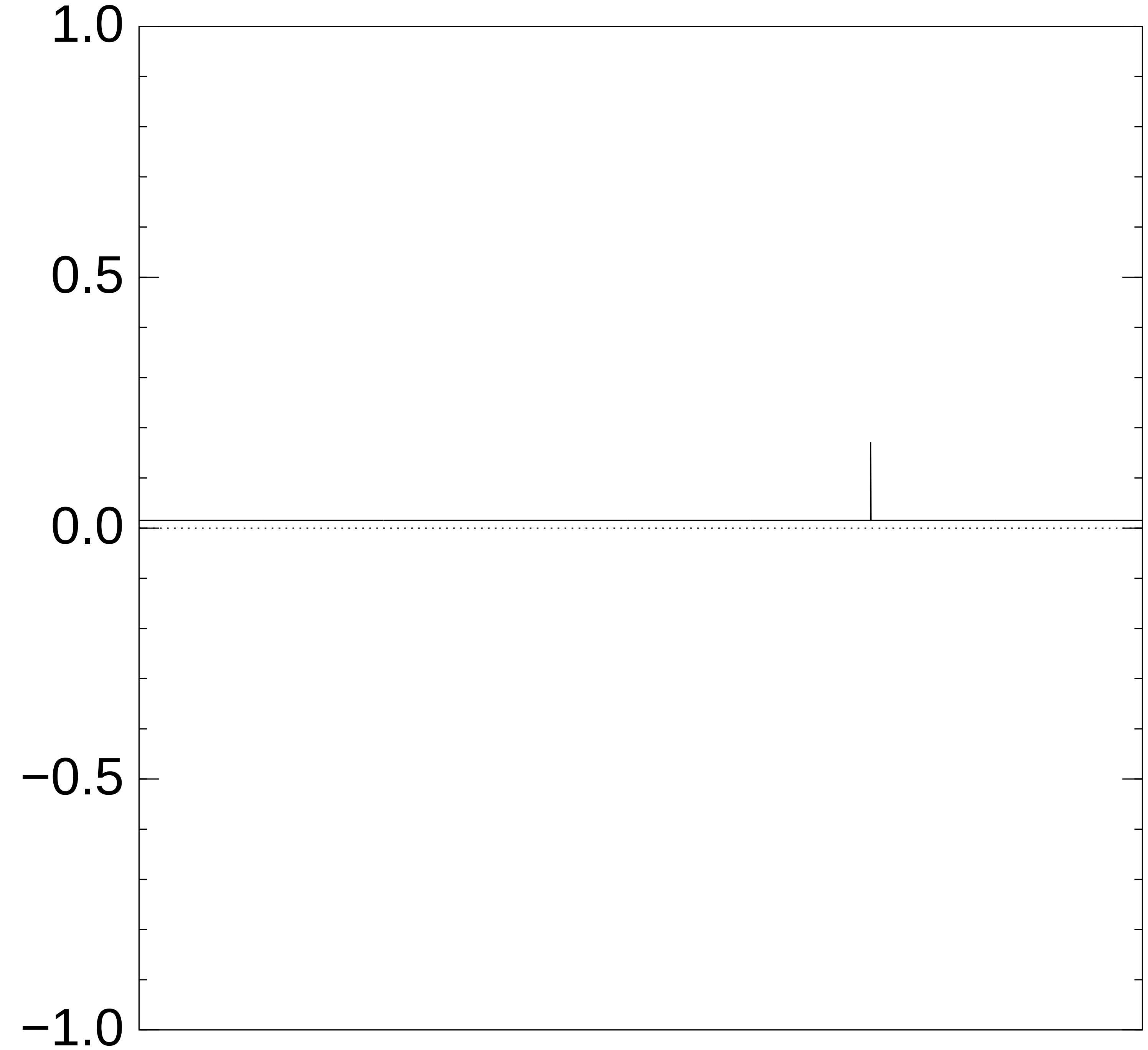
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $5 \times$ (Step 1 + Step 2):



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Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
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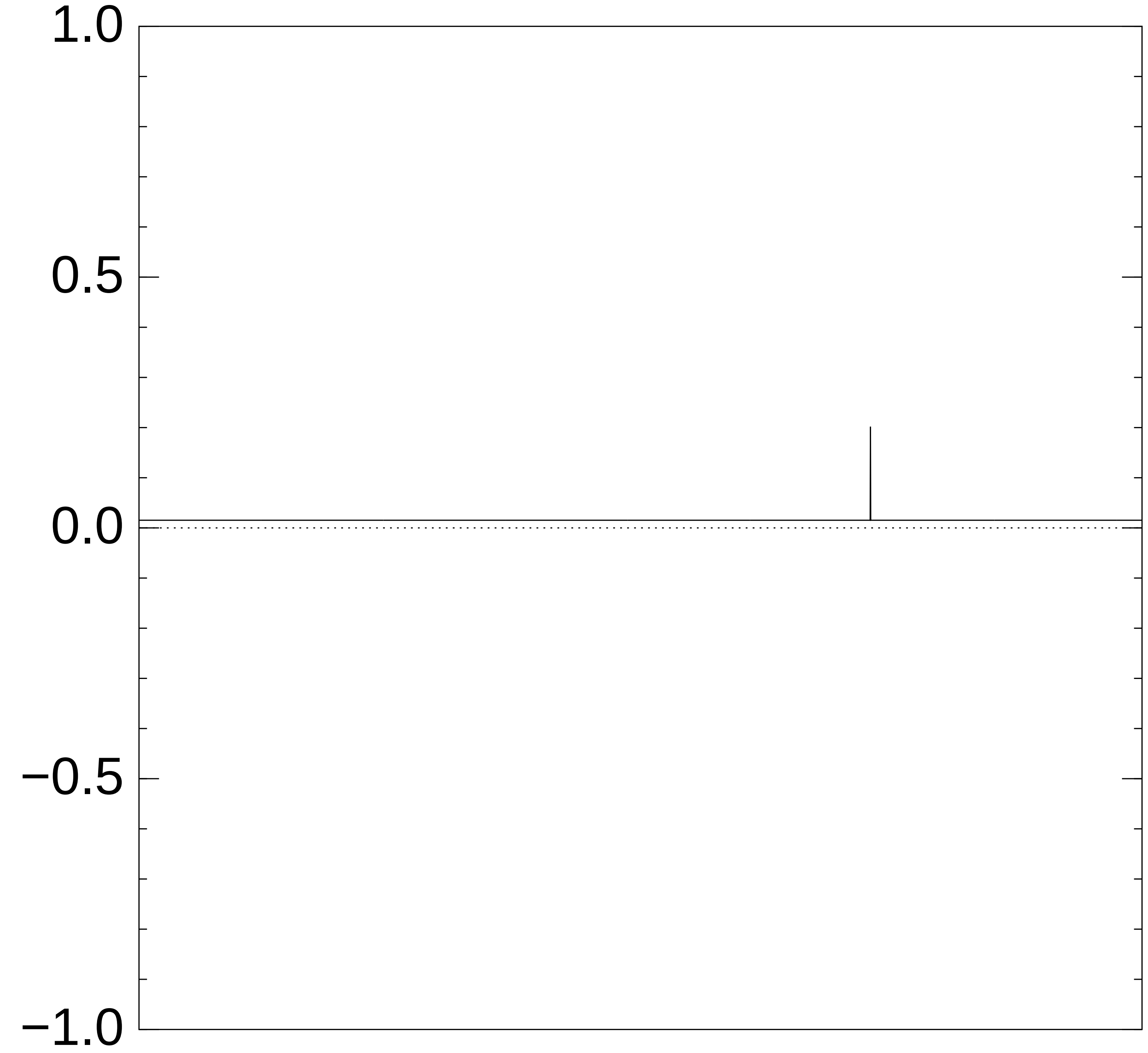
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $6 \times$ (Step 1 + Step 2):



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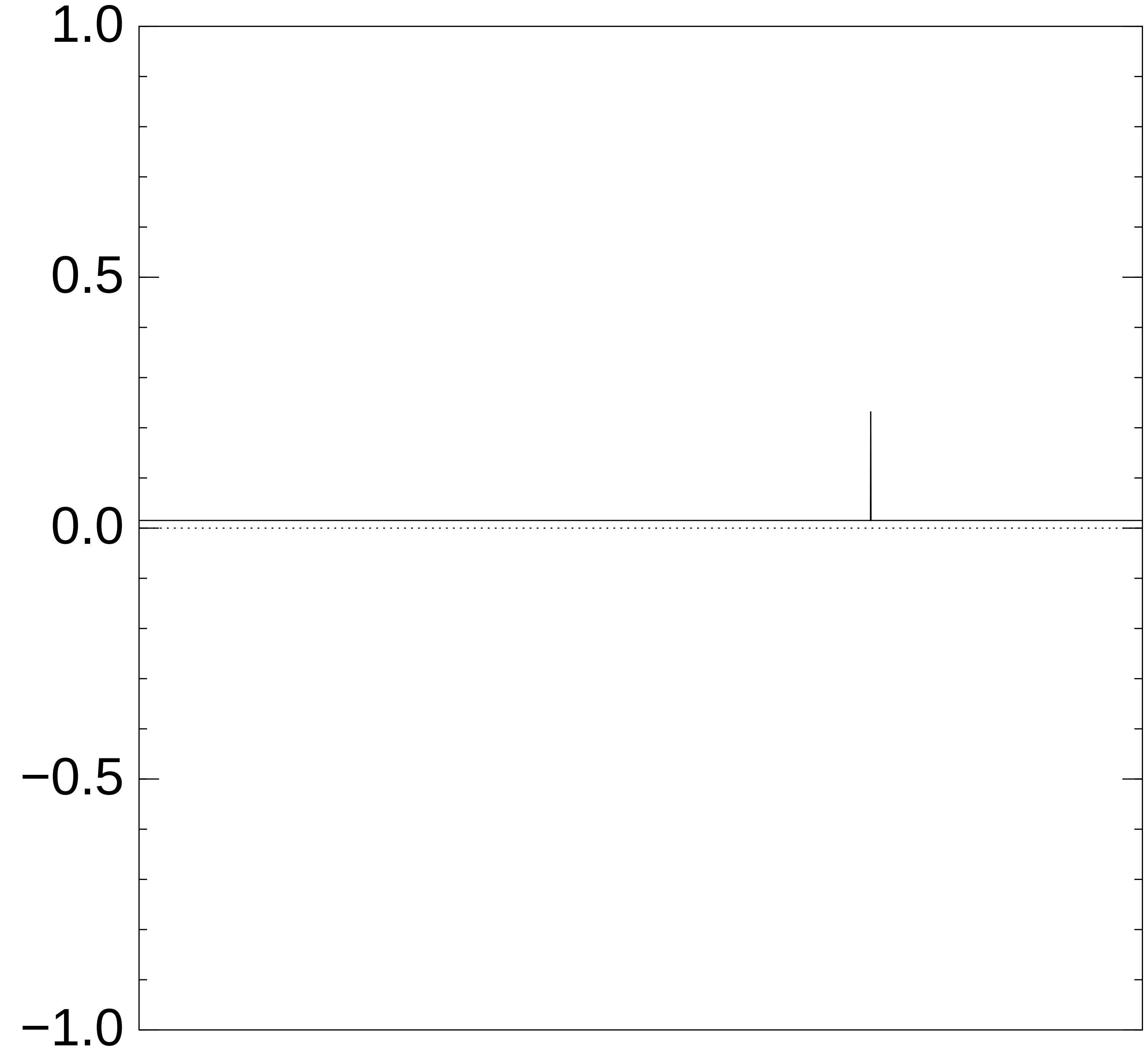
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $7 \times$ (Step $1 +$ Step 2):



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This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
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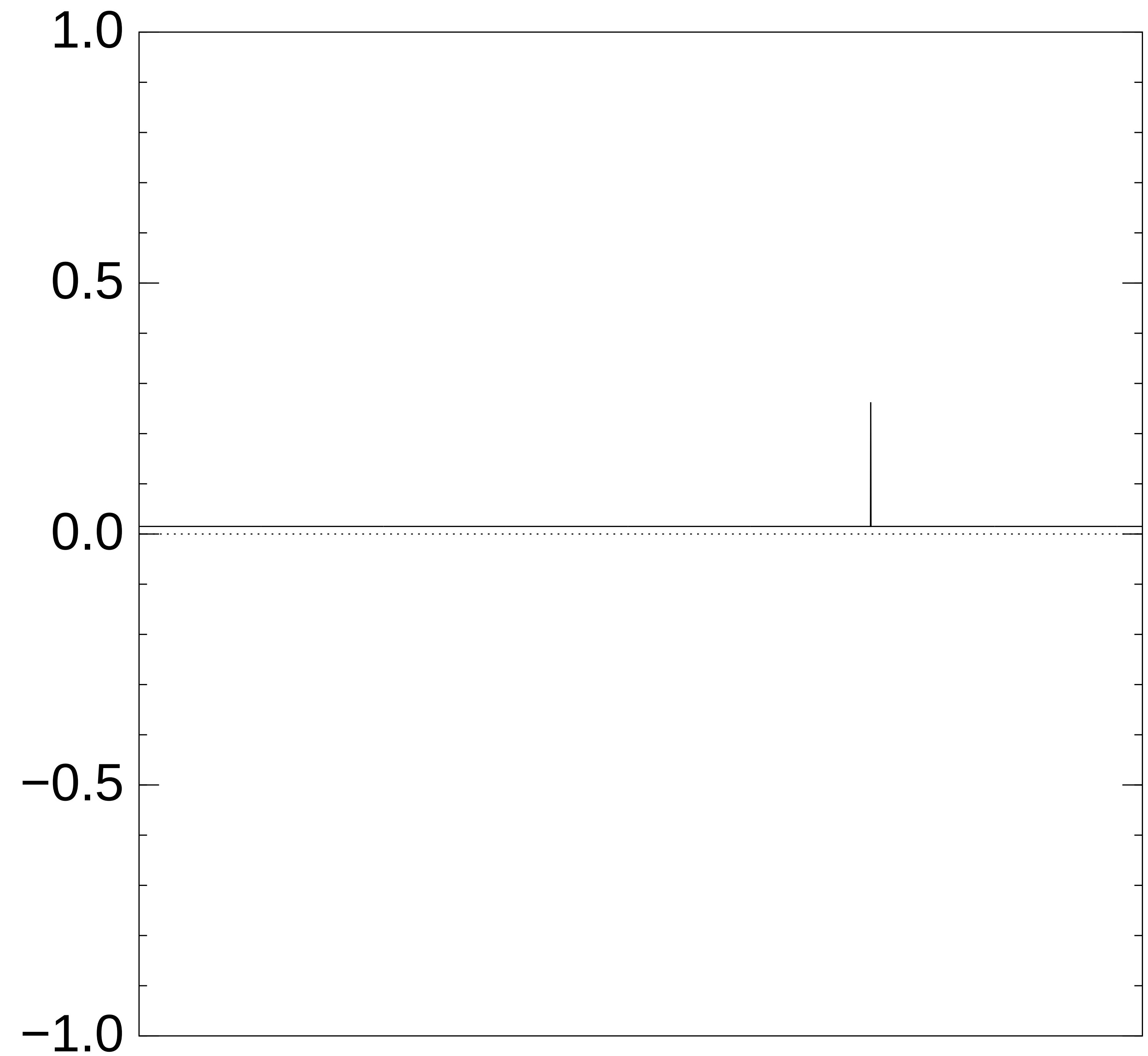
Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $8 \times$ (Step $1 +$ Step 2):



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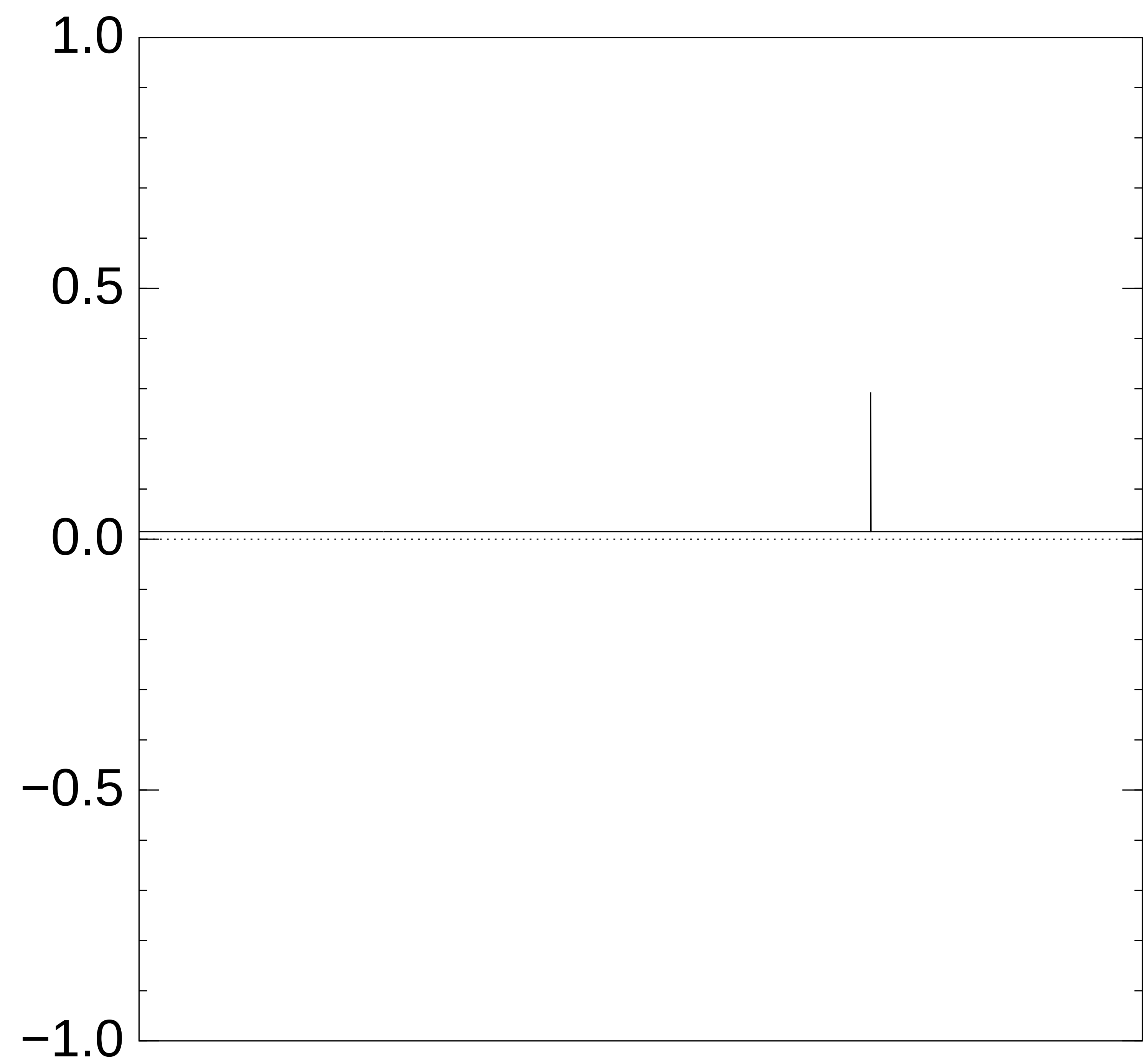
Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $9 \times$ (Step 1 + Step 2):



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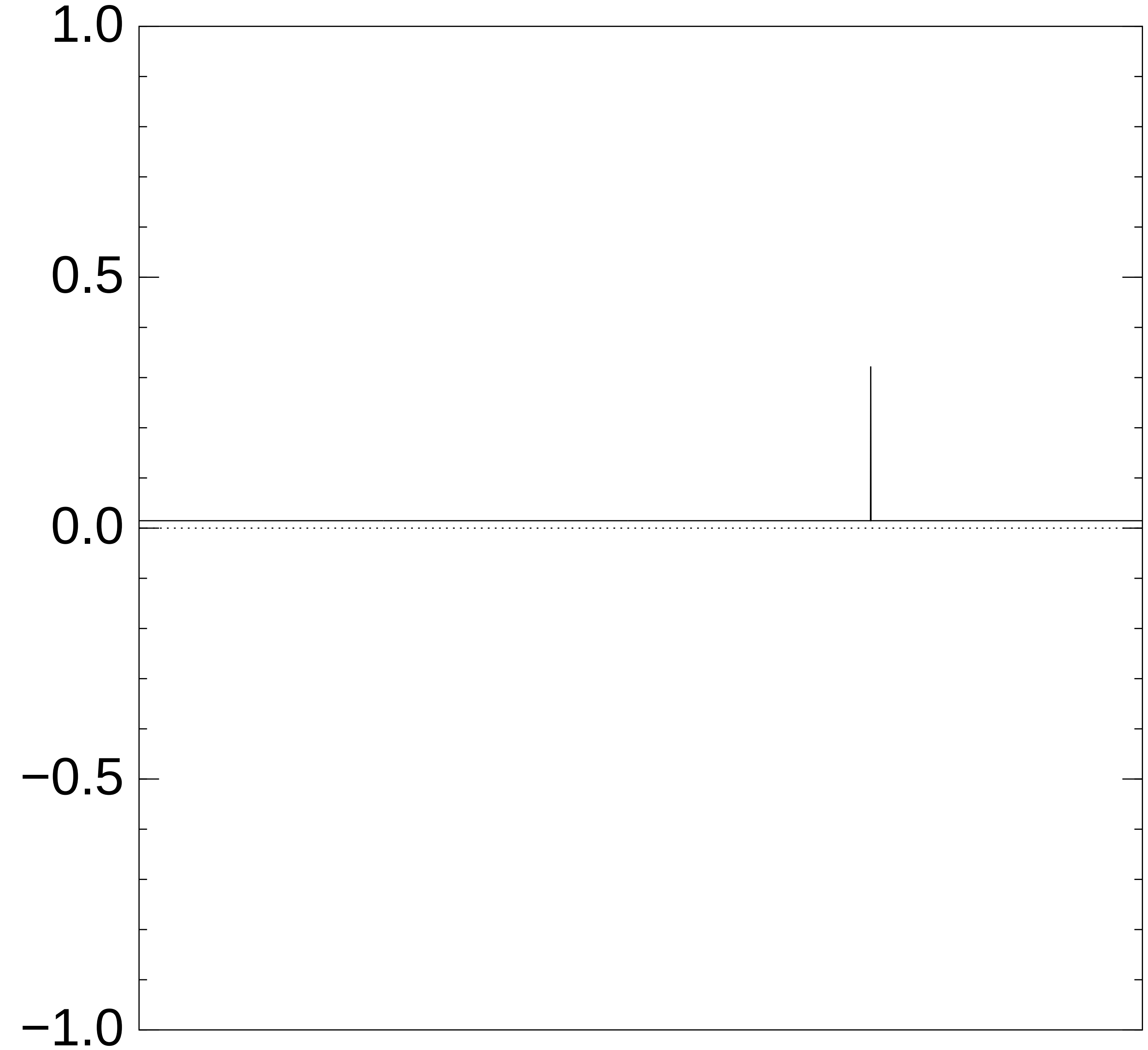
This is also easy.

Repeat steps 1 and 2 about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.

With high probability this finds the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$ for 36634 example with $n = 12$ after $10 \times$ (Step 1 + Step 2):



Step 1: Set $a \leftarrow b$ where

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Step 2: "Grover diffusion".

Set $a \leftarrow b$ where

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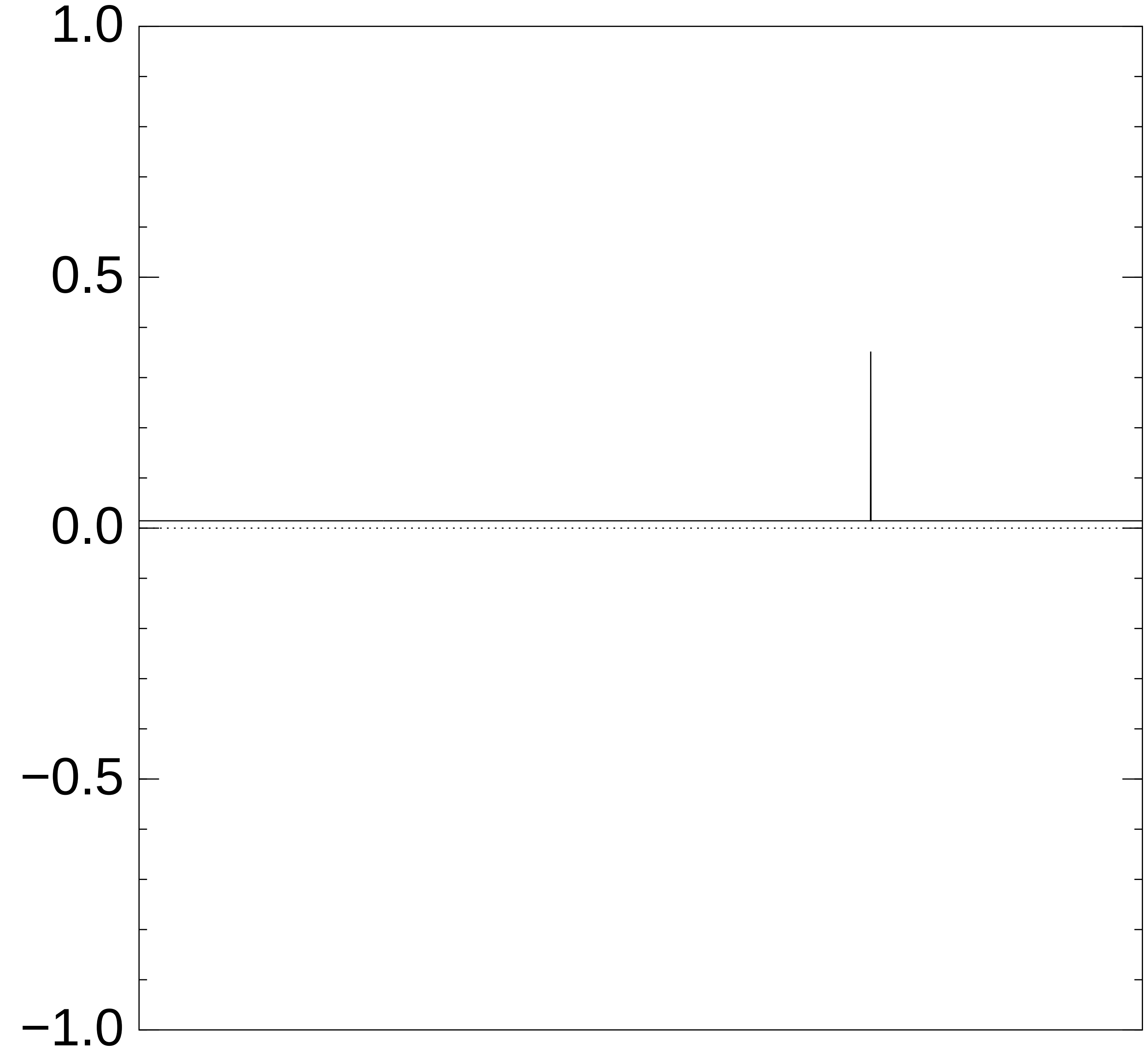
This is also easy.

Repeat steps 1 and 2 about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.

With high probability this finds the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$ for 36634 example with $n = 12$ after $11 \times$ (Step $1 +$ Step 2):



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Measure the n qubits.

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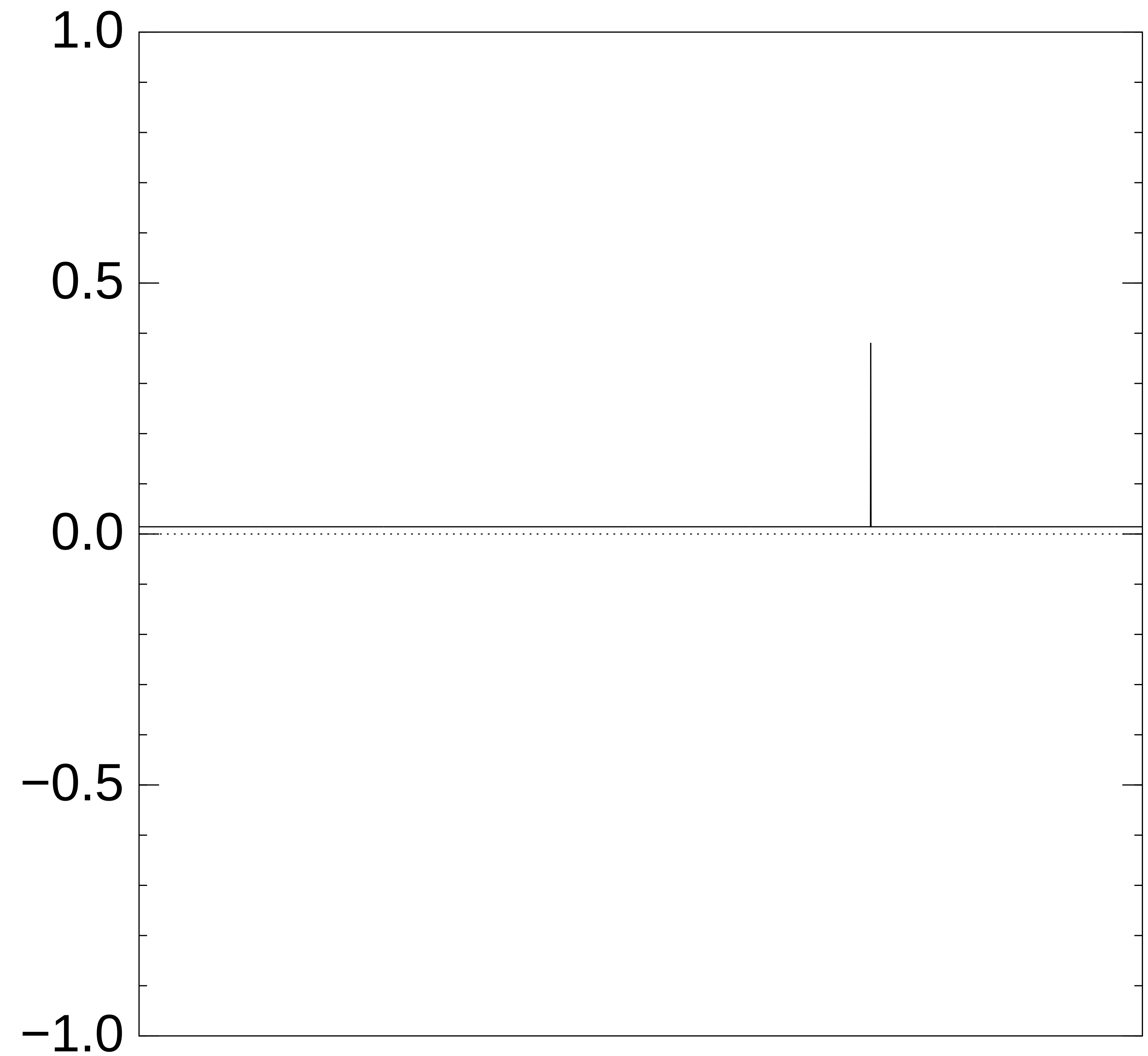
Graph of $J \mapsto a_J$ for 36634 example with $n = 12$ after $12 \times$ (Step 1 + Step 2):



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Measure the n qubits.
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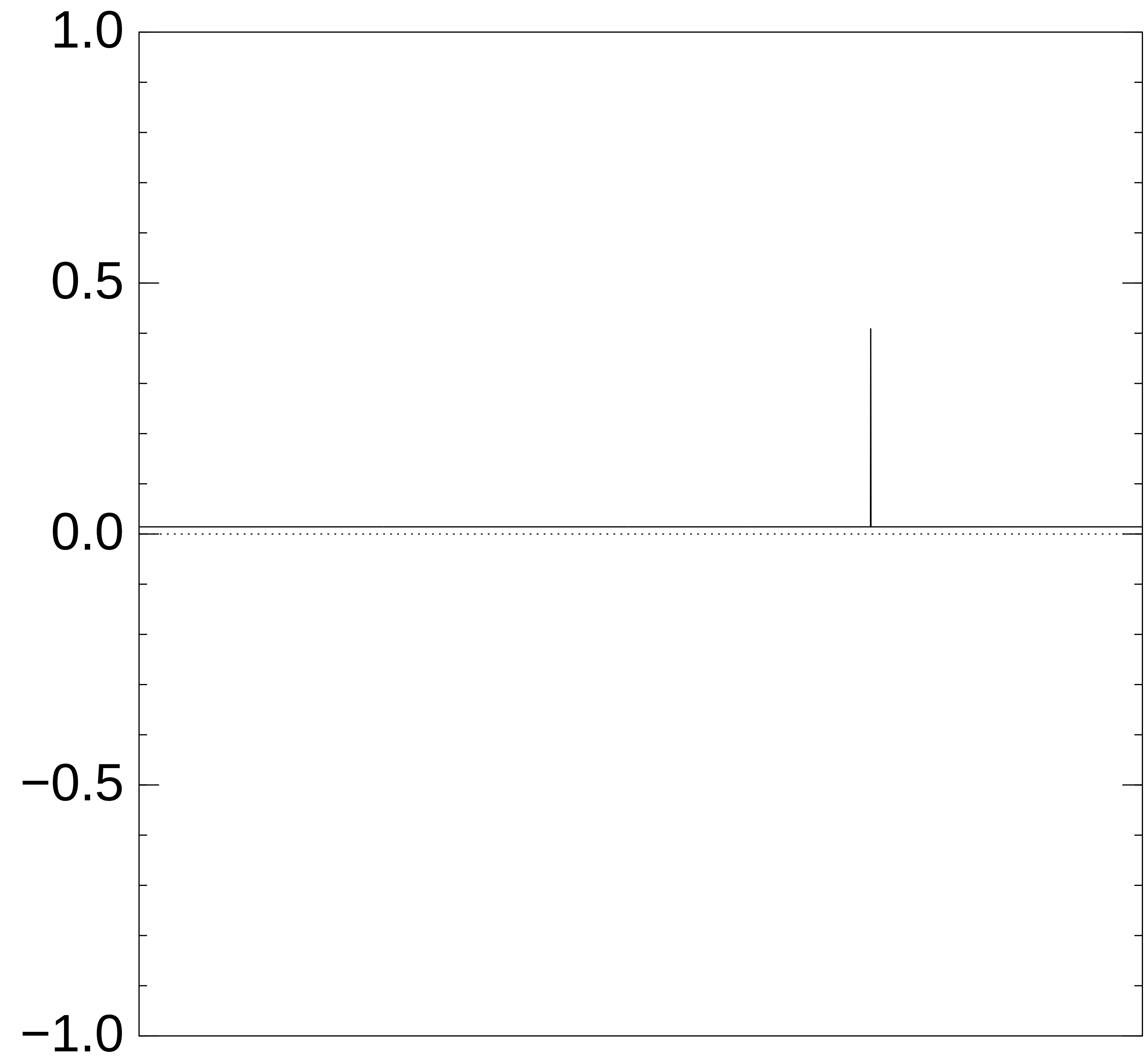
Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $13 \times$ (Step 1 + Step 2):



Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
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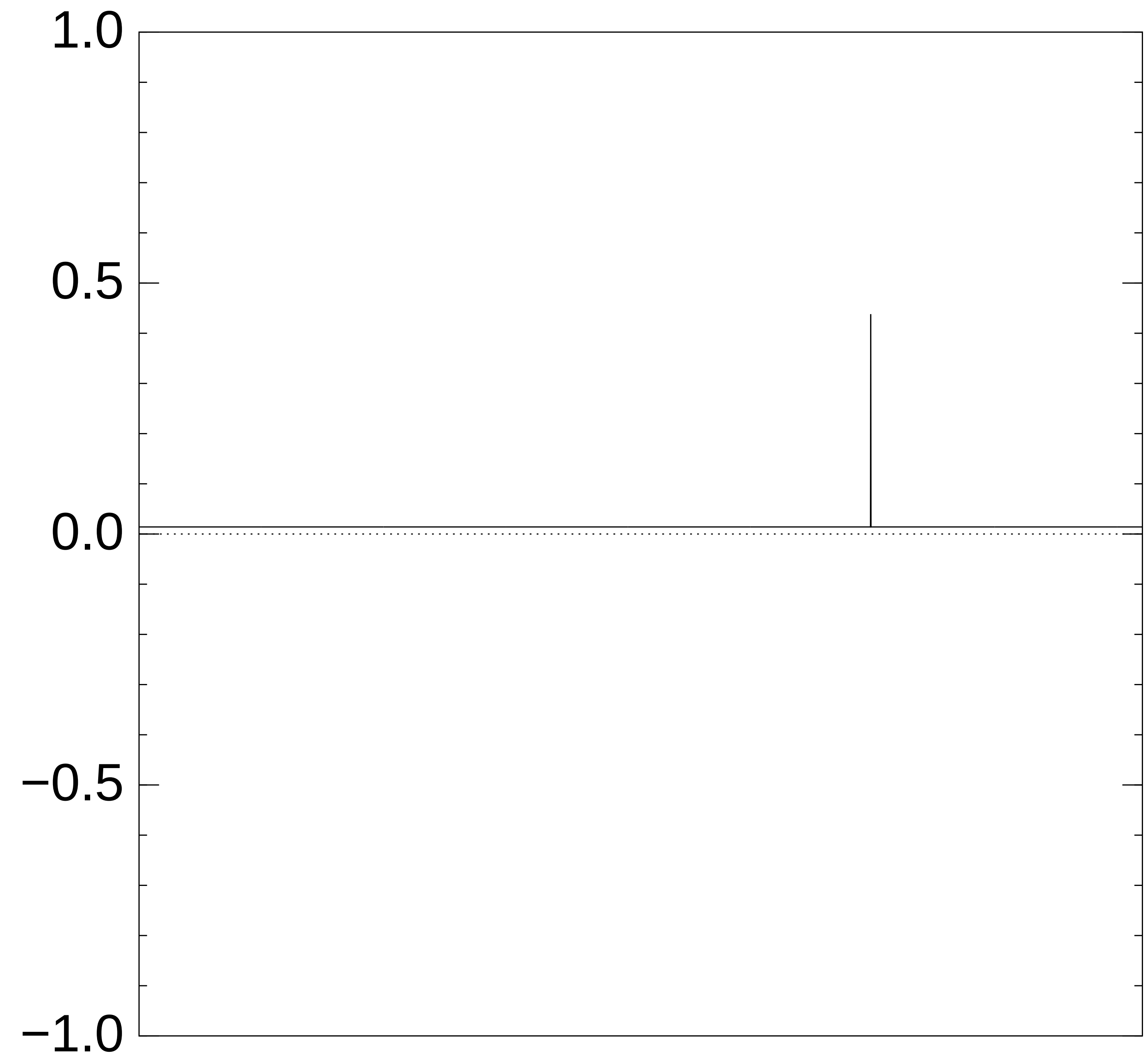
Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $14 \times$ (Step 1 + Step 2):



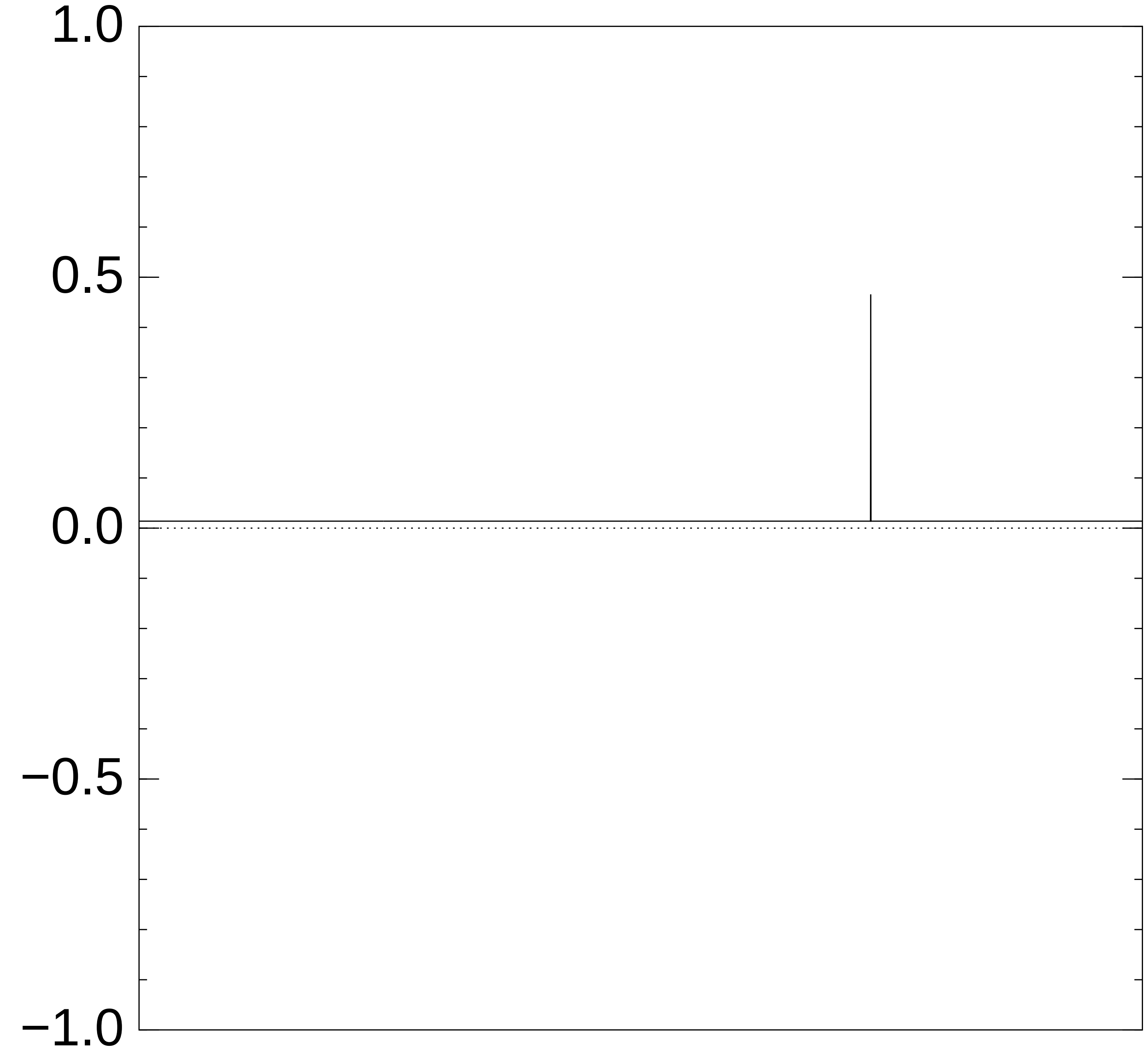
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $15 \times$ (Step 1 + Step 2):



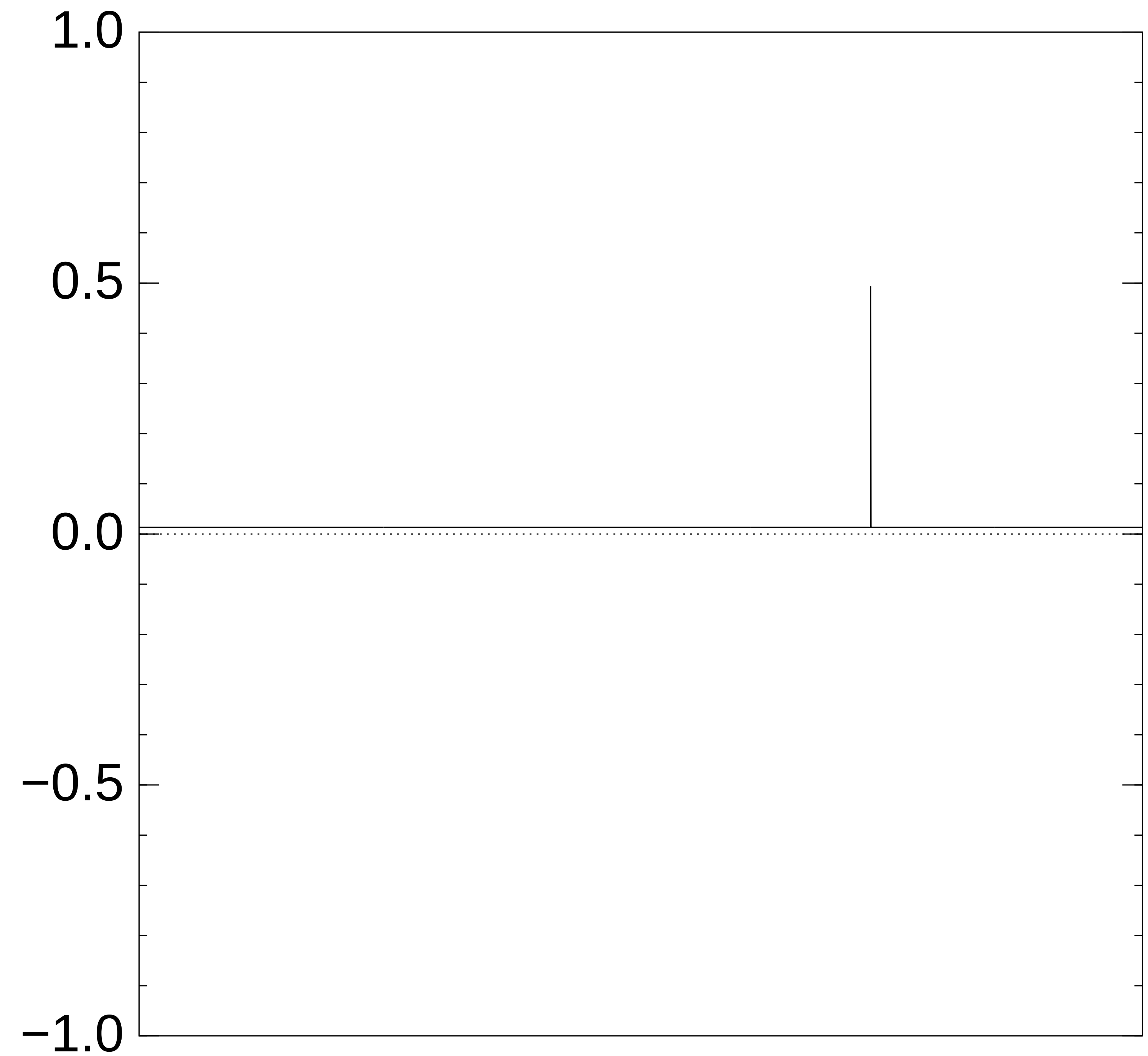
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n)\sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $16 \times$ (Step 1 + Step 2):



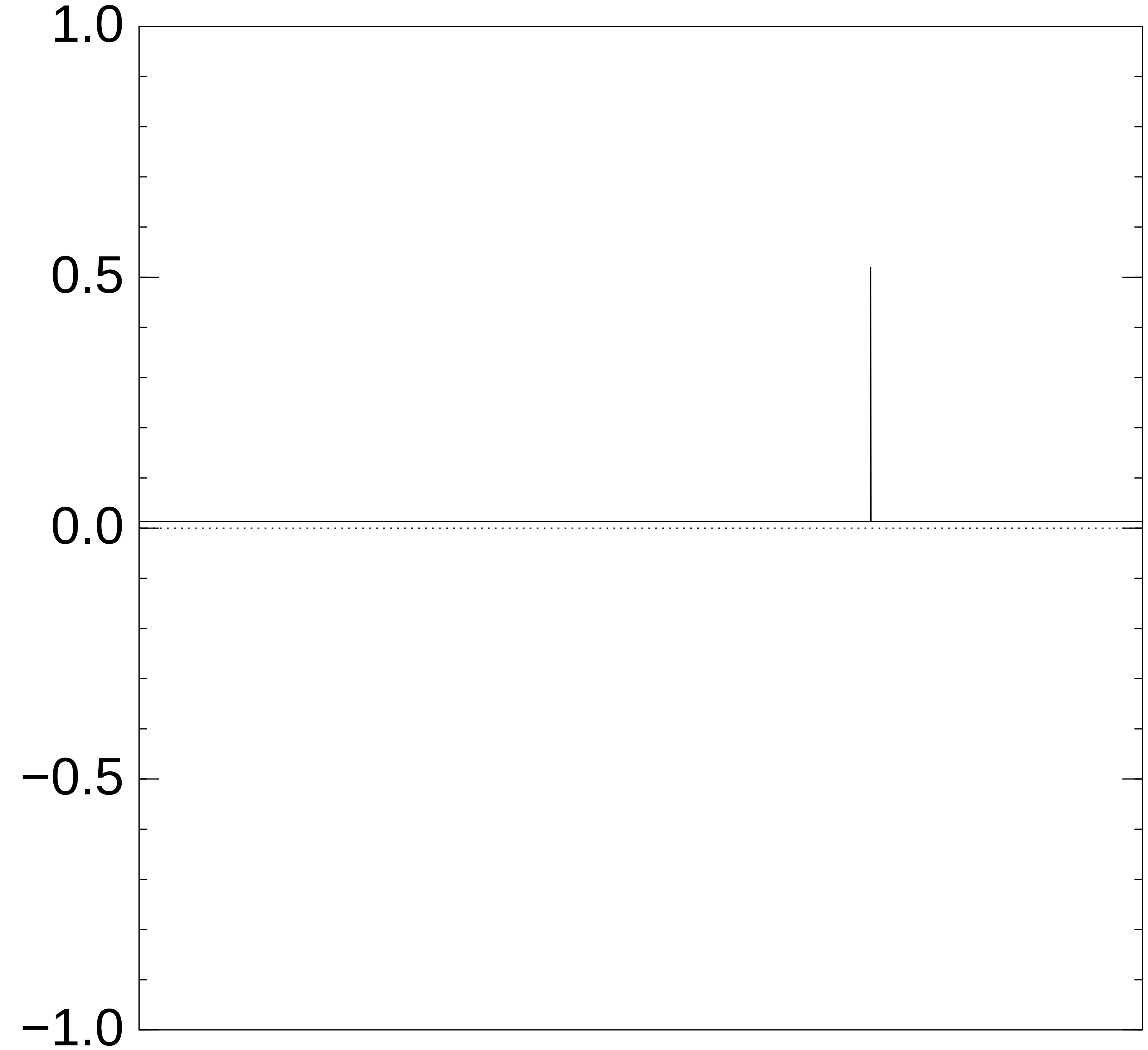
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $17 \times$ (Step 1 + Step 2):



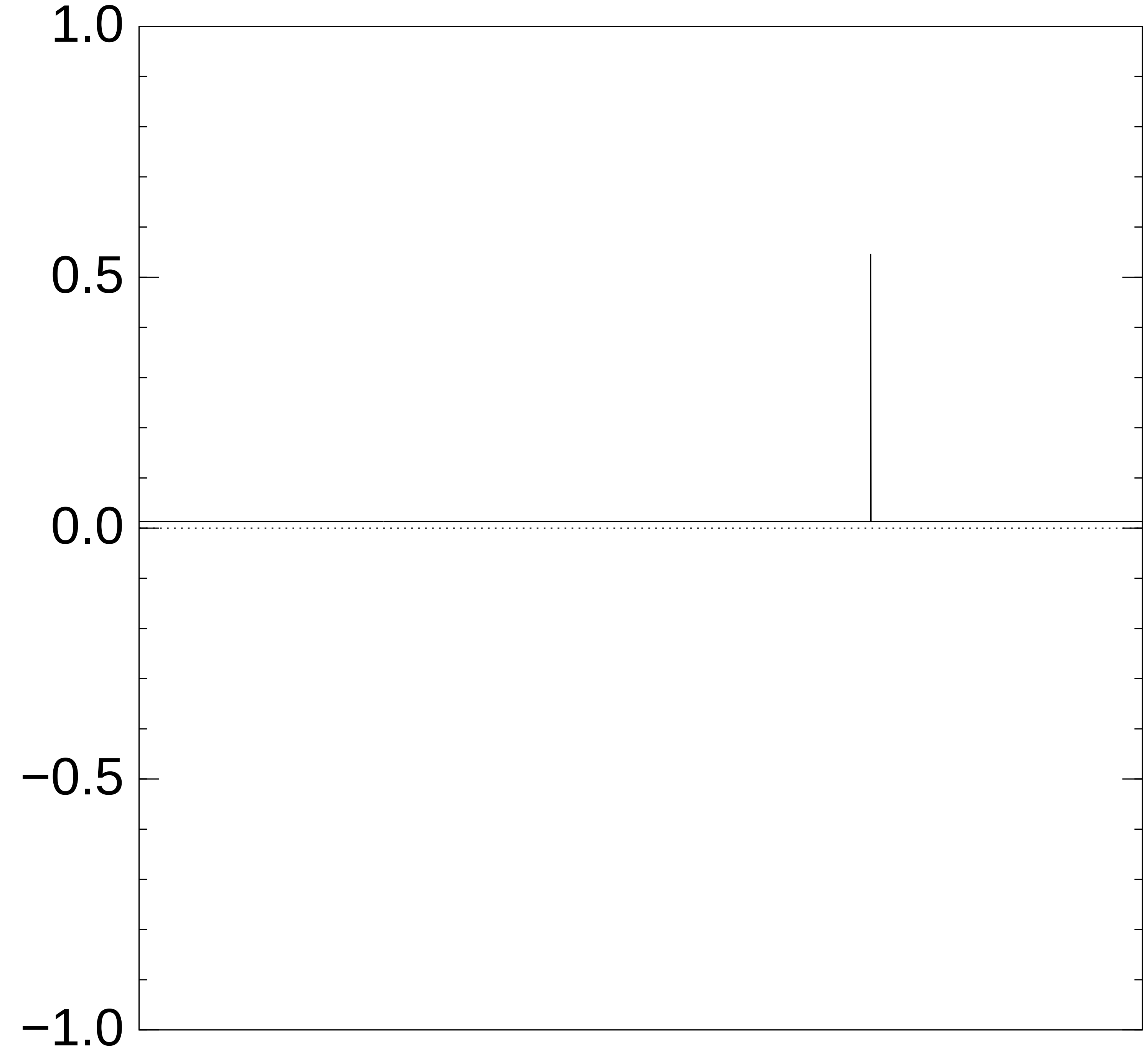
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $18 \times$ (Step 1 + Step 2):



Step 1: Set $a \leftarrow b$ where

$b_J = -a_J$ if $\Sigma(J) = t$,

$b_J = a_J$ otherwise.

This is about as easy

as computing Σ.

Step 2: "Grover diffusion".

Set $a \leftarrow b$ where

$b_J = -a_J + (2/2^n) \sum_I a_I$.

This is also easy.

Repeat steps 1 and 2

about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.

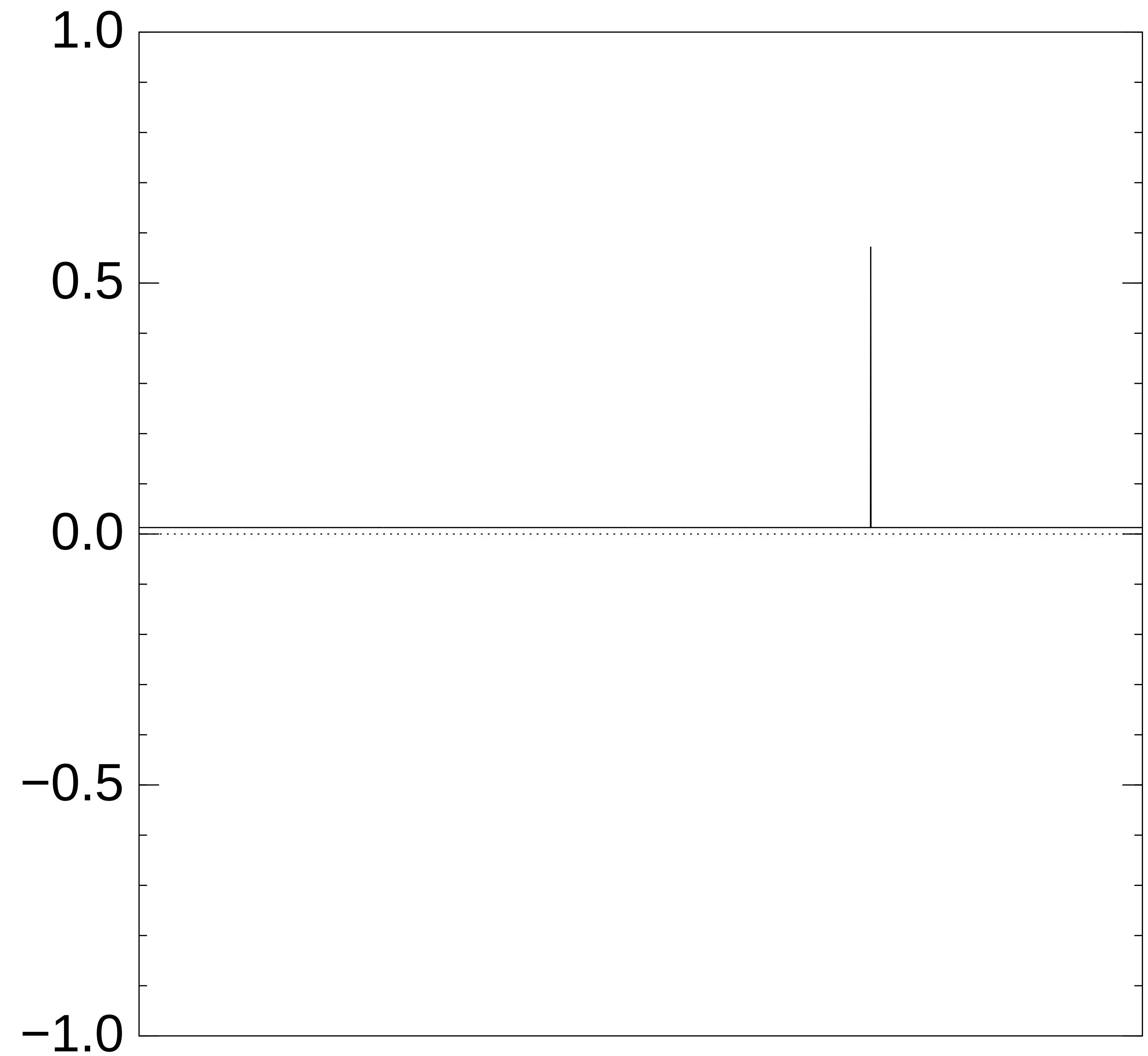
With high probability this finds

the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$

for 36634 example with $n = 12$

after $19 \times$ (Step 1 + Step 2):



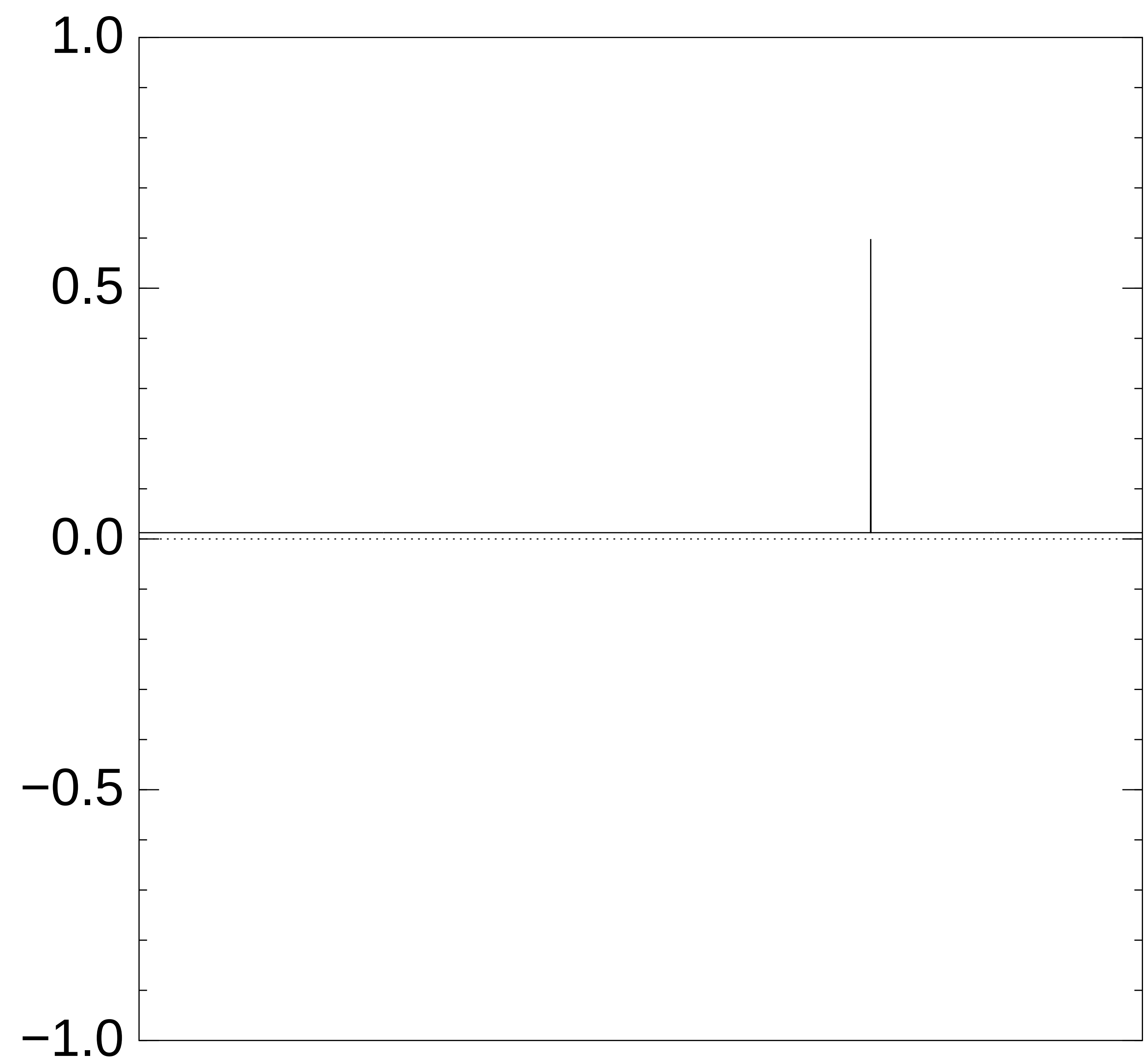
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $20 \times$ (Step 1 + Step 2):



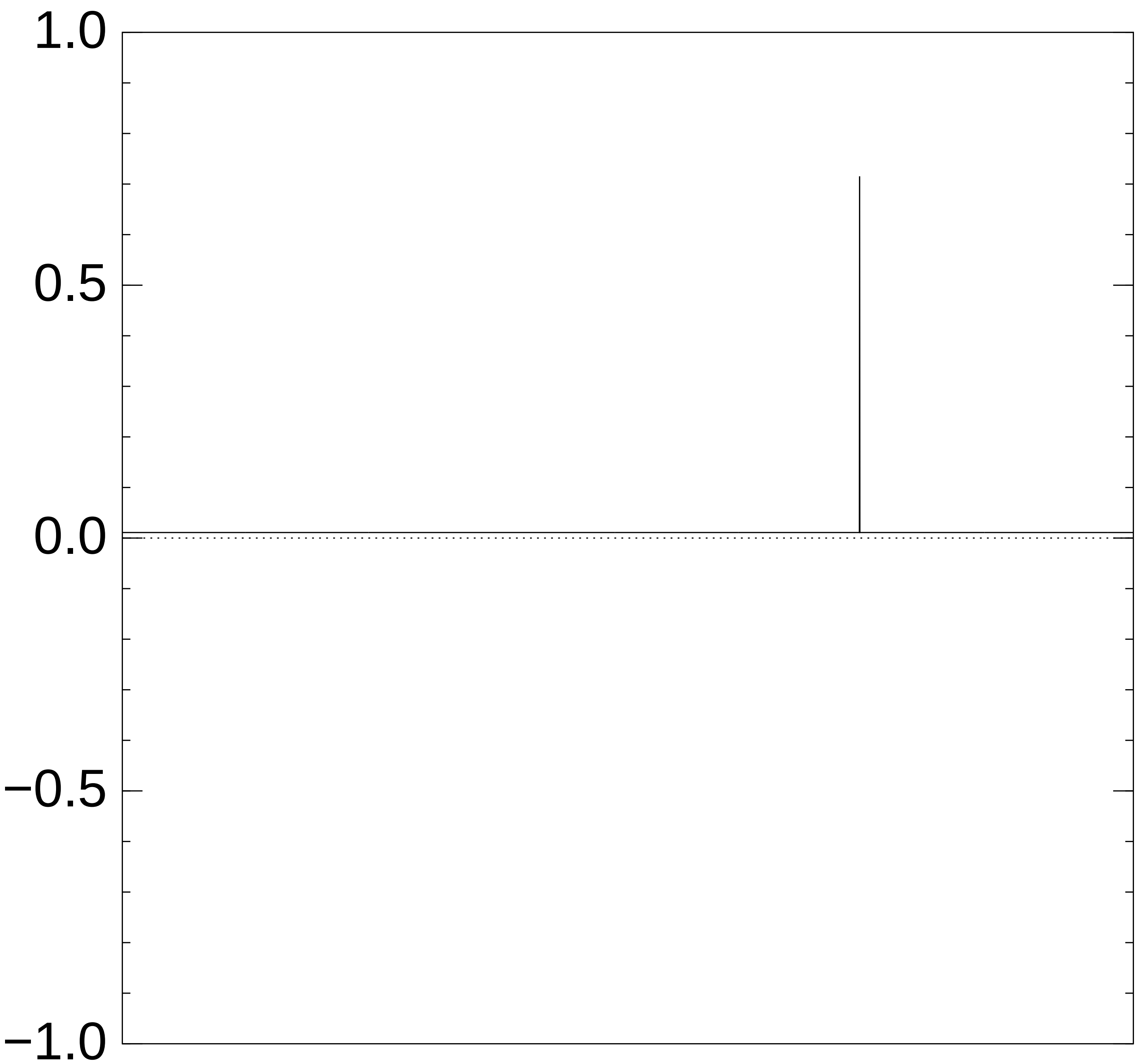
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $25 \times$ (Step 1 + Step 2):



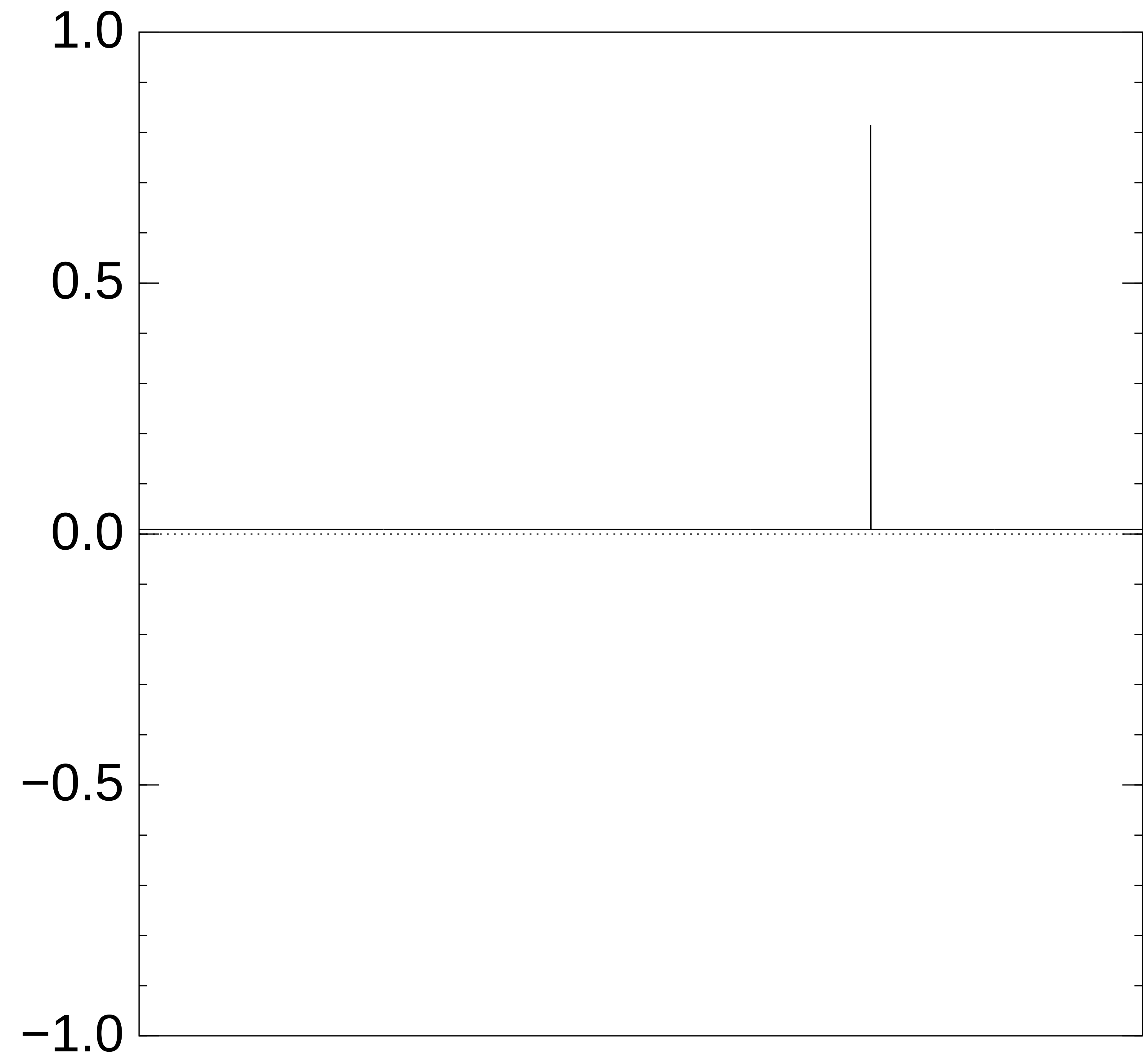
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n)\sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $30 \times$ (Step 1 + Step 2):



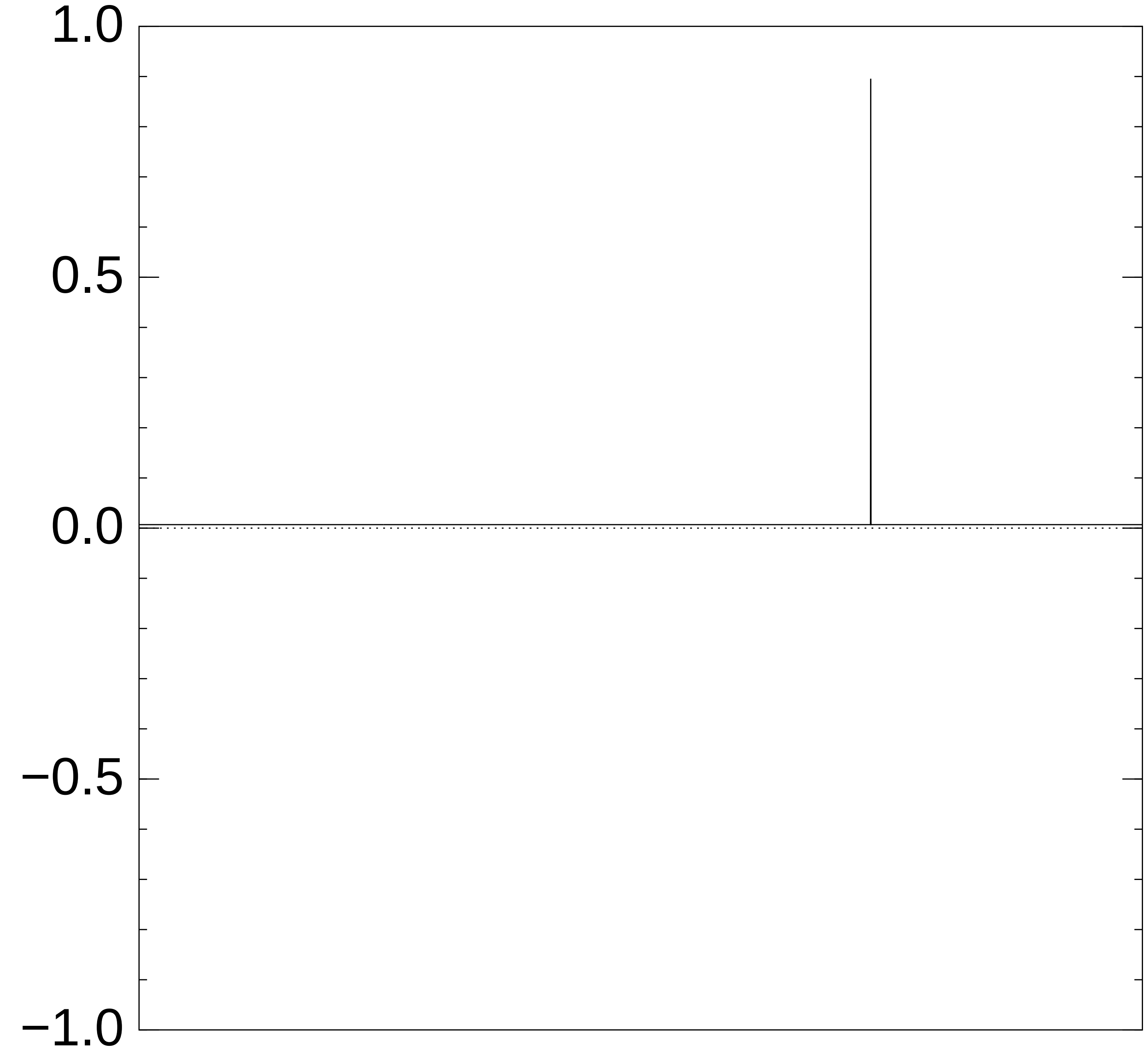
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $35 \times$ (Step $1 +$ Step 2):



Good moment to stop, measure.

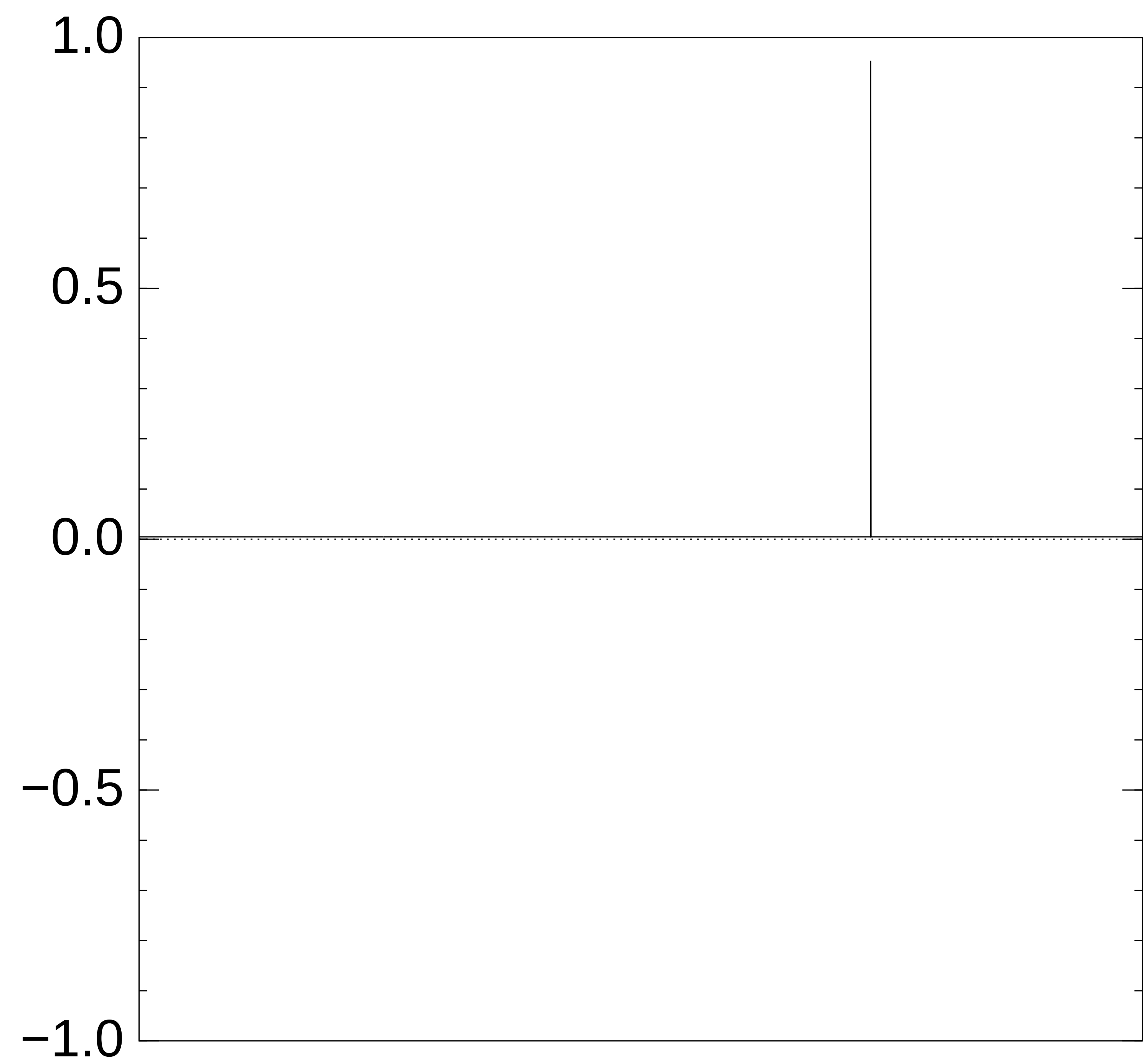
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $40 \times$ (Step 1 + Step 2):



Step 1: Set $a \leftarrow b$ where

$b_J = -a_J$ if $\Sigma(J) = t$,

$b_J = a_J$ otherwise.

This is about as easy

as computing Σ.

Step 2: "Grover diffusion".

Set $a \leftarrow b$ where

$b_J = -a_J + (2/2^n) \sum_I a_I$.

This is also easy.

Repeat steps 1 and 2

about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.

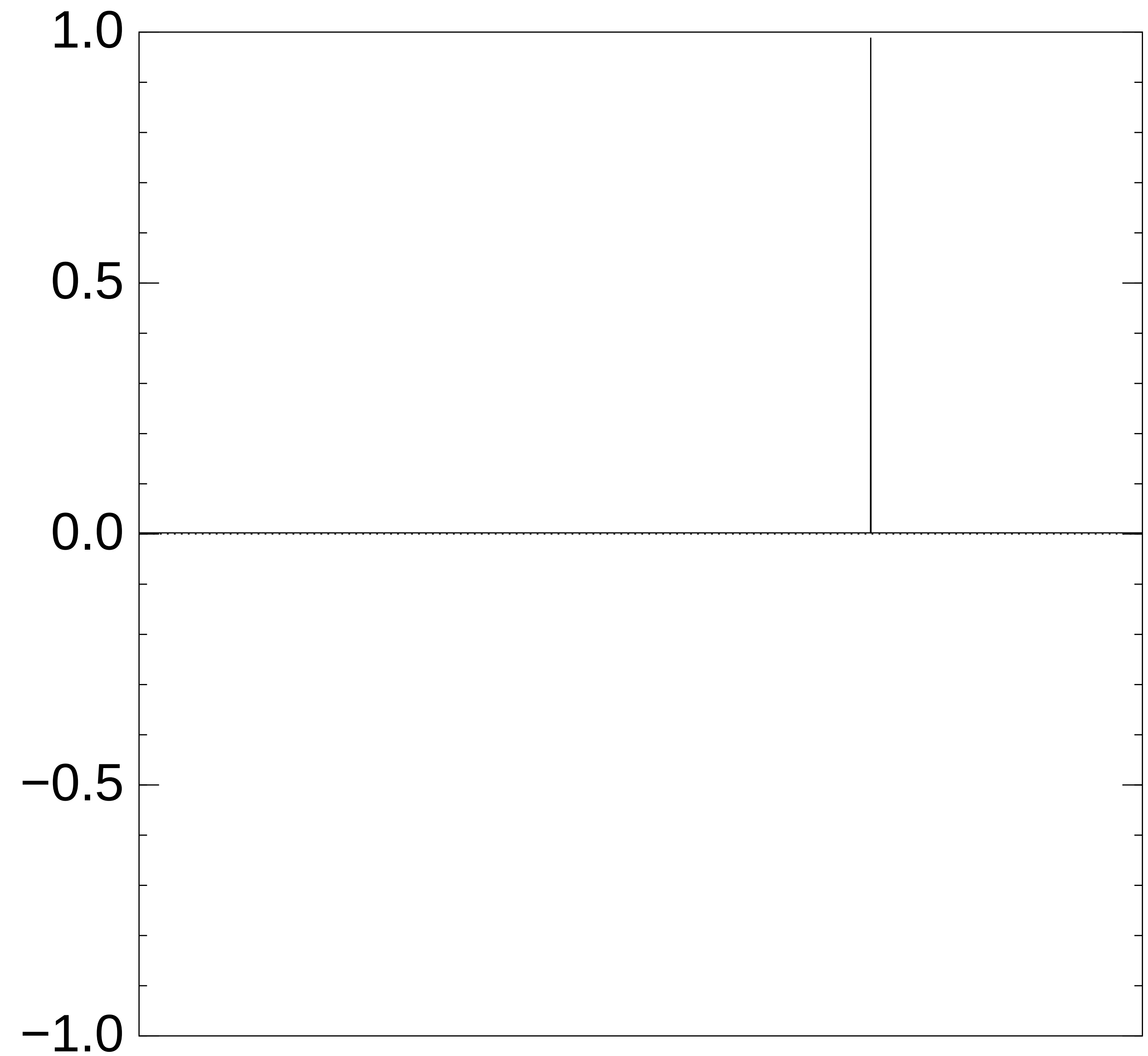
With high probability this finds

the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$

for 36634 example with $n = 12$

after $45 \times$ (Step 1 + Step 2):



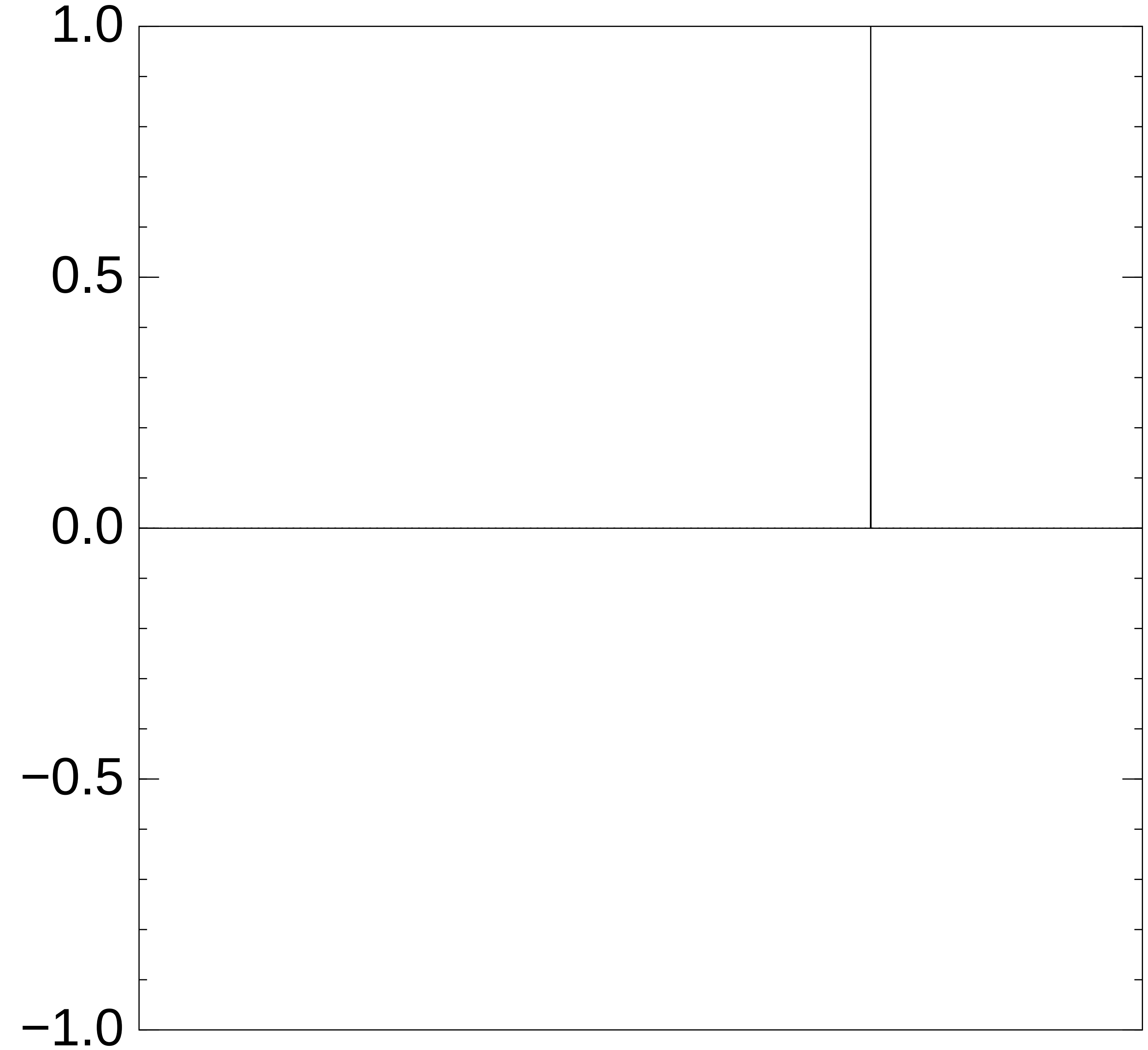
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n)\sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $50 \times$ (Step $1 +$ Step 2):



Traditional stopping point.

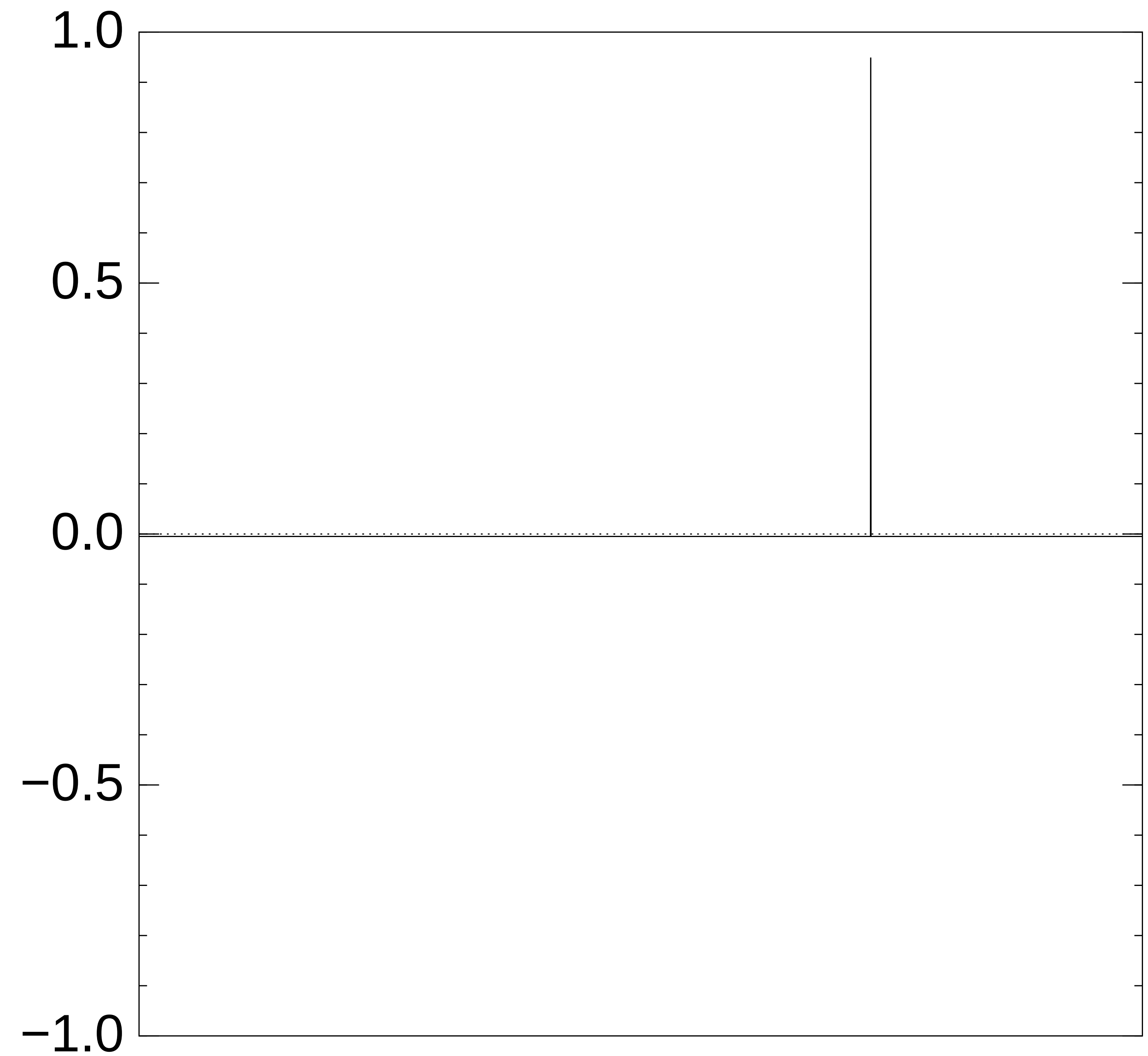
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $60 \times$ (Step 1 + Step 2):



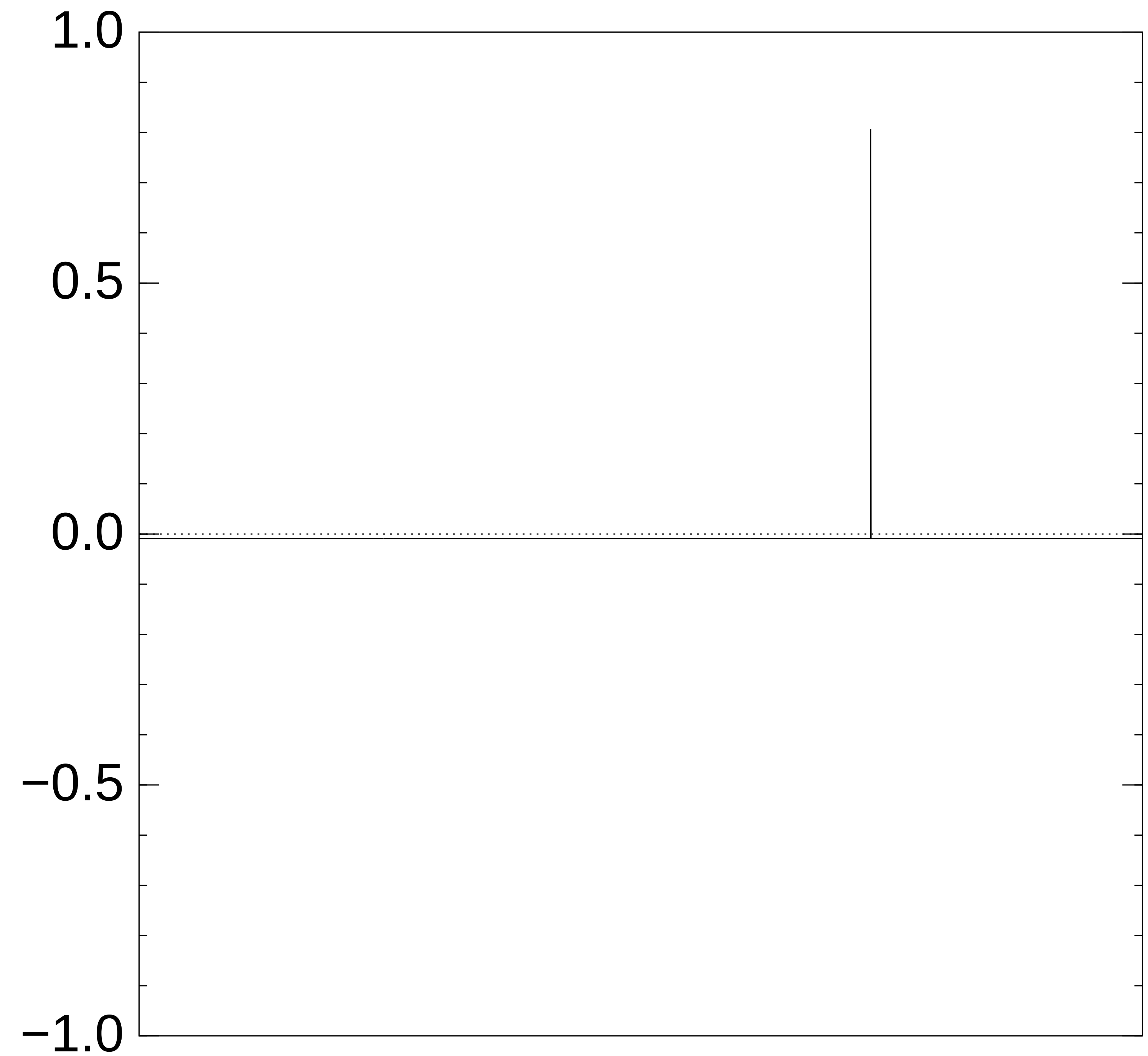
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $70 \times$ (Step 1 + Step 2):



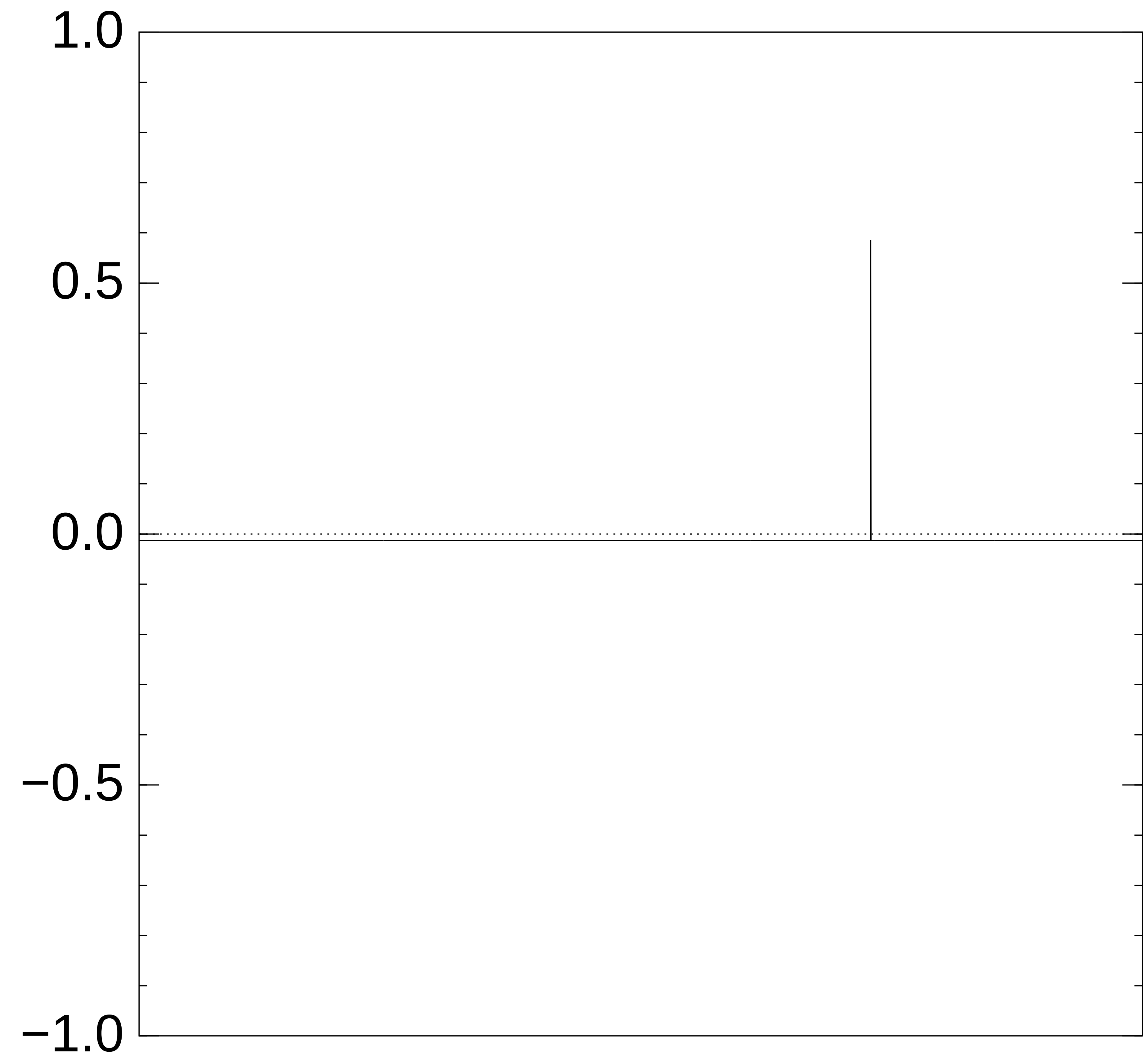
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $80 \times$ (Step 1 + Step 2):



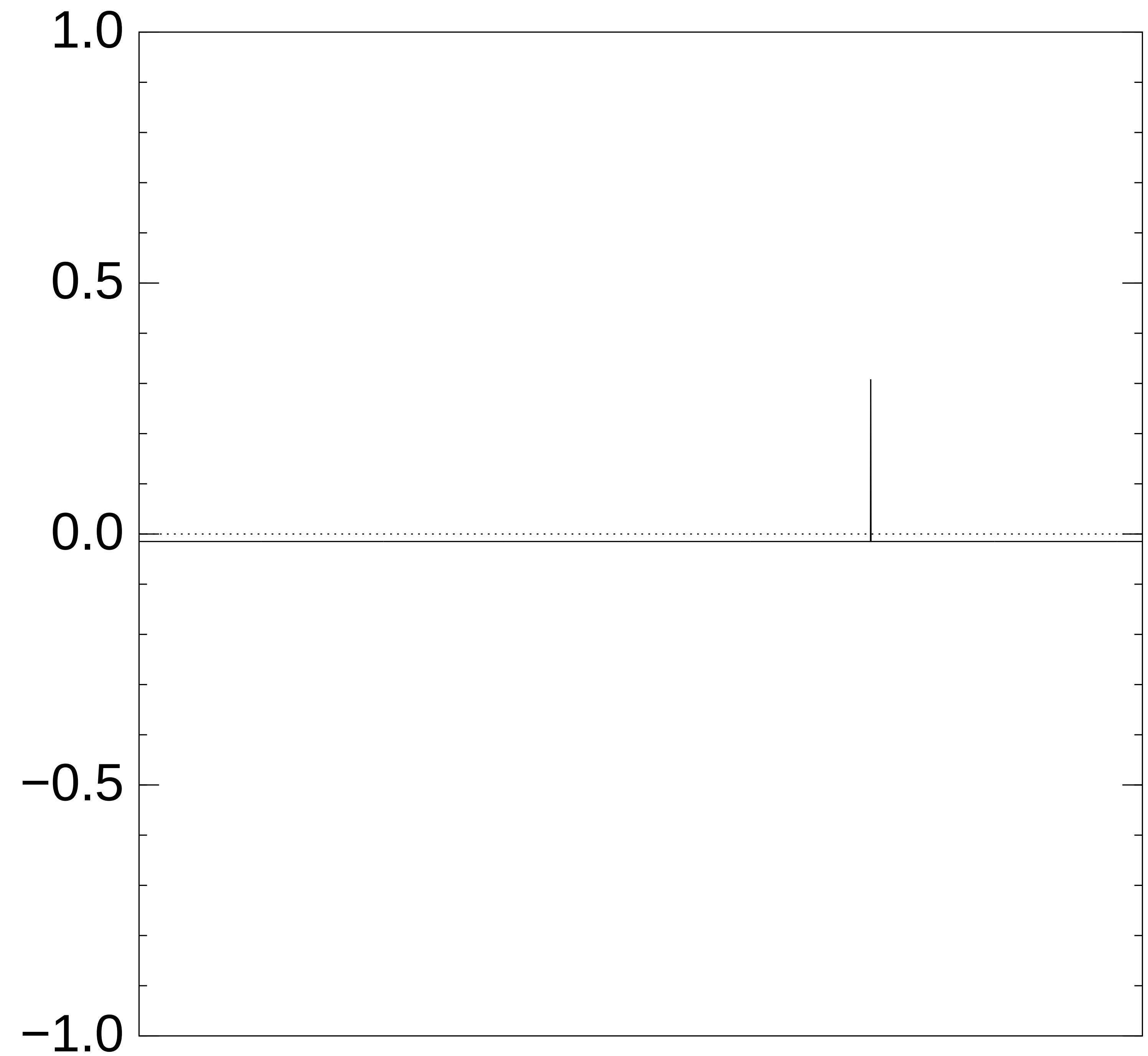
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n)\sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $90 \times$ (Step 1 + Step 2):



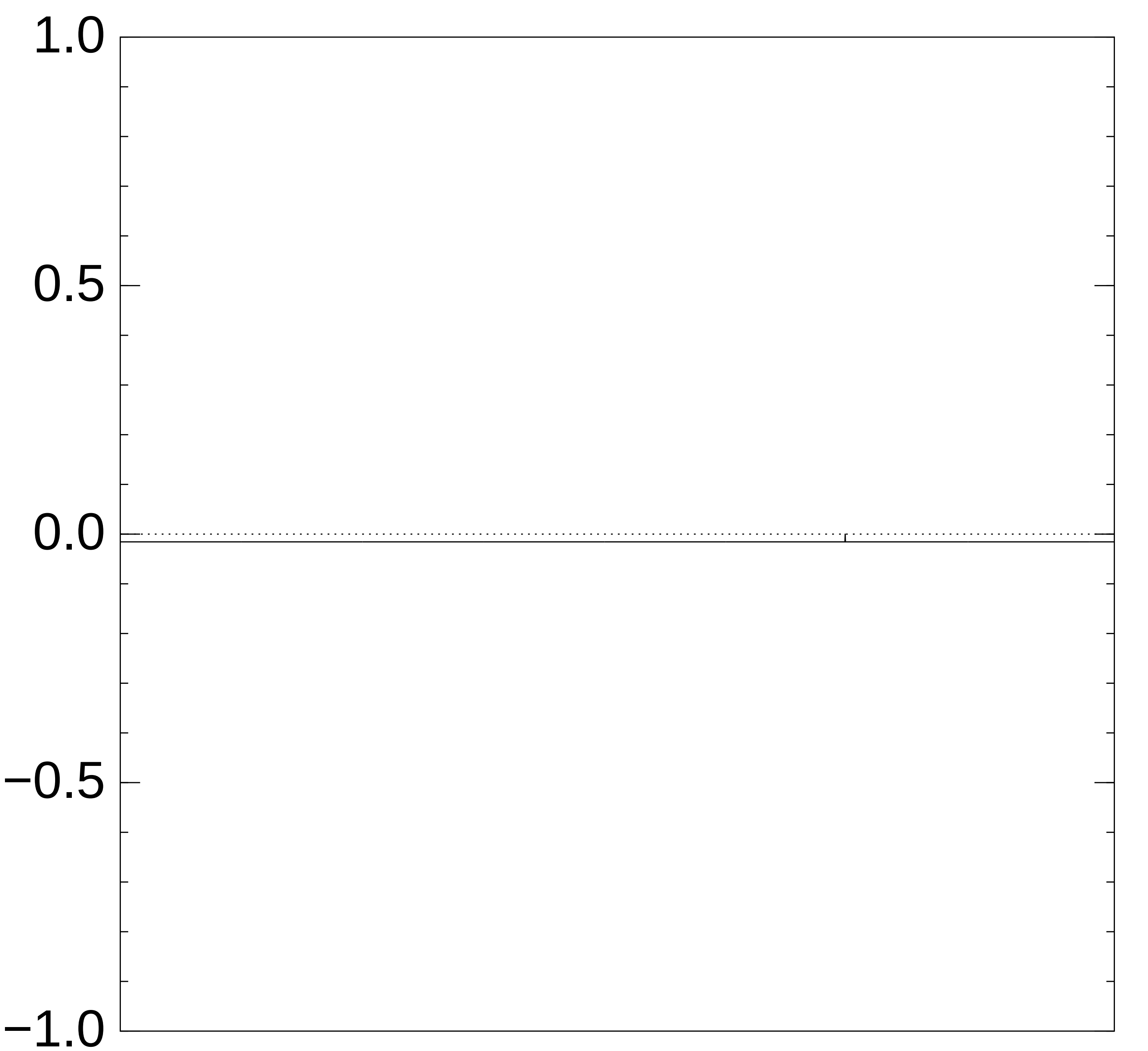
Step 1: Set $a \leftarrow b$ where
$b_J = -a_J$ if $\Sigma(J) = t$,
$b_J = a_J$ otherwise.
This is about as easy
as computing Σ.

Step 2: "Grover diffusion".
Set $a \leftarrow b$ where
$b_J = -a_J + (2/2^n) \sum_I a_I$.
This is also easy.

Repeat steps 1 and 2
about $0.58 \cdot 2^{0.5n}$ times.

Measure the n qubits.
With high probability this finds
the unique J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $100 \times$ (Step 1 + Step 2):



Very bad stopping point.

Set $a \leftarrow b$ where
$_J$ if $\Sigma(J) = t$,
otherwise.
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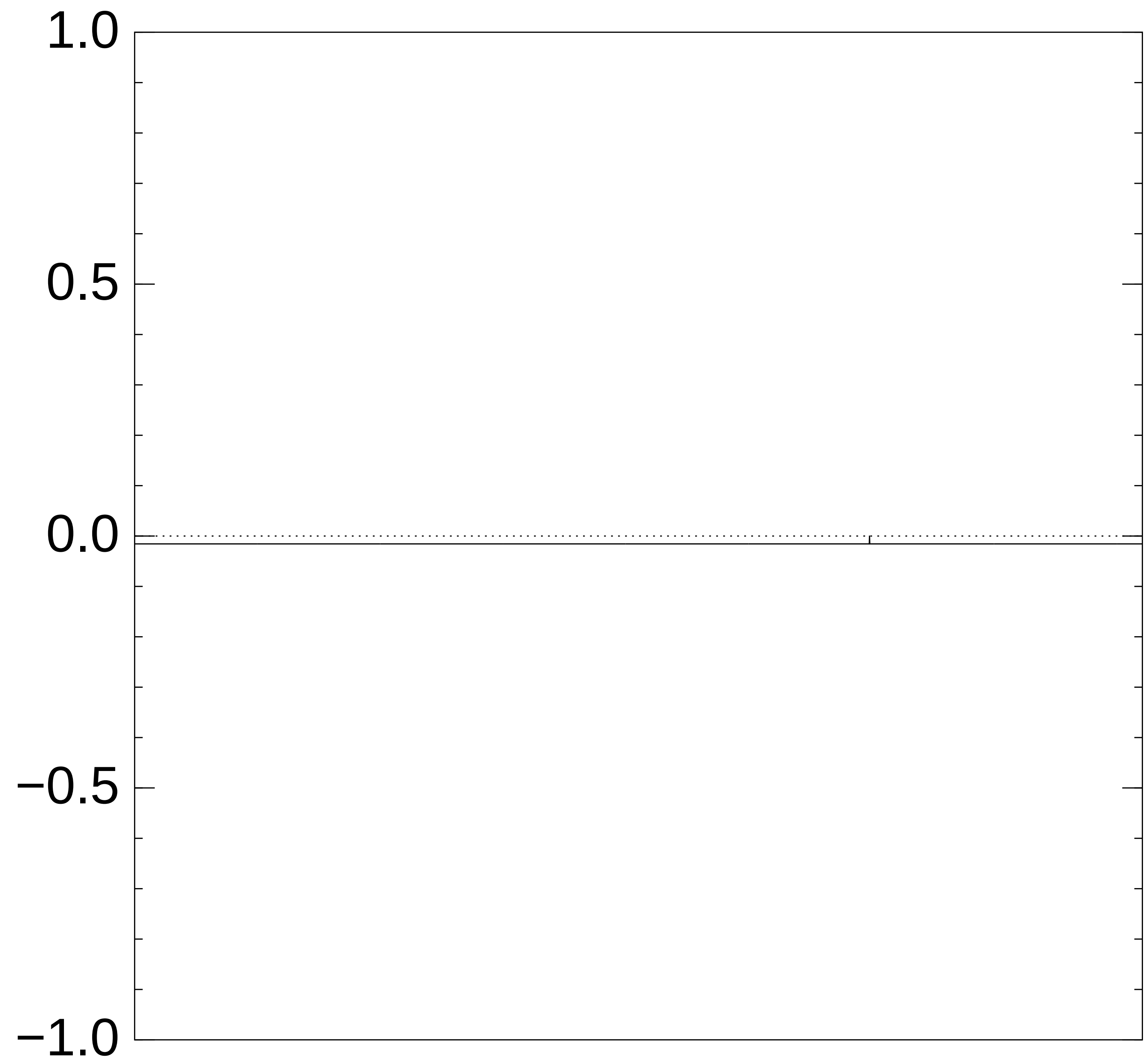
ue J such that $\Sigma(J) = t$.

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $100 \times$ (Step 1 + Step 2):



Very bad stopping point.

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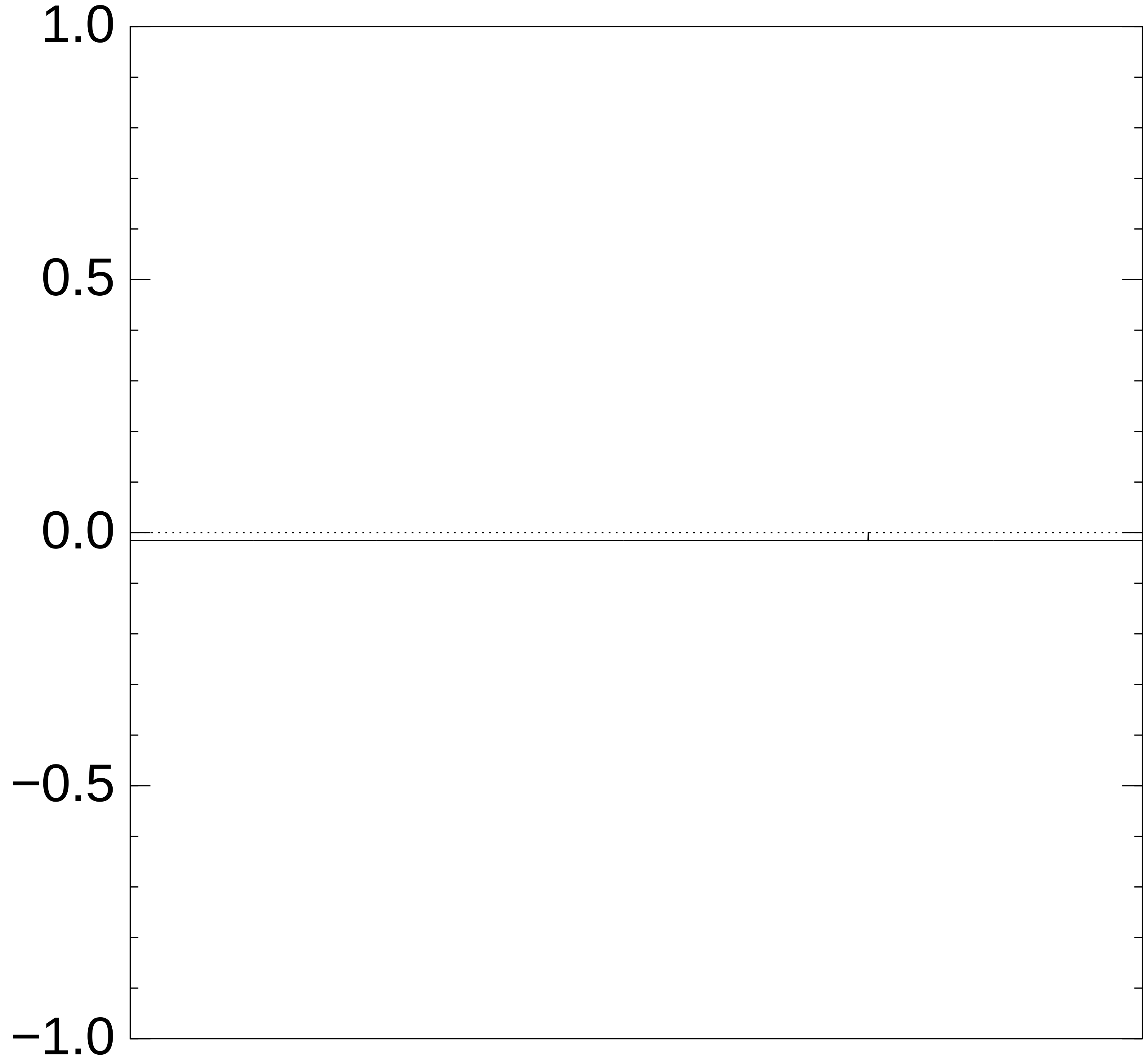
Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $100 \times$ (Step $1 +$ Step 2):



Very bad stopping point.

$J \mapsto a_J$ is complet
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(1) a_J for roots J
(2) a_J for non-roo

Step $1 +$ Step 2
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$\Rightarrow$ Probability is $\approx$
after $\approx(\pi/4)2^{0.5n}$

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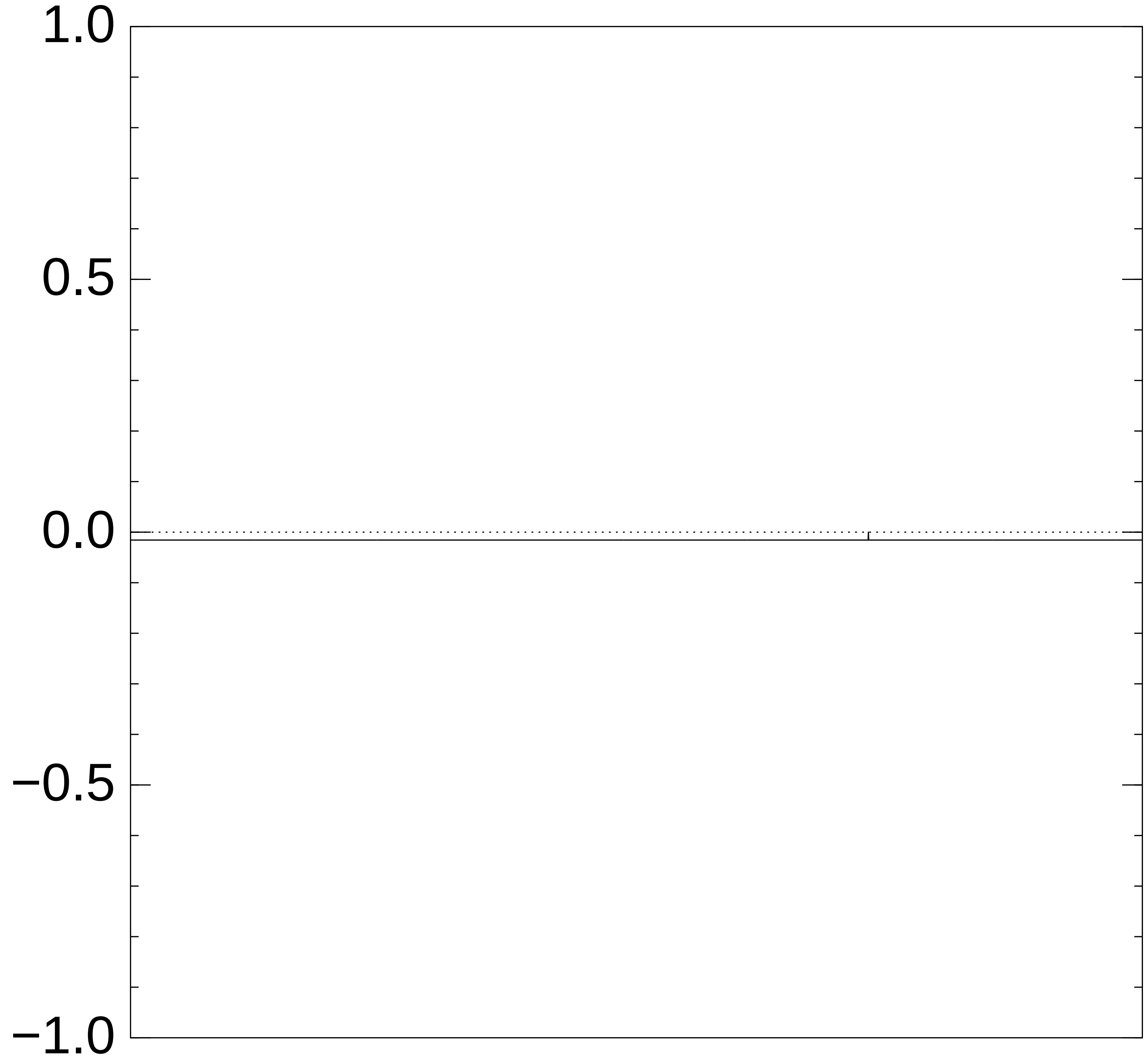
) $= t$.

Graph of $J \mapsto a_J$

for 36634 example with $n = 12$

after $100 \times$ (Step $1 +$ Step 2):



Very bad stopping point.

$J \mapsto a_J$ is completely descri

by a vector of two numbers

(with fixed multiplicities):

(1) a_J for roots J;

(2) a_J for non-roots J.

Step $1 +$ Step 2

act linearly on this vector.

Easily compute eigenvalues

and powers of this linear ma

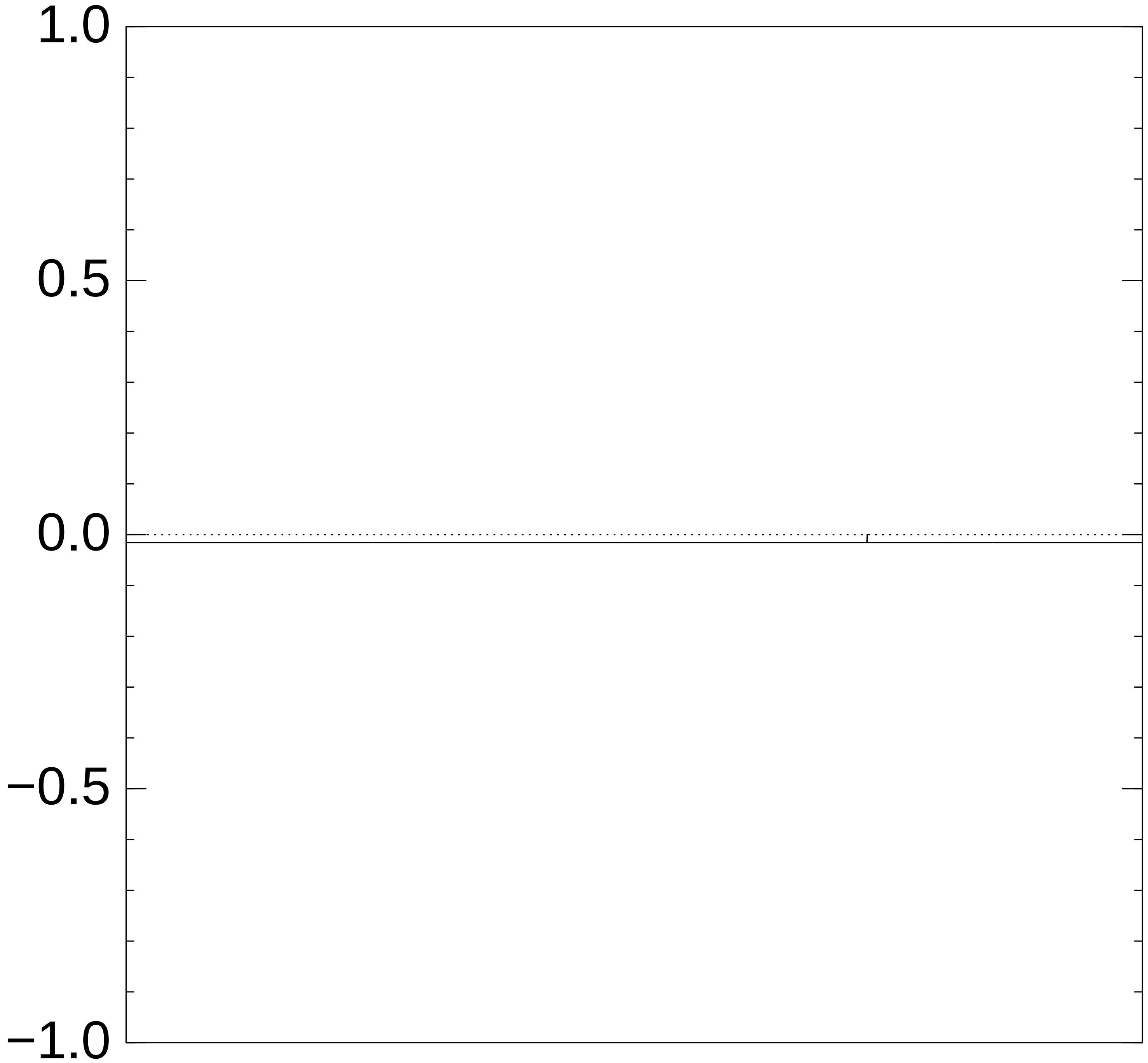
to understand evolution

of state of Grover's algorith

$\Rightarrow$ Probability is ≈ 1

after $\approx (\pi/4) 2^{0.5n}$ iterations

Graph of $J \mapsto a_J$
for 36634 example with $n = 12$
after $100 \times$ (Step $1 +$ Step 2):



Very bad stopping point.

$J \mapsto a_J$ is completely described
by a vector of two numbers
(with fixed multiplicities):
(1) a_J for roots J;
(2) a_J for non-roots J.

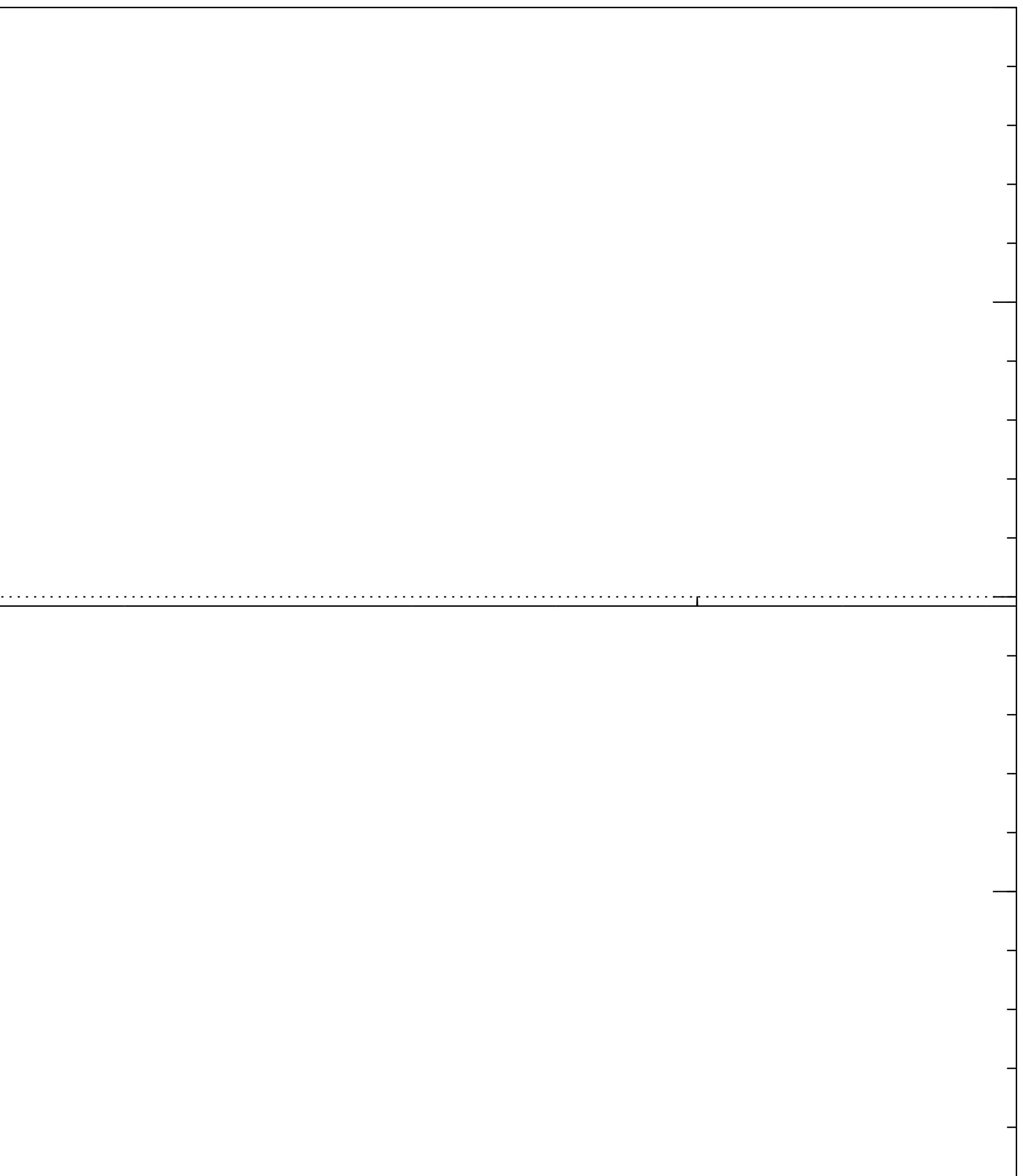
Step $1 +$ Step 2
act linearly on this vector.

Easily compute eigenvalues
and powers of this linear map
to understand evolution
of state of Grover's algorithm.
$\Rightarrow$ Probability is ≈ 1
after $\approx (\pi/4)2^{0.5n}$ iterations.

f $J \mapsto a_J$

4 example with $n = 12$

0 $\times$ (Step 1 + Step 2):



d stopping point.

$J \mapsto a_J$ is completely described
by a vector of two numbers
(with fixed multiplicities):
(1) a_J for roots J;
(2) a_J for non-roots J.

Step 1 + Step 2
act linearly on this vector.

Easily compute eigenvalues
and powers of this linear map
to understand evolution
of state of Grover's algorithm.
$\Rightarrow$ Probability is $\approx$1
after $\approx(\pi/4)2^{0.5n}$ iterations.

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Sort list
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and list
for all J_2
Merge t
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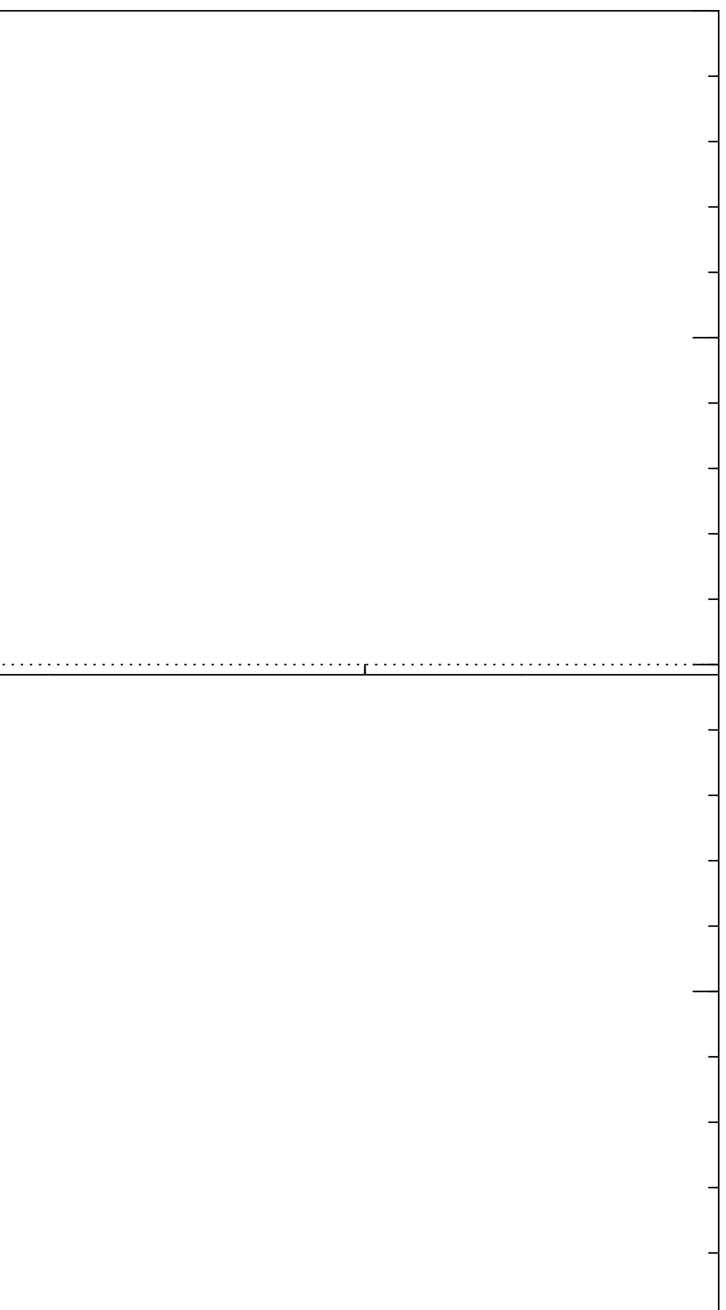
i.e., $\Sigma(J$

e with $n = 12$
$1 +$ Step 2):



point.

$J \mapsto a_J$ is completely described
by a vector of two numbers
(with fixed multiplicities):
(1) a_J for roots J;
(2) a_J for non-roots J.

Step 1 + Step 2
act linearly on this vector.

Easily compute eigenvalues
and powers of this linear map
to understand evolution
of state of Grover's algorithm.
$\Rightarrow$ Probability is ≈ 1
after $\approx(\pi/4)2^{0.5n}$ iterations.

Left-right split (0.

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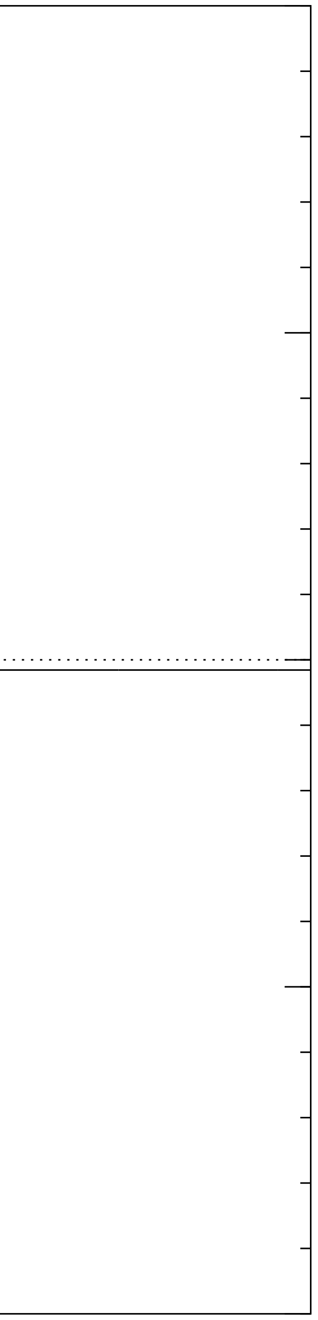
1974 Horowitz–Sa
Sort list of $\Sigma(J_1)$
for all $J_1 \subseteq \{1, \ldots$
and list of $t - \Sigma($
for all $J_2 \subseteq \{n/2$
Merge to find colli
$\Sigma(J_1) = t - \Sigma(J_2$
i.e., $\Sigma(J_1 \cup J_2) =$

= 12

2):



$J \mapsto a_J$ is completely described by a vector of two numbers (with fixed multiplicities):

(1) a_J for roots J;

(2) a_J for non-roots J.

Step 1 + Step 2 act linearly on this vector.

Easily compute eigenvalues and powers of this linear map to understand evolution of state of Grover's algorithm.

$\Rightarrow$ Probability is $\approx$1 after $\approx(\pi/4)2^{0.5n}$ iterations.

Left-right split (0.5)

Don't need quantum compu

to achieve exponent 0.5.

For simplicity assume $n \in 2$

1974 Horowitz–Sahni:
Sort list of $\Sigma(J_1)$ for all $J_1 \subseteq \{1,\ldots,n/2\}$ and list of $t - \Sigma(J_2)$ for all $J_2 \subseteq \{n/2+1,\ldots,n$
Merge to find collisions $\Sigma(J_1) = t - \Sigma(J_2)$, i.e., $\Sigma(J_1 \cup J_2) = t$.

$J \mapsto a_J$ is completely described by a vector of two numbers (with fixed multiplicities):
(1) a_J for roots J;
(2) a_J for non-roots J.

Step 1 + Step 2 act linearly on this vector.

Easily compute eigenvalues and powers of this linear map to understand evolution of state of Grover's algorithm.
$\Rightarrow$ Probability is ≈ 1 after $\approx(\pi/4)2^{0.5n}$ iterations.

Left-right split (0.5)

Don't need quantum computers to achieve exponent 0.5.

For simplicity assume $n \in 2\mathbf{Z}$.

1974 Horowitz–Sahni:
Sort list of $\Sigma(J_1)$ for all $J_1 \subseteq \{1, \ldots, n/2\}$ and list of $t - \Sigma(J_2)$ for all $J_2 \subseteq \{n/2+1, \ldots, n\}$.
Merge to find collisions $\Sigma(J_1) = t - \Sigma(J_2)$, i.e., $\Sigma(J_1 \cup J_2) = t$.

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Left-right split (0.5)

Don't need quantum computers
to achieve exponent 0.5.

For simplicity assume $n \in 2\mathbf{Z}$.

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Cost $2^{0.5}$
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i.e., $\Sigma(J_1 \cup J_2) = t$.

Cost $2^{0.5n}$ for sort
We assign cost 1 t

e.g. 36634 as sum
$(499, 852, 1927, 25$
$4688, 5989, 6385, 7$

Sort the 64 sums
$0, 499, 852, 499 +$
$499 + 852 + 1927$
and the 64 differen
$36634 - 0, 36634 -$
$36634 - 4688 - \cdots$
to see that
$499 + 852 + 2535$
$36634 - 5989 - 638$

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Left-right split (0.5)

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1974 Horowitz–Sahni:
Sort list of $\Sigma(J_1)$
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i.e., $\Sigma(J_1 \cup J_2) = t$.

Cost $2^{0.5n}$ for sorting, mergi
We assign cost 1 to RAM.

e.g. 36634 as sum of
$(499, 852, 1927, 2535, 3596,$
$4688, 5989, 6385, 7353, 7650$

Sort the 64 sums
$0, 499, 852, 499 + 852, \ldots,$
$499 + 852 + 1927 + \cdots + 3$
and the 64 differences
$36634 - 0, 36634 - 4688, \ldots$
$36634 - 4688 - \cdots - 9413$
to see that
$499 + 852 + 2535 + 3608 =$
$36634 - 5989 - 6385 - 7353 -$

Left-right split (0.5)

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For simplicity assume $n \in 2\mathbf{Z}$.

1974 Horowitz–Sahni:
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e.g. 36634 as sum of
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Sort the 64 sums
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$499 + 852 + 1927 + \cdots + 3608$
and the 64 differences
$36634 - 0, 36634 - 4688, \ldots,$
$36634 - 4688 - \cdots - 9413$
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$_2 \subseteq \{n/2+1, \ldots, n\}$.
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Sort the 64 sums
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Moduli (0.5)

For simplicity assu

Choose $M \approx 2^{0.25}$
Choose $t_1 \in \{0, 1,$
Define $t_2 = t - t_1$

Find all $J_1 \subseteq \{1, .$
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<u>Moduli (0.5)</u>

For simplicity assume $n \in 4$

Choose $M \approx 2^{0.25n}$.
Choose $t_1 \in \{0, 1, \ldots, M -$
Define $t_2 = t - t_1$.

Find all $J_1 \subseteq \{1, \ldots, n/2\}$
such that $\Sigma(J_1) \equiv t_1 \quad$ (mo
How? Split J_1 as $J_{11} \cup J_{12}$.

Find all $J_2 \subseteq \{n/2 + 1, \ldots,$
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Sort and merge to find all
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Cost $2^{0.5n}$ for sorting, merging.
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e.g. 36634 as sum of
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Moduli (0.5)

For simplicity assume $n \in 4\mathbf{Z}$.

Choose $M \approx 2^{0.25n}$.
Choose $t_1 \in \{0, 1, \ldots, M-1\}$.
Define $t_2 = t - t_1$.

Find all $J_1 \subseteq \{1, \ldots, n/2\}$
such that $\Sigma(J_1) \equiv t_1 \pmod{M}$.
How? Split J_1 as $J_{11} \cup J_{12}$.

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Sort and merge to find all
collisions $\Sigma(J_1) = t - \Sigma(J_2)$,
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34 as sum of
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64 sums
$52, 499 + 852, \ldots,$
$52 + 1927 + \cdots + 3608$
64 differences
$0, 36634 - 4688, \ldots,$
$4688 - \cdots - 9413$
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$52 + 2535 + 3608 =$
$5989 - 6385 - 7353 - 9413.$

Moduli (0.5)

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Choose $t_1 \in \{0, 1, \ldots, M-1\}$.
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Moduli (0.5)

For simplicity assume $n \in 4\mathbf{Z}$.

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Choose $t_1 \in \{0, 1, \ldots, M-1\}$.
Define $t_2 = t - t_1$.

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Sort and merge to find all collisions $\Sigma(J_1) = t - \Sigma(J_2)$, i.e., $\Sigma(J_1 \cup J_2) = t$.

Finds J iff $\Sigma(J_1) \equiv$
There are $\approx 2^{0.25n}$
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There are $\approx 2^{0.25n}$ choices o
Each choice costs $2^{0.25n}$.
Total cost $2^{0.5n}$.

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Algorithm has been
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2006 Elsenhans–Jahnel;
2010 Howgrave-Graham–Jou

Different technique
for similar space reduction:
1981 Schroeppel–Shamir.

Moduli (0.5)

For simplicity assume $n \in 4\mathbf{Z}$.

Choose $M \approx 2^{0.25n}$.
Choose $t_1 \in \{0, 1, \ldots, M-1\}$.
Define $t_2 = t - t_1$.

Find all $J_1 \subseteq \{1, \ldots, n/2\}$
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How? Split J_1 as $J_{11} \cup J_{12}$.

Find all $J_2 \subseteq \{n/2+1, \ldots, n\}$
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$t_1 \in \{0, 1, \ldots, M-1\}$.

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1981 Schroeppel–Shamir.

e.g. $M = 8$, $t = 3$

$(499, 852, 1927, 25$

$4688, 5989, 6385, 7$

Try each $t_1 \in \{0,$

In particular try t_1

There are 12 subs

$(499, 852, 1927, 25$

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$(4688, 5989, 6385,$

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Sort and merge to

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Finds J iff $\Sigma(J_1) \equiv t_1$.
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Different technique
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e.g. $M = 8$, $t = 36634$, $x =$
$(499, 852, 1927, 2535, 3596,$
$4688, 5989, 6385, 7353, 7650$

Try each $t_1 \in \{0, 1, \ldots, 7\}$.

In particular try $t_1 = 6$.
There are 12 subsequences o
$(499, 852, 1927, 2535, 3596,$
with sum 6 modulo 8.
There are 6 subsequences of
$(4688, 5989, 6385, 7353, 765$
with sum $36634 - 6$ modulo
Sort and merge to find
$499 + 852 + 2535 + 3608 =$
$36634 - 5989 - 6385 - 7353 -$

Finds J iff $\Sigma(J_1) \equiv t_1$.
There are $\approx 2^{0.25n}$ choices of t_1.
Each choice costs $2^{0.25n}$.
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$(499, 852, 1927, 2535, 3596, 3608)$
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There are 6 subsequences of
$(4688, 5989, 6385, 7353, 7650, 9413)$
with sum $36634 - 6$ modulo 8.
Sort and merge to find
$499 + 852 + 2535 + 3608 =$
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Try each $t_1 \in \{0, 1, \ldots, 7\}$.

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There are 12 subsequences of $(499, 852, 1927, 2535, 3596, 3608)$ with sum 6 modulo 8.
There are 6 subsequences of $(4688, 5989, 6385, 7353, 7650, 9413)$ with sum $36634 - 6$ modulo 8.
Sort and merge to find
$499 + 852 + 2535 + 3608 = 36634 - 5989 - 6385 - 7353 - 9413$.

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e.g. $M = 8$, $t = 36634$, $x =$ (499, 852, 1927, 2535, 3596, 3608, 4688, 5989, 6385, 7353, 7650, 9413):

Try each $t_1 \in \{0, 1, \ldots, 7\}$.

In particular try $t_1 = 6$.
There are 12 subsequences of (499, 852, 1927, 2535, 3596, 3608) with sum 6 modulo 8.
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Sort and merge to find
$499 + 852 + 2535 + 3608 =$
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Compute $\Sigma(J_1)$ fo
$J_1 \subseteq \{1, 2, \ldots, n/$
Sort $L = \{\Sigma(J_1)\}$.

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$J_2 \mapsto [t - \Sigma(J_2) \notin$
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e.g. $M = 8$, $t = 36634$, $x =$
$(499, 852, 1927, 2535, 3596, 3608,$
$4688, 5989, 6385, 7353, 7650, 9413)$:

Try each $t_1 \in \{0, 1, \ldots, 7\}$.

In particular try $t_1 = 6$.
There are 12 subsequences of
$(499, 852, 1927, 2535, 3596, 3608)$
with sum 6 modulo 8.
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with sum $36634 - 6$ modulo 8.
Sort and merge to find
$499 + 852 + 2535 + 3608 =$
$36634 - 5989 - 6385 - 7353 - 9413$.

Quantum left-right split (0.3

Cost $2^{n/3}$, imitating
1998 Brassard–Høyer–Tapp:

For simplicity assume $n \in 3$

Compute $\Sigma(J_1)$ for all
$J_1 \subseteq \{1, 2, \ldots, n/3\}$.
Sort $L = \{\Sigma(J_1)\}$.

Can now efficiently compute
$J_2 \mapsto [t - \Sigma(J_2) \notin L]$
for $J_2 \subseteq \{n/3 + 1, \ldots, n\}$.
Recall: we assign cost 1 to

Use Grover's method to see
whether this function has a

e.g. $M = 8$, $t = 36634$, $x =$
$(499, 852, 1927, 2535, 3596, 3608,$
$4688, 5989, 6385, 7353, 7650, 9413)$:

Try each $t_1 \in \{0, 1, \ldots, 7\}$.

In particular try $t_1 = 6$.
There are 12 subsequences of
$(499, 852, 1927, 2535, 3596, 3608)$
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$(4688, 5989, 6385, 7353, 7650, 9413)$
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Sort and merge to find
$499 + 852 + 2535 + 3608 =$
$36634 - 5989 - 6385 - 7353 - 9413$.

<u>Quantum left-right split (0.333...)</u>

Cost $2^{n/3}$, imitating
1998 Brassard–Høyer–Tapp:

For simplicity assume $n \in 3\mathbf{Z}$.

Compute $\Sigma(J_1)$ for all
$J_1 \subseteq \{1, 2, \ldots, n/3\}$.
Sort $L = \{\Sigma(J_1)\}$.

Can now efficiently compute
$J_2 \mapsto [t - \Sigma(J_2) \notin L]$
for $J_2 \subseteq \{n/3 + 1, \ldots, n\}$.
Recall: we assign cost 1 to RAM.

Use Grover's method to see
whether this function has a root.

= 8, $t = 36634$, $x =$
2, 1927, 2535, 3596, 3608,
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n $t_1 \in \{0, 1, \ldots, 7\}$.

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$52 + 2535 + 3608 =$
$5989 - 6385 - 7353 - 9413$.

Quantum left-right split (0.333 . . .)

Cost $2^{n/3}$, imitating
1998 Brassard–Høyer–Tapp:

For simplicity assume $n \in 3\mathbf{Z}$.

Compute $\Sigma(J_1)$ for all
$J_1 \subseteq \{1, 2, \ldots, n/3\}$.
Sort $L = \{\Sigma(J_1)\}$.

Can now efficiently compute
$J_2 \mapsto [t - \Sigma(J_2) \notin L]$
for $J_2 \subseteq \{n/3 + 1, \ldots, n\}$.
Recall: we assign cost 1 to RAM.

Use Grover's method to see
whether this function has a root.

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Problem: find this

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Quantum walk

Unique-collision-finding prob
Say f has n-bit inputs,
exactly one collision $\{p, q\}$:
i.e., $p \neq q$, $f(p) = f(q)$.
Problem: find this collision.

Cost 2^n: Define S as
the set of n-bit strings.
Compute $f(S)$, sort.

Generalize to cost r,
success probability $\approx(r/2^n)$
Choose a set S of size r.
Compute $f(S)$, sort.

Quantum left-right split ($0.333\ldots$)

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Say f has n-bit inputs,
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Problem: find this collision.

Cost 2^n: Define S as
the set of n-bit strings.
Compute $f(S)$, sort.

Generalize to cost r,
success probability $\approx(r/2^n)^2$:
Choose a set S of size r.
Compute $f(S)$, sort.

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Very efficient to move from
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Create superposition of stat
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find S containing a collision

Quantum walk

Unique-collision-finding problem:
Say f has n-bit inputs, exactly one collision $\{p, q\}$: i.e., $p \neq q$, $f(p) = f(q)$.
Problem: find this collision.

Cost 2^n: Define S as the set of n-bit strings.
Compute $f(S)$, sort.

Generalize to cost r, success probability $\approx(r/2^n)^2$:
Choose a set S of size r.
Compute $f(S)$, sort.

Data structure $D(S)$ capturing the generalized computation: the set S; the multiset $f(S)$; the number of collisions in S.

Very efficient to move from $D(S)$ to $D(T)$ if T is an **adjacent** set: $\#S = \#T = r$, $\#(S \cap T) = r - 1$.

2003 Ambainis, simplified 2007 Magniez–Nayak–Roland–Santha:
Create superposition of states $(D(S), D(T))$ with adjacent S, T.
By a quantum walk find S containing a collision.

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How the quantum

Start from uniform
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Create superposition of states
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How the quantum walk work

Start from uniform superpos
Repeat $\approx 0.6 \cdot 2^n/r$ times:
  Negate $a_{S,T}$
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  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
    For each T:
      Diffuse $a_{S,T}$ across a
    For each S:
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Now high probability
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Cost $r + 2^n/\sqrt{r}$. Optimize:

Data structure $D(S)$ capturing the generalized computation: the set S; the multiset $f(S)$; the number of collisions in S.

Very efficient to move from $D(S)$ to $D(T)$ if T is an **adjacent** set: $\#S = \#T = r$, $\#(S \cap T) = r - 1$.

2003 Ambainis, simplified 2007 Magniez–Nayak–Roland–Santha: Create superposition of states $(D(S), D(T))$ with adjacent S, T. By a quantum walk find S containing a collision.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n/r$ times:
 Negate $a_{S,T}$
  if S contains collision.
 Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
  For each T:
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  For each S:
   Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.
Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

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ontaining a collision.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n / r$ times:
  Negate $a_{S,T}$
    if S contains collision.
  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
    For each T:
      Diffuse $a_{S,T}$ across all S.
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      Diffuse $a_{S,T}$ across all T.

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Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) acc
$(\#(S \cap \{p, q\}), \#$
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Analyze evolution

e.g. $n = 15$, $r = 1$
0 negations and 0

Pr[class $(0, 0)$] $\approx$ 0
Pr[class $(0, 1)$] $\approx$ 0
Pr[class $(1, 0)$] $\approx$ 0
Pr[class $(1, 1)$] $\approx$ 0
Pr[class $(1, 2)$] $\approx$ 0
Pr[class $(2, 1)$] $\approx$ 0
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How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n / r$ times:

Negate $a_{S,T}$

if S contains collision.

Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

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Now high probability

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Cost $r + 2^n / \sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q$

reduce a to low-dim vector.

Analyze evolution of this vec

e.g. $n = 15$, $r = 1024$, after

0 negations and 0 diffusions

Pr[class $(0, 0)$] $\approx 0.938; +$

Pr[class $(0, 1)$] $\approx 0.000; +$

Pr[class $(1, 0)$] $\approx 0.000; +$

Pr[class $(1, 1)$] $\approx 0.060; +$

Pr[class $(1, 2)$] $\approx 0.000; +$

Pr[class $(2, 1)$] $\approx 0.000; +$

Pr[class $(2, 2)$] $\approx 0.001; +$

Right column is sign of $a_{S,T}$

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

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Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

0 negations and 0 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.938; +$

$\Pr[\text{class } (0, 1)] \approx 0.000; +$

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$\Pr[\text{class } (1, 1)] \approx 0.060; +$

$\Pr[\text{class } (1, 2)] \approx 0.000; +$

$\Pr[\text{class } (2, 1)] \approx 0.000; +$

$\Pr[\text{class } (2, 2)] \approx 0.001; +$

Right column is sign of $a_{S,T}$.

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Start from uniform superposition.

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  Negate $a_{S,T}$

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  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

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Now high probability
that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
1 negation and 46 diffusions:

Pr[class $(0, 0)$] $\approx 0.935; +$
Pr[class $(0, 1)$] $\approx 0.000; +$
Pr[class $(1, 0)$] $\approx 0.000; -$
Pr[class $(1, 1)$] $\approx 0.057; +$
Pr[class $(1, 2)$] $\approx 0.000; +$
Pr[class $(2, 1)$] $\approx 0.000; -$
Pr[class $(2, 2)$] $\approx 0.008; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 2 negations and 92 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.918; +$

$\Pr[\text{class } (0,1)] \approx 0.001; +$

$\Pr[\text{class } (1,0)] \approx 0.000; -$

$\Pr[\text{class } (1,1)] \approx 0.059; +$

$\Pr[\text{class } (1,2)] \approx 0.001; +$

$\Pr[\text{class } (2,1)] \approx 0.000; -$

$\Pr[\text{class } (2,2)] \approx 0.022; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

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Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 3 negations and 138 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.897; +$

$\Pr[\text{class } (0, 1)] \approx 0.001; +$

$\Pr[\text{class } (1, 0)] \approx 0.000; -$

$\Pr[\text{class } (1, 1)] \approx 0.058; +$

$\Pr[\text{class } (1, 2)] \approx 0.002; +$

$\Pr[\text{class } (2, 1)] \approx 0.000; +$

$\Pr[\text{class } (2, 2)] \approx 0.042; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

4 negations and 184 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.873; +$

$\Pr[\text{class } (0,1)] \approx 0.001; +$

$\Pr[\text{class } (1,0)] \approx 0.000; -$

$\Pr[\text{class } (1,1)] \approx 0.054; +$

$\Pr[\text{class } (1,2)] \approx 0.002; +$

$\Pr[\text{class } (2,1)] \approx 0.000; +$

$\Pr[\text{class } (2,2)] \approx 0.070; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability
that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
5 negations and 230 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.838; +$
$\Pr[\text{class } (0,1)] \approx 0.001; +$
$\Pr[\text{class } (1,0)] \approx 0.001; -$
$\Pr[\text{class } (1,1)] \approx 0.054; +$
$\Pr[\text{class } (1,2)] \approx 0.003; +$
$\Pr[\text{class } (2,1)] \approx 0.000; +$
$\Pr[\text{class } (2,2)] \approx 0.104; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

6 negations and 276 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.800; +$

$\Pr[\text{class } (0,1)] \approx 0.001; +$

$\Pr[\text{class } (1,0)] \approx 0.001; -$

$\Pr[\text{class } (1,1)] \approx 0.051; +$

$\Pr[\text{class } (1,2)] \approx 0.006; +$

$\Pr[\text{class } (2,1)] \approx 0.000; +$

$\Pr[\text{class } (2,2)] \approx 0.141; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

- Negate $a_{S,T}$
  - if S contains collision.
- Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
  - For each T:
    - Diffuse $a_{S,T}$ across all S.
  - For each S:
    - Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 7 negations and 322 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.758; +$

$\Pr[\text{class } (0,1)] \approx 0.002; +$

$\Pr[\text{class } (1,0)] \approx 0.001; -$

$\Pr[\text{class } (1,1)] \approx 0.047; +$

$\Pr[\text{class } (1,2)] \approx 0.007; +$

$\Pr[\text{class } (2,1)] \approx 0.000; +$

$\Pr[\text{class } (2,2)] \approx 0.184; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

8 negations and 368 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.708; +$

$\Pr[\text{class } (0,1)] \approx 0.003; +$

$\Pr[\text{class } (1,0)] \approx 0.001; -$

$\Pr[\text{class } (1,1)] \approx 0.046; +$

$\Pr[\text{class } (1,2)] \approx 0.007; +$

$\Pr[\text{class } (2,1)] \approx 0.000; +$

$\Pr[\text{class } (2,2)] \approx 0.234; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

9 negations and 414 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.658; +$

$\Pr[\text{class } (0,1)] \approx 0.003; +$

$\Pr[\text{class } (1,0)] \approx 0.001; -$

$\Pr[\text{class } (1,1)] \approx 0.042; +$

$\Pr[\text{class } (1,2)] \approx 0.009; +$

$\Pr[\text{class } (2,1)] \approx 0.000; +$

$\Pr[\text{class } (2,2)] \approx 0.287; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 10 negations and 460 diffusions:

Pr[class $(0, 0)$] $\approx 0.606; +$

Pr[class $(0, 1)$] $\approx 0.003; +$

Pr[class $(1, 0)$] $\approx 0.002; -$

Pr[class $(1, 1)$] $\approx 0.037; +$

Pr[class $(1, 2)$] $\approx 0.013; +$

Pr[class $(2, 1)$] $\approx 0.000; +$

Pr[class $(2, 2)$] $\approx 0.338; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 11 negations and 506 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.547; +$

$\Pr[\text{class } (0,1)] \approx 0.004; +$

$\Pr[\text{class } (1,0)] \approx 0.003; -$

$\Pr[\text{class } (1,1)] \approx 0.036; +$

$\Pr[\text{class } (1,2)] \approx 0.015; +$

$\Pr[\text{class } (2,1)] \approx 0.001; +$

$\Pr[\text{class } (2,2)] \approx 0.394; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

12 negations and 552 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.491; +$

$\Pr[\text{class } (0,1)] \approx 0.004; +$

$\Pr[\text{class } (1,0)] \approx 0.003; -$

$\Pr[\text{class } (1,1)] \approx 0.032; +$

$\Pr[\text{class } (1,2)] \approx 0.014; +$

$\Pr[\text{class } (2,1)] \approx 0.001; +$

$\Pr[\text{class } (2,2)] \approx 0.455; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S,T) according to $(\#(S \cap \{p,q\}), \#(T \cap \{p,q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 13 negations and 598 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.436; +$

$\Pr[\text{class } (0,1)] \approx 0.005; +$

$\Pr[\text{class } (1,0)] \approx 0.003; -$

$\Pr[\text{class } (1,1)] \approx 0.026; +$

$\Pr[\text{class } (1,2)] \approx 0.017; +$

$\Pr[\text{class } (2,1)] \approx 0.000; +$

$\Pr[\text{class } (2,2)] \approx 0.513; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 14 negations and 644 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.377; +$

$\Pr[\text{class } (0,1)] \approx 0.006; +$

$\Pr[\text{class } (1,0)] \approx 0.004; -$

$\Pr[\text{class } (1,1)] \approx 0.025; +$

$\Pr[\text{class } (1,2)] \approx 0.022; +$

$\Pr[\text{class } (2,1)] \approx 0.001; +$

$\Pr[\text{class } (2,2)] \approx 0.566; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

15 negations and 690 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.322; +$

$\Pr[\text{class } (0, 1)] \approx 0.005; +$

$\Pr[\text{class } (1, 0)] \approx 0.004; -$

$\Pr[\text{class } (1, 1)] \approx 0.021; +$

$\Pr[\text{class } (1, 2)] \approx 0.023; +$

$\Pr[\text{class } (2, 1)] \approx 0.001; +$

$\Pr[\text{class } (2, 2)] \approx 0.623; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

16 negations and 736 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.270; +$

$\Pr[\text{class } (0, 1)] \approx 0.006; +$

$\Pr[\text{class } (1, 0)] \approx 0.005; -$

$\Pr[\text{class } (1, 1)] \approx 0.017; +$

$\Pr[\text{class } (1, 2)] \approx 0.022; +$

$\Pr[\text{class } (2, 1)] \approx 0.001; +$

$\Pr[\text{class } (2, 2)] \approx 0.680; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n/r$ times:
- Negate $a_{S,T}$
  - if S contains collision.
- Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
  - For each T:
    - Diffuse $a_{S,T}$ across all S.
  - For each S:
    - Diffuse $a_{S,T}$ across all T.

Now high probability
that T contains collision.
Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
17 negations and 782 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.218; +$
$\Pr[\text{class } (0,1)] \approx 0.007; +$
$\Pr[\text{class } (1,0)] \approx 0.005; -$
$\Pr[\text{class } (1,1)] \approx 0.015; +$
$\Pr[\text{class } (1,2)] \approx 0.024; +$
$\Pr[\text{class } (2,1)] \approx 0.001; +$
$\Pr[\text{class } (2,2)] \approx 0.730; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n/r$ times:
  Negate $a_{S,T}$
    if S contains collision.
  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
    For each T:
      Diffuse $a_{S,T}$ across all S.
    For each S:
      Diffuse $a_{S,T}$ across all T.

Now high probability
that T contains collision.
Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
18 negations and 828 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.172; +$
$\Pr[\text{class } (0,1)] \approx 0.006; +$
$\Pr[\text{class } (1,0)] \approx 0.005; -$
$\Pr[\text{class } (1,1)] \approx 0.011; +$
$\Pr[\text{class } (1,2)] \approx 0.029; +$
$\Pr[\text{class } (2,1)] \approx 0.001; +$
$\Pr[\text{class } (2,2)] \approx 0.775; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

- Negate $a_{S,T}$
  - if S contains collision.
- Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
  - For each T:
    - Diffuse $a_{S,T}$ across all S.
  - For each S:
    - Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 19 negations and 874 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.131; +$

$\Pr[\text{class } (0, 1)] \approx 0.007; +$

$\Pr[\text{class } (1, 0)] \approx 0.006; -$

$\Pr[\text{class } (1, 1)] \approx 0.008; +$

$\Pr[\text{class } (1, 2)] \approx 0.030; +$

$\Pr[\text{class } (2, 1)] \approx 0.002; +$

$\Pr[\text{class } (2, 2)] \approx 0.816; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

20 negations and 920 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.093; +$

$\Pr[\text{class } (0,1)] \approx 0.007; +$

$\Pr[\text{class } (1,0)] \approx 0.007; -$

$\Pr[\text{class } (1,1)] \approx 0.007; +$

$\Pr[\text{class } (1,2)] \approx 0.027; +$

$\Pr[\text{class } (2,1)] \approx 0.002; +$

$\Pr[\text{class } (2,2)] \approx 0.857; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to $(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$; reduce a to low-dim vector. Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after 21 negations and 966 diffusions:

$\Pr[\text{class } (0, 0)] \approx 0.062; +$

$\Pr[\text{class } (0, 1)] \approx 0.007; +$

$\Pr[\text{class } (1, 0)] \approx 0.006; -$

$\Pr[\text{class } (1, 1)] \approx 0.004; +$

$\Pr[\text{class } (1, 2)] \approx 0.030; +$

$\Pr[\text{class } (2, 1)] \approx 0.001; +$

$\Pr[\text{class } (2, 2)] \approx 0.890; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n/r$ times:
  Negate $a_{S,T}$
    if S contains collision.
  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
    For each T:
      Diffuse $a_{S,T}$ across all S.
    For each S:
      Diffuse $a_{S,T}$ across all T.

Now high probability
that T contains collision.
Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
22 negations and 1012 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.037; +$
$\Pr[\text{class } (0,1)] \approx 0.008; +$
$\Pr[\text{class } (1,0)] \approx 0.007; -$
$\Pr[\text{class } (1,1)] \approx 0.002; +$
$\Pr[\text{class } (1,2)] \approx 0.034; +$
$\Pr[\text{class } (2,1)] \approx 0.001; +$
$\Pr[\text{class } (2,2)] \approx 0.910; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n/r$ times:
  Negate $a_{S,T}$
    if S contains collision.
  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
    For each T:
      Diffuse $a_{S,T}$ across all S.
    For each S:
      Diffuse $a_{S,T}$ across all T.

Now high probability
that T contains collision.
Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
23 negations and 1058 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.017; +$
$\Pr[\text{class } (0,1)] \approx 0.008; +$
$\Pr[\text{class } (1,0)] \approx 0.007; -$
$\Pr[\text{class } (1,1)] \approx 0.002; +$
$\Pr[\text{class } (1,2)] \approx 0.034; +$
$\Pr[\text{class } (2,1)] \approx 0.002; +$
$\Pr[\text{class } (2,2)] \approx 0.930; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n/r$ times:
  Negate $a_{S,T}$
    if S contains collision.
  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
    For each T:
      Diffuse $a_{S,T}$ across all S.
    For each S:
      Diffuse $a_{S,T}$ across all T.

Now high probability
that T contains collision.
Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
24 negations and 1104 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.005; +$
$\Pr[\text{class } (0,1)] \approx 0.007; +$
$\Pr[\text{class } (1,0)] \approx 0.007; -$
$\Pr[\text{class } (1,1)] \approx 0.000; +$
$\Pr[\text{class } (1,2)] \approx 0.030; +$
$\Pr[\text{class } (2,1)] \approx 0.002; +$
$\Pr[\text{class } (2,2)] \approx 0.948; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.

Repeat $\approx 0.6 \cdot 2^n/r$ times:

  Negate $a_{S,T}$

    if S contains collision.

  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:

    For each T:

      Diffuse $a_{S,T}$ across all S.

    For each S:

      Diffuse $a_{S,T}$ across all T.

Now high probability

that T contains collision.

Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

25 negations and 1150 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.000; +$

$\Pr[\text{class } (0,1)] \approx 0.008; +$

$\Pr[\text{class } (1,0)] \approx 0.008; -$

$\Pr[\text{class } (1,1)] \approx 0.000; +$

$\Pr[\text{class } (1,2)] \approx 0.031; +$

$\Pr[\text{class } (2,1)] \approx 0.001; +$

$\Pr[\text{class } (2,2)] \approx 0.952; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n/r$ times:
  Negate $a_{S,T}$
    if S contains collision.
  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
    For each T:
      Diffuse $a_{S,T}$ across all S.
    For each S:
      Diffuse $a_{S,T}$ across all T.

Now high probability
that T contains collision.
Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
26 negations and 1196 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.002; -$
$\Pr[\text{class } (0,1)] \approx 0.008; +$
$\Pr[\text{class } (1,0)] \approx 0.008; -$
$\Pr[\text{class } (1,1)] \approx 0.000; -$
$\Pr[\text{class } (1,2)] \approx 0.035; +$
$\Pr[\text{class } (2,1)] \approx 0.002; +$
$\Pr[\text{class } (2,2)] \approx 0.945; +$

Right column is sign of $a_{S,T}$.

How the quantum walk works:

Start from uniform superposition.
Repeat $\approx 0.6 \cdot 2^n/r$ times:
  Negate $a_{S,T}$
    if S contains collision.
  Repeat $\approx 0.7 \cdot \sqrt{r}$ times:
    For each T:
      Diffuse $a_{S,T}$ across all S.
    For each S:
      Diffuse $a_{S,T}$ across all T.

Now high probability
that T contains collision.
Cost $r + 2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to
$(\#(S \cap \{p, q\}), \#(T \cap \{p, q\}))$;
reduce a to low-dim vector.
Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after
27 negations and 1242 diffusions:

$\Pr[\text{class } (0,0)] \approx 0.011; -$
$\Pr[\text{class } (0,1)] \approx 0.007; +$
$\Pr[\text{class } (1,0)] \approx 0.007; -$
$\Pr[\text{class } (1,1)] \approx 0.001; -$
$\Pr[\text{class } (1,2)] \approx 0.034; +$
$\Pr[\text{class } (2,1)] \approx 0.003; +$
$\Pr[\text{class } (2,2)] \approx 0.938; +$

Right column is sign of $a_{S,T}$.

e quantum walk works:

om uniform superposition.

≈$0.6 \cdot 2^n/r$ times:

ze $a_{S,T}$

T contains collision.

nt ≈$0.7 \cdot \sqrt{r}$ times:

each T:

Diffuse $a_{S,T}$ across all S.

each S:

Diffuse $a_{S,T}$ across all T.

h probability

contains collision.

$2^n/\sqrt{r}$. Optimize: $2^{2n/3}$.

Classify (S, T) according to

$(\#(S \cap \{p,q\}), \#(T \cap \{p,q\}))$;

reduce a to low-dim vector.

Analyze evolution of this vector.

e.g. $n = 15$, $r = 1024$, after

27 negations and 1242 diffusions:

Pr[class $(0,0)$] $\approx 0.011; -$

Pr[class $(0,1)$] $\approx 0.007; +$

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Subset-s

Consider

$f(1, J_1)$

for $J_1 \subseteq$

$f(2, J_2)$

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Optimize: $2^{2n/3}$.

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Good chance of un

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<u>Subset-sum walk (0.333...)</u>

Consider f defined by
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Good chance of unique collision $\Sigma(J_1)=t-\Sigma(J_2)$.

$n/2+1$ bits of input, so quantum walk costs $2^{n/3}$

Easily tweak quantum walk to handle more collisions, ignore $\Sigma(J_1)=\Sigma(J_1')$, etc.

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Similarly take a set S_{12} of r subsets of $\{n/4 + 1, \ldots, n/2\}$. Compute $t_1 - \Sigma(J_{12}) \bmod M$ for each $J_{12} \in S_{12}$.

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Find all collisions $\Sigma(J_{11}) \equiv t_1 - \Sigma(J_{12})$,
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Compute each $\Sigma(J_1)$.

Similarly S_{21}, $S_{22} \Rightarrow$
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Find collisions $\Sigma(J_1) = t - \Sigma$

Success probability $r^4/2^n$
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Quantum moduli (

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<u>Quantum moduli (0.3)</u>

Capture execution of
generalized moduli algorithm
as data structure
$D(S_{11}, S_{12}, S_{21}, S_{22})$.
Easy to move
from S_{ij} to adjacent T_{ij}.

Convert into quantum walk:
cost $r + \sqrt{r}2^{n/2}/r^2$.
$2^{0.2n}$ for $r \approx 2^{0.2n}$.

Use "amplitude amplification
to search for correct t_1.
Total cost $2^{0.3n}$.

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Total cost $2^{0.3n}$.

Quantum reps (0.2

Central result of th
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Subset-sum expon
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Ambainis uses ad-
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Much easier, presu

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Quantum moduli (0.3)

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Use "amplitude amplification" to search for correct t_1.
Total cost $2^{0.3n}$.

Quantum reps (0.241...)

Central result of the paper:
Combine quantum walk with "representations" idea
2010 Howgrave-Graham–Jou
Subset-sum exponent 0.241
new record.

Lower-level improvement:
Ambainis uses ad-hoc "combination of a hash tabl and a skip list" to ensure history-independence.
We use radix trees.
Much easier, presumably fas

Quantum moduli (0.3)

Capture execution of generalized moduli algorithm as data structure $D(S_{11}, S_{12}, S_{21}, S_{22})$. Easy to move from S_{ij} to adjacent T_{ij}.

Convert into quantum walk: cost $r + \sqrt{r}2^{n/2}/r^2$. $2^{0.2n}$ for $r \approx 2^{0.2n}$.

Use "amplitude amplification" to search for correct t_1. Total cost $2^{0.3n}$.

Quantum reps (0.241...)

Central result of the paper: Combine quantum walk with "representations" idea of 2010 Howgrave-Graham–Joux. Subset-sum exponent 0.241...; new record.

Lower-level improvement: Ambainis uses ad-hoc "combination of a hash table and a skip list" to ensure history-independence. We use radix trees. Much easier, presumably faster.