Fast arithmetic on elliptic curves

D. J. Bernstein
University of Illinois at Chicago

EC point counting

1983 (published 1985) Schoof:
Algorithm to count points on
elliptic curves over finite fields.

Input: prime power q; $a, b \in \mathbf{F}_q$
such that $6(4a^3 + 27b^2) \neq 0$.

Output: $\#\{(x, y) \in \mathbf{F}_q \times \mathbf{F}_q :$
$y^2 = x^3 + ax + b\} + 1$;
i.e., $\#E(\mathbf{F}_q)$ where E is the
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Time: $(\log q)^{O(1)}$.

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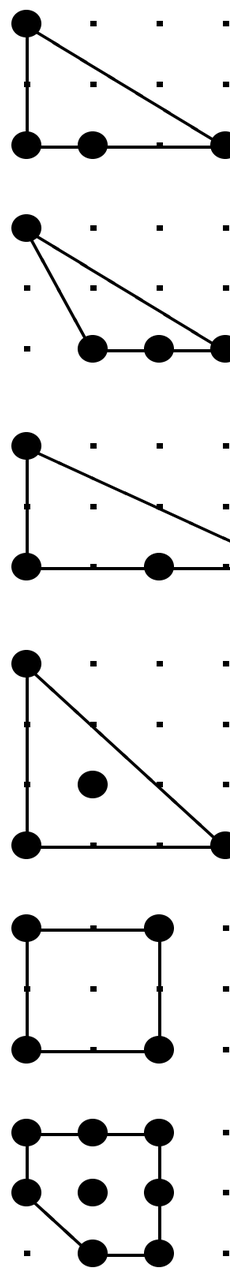

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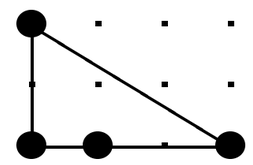

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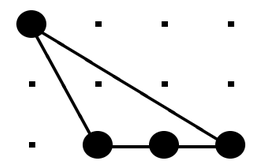

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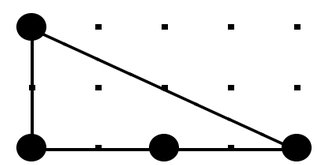

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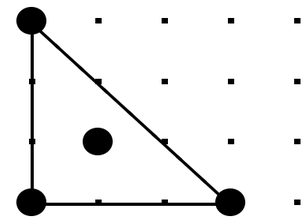

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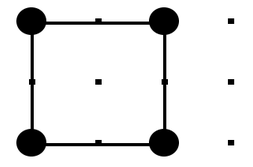

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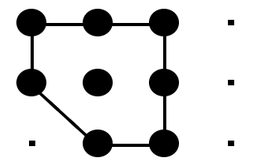

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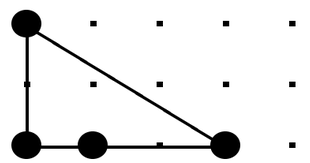

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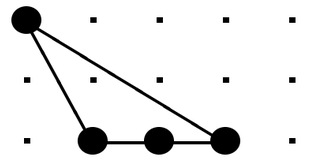

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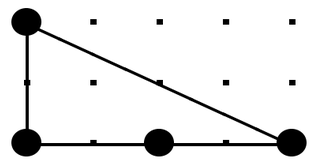
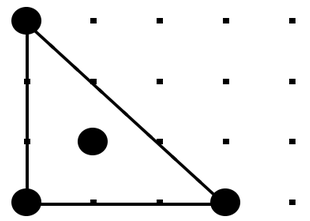


Jacobi quartic



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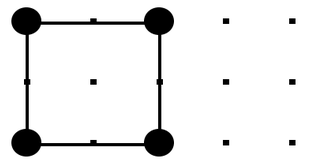

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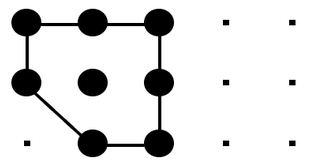

Binary Edwards

"It is preferable to use
models of elliptic curves
lying in low-dimensional spaces,
for otherwise the number of
coordinates and operations is
increasing. This limits us ... to
4 basic models of elliptic curves."

Short Weierstrass:
$y^2 = x^3 + ax + b$.

Jacobi intersection:
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

Jacobi quartic: $y^2 = x^4 + 2ax^2 + 1$.

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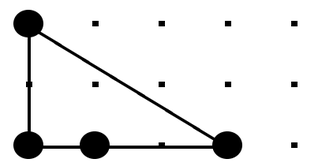
Some Newton polygons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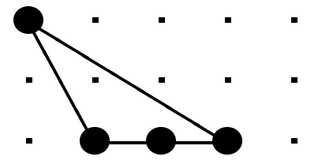
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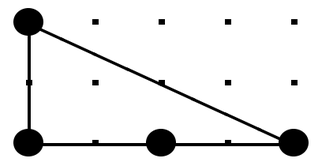
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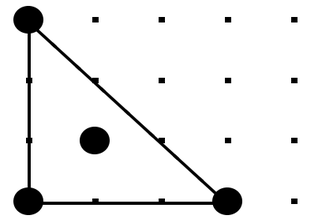
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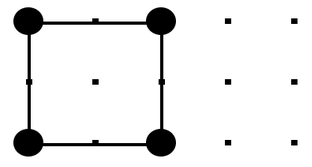
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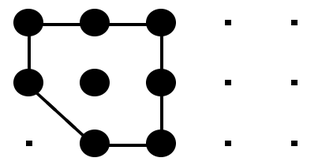
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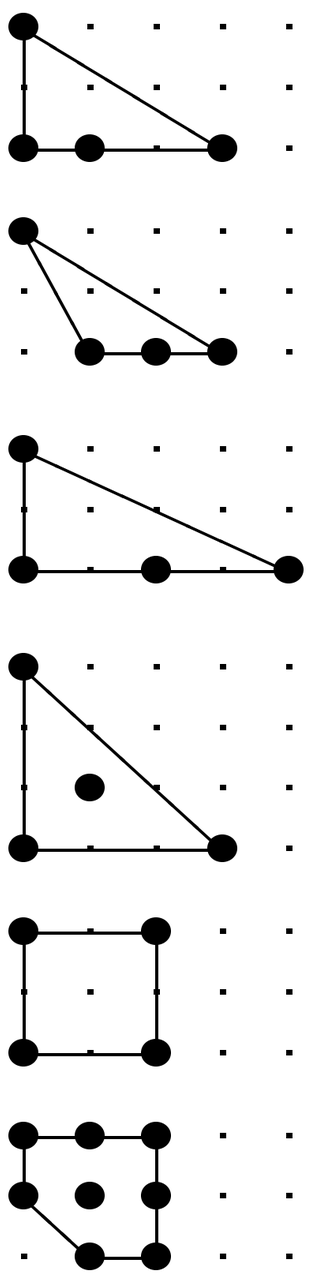

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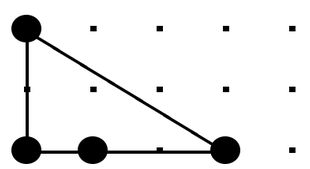

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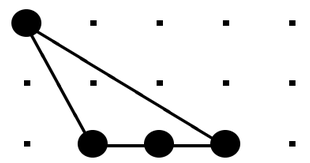

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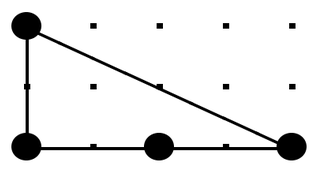

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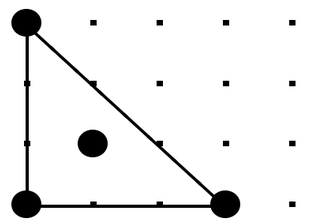

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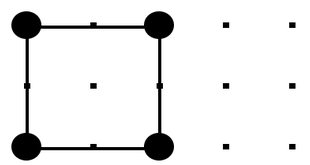

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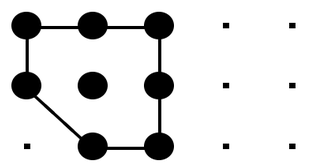

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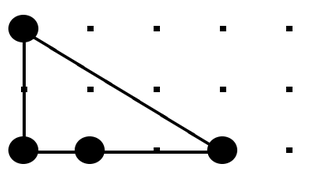

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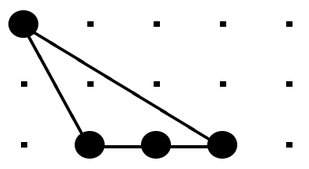

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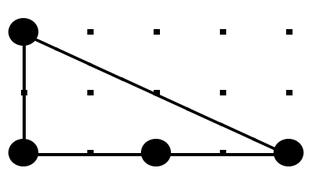

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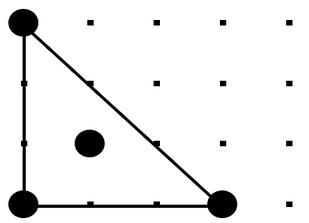

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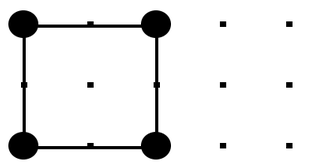

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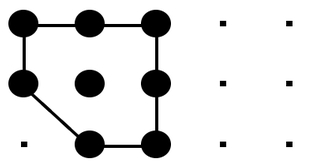

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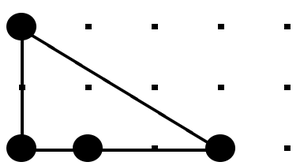

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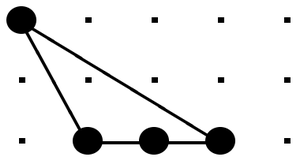

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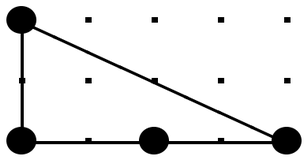

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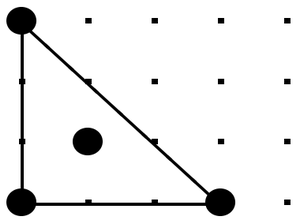

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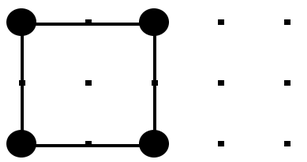

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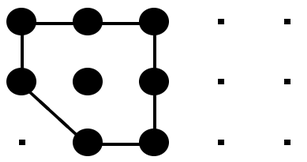

Binary Edwards

Optimizing Jacobian coordinates

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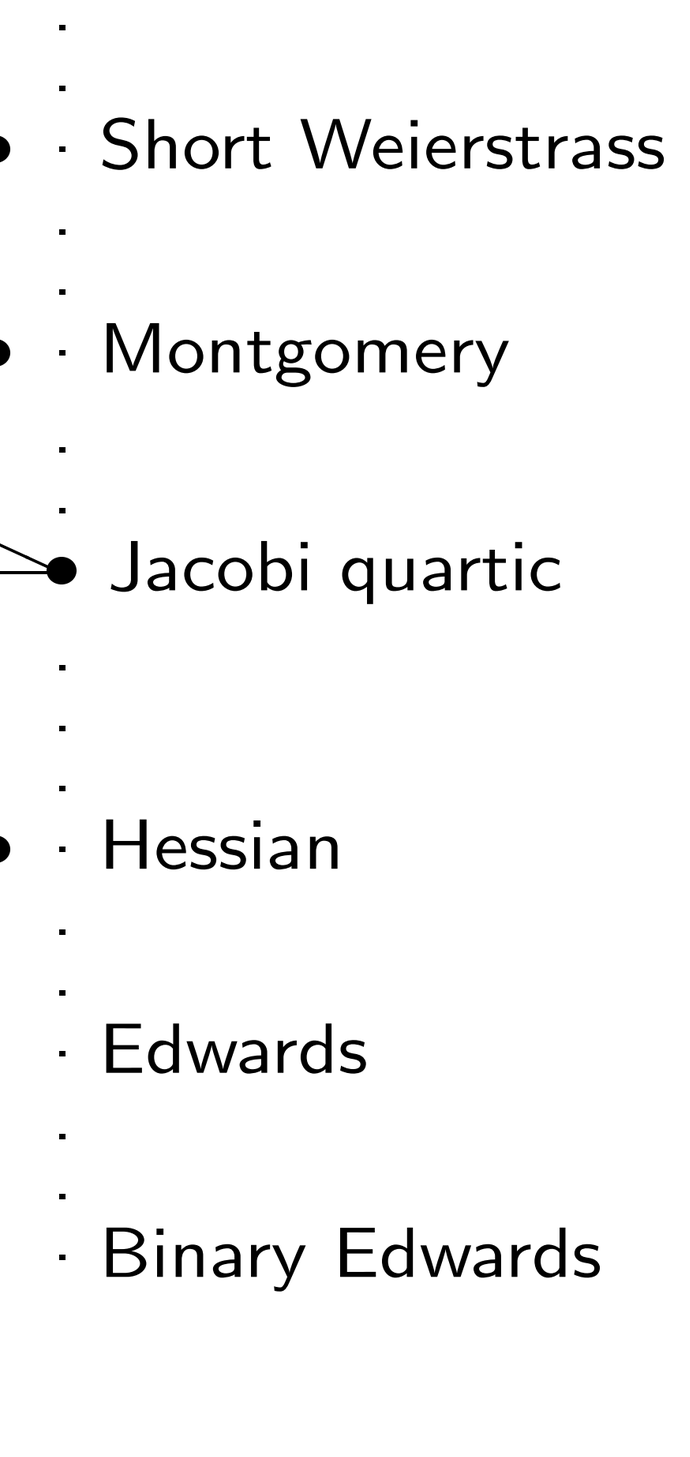
Newton polygons



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Total cost 3**M** +
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$X_1^2, Y_1^2, Y_1^4, Z_1^2, Z$

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$Z_3 = 2Y_1 \cdot Z_1$.

Total cost 3**M** + 6**S** + 1**D**
S is the cost of squaring in
D is the cost of multiplying

The squarings produce
$X_1^2, Y_1^2, Y_1^4, Z_1^2, Z_1^4, M^2$.

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Most ECC standards choos
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Curve-choice advice from
1986 Chudnovsky–Chudno

Can eliminate the 1**D**
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But "it is even smarter"
to choose curve with $a =$

If $a = -3$ then $M = 3(X_1^2$
$= 3(X_1 - Z_1^2) \cdot (X_1 + Z_1^2)$
Replace 2**S** with 1**M**.

Now DBL costs $4\mathbf{M} + 4\mathbf{S}$.

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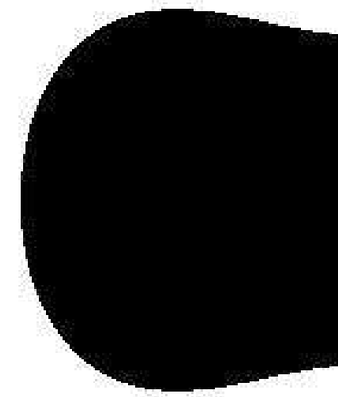

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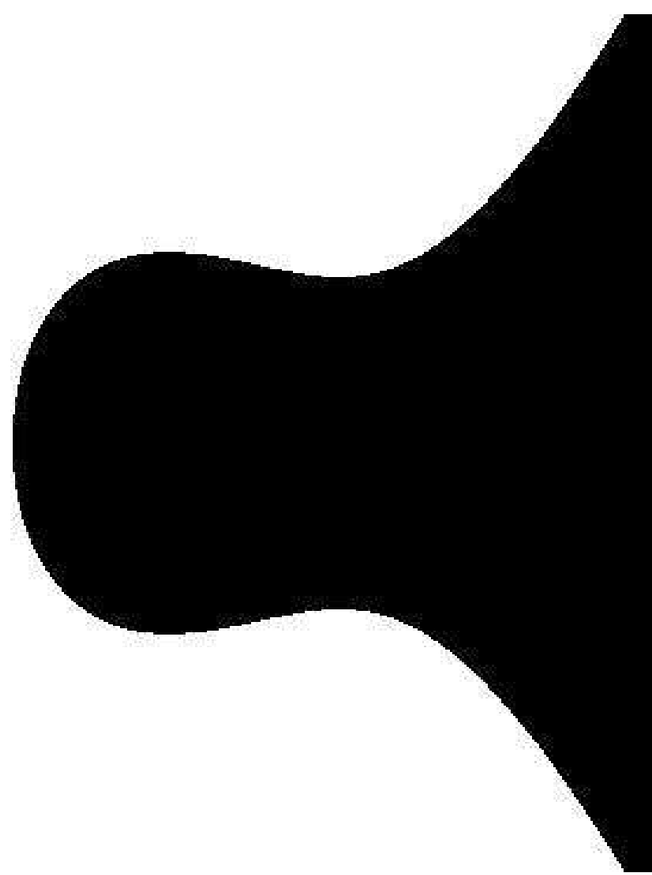

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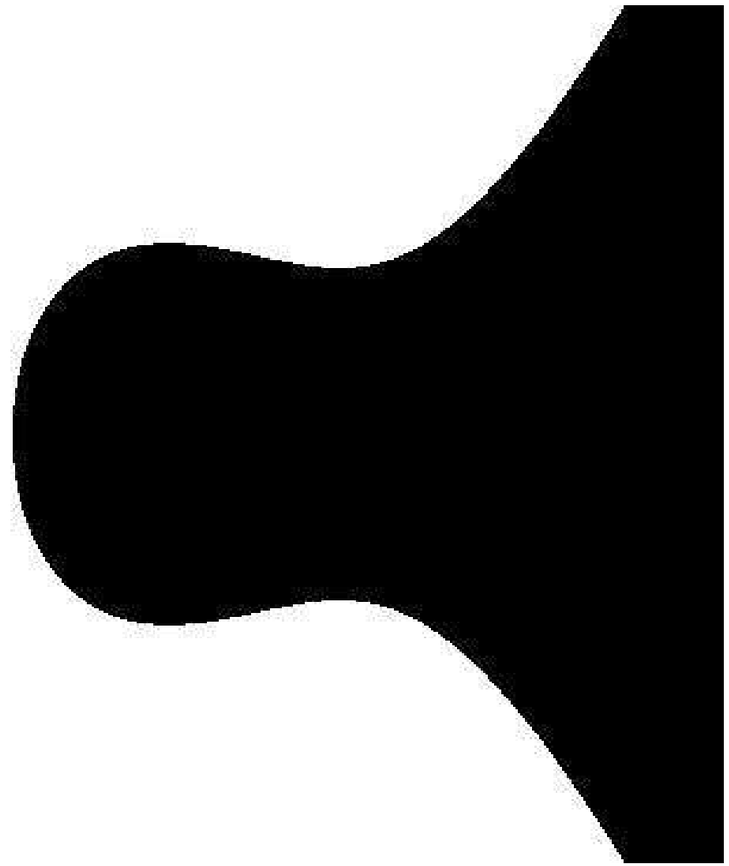

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$+ 1\mathbf{S} + 1\mathbf{D}$ for ADD.
$1\mathbf{S} + 1\mathbf{D}$ for mADD.

d Edwards coordinates
Bernstein–Lange):
$4\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{S} + 1\mathbf{D}$ for ADD.
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Edwards speed news:
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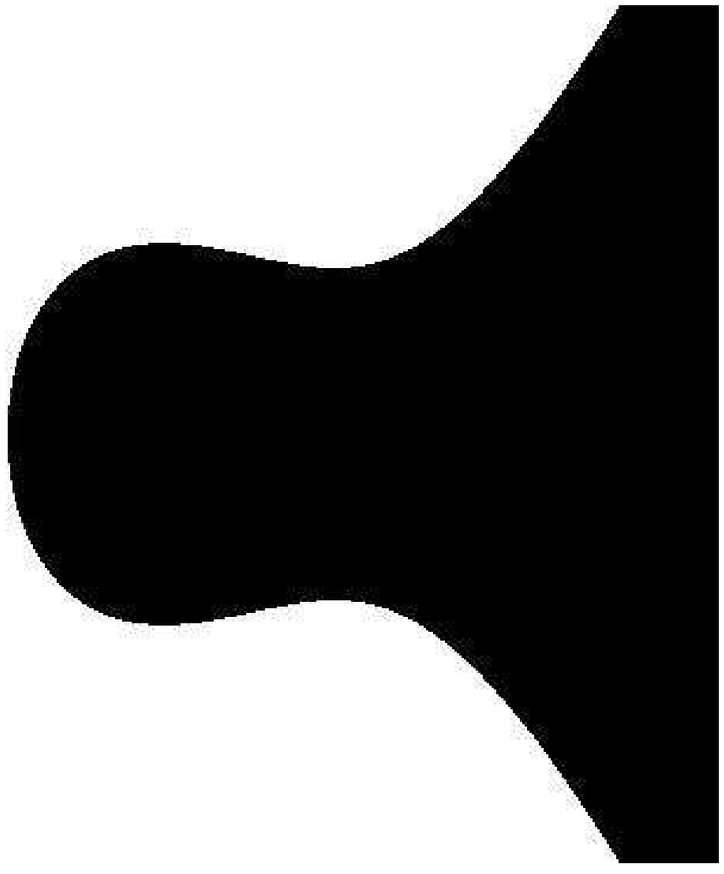

$y^2 = x^3 - 0.4x + 0.7$

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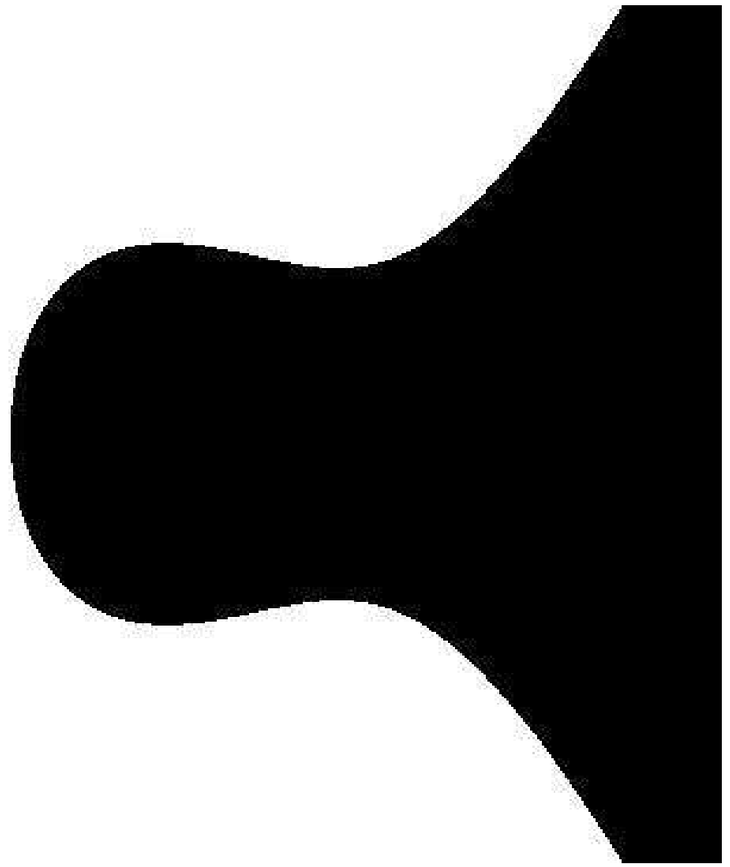

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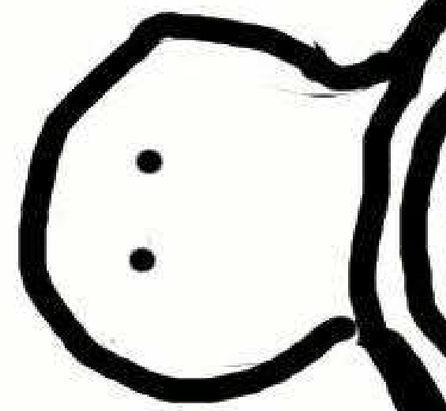

The Weierstrass-
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(Thanks to Tanja
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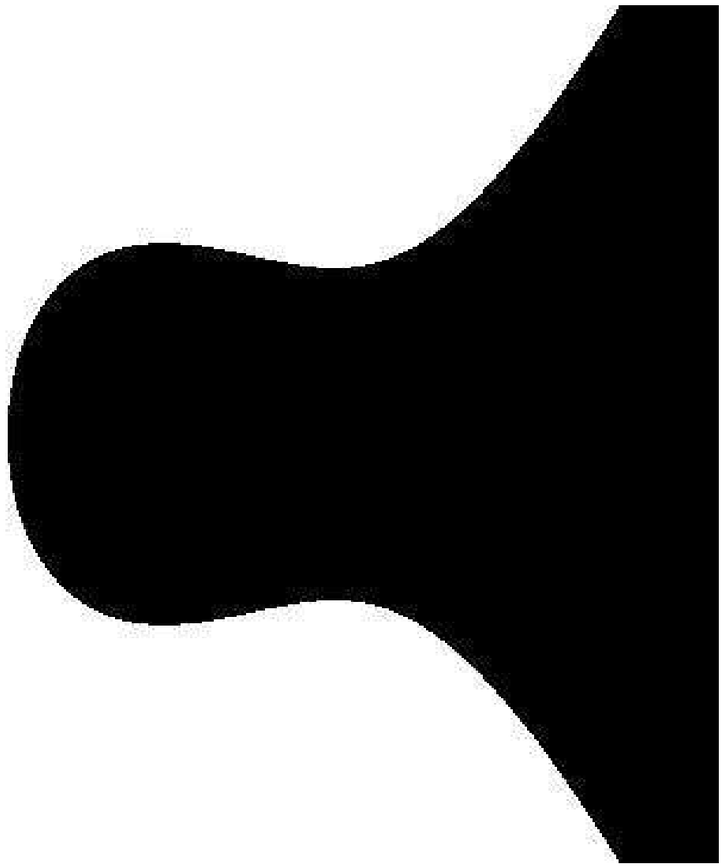

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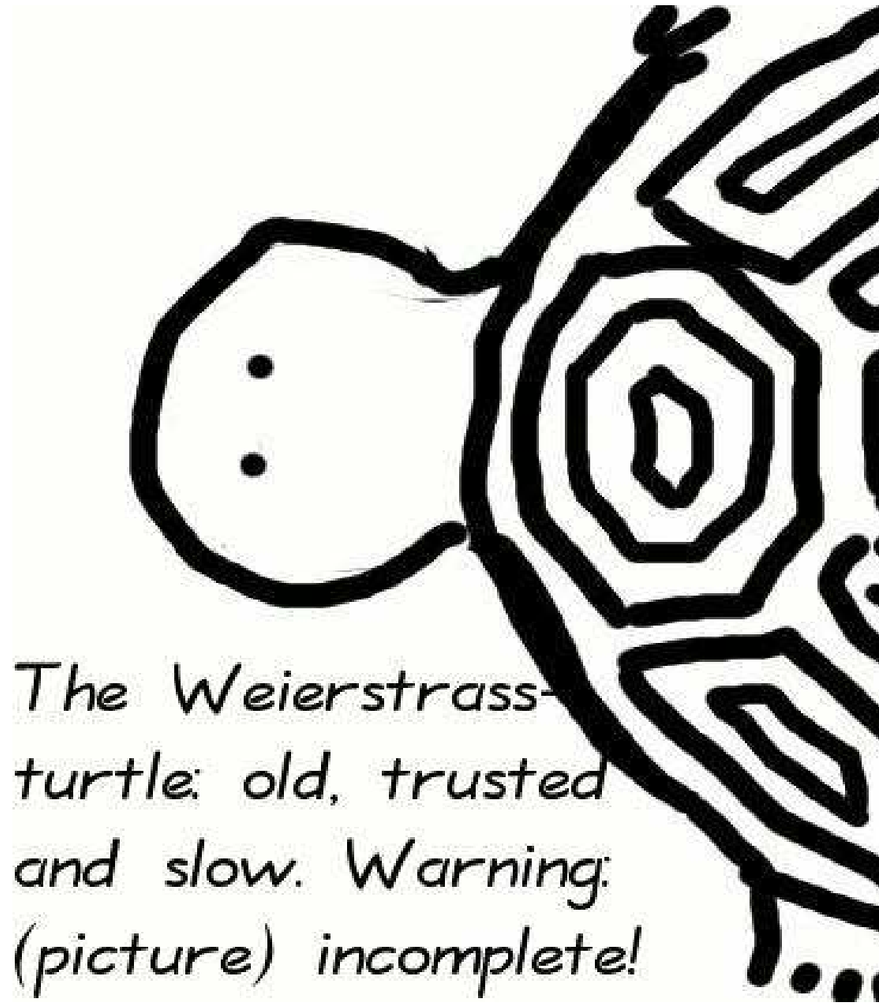

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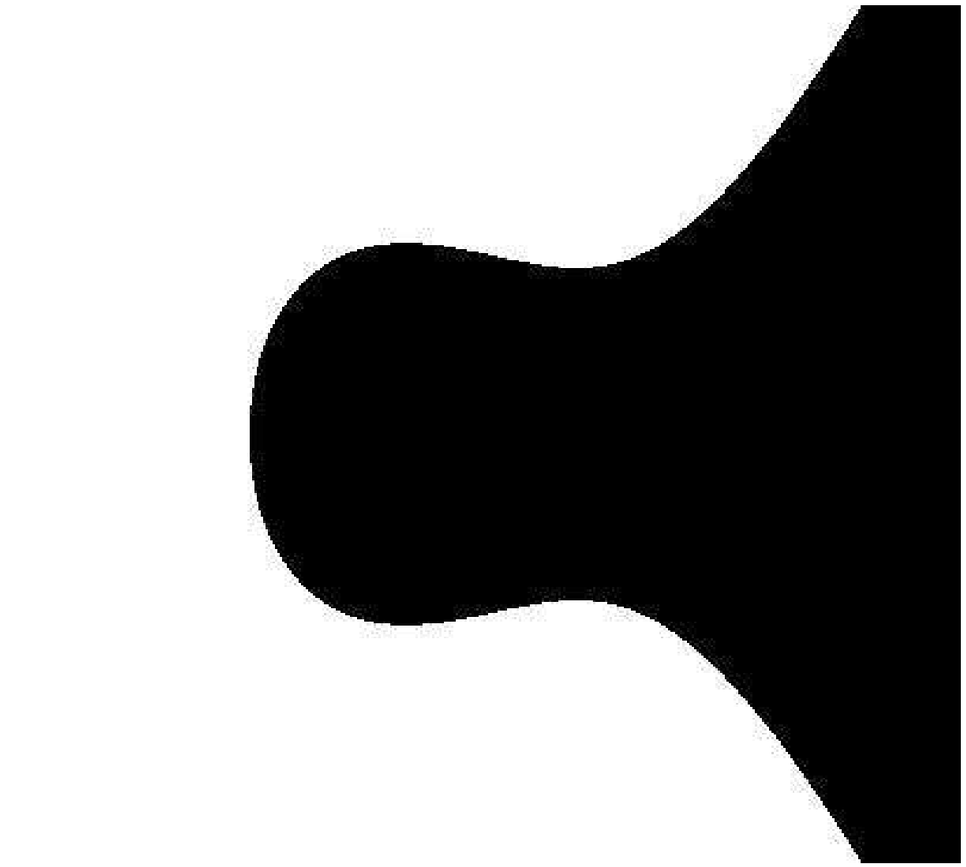

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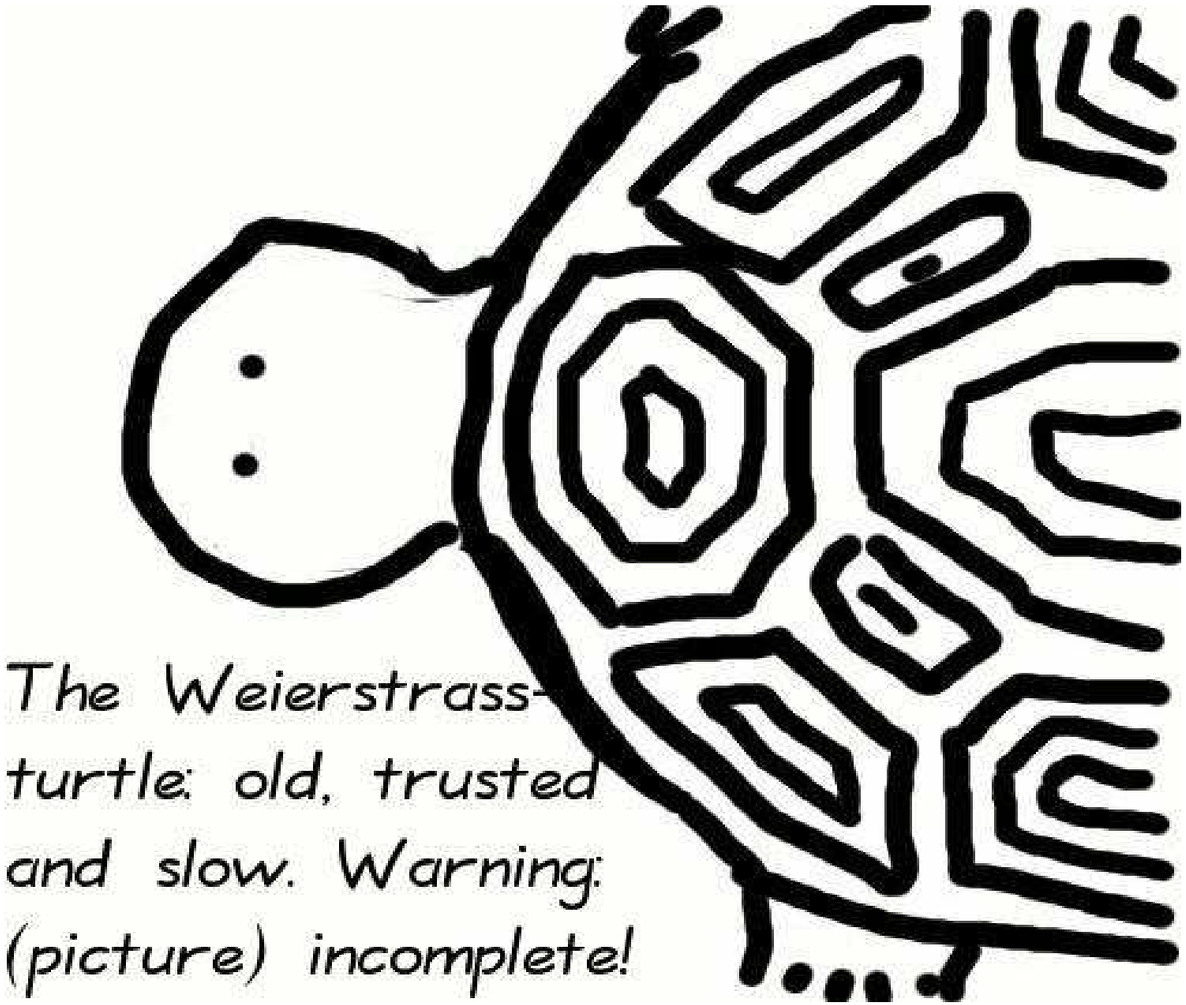

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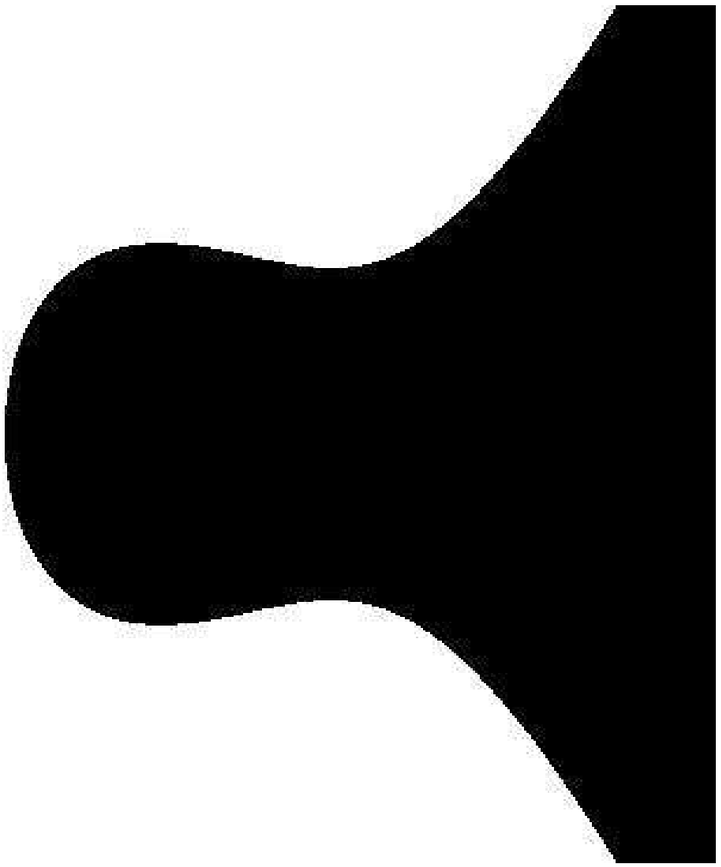

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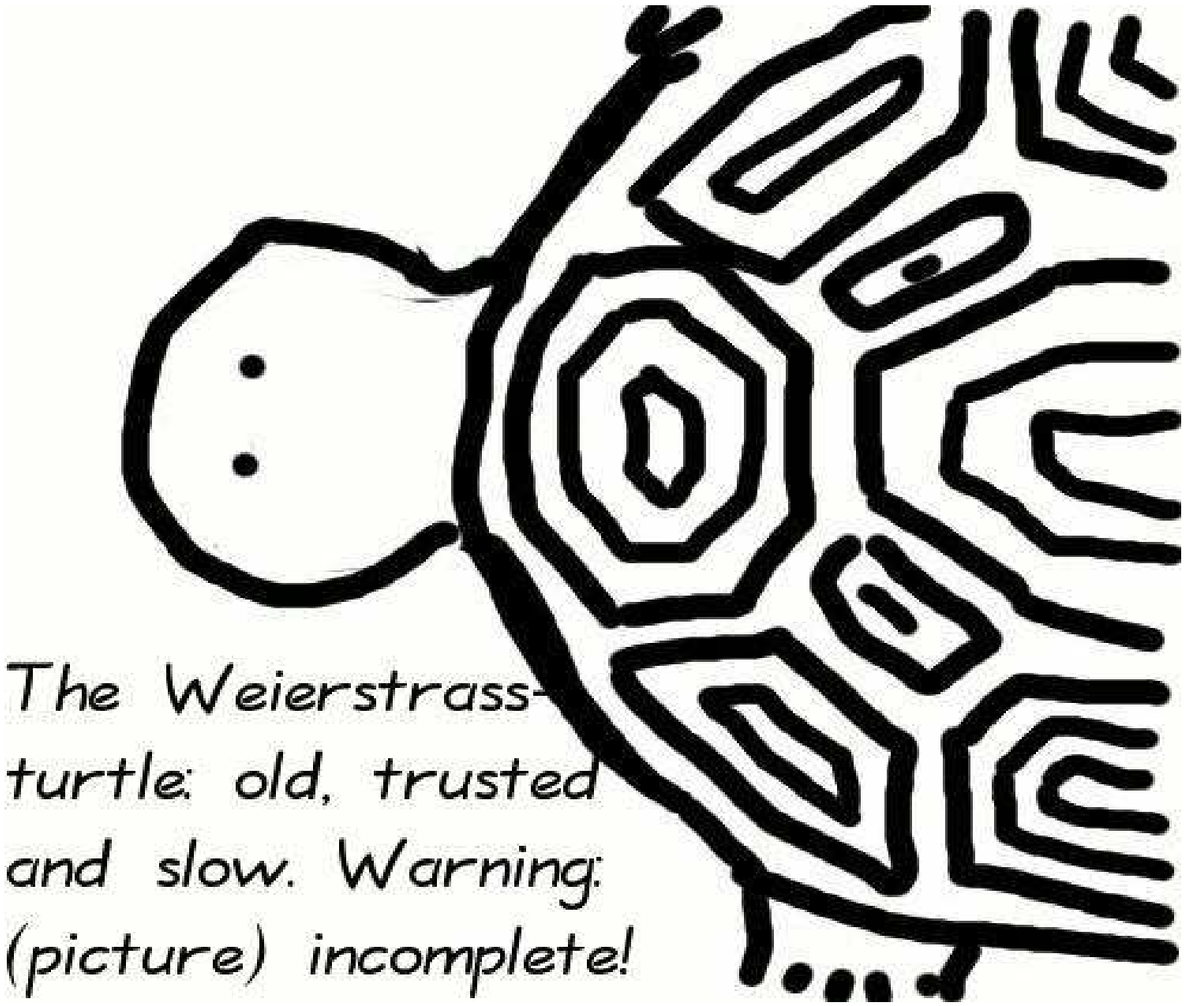

(Thanks to Tanja Lange
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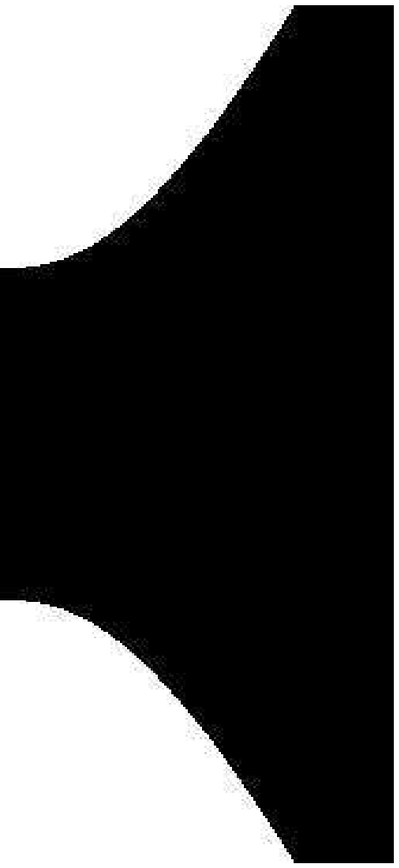
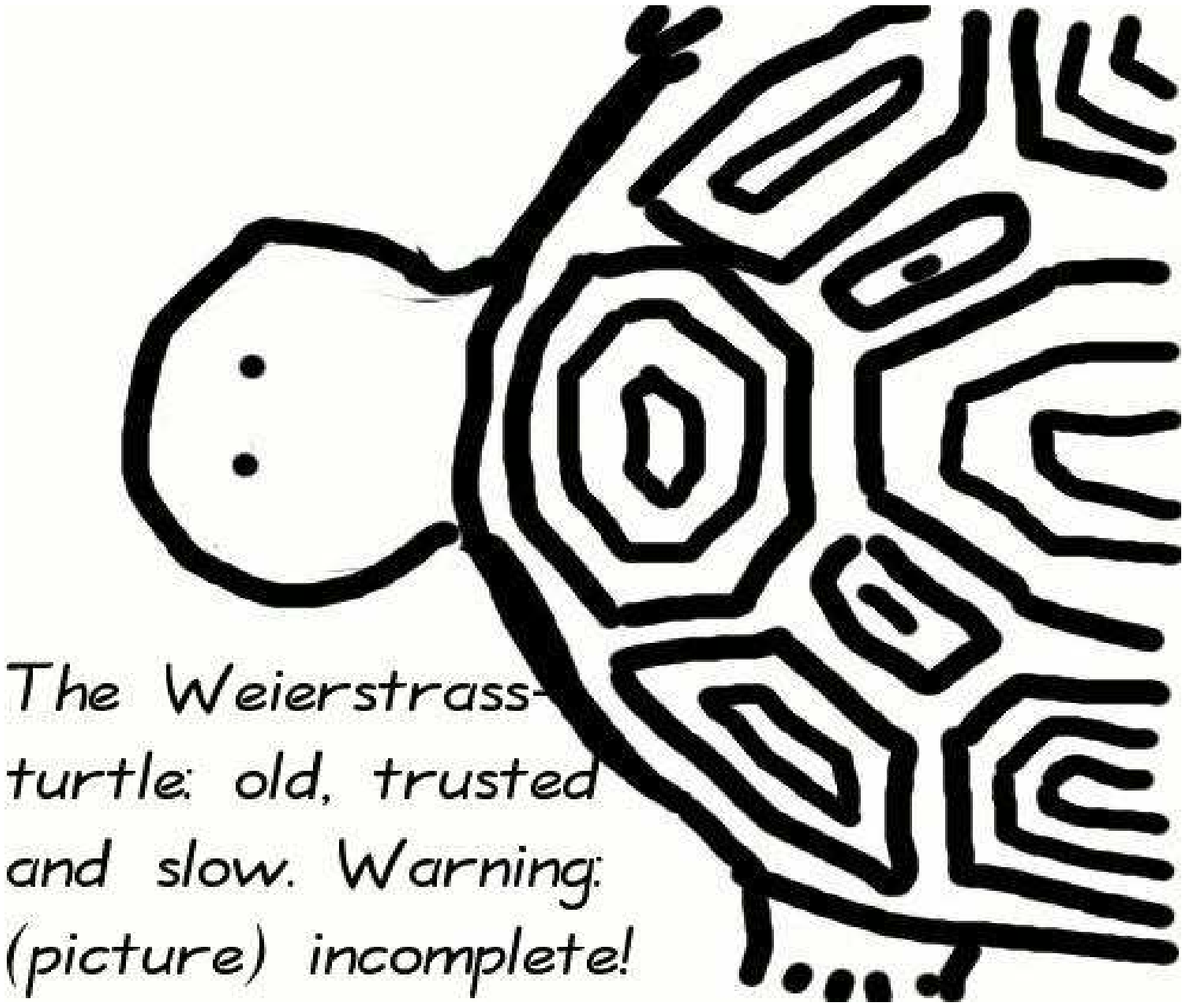
$x^2 + y$



$+ 0.7$



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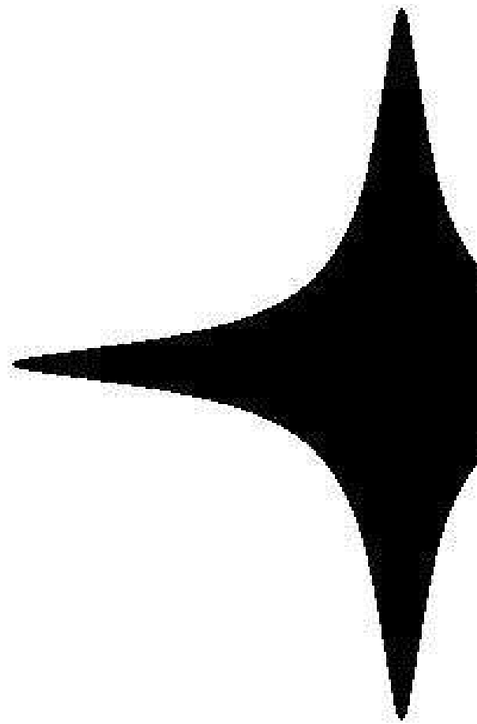
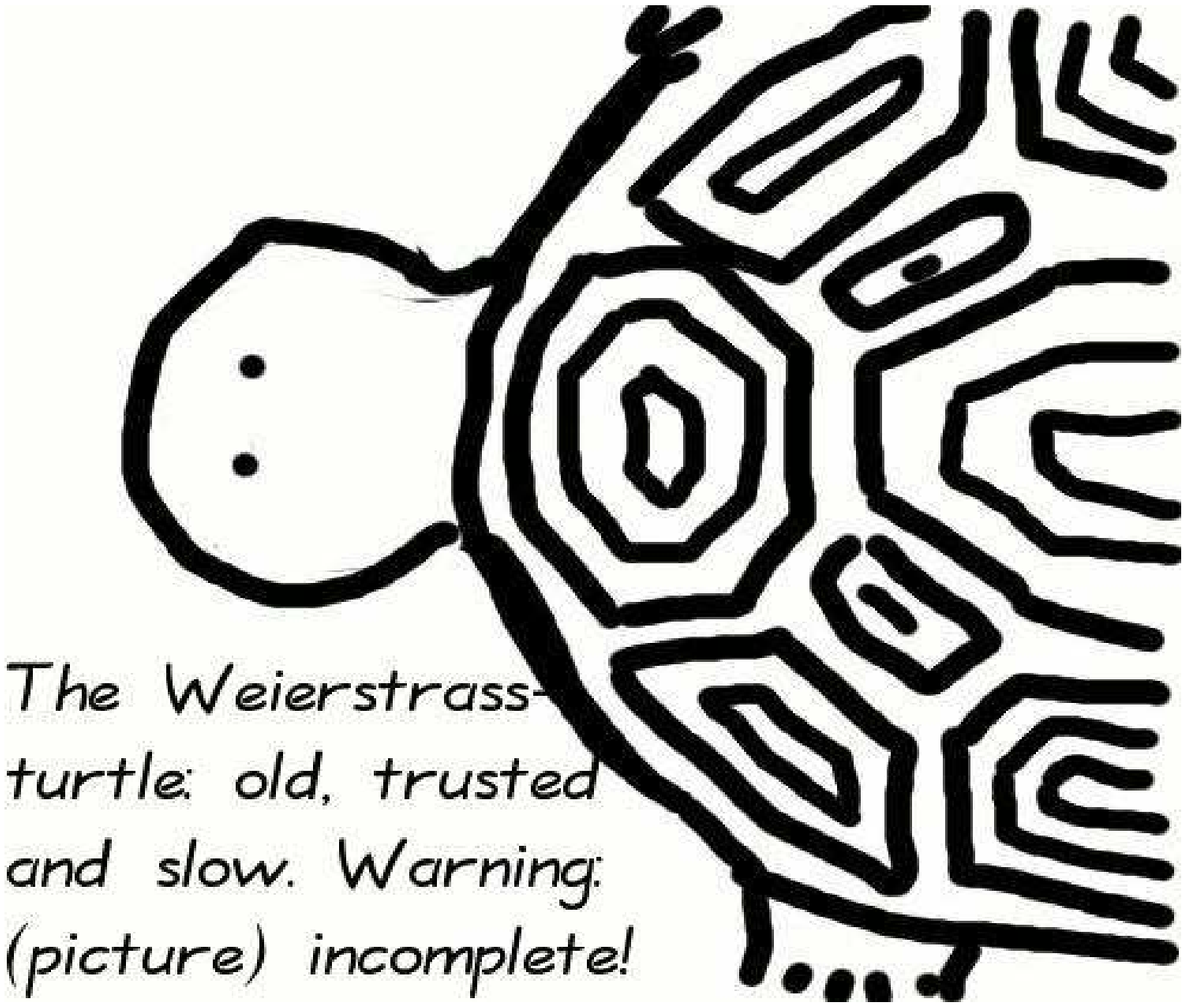


$x^2 + y^2 = 1 - 30$



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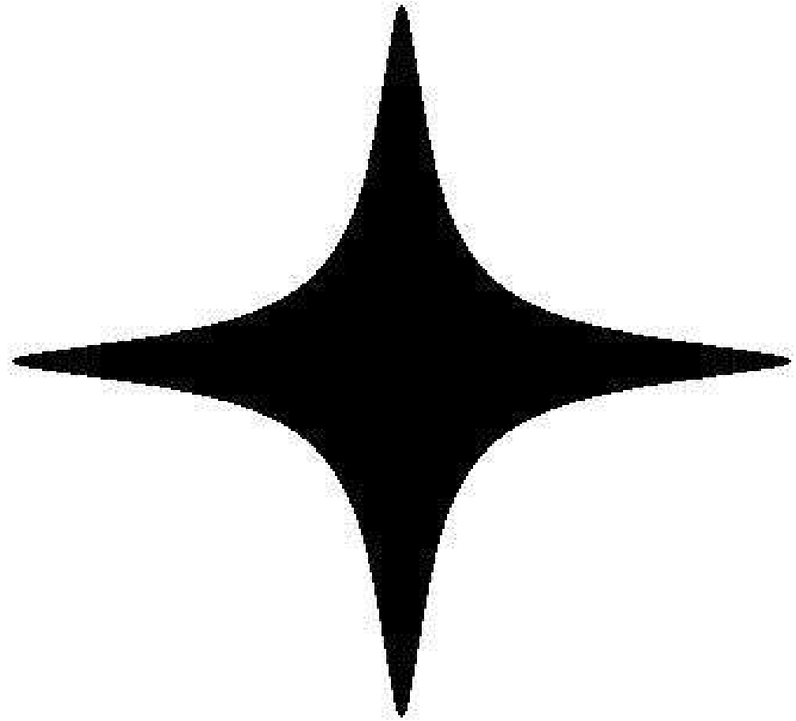
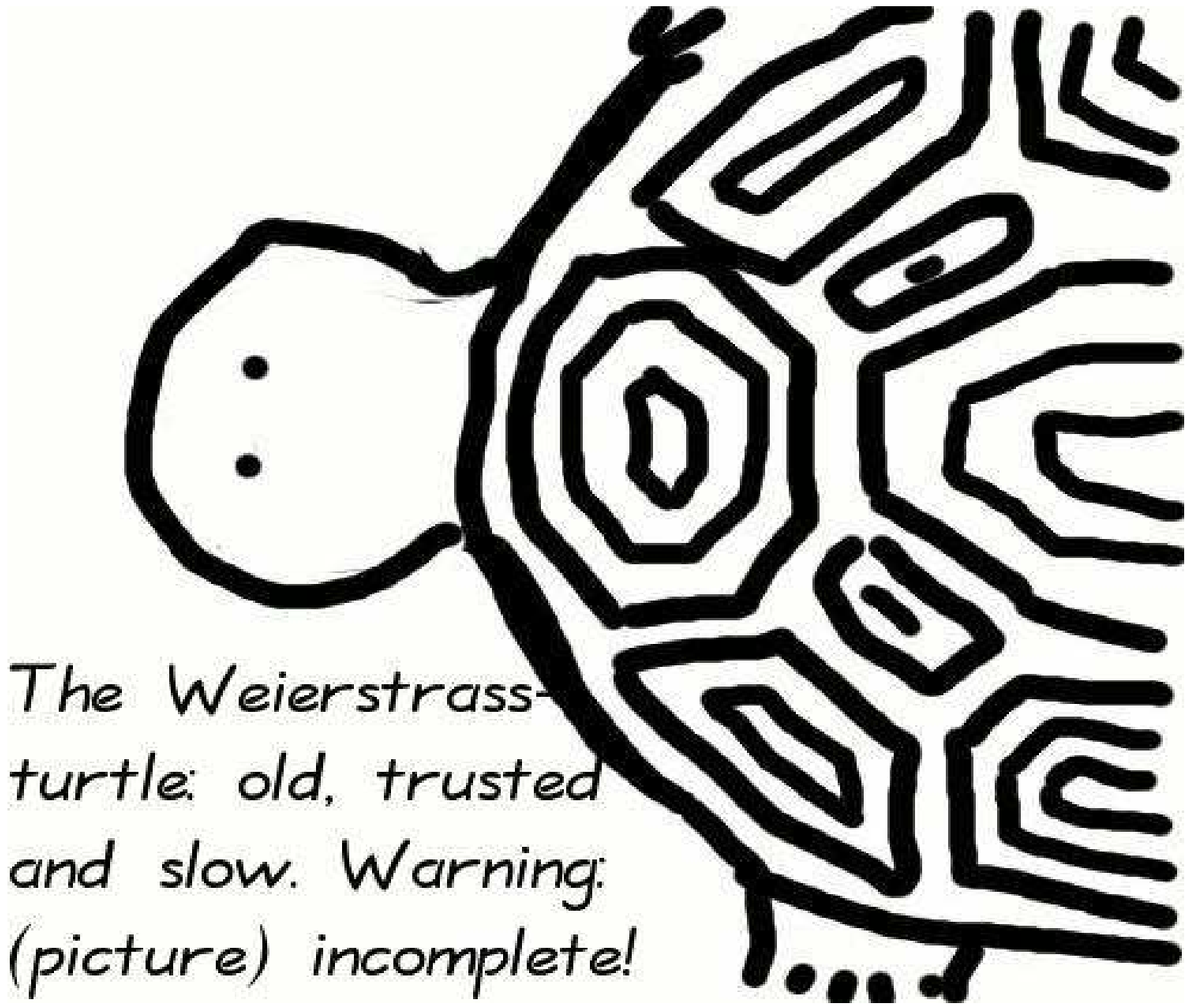


$$x^2 + y^2 = 1 - 300x^2y^2$$



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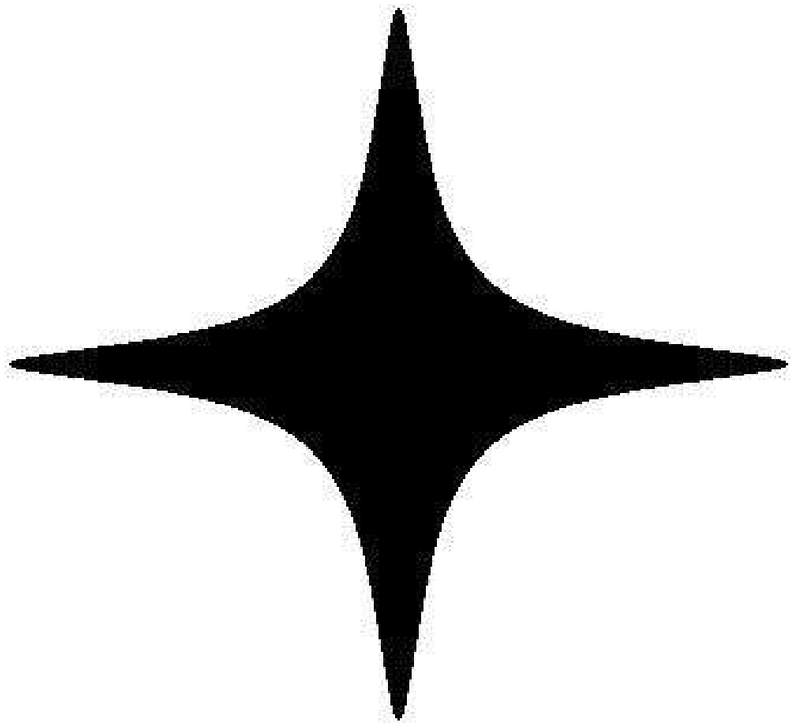

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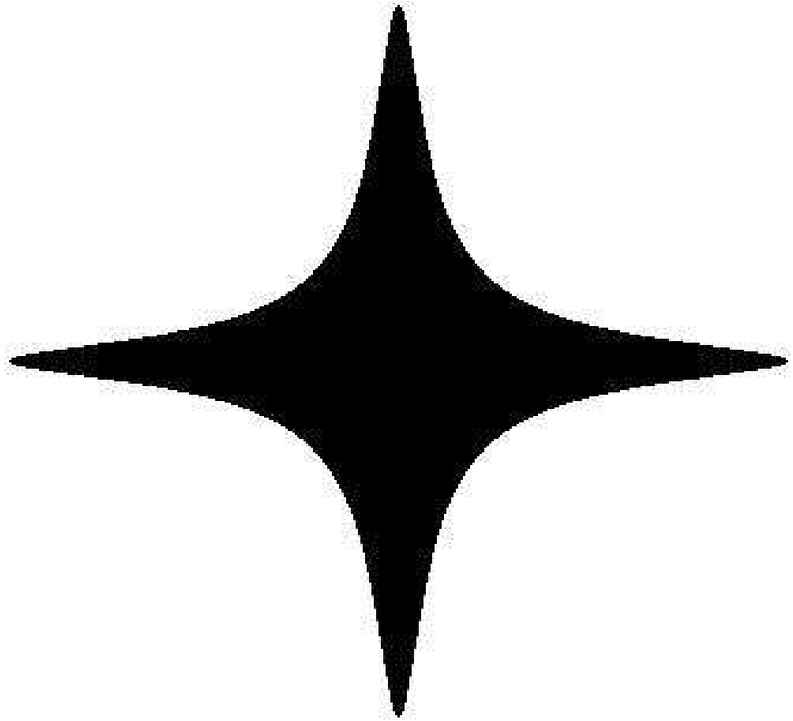

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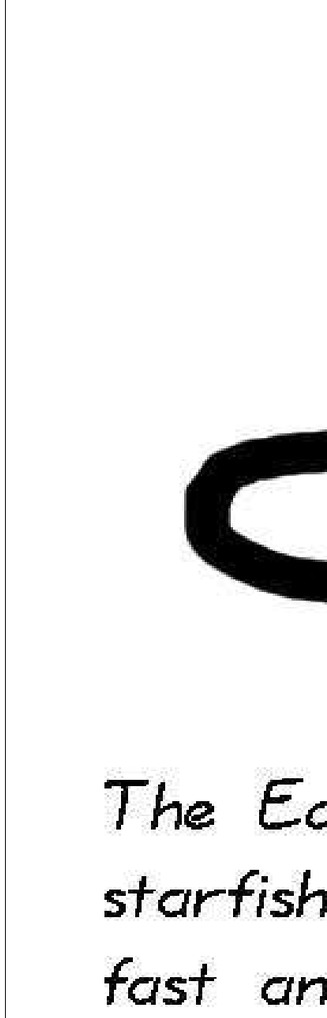

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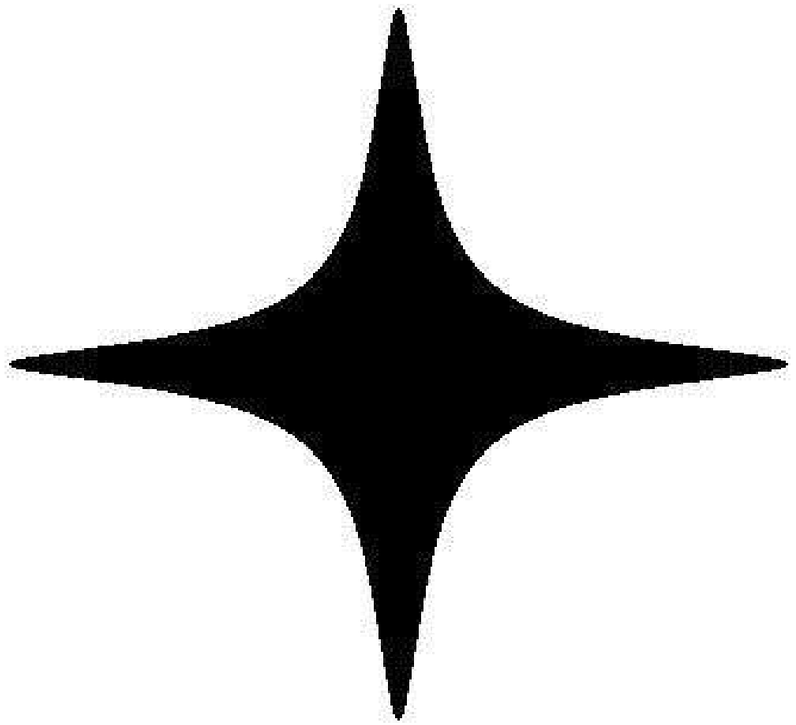

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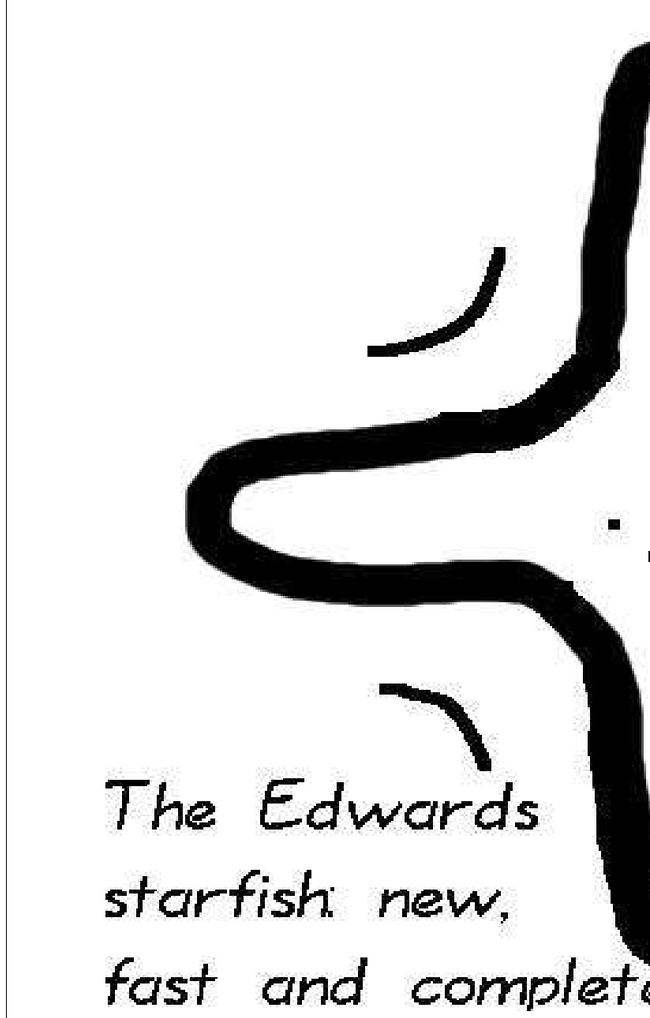

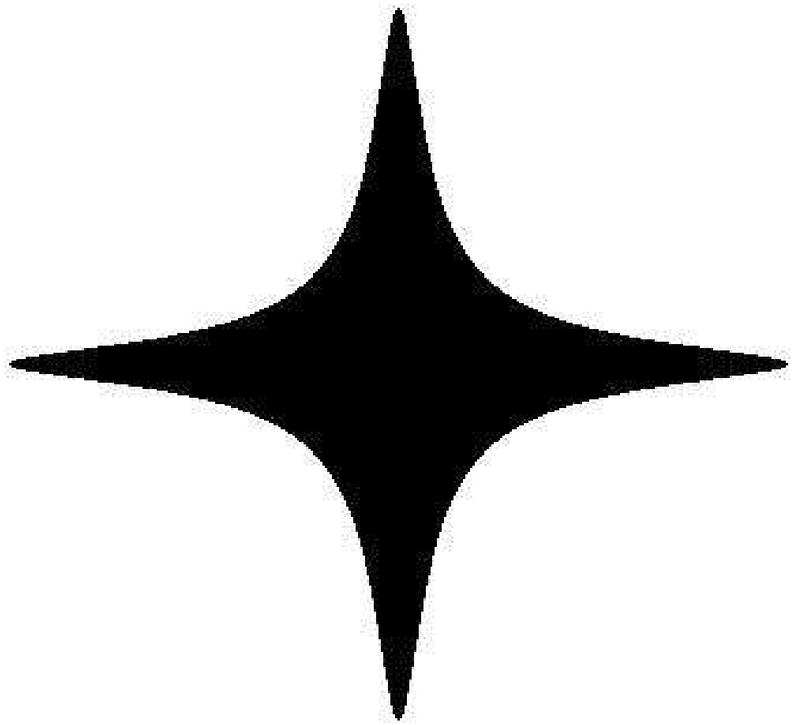

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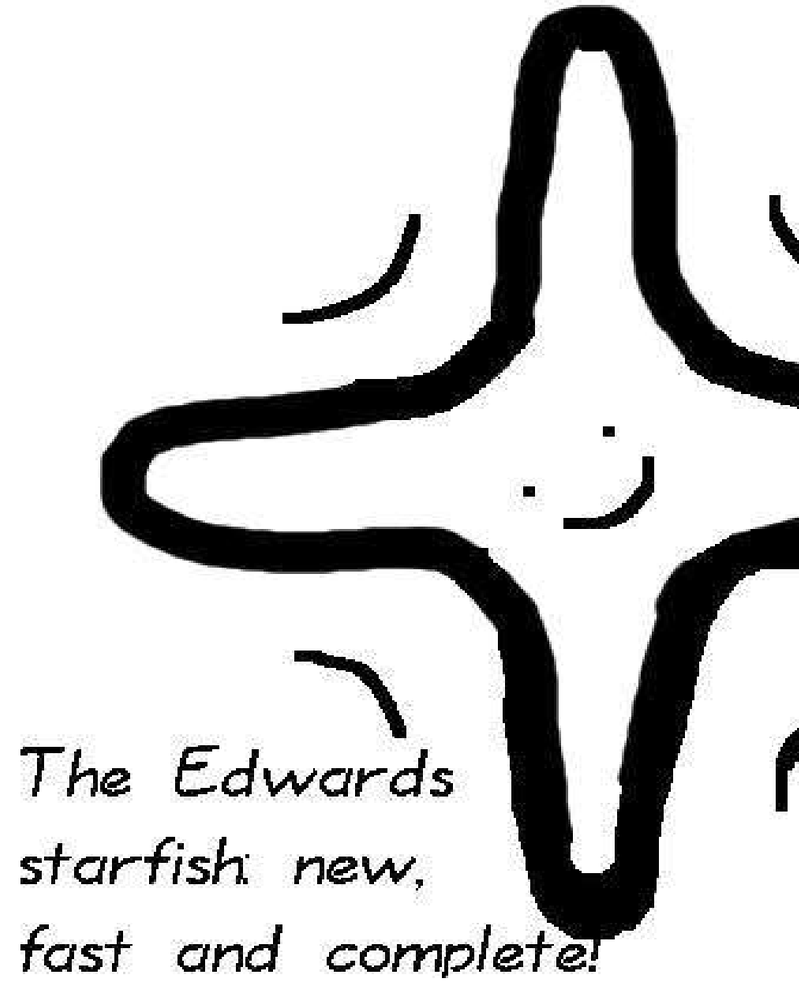

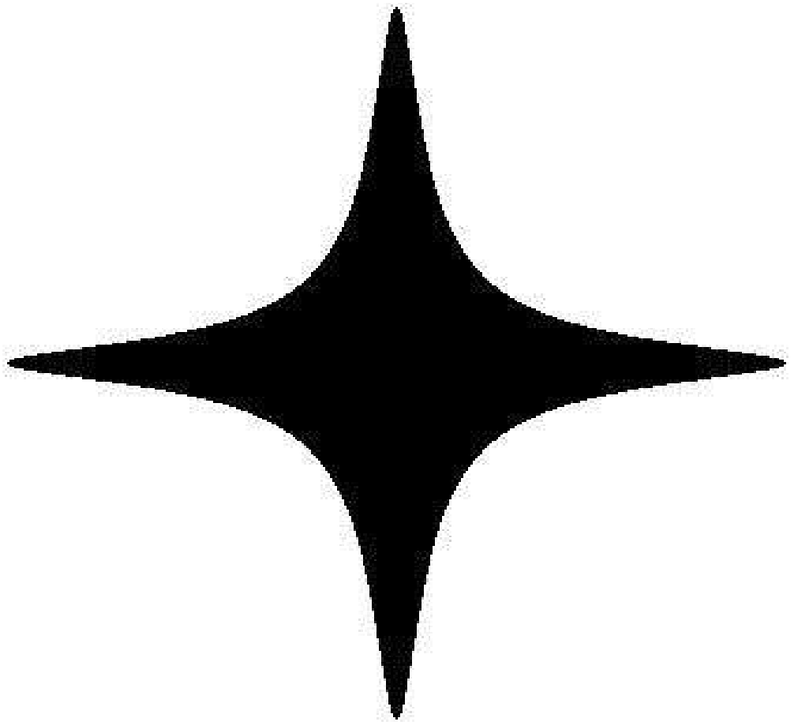

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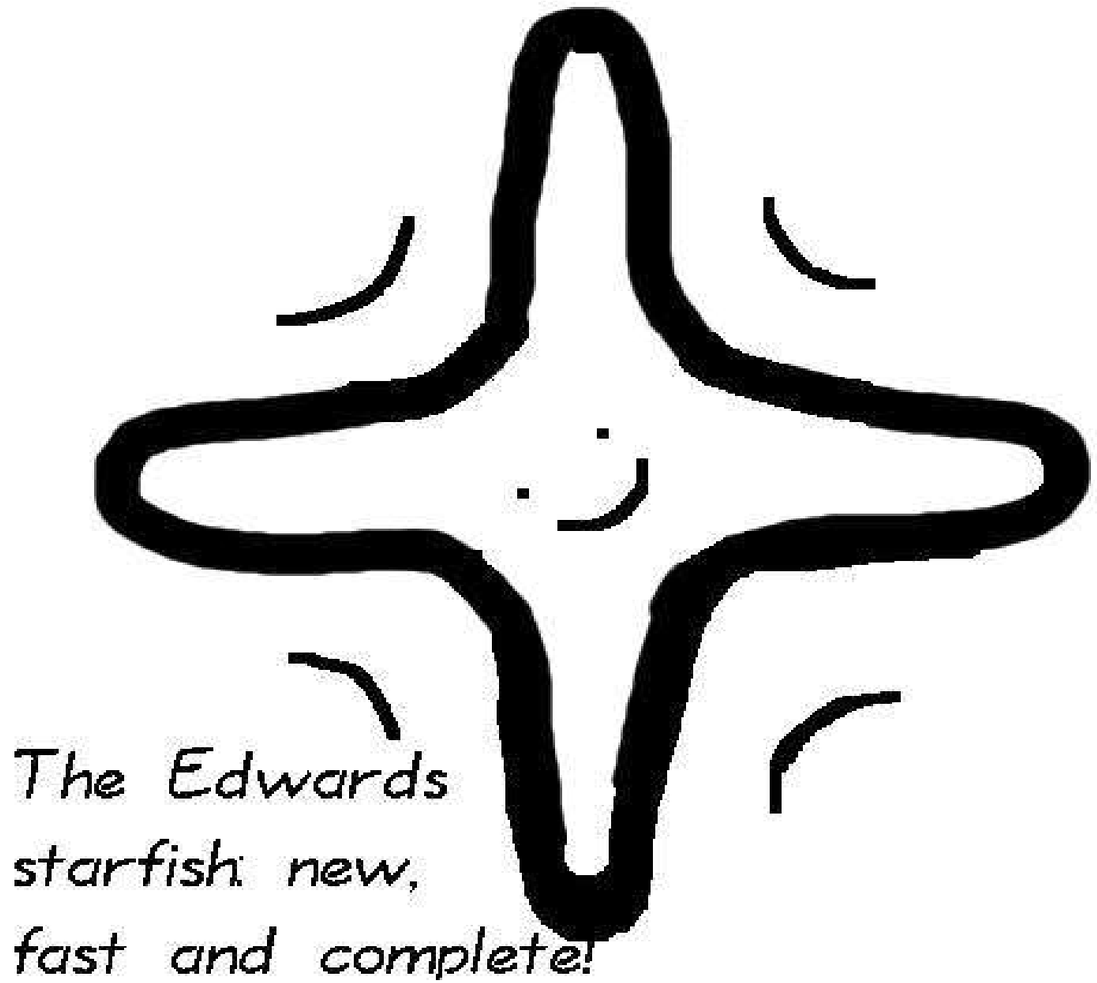

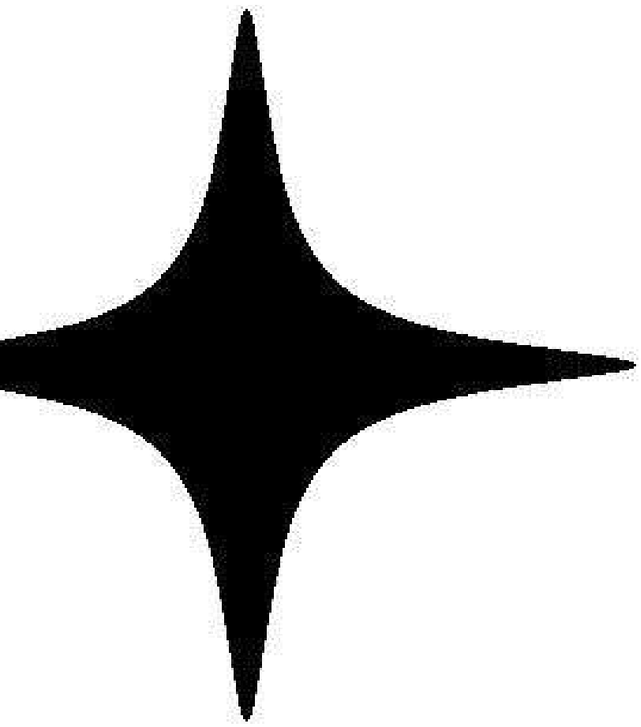

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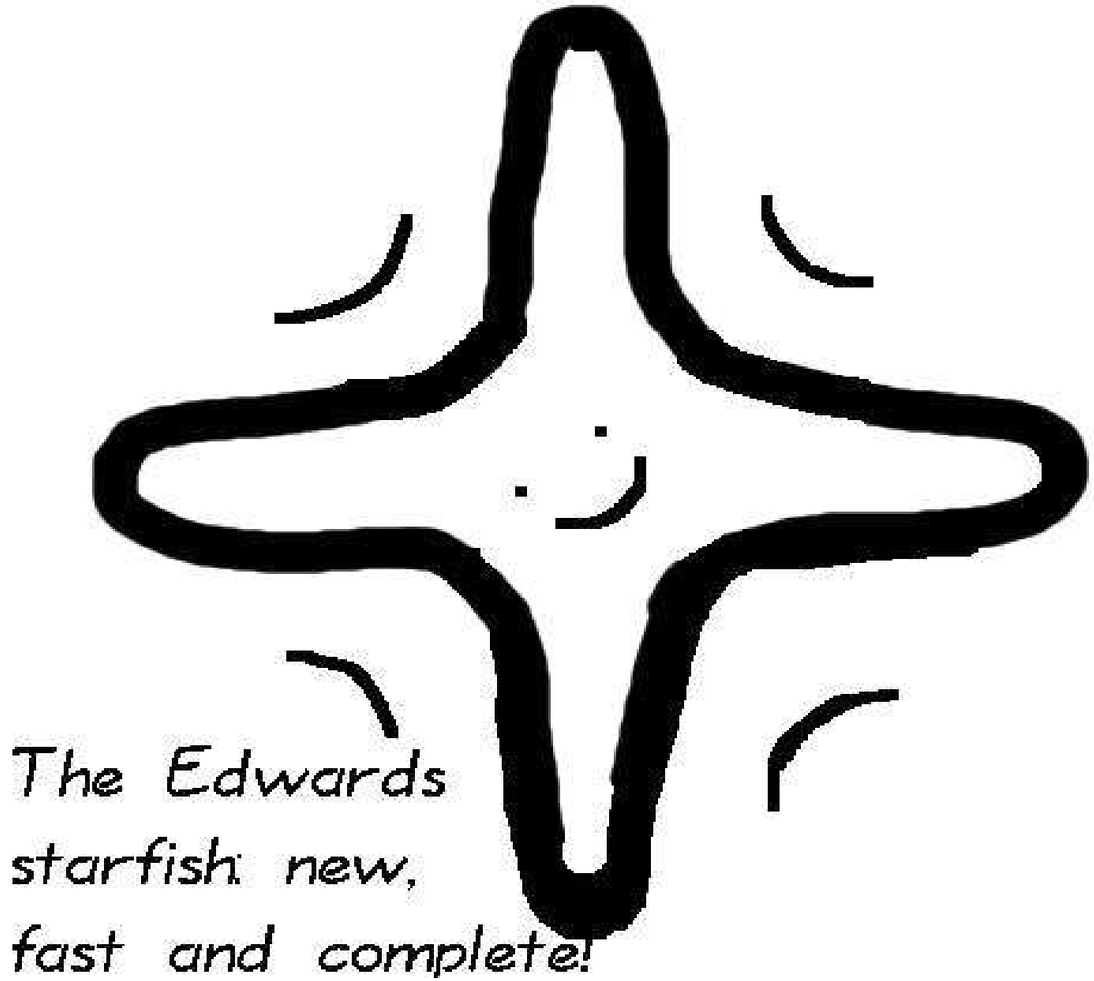

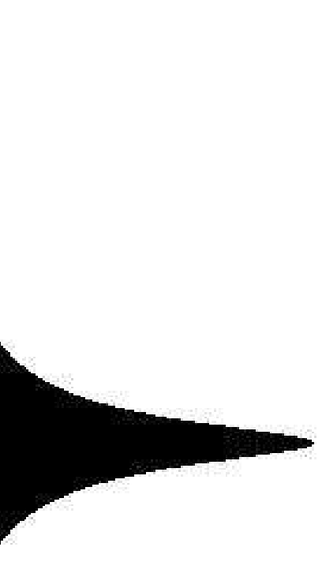

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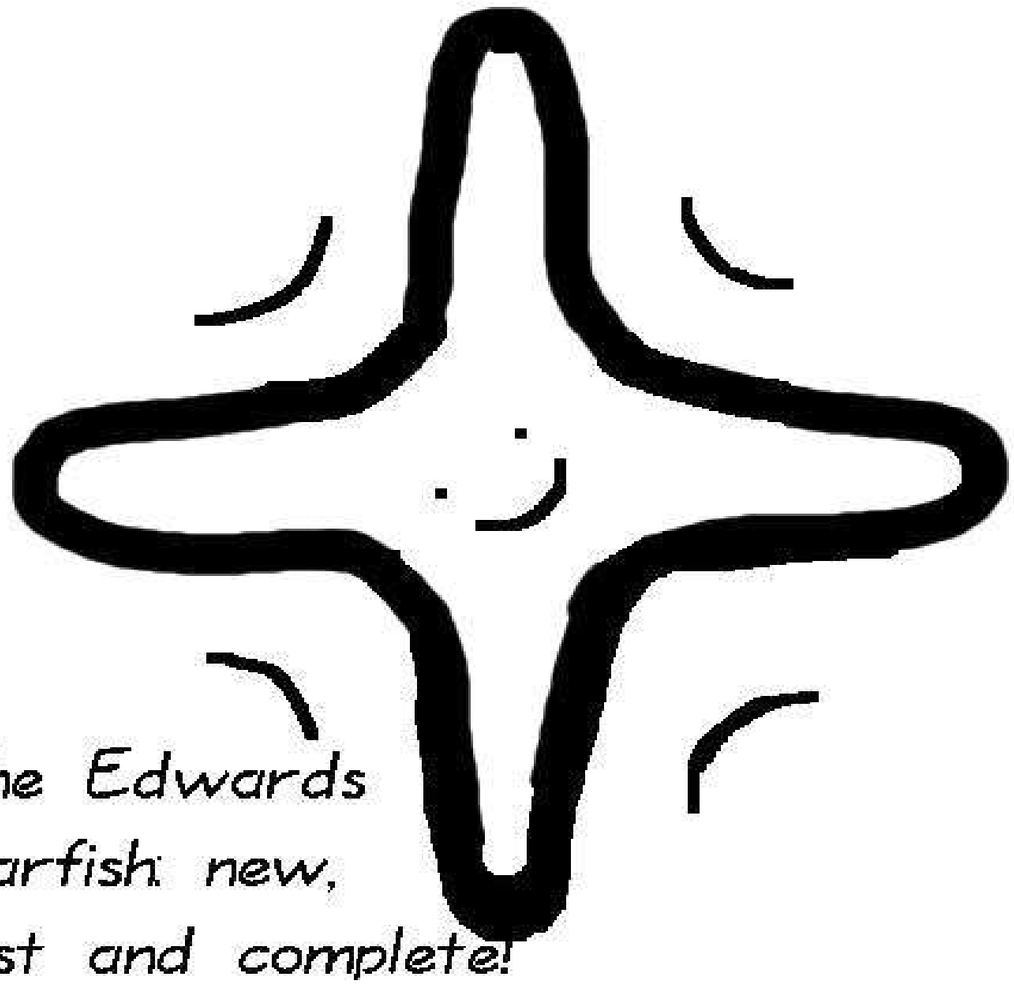
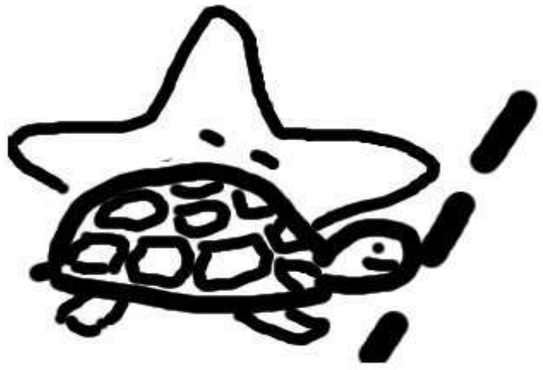


The Edwards starfish: new, fast and complete!



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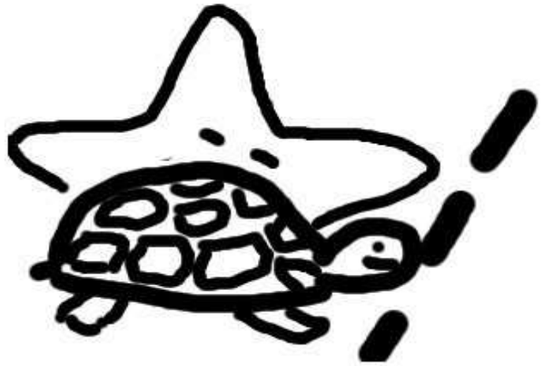
The Edwards starfish: new, fast and complete!



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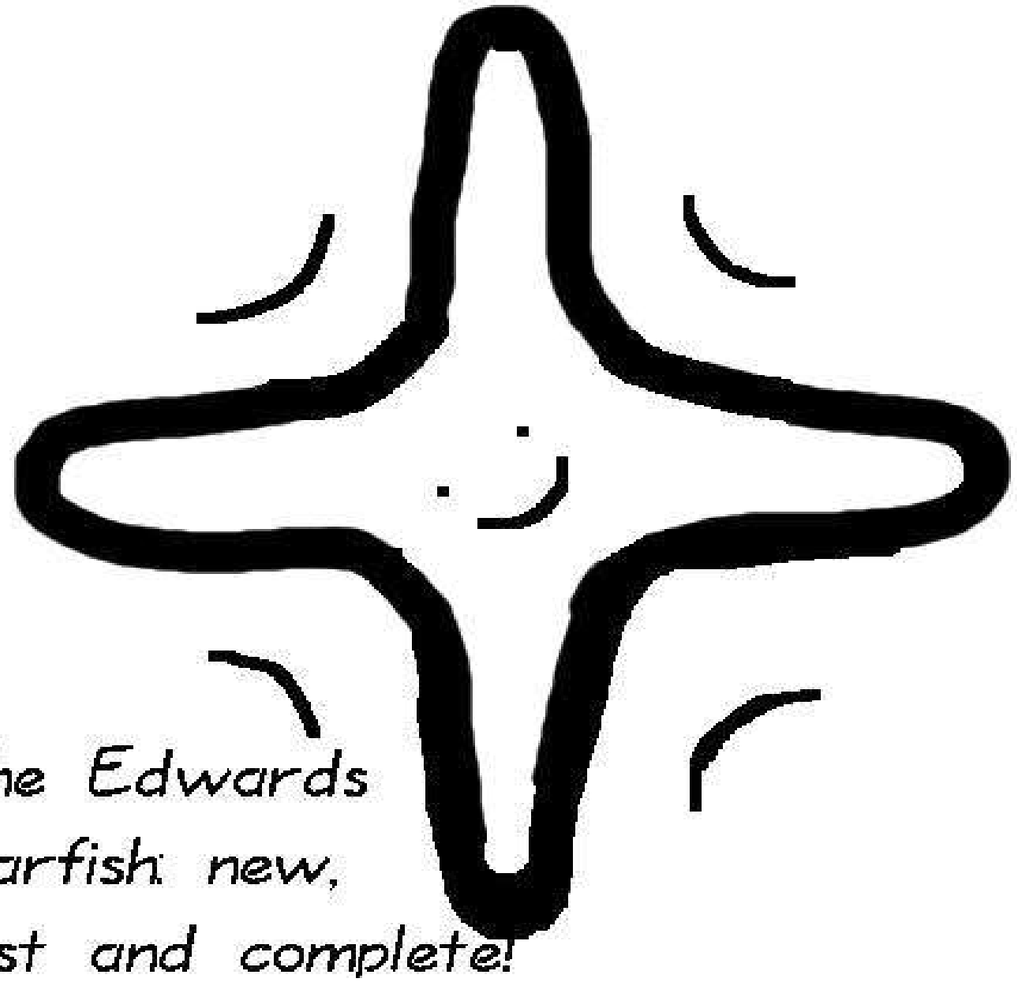

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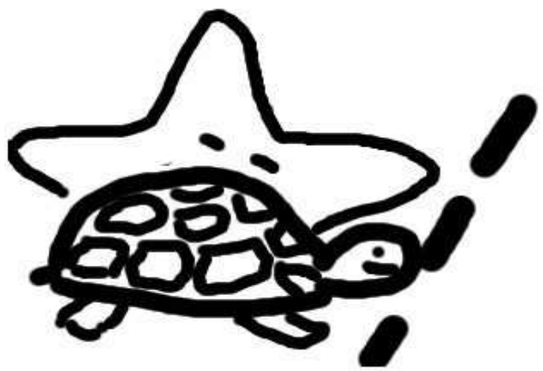

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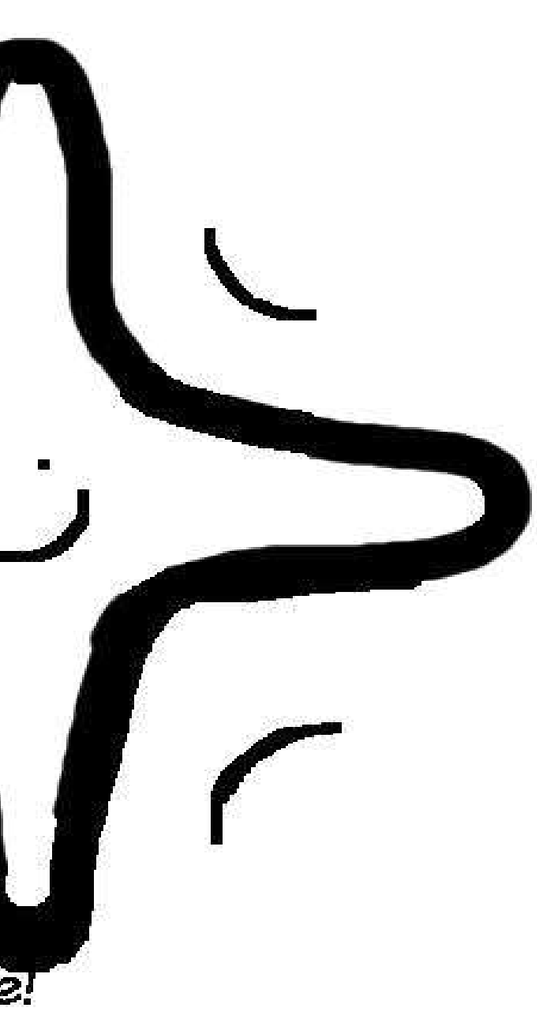

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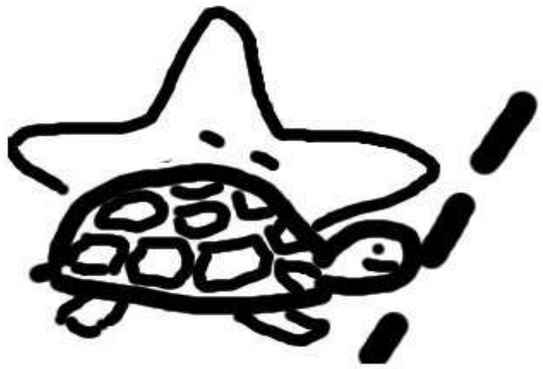

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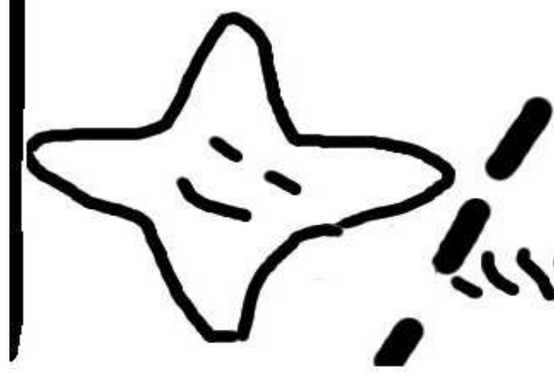

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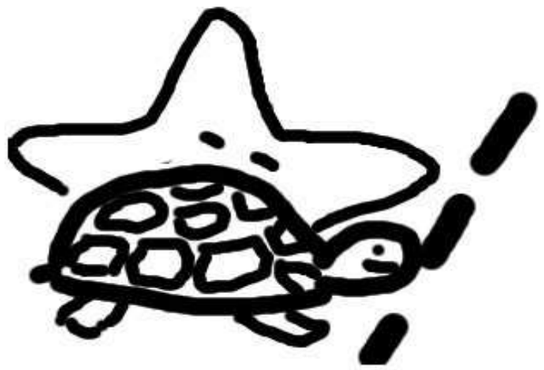

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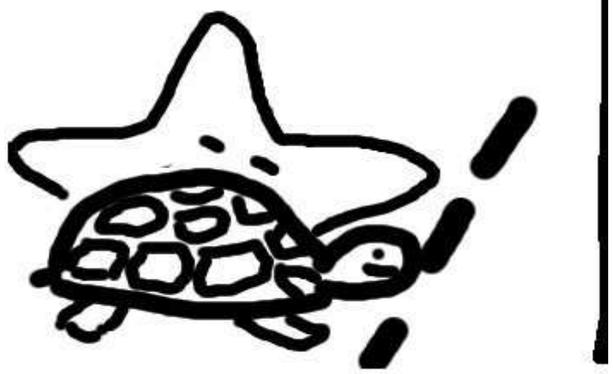

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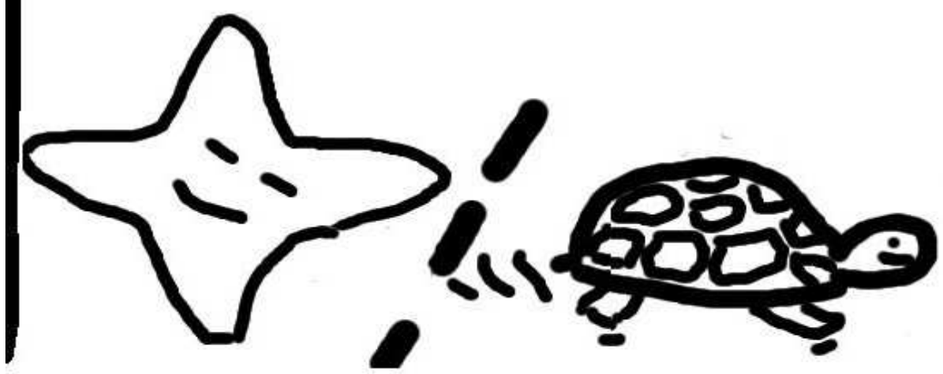

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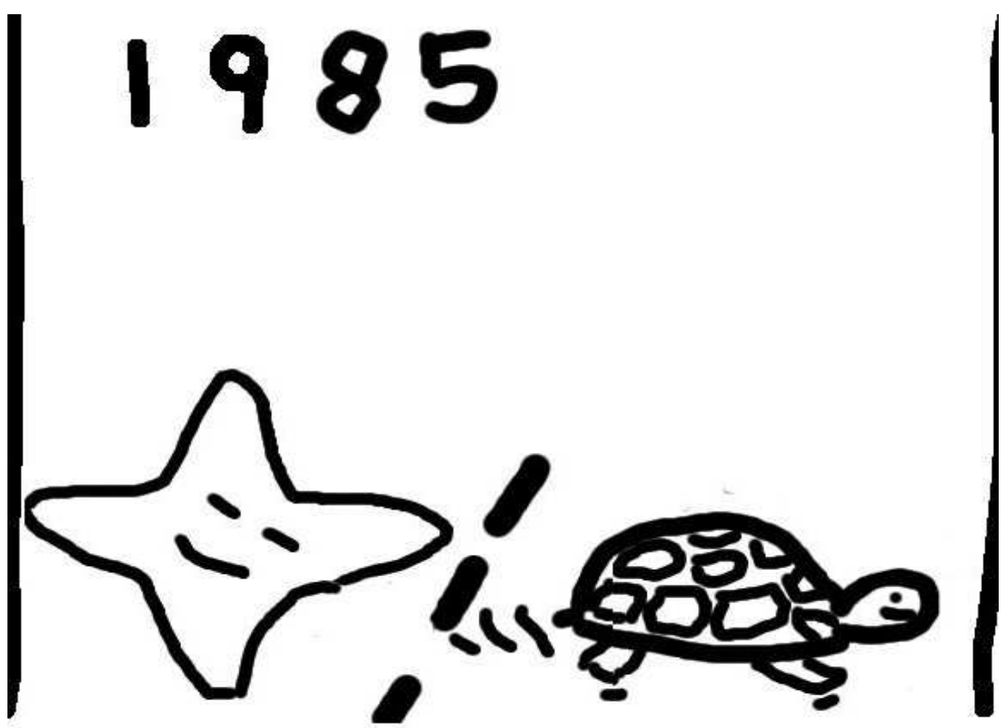

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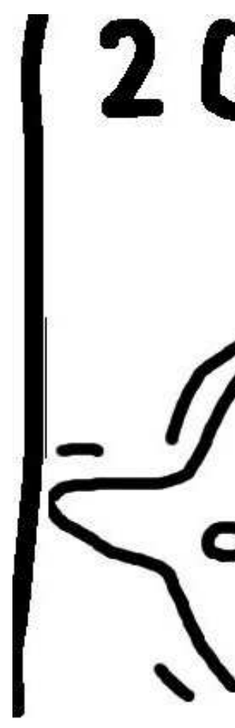

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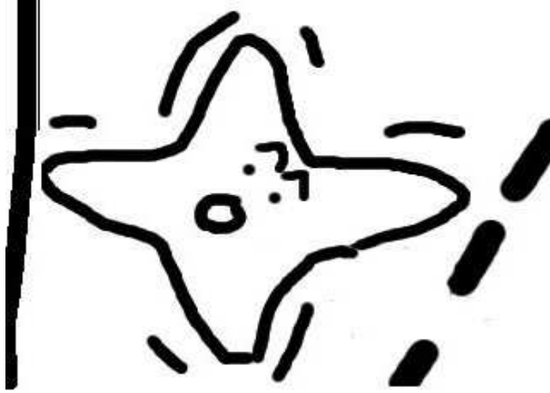

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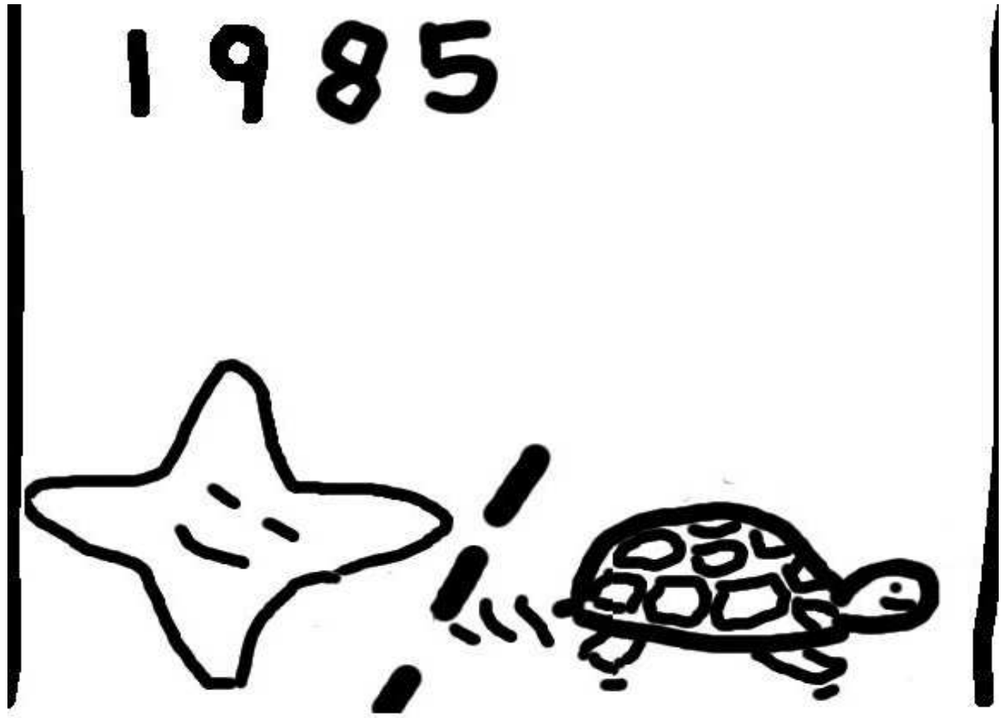

Weierstrass sets off, Edwards left behind sleeping

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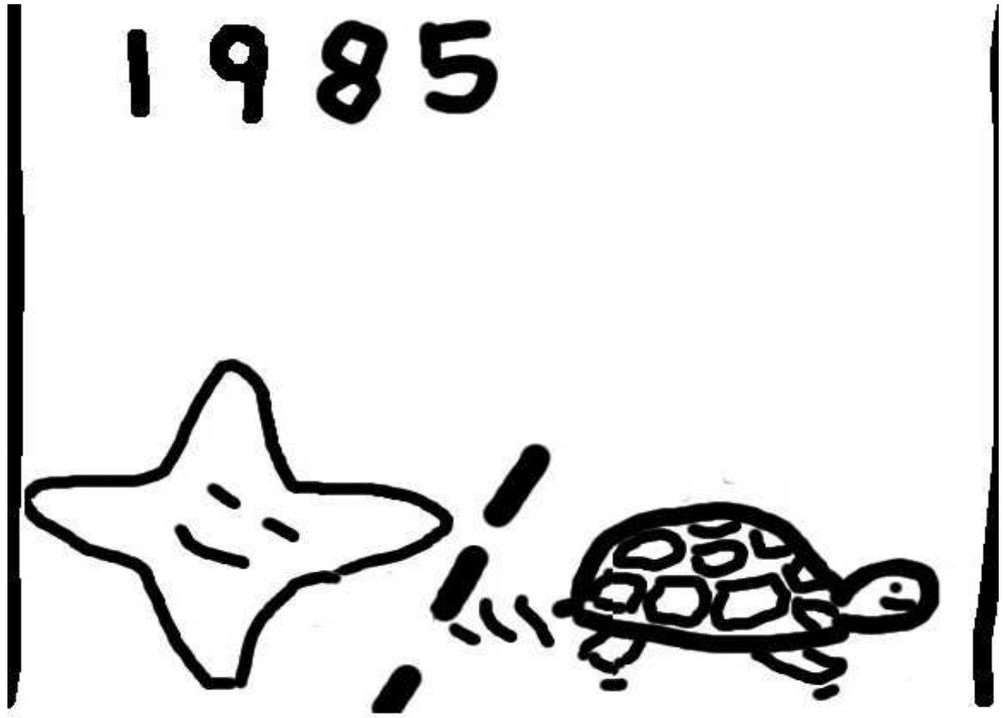

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Weierstrass has made some progress - finally Edwards wakes up.

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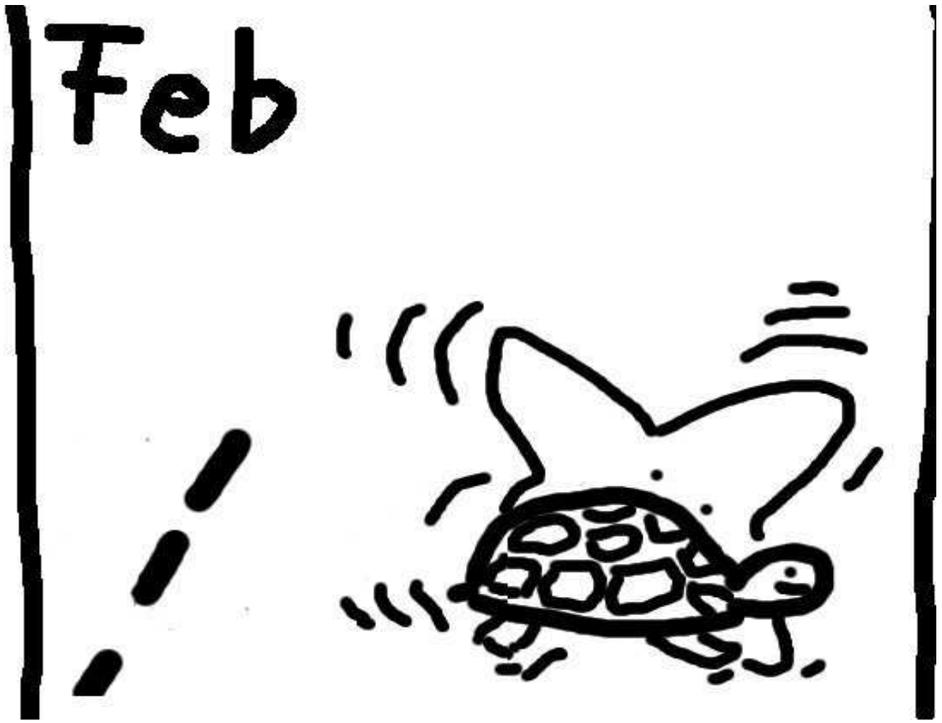
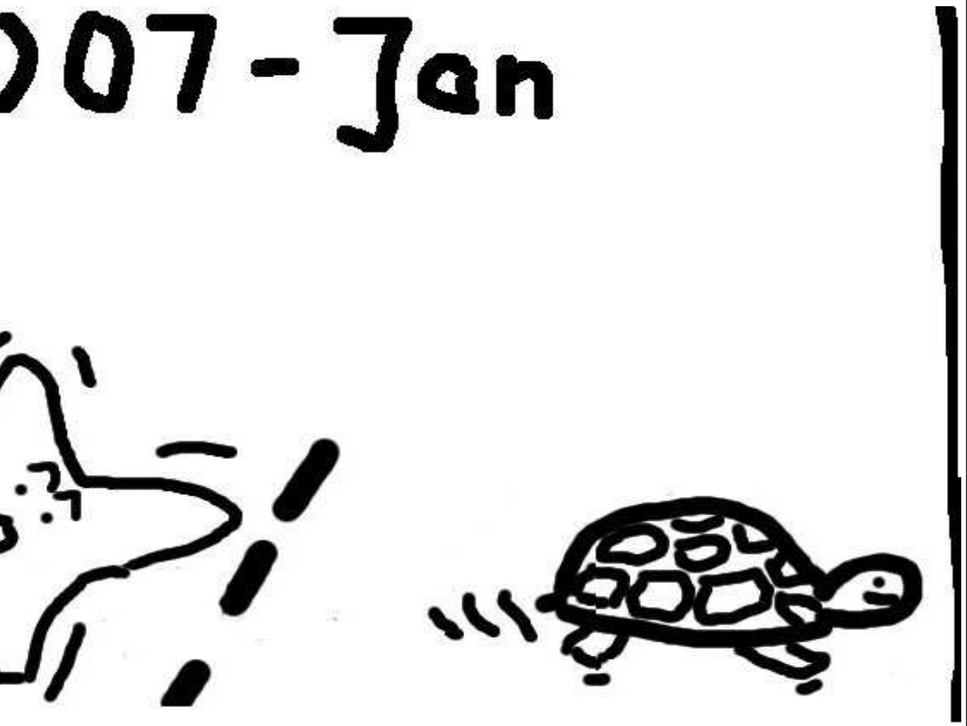
Exciting progress: Edwards about to overtake!!

Weierstrass has made some progress - finally Edwards wakes up.



Exciting progress: Edwards about to overtake!!



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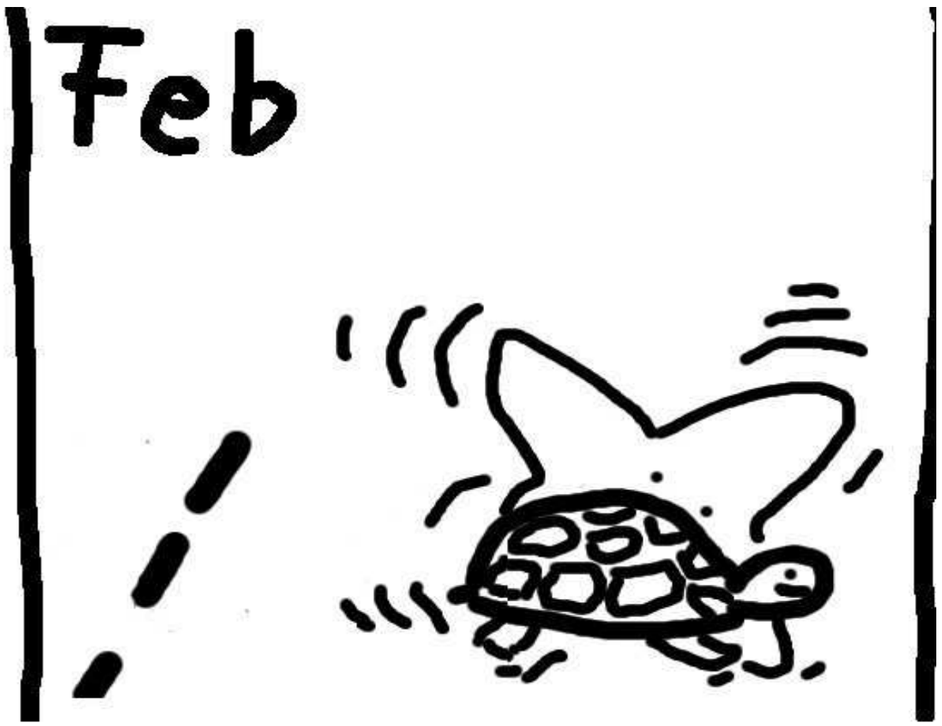

Exciting progress: Edwards
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Exciting progress: Edwards about to overtake!!

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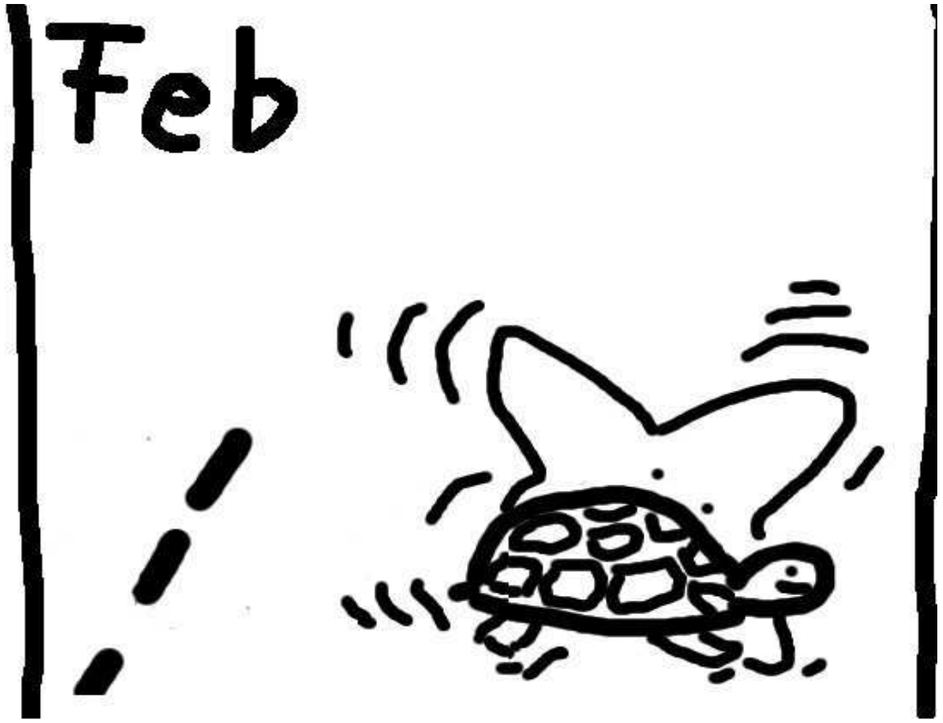
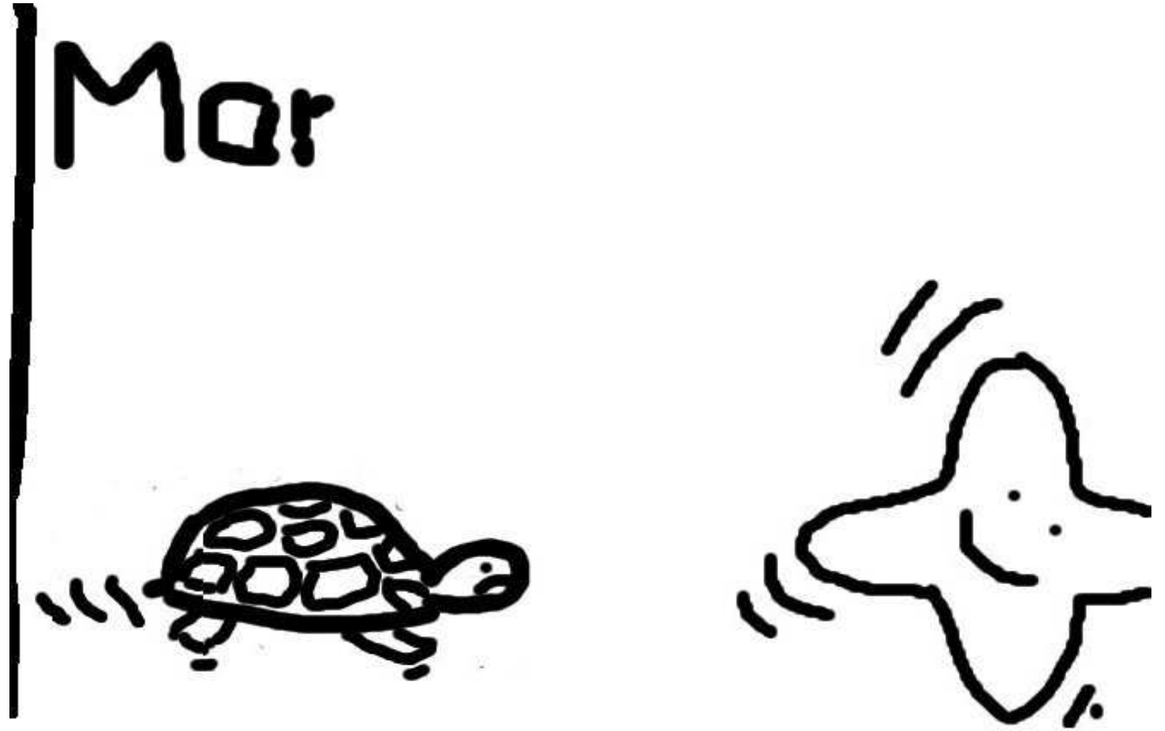


Exciting progress: Edwards about to overtake!!



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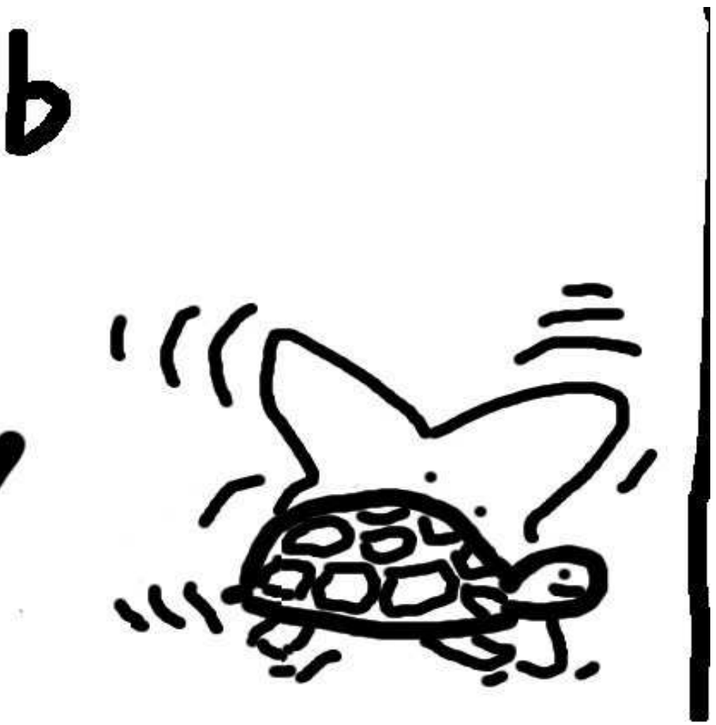

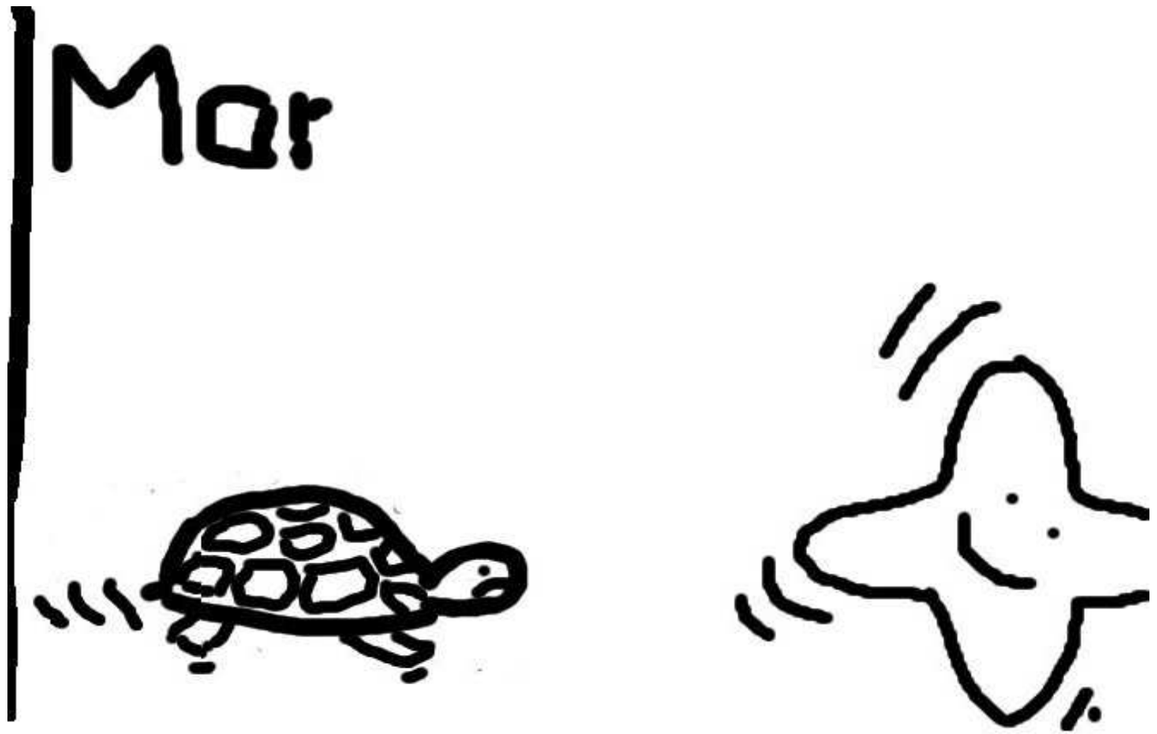

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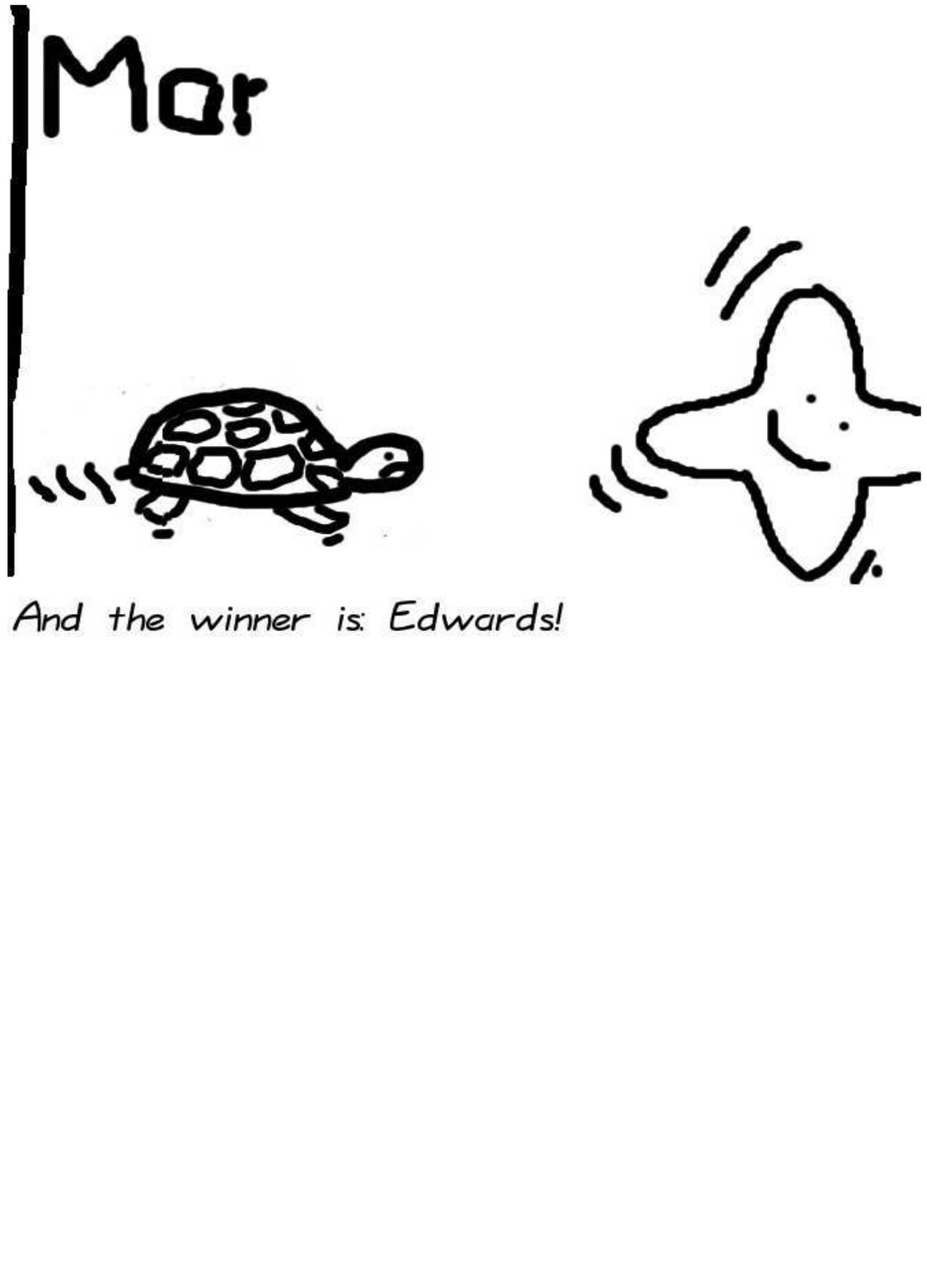

And the winner is: Edwards!

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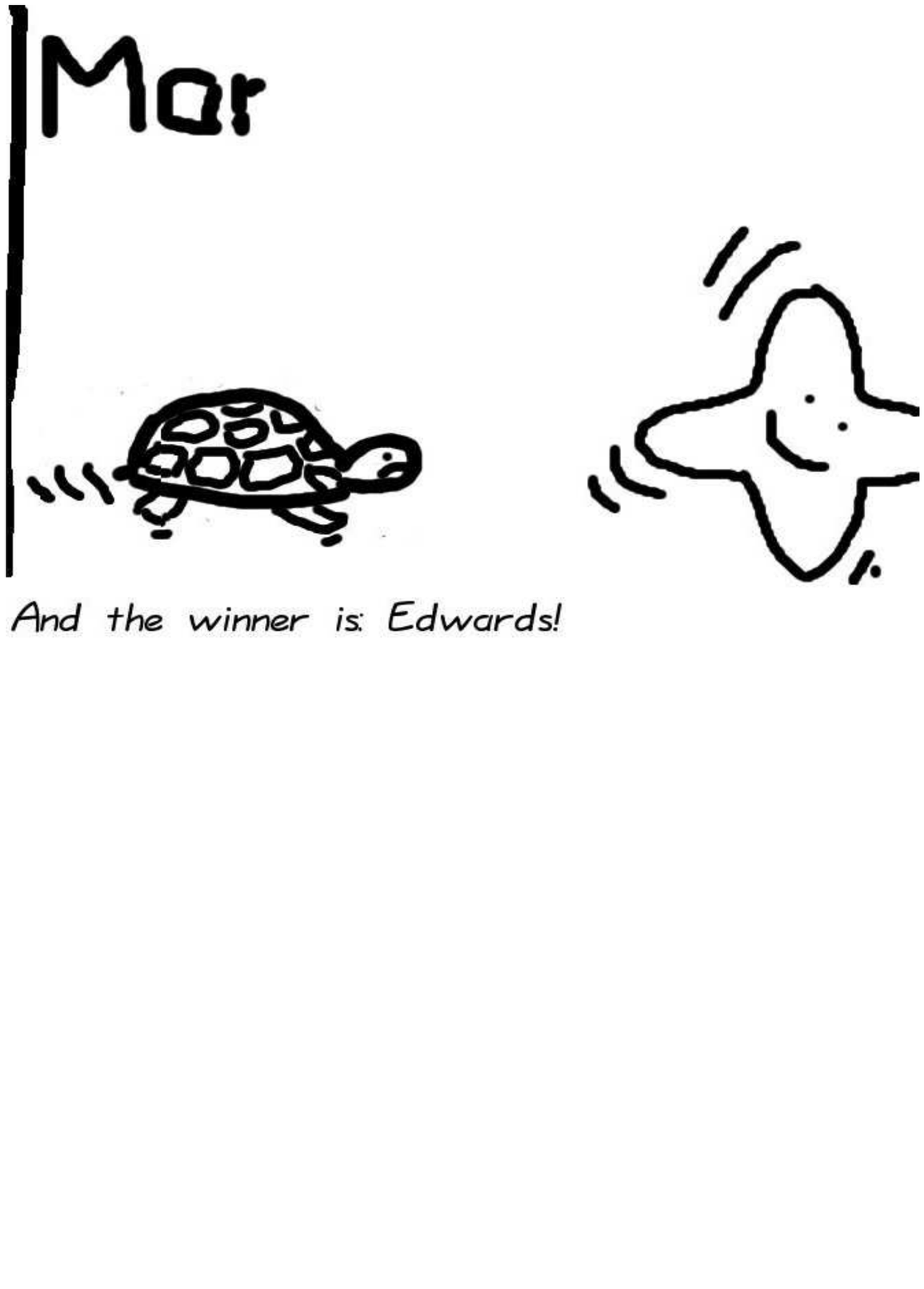

Speed-oriented Jacobian st

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in Jacobian coordinates
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Also specifies a method of
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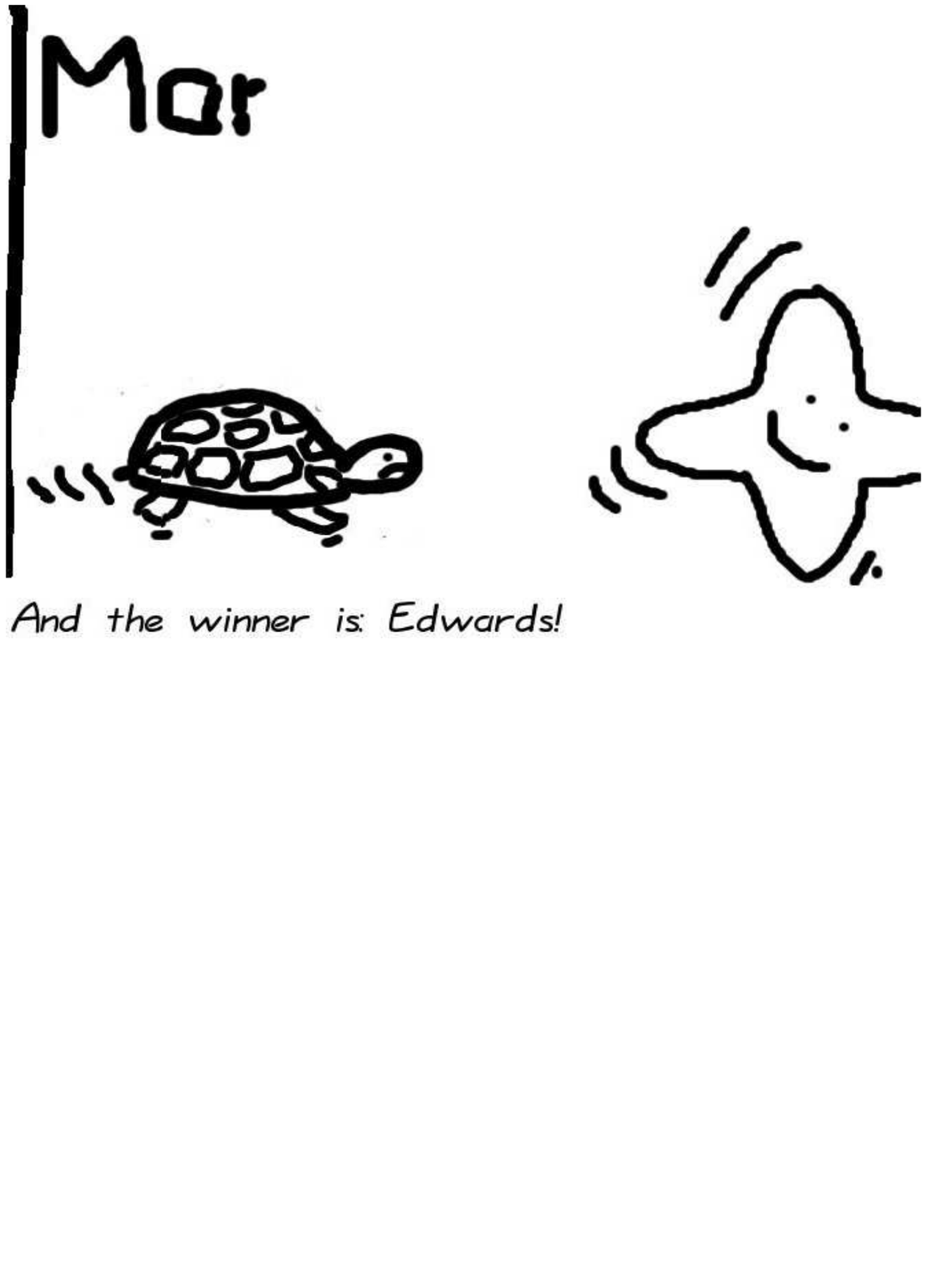

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1986 Chudnovsky
Speed up ADD b
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Speed up ADD by switchin
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$7\mathbf{M} + 3\mathbf{S}$ for DBL if $a = -$
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$2(x_2, y_2) = (x_4,$

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$(x_3, y_3) - (x_2, y_2$
$(x_3, y_3) + (x_2, y_2$

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$, Y/Z^3)$ to $(X/Z, Y/Z)$.
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$F = (X$
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This re
does n
DADD
Q, R, Q

e.g. $2R$
e.g. $3R$
e.g. $6R$

$2\mathbf{M} + 2$
$4\mathbf{M} + 2$
Save 1

Easily
$\approx \lg n$
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$ax^2 + x.$

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DADD, "differen
$Q, R, Q - R \mapsto Q$

e.g. $2P, P, P \mapsto$
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$2\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ f
$4\mathbf{M} + 2\mathbf{S}$ for DA
Save $1\mathbf{M}$ if $Z_1 =$

Easily compute n
$\approx \lg n$ DBL, $\approx$ lg
Almost as fast as
Relatively slow fo

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e.g. $6P, 5P, P \mapsto 11P$.

$2\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for DBL.
$4\mathbf{M} + 2\mathbf{S}$ for DADD.
Save $1\mathbf{M}$ if $Z_1 = 1$.

Easily compute $n(X_1 : Z_1)$
$\approx \lg n$ DBL, $\approx \lg n$ DADD
Almost as fast as Edwards
Relatively slow for $mP + n$

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Relatively slow for $mP + nQ$ etc.

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Doubli

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Use y^2

Choose

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$Z_2) = (X_1{:}Z_1)$,
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Doubling-oriente

2006 Doche–Icar

Use $y^2 = x^3 + a$
Choose small a.

Use $(X : Y : Z :$
to represent $(X/$

$3\mathbf{M} + 4\mathbf{S} + 2\mathbf{D}$ f
How? Factor DB
where φ is a 2-is

2007 Bernstein–L
$2\mathbf{M} + 5\mathbf{S} + 2\mathbf{D}$ f
on the same curv

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$) \Rightarrow$

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$Z_5)$.

This representation
does not allow ADD but it allows
DADD, "differential addition":
$Q, R, Q - R \mapsto Q + R$.

e.g. $2P, P, P \mapsto 3P$.
e.g. $3P, 2P, P \mapsto 5P$.
e.g. $6P, 5P, P \mapsto 11P$.

$2\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for DBL.
$4\mathbf{M} + 2\mathbf{S}$ for DADD.
Save $1\mathbf{M}$ if $Z_1 = 1$.

Easily compute $n(X_1 : Z_1)$ using
$\approx \lg n$ DBL, $\approx \lg n$ DADD.
Almost as fast as Edwards nP.
Relatively slow for $mP + nQ$ etc.

<u>Doubling-oriented curves</u>

2006 Doche–Icart–Kohel:

Use $y^2 = x^3 + ax^2 + 16ax$
Choose small a.

Use $(X : Y : Z : Z^2)$
to represent $(X/Z, Y/Z^2)$.

$3\mathbf{M} + 4\mathbf{S} + 2\mathbf{D}$ for DBL.
How? Factor DBL as $\hat{\varphi}(\varphi$
where φ is a 2-isogeny.

2007 Bernstein–Lange:
$2\mathbf{M} + 5\mathbf{S} + 2\mathbf{D}$ for DBL
on the same curves.

This representation
does not allow ADD but it allows
DADD, "differential addition":
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e.g. $3P, 2P, P \mapsto 5P$.
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fast DBL+DADD on Jacob
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$12\mathbf{M}$ for ADD:
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$+ y^3 + 1 = 3dxy$.

or ADD:

$Y_1X_2 \cdot Y_1Z_2 - Z_1Y_2 \cdot X_1Y_2$,

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3**S** for DBL.

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$3\mathbf{M} + 6\mathbf{S}$ for DBL.

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$= 3dxy$.

$Y_2 - Z_1Y_2 \cdot X_1Y_2$,

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$3\mathbf{M} + 6\mathbf{S}$ for DBL.

$x^3 - y^3 + 1 = 0$.

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$+ X_1Y_2,$

$\cdot Z_1X_2,$

$\cdot Y_1Z_2.$

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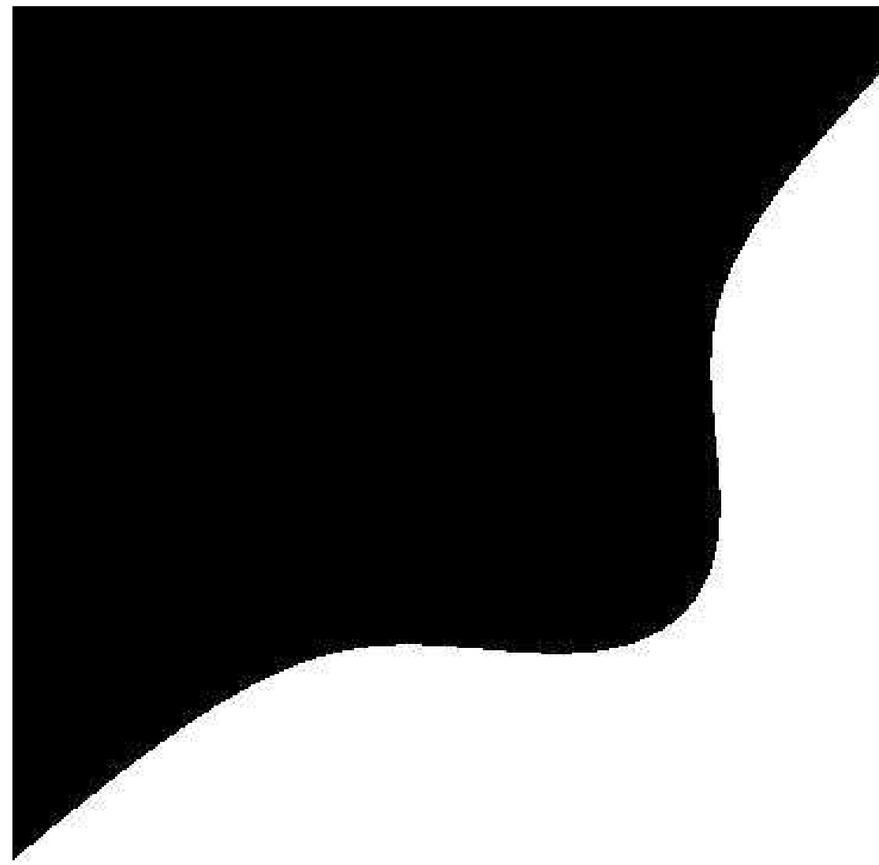

$x^3 - y^3 + 1 = 0.3xy$

2001 Joye–Quisquater:
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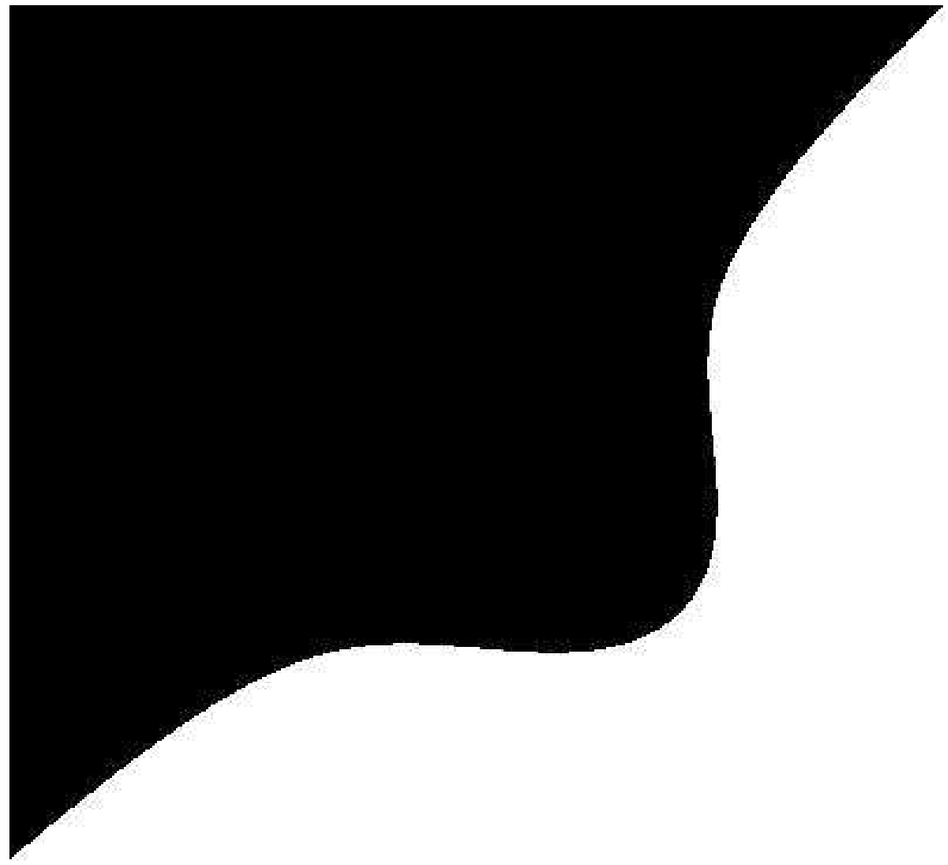

$x^3 - y^3 + 1 = 0.3xy$

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$Y_1 : Z_1) =$

$X_1 : Y_1) + (Y_1 : Z_1 : X_1)$

use ADD to double.

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2 Hisil–Wong–Carter–Dawson:

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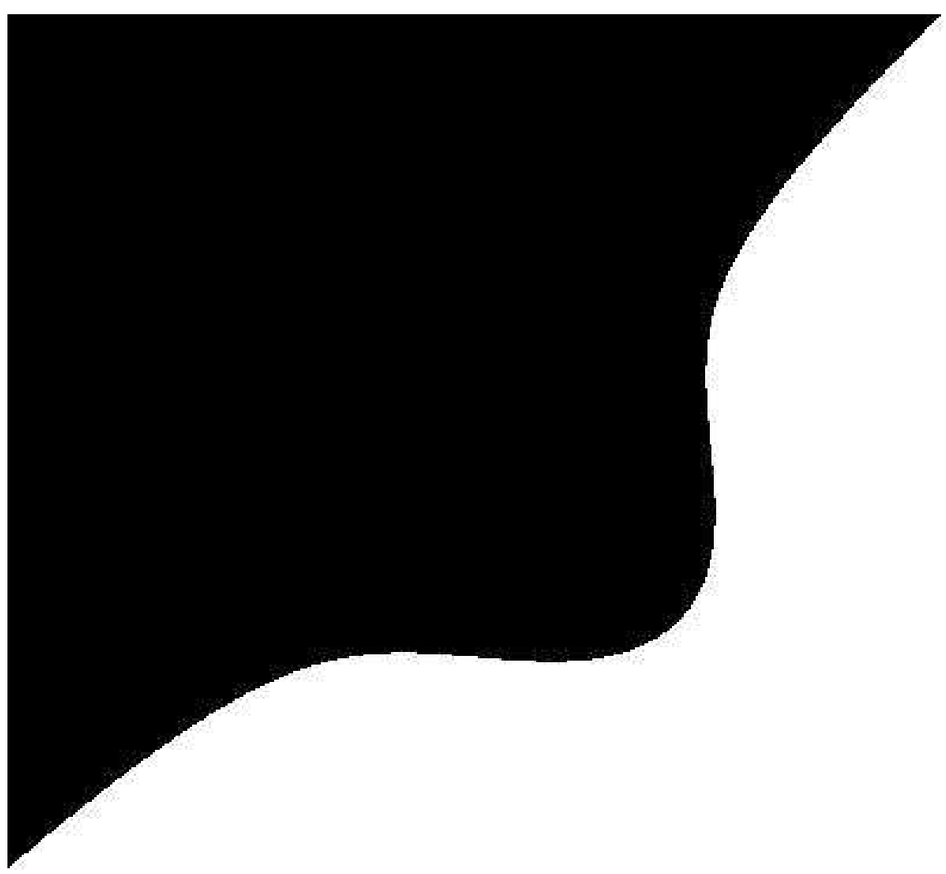

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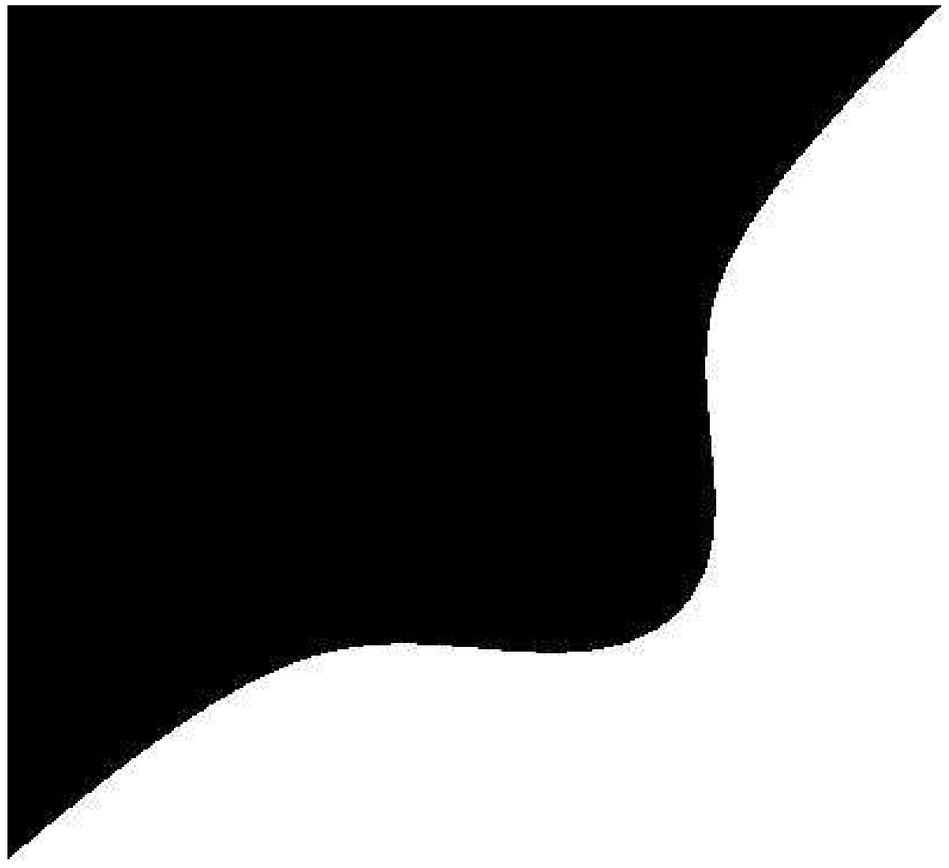

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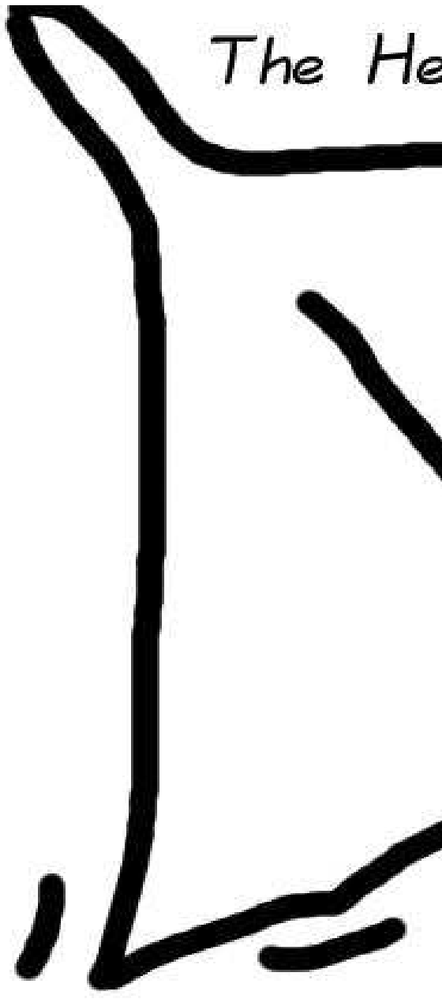

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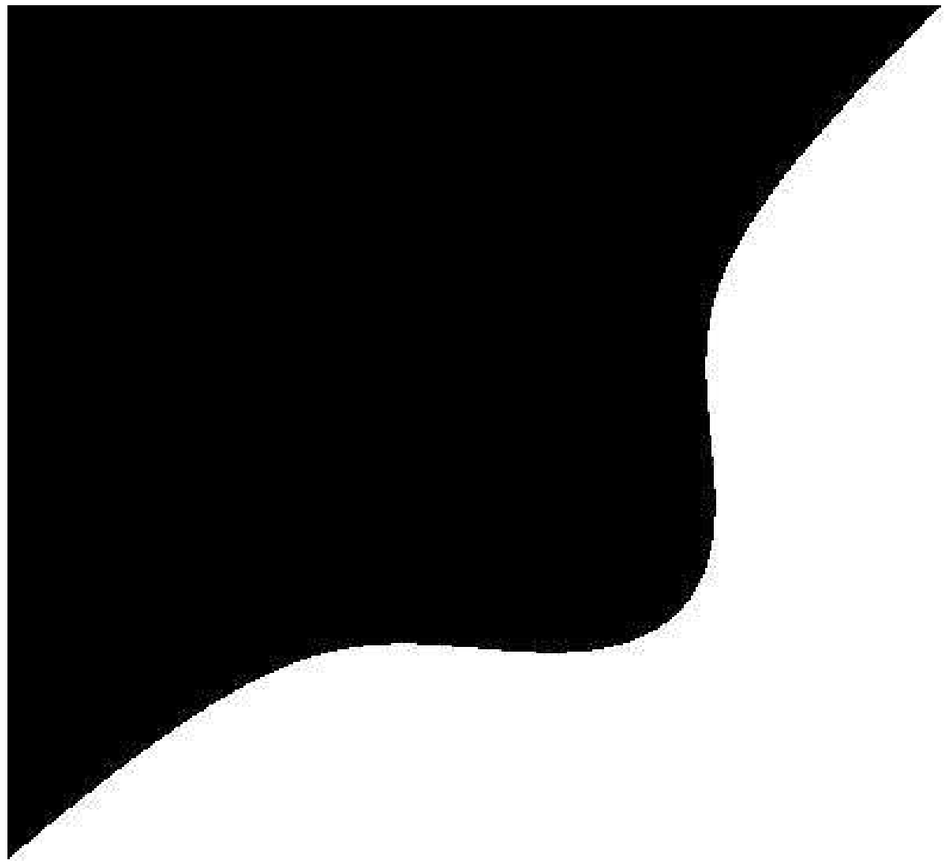

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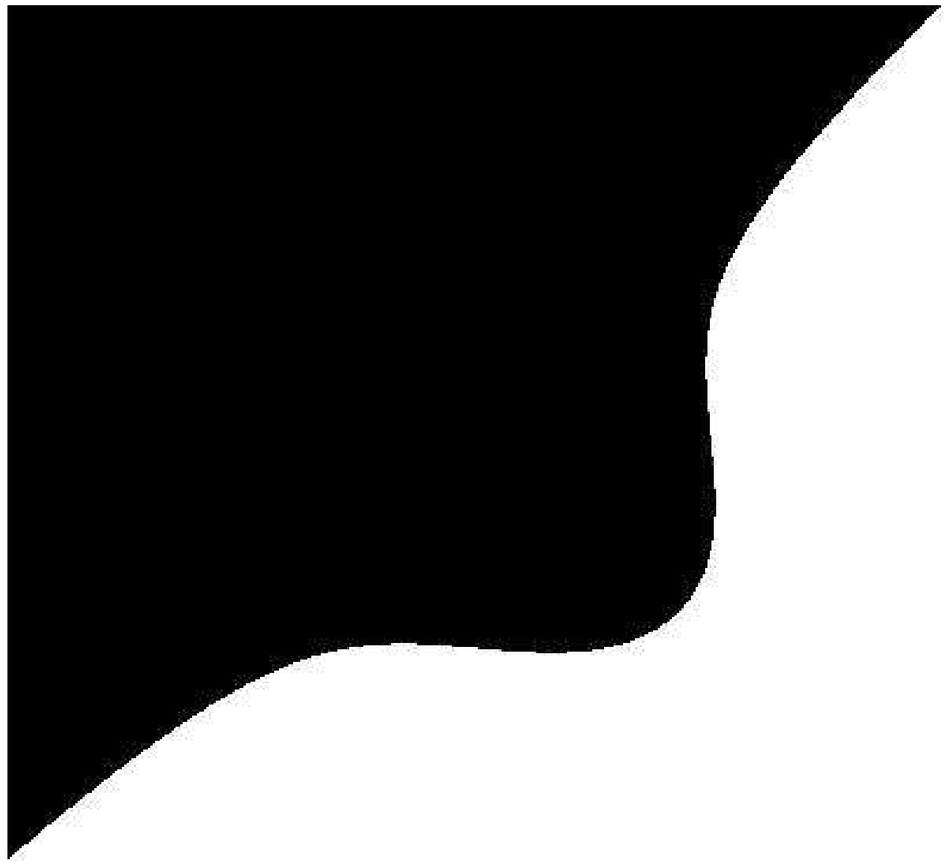
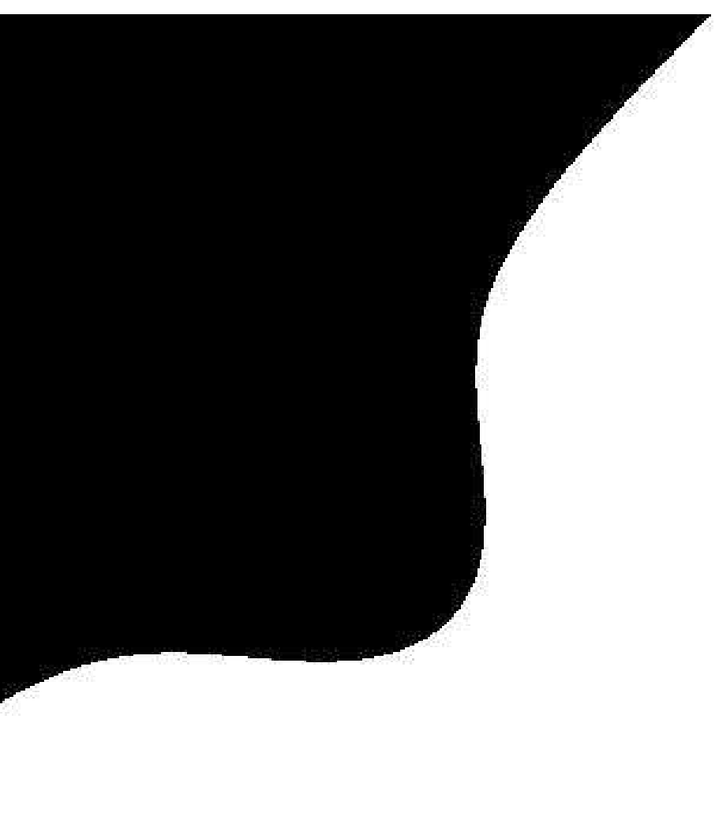


$$x^3 - y^3 + 1 = 0.3xy$$



$$s^3 + 1 = 0.3xy$$

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Jacobi intersections

1986 Chudnovsky–Chudnov

$(S : C : D : Z)$ represent
$(S/Z, C/Z, D/Z)$ on
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

$14\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
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$(S/Z, C/Z, D/Z)$ on
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

$14\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
"Tremendous advantage"
of being strongly unified.

$5\mathbf{M} + 3\mathbf{S}$ for DBL.
"Perhaps (?) ... the most
efficient duplication formulas
which do not depend on the
coefficients of an elliptic curve."

Jacobi intersections

1986 Chudnovsky–Chudnovsky:

$(S : C : D : Z)$ represent
$(S/Z, C/Z, D/Z)$ on
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

$14\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
"Tremendous advantage"
of being strongly unified.

$5\mathbf{M} + 3\mathbf{S}$ for DBL.
"Perhaps (?) . . . the most
efficient duplication formulas
which do not depend on the
coefficients of an elliptic curve."

2001 L
$13\mathbf{M} +$
$4\mathbf{M} + 3$

2007 B
$3\mathbf{M} + 4$

2008.0
$13\mathbf{M} +$
$2\mathbf{M} + 5$
Also (S
$11\mathbf{M} +$
$2\mathbf{M} + 5$

Jacobi intersections

1986 Chudnovsky–Chudnovsky:

$(S : C : D : Z)$ represent
$(S/Z, C/Z, D/Z)$ on
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

$14\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
"Tremendous advantage"
of being strongly unified.

$5\mathbf{M} + 3\mathbf{S}$ for DBL.
"Perhaps (?) ... the most
efficient duplication formulas
which do not depend on the
coefficients of an elliptic curve."

2001 Liardet–Sm
$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$
$4\mathbf{M} + 3\mathbf{S}$ for DB

2007 Bernstein–L
$3\mathbf{M} + 4\mathbf{S}$ for DB

2008.02 Hisil–Wo
$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ f
Also $(S : C : D :$
$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ f

Jacobi intersections

1986 Chudnovsky–Chudnovsky:

$(S : C : D : Z)$ represent
$(S/Z, C/Z, D/Z)$ on
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

$14\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
"Tremendous advantage"
of being strongly unified.

$5\mathbf{M} + 3\mathbf{S}$ for DBL.
"Perhaps (?) . . . the most
efficient duplication formulas
which do not depend on the
coefficients of an elliptic curve."

2001 Liardet–Smart:
$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
$4\mathbf{M} + 3\mathbf{S}$ for DBL.

2007 Bernstein–Lange:
$3\mathbf{M} + 4\mathbf{S}$ for DBL.

2008.02 Hisil–Wong–Carter
$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.
Also $(S : C : D : Z : SC :$
$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

Jacobi intersections

1986 Chudnovsky–Chudnovsky:

$(S : C : D : Z)$ represent
$(S/Z, C/Z, D/Z)$ on
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

$14\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
"Tremendous advantage"
of being strongly unified.

$5\mathbf{M} + 3\mathbf{S}$ for DBL.
"Perhaps (?) ... the most
efficient duplication formulas
which do not depend on the
coefficients of an elliptic curve."

2001 Liardet–Smart:
$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
$4\mathbf{M} + 3\mathbf{S}$ for DBL.

2007 Bernstein–Lange:
$3\mathbf{M} + 4\mathbf{S}$ for DBL.

2008.02 Hisil–Wong–Carter–Dawson:
$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.
Also $(S : C : D : Z : SC : DZ)$:
$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

intersections

Chudnovsky–Chudnovsky:

$: D : Z)$ represent
$C/Z, D/Z)$ on
$^2 = 1$, $as^2 + d^2 = 1$.

$+ 2\mathbf{S} + 1\mathbf{D}$ for ADD.
endous advantage"
g strongly unified.

$3\mathbf{S}$ for DBL.
ps (?) . . . the most
t duplication formulas
do not depend on the
ents of an elliptic curve."

2001 Liardet–Smart:
$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
$4\mathbf{M} + 3\mathbf{S}$ for DBL.

2007 Bernstein–Lange:
$3\mathbf{M} + 4\mathbf{S}$ for DBL.

2008.02 Hisil–Wong–Carter–Dawson:
$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.
Also $(S : C : D : Z : SC : DZ)$:
$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

Jacobi

$(X:Y:Z$
on $y^2 =$

1986 C
$3\mathbf{M} + 6$
Slow A

2002 B
New ch
$10\mathbf{M} +$
strongl

2007 B
$1\mathbf{M} + 9$

ons

y–Chudnovsky:

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) on
$+ d^2 = 1$.

for ADD.
vantage"
unified.

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the most
ion formulas
pend on the
elliptic curve."

2001 Liardet–Smart:
$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
$4\mathbf{M} + 3\mathbf{S}$ for DBL.

2007 Bernstein–Lange:
$3\mathbf{M} + 4\mathbf{S}$ for DBL.

2008.02 Hisil–Wong–Carter–Dawson:
$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.
Also $(S : C : D : Z : SC : DZ)$:
$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

Jacobi quartics

$(X:Y:Z)$ represen
on $y^2 = x^4 + 2a$

1986 Chudnovsky
$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ f
Slow ADD.

2002 Billet–Joye:
New choice of ne
$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$
strongly unified.

2007 Bernstein–L
$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ f

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ne
urve."

2001 Liardet–Smart:
$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
$4\mathbf{M} + 3\mathbf{S}$ for DBL.

2007 Bernstein–Lange:
$3\mathbf{M} + 4\mathbf{S}$ for DBL.

2008.02 Hisil–Wong–Carter–Dawson:
$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.
Also $(S : C : D : Z : SC : DZ)$:
$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

<u>Jacobi quartics</u>

$(X{:}Y{:}Z)$ represent $(X/Z, Y$
on $y^2 = x^4 + 2ax^2 + 1$.

1986 Chudnovsky–Chudnov
$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.
Slow ADD.

2002 Billet–Joye:
New choice of neutral elem
$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$ for ADD,
strongly unified.

2007 Bernstein–Lange:
$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ for DBL.

2001 Liardet–Smart:
$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
$4\mathbf{M} + 3\mathbf{S}$ for DBL.

2007 Bernstein–Lange:
$3\mathbf{M} + 4\mathbf{S}$ for DBL.

2008.02 Hisil–Wong–Carter–Dawson:
$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.
Also $(S : C : D : Z : SC : DZ)$:
$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

Jacobi quartics

$(X:Y:Z)$ represent $(X/Z, Y/Z^2)$
on $y^2 = x^4 + 2ax^2 + 1$.

1986 Chudnovsky–Chudnovsky:
$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.
Slow ADD.

2002 Billet–Joye:
New choice of neutral element.
$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$ for ADD,
strongly unified.

2007 Bernstein–Lange:
$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ for DBL.

iardet–Smart:
$+ 2\mathbf{S} + 1\mathbf{D}$ for ADD.
$3\mathbf{S}$ for DBL.

Bernstein–Lange:
$4\mathbf{S}$ for DBL.

2 Hisil–Wong–Carter–Dawson:
$+ 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$5\mathbf{S} + 1\mathbf{D}$ for DBL.
$S : C : D : Z : SC : DZ$):
$+ 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$5\mathbf{S} + 1\mathbf{D}$ for DBL.

Jacobi quartics

$(X{:}Y{:}Z)$ represent $(X/Z, Y/Z^2)$
on $y^2 = x^4 + 2ax^2 + 1$.

1986 Chudnovsky–Chudnovsky:
$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.
Slow ADD.

2002 Billet–Joye:
New choice of neutral element.
$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$ for ADD,
strongly unified.

2007 Bernstein–Lange:
$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ for DBL.

2007 H
$2\mathbf{M} + 6$

2007 F
$2\mathbf{M} + 6$
$1\mathbf{M} +$
on curv

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2008.0
use $(X$
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for ADD.
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Lange:
L.

ong–Carter–Dawson:
for ADD.
or DBL.
$Z : SC : DZ$):
for ADD.
or DBL.

Jacobi quartics

$(X{:}Y{:}Z)$ represent $(X/Z, Y/Z^2)$
on $y^2 = x^4 + 2ax^2 + 1$.

1986 Chudnovsky–Chudnovsky:
$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.
Slow ADD.

2002 Billet–Joye:
New choice of neutral element.
$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$ for ADD,
strongly unified.

2007 Bernstein–Lange:
$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ for DBL.

2007 Hisil–Carter
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ f

2007 Feng–Wu:
$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ f
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ f
on curves chosen

More speedups:
2007 Hisil–Carter
2008.02 Hisil–Wo
use $(X : Y : Z : $
or $(X : Y : Z : X$
Can combine wit
Competitive with

r–Dawson:

DZ):

Jacobi quartics

$(X{:}Y{:}Z)$ represent $(X/Z, Y/Z^2)$
on $y^2 = x^4 + 2ax^2 + 1$.

1986 Chudnovsky–Chudnovsky:
$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.
Slow ADD.

2002 Billet–Joye:
New choice of neutral element.
$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$ for ADD,
strongly unified.

2007 Bernstein–Lange:
$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ for DBL.

2007 Hisil–Carter–Dawson:
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

2007 Feng–Wu:
$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL
on curves chosen with $a^2 +$

More speedups: 2007 Duq
2007 Hisil–Carter–Dawson,
2008.02 Hisil–Wong–Carter
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2$
Can combine with Feng–W
Competitive with Edwards!

Jacobi quartics

$(X{:}Y{:}Z)$ represent $(X/Z, Y/Z^2)$
on $y^2 = x^4 + 2ax^2 + 1$.

1986 Chudnovsky–Chudnovsky:
$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.
Slow ADD.

2002 Billet–Joye:
New choice of neutral element.
$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$ for ADD,
strongly unified.

2007 Bernstein–Lange:
$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ for DBL.

2007 Hisil–Carter–Dawson:
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

2007 Feng–Wu:
$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL
on curves chosen with $a^2 + c^2 = 1$.

More speedups: 2007 Duquesne,
2007 Hisil–Carter–Dawson,
2008.02 Hisil–Wong–Carter–Dawson:
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2XZ)$.
Can combine with Feng–Wu.
Competitive with Edwards!

quartics

Z) represent $(X/Z, Y/Z^2)$
$= x^4 + 2ax^2 + 1$.

Chudnovsky–Chudnovsky:
$6\mathbf{S} + 2\mathbf{D}$ for DBL.
ADD.

Billet–Joye:
hoice of neutral element.
$+ 3\mathbf{S} + 1\mathbf{D}$ for ADD,
y unified.

Bernstein–Lange:
$9\mathbf{S} + 1\mathbf{D}$ for DBL.

2007 Hisil–Carter–Dawson:
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

2007 Feng–Wu:
$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL
on curves chosen with $a^2 + c^2 = 1$.

More speedups: 2007 Duquesne,
2007 Hisil–Carter–Dawson,
2008.02 Hisil–Wong–Carter–Dawson:
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2XZ)$.
Can combine with Feng–Wu.
Competitive with Edwards!

$x^2 = y$

nt $(X/Z, Y/Z^2)$
$x^2 + 1$.

y–Chudnovsky:
or DBL.

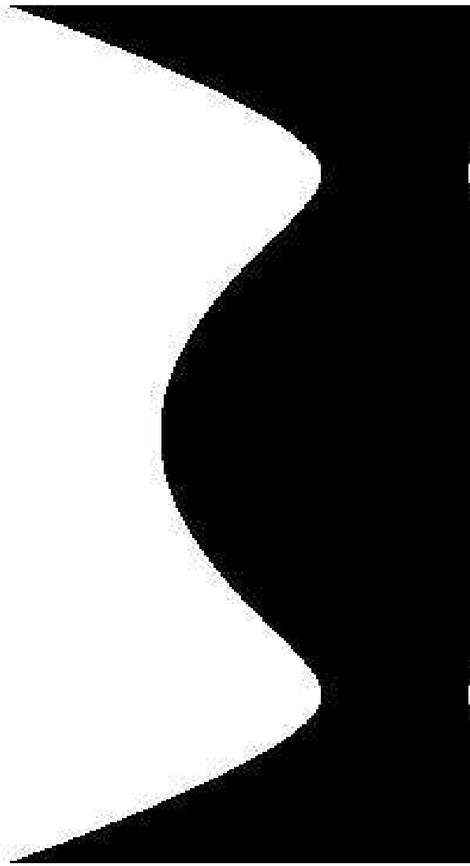
eutral element.
for ADD,

Lange:
or DBL.

2007 Hisil–Carter–Dawson:
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

2007 Feng–Wu:
$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL
on curves chosen with $a^2 + c^2 = 1$.

More speedups: 2007 Duquesne,
2007 Hisil–Carter–Dawson,
2008.02 Hisil–Wong–Carter–Dawson:
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2XZ)$.
Can combine with Feng–Wu.
Competitive with Edwards!



$x^2 = y^4 - 1.9y^2$

Y/Z^2)

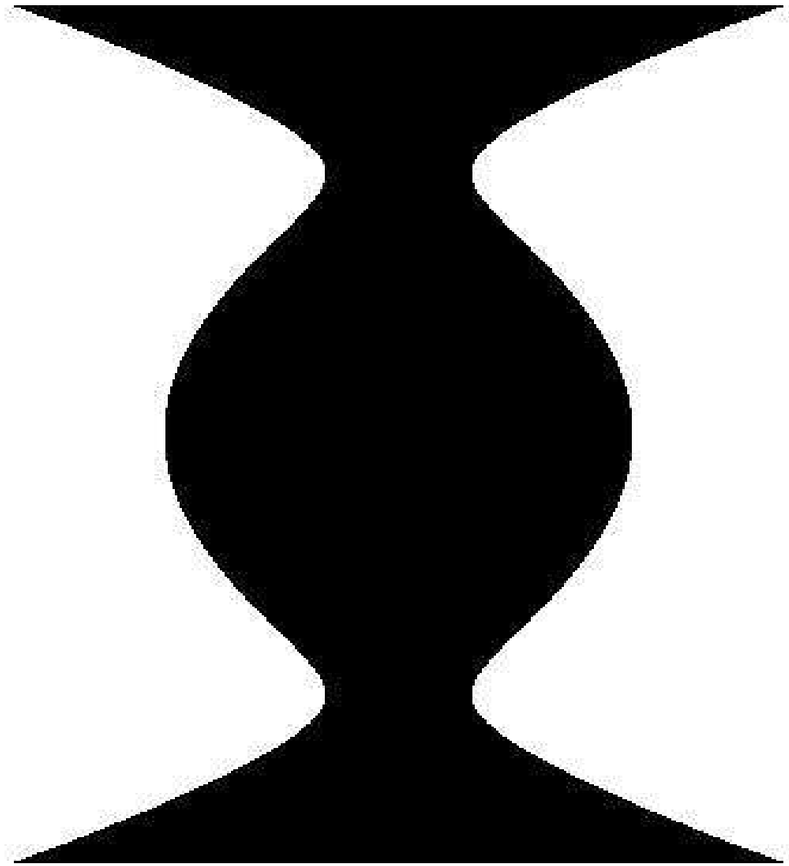
vsky:

nent.

2007 Hisil–Carter–Dawson:
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

2007 Feng–Wu:
$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL
on curves chosen with $a^2 + c^2 = 1$.

More speedups: 2007 Duquesne,
2007 Hisil–Carter–Dawson,
2008.02 Hisil–Wong–Carter–Dawson:
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2XZ)$.
Can combine with Feng–Wu.
Competitive with Edwards!

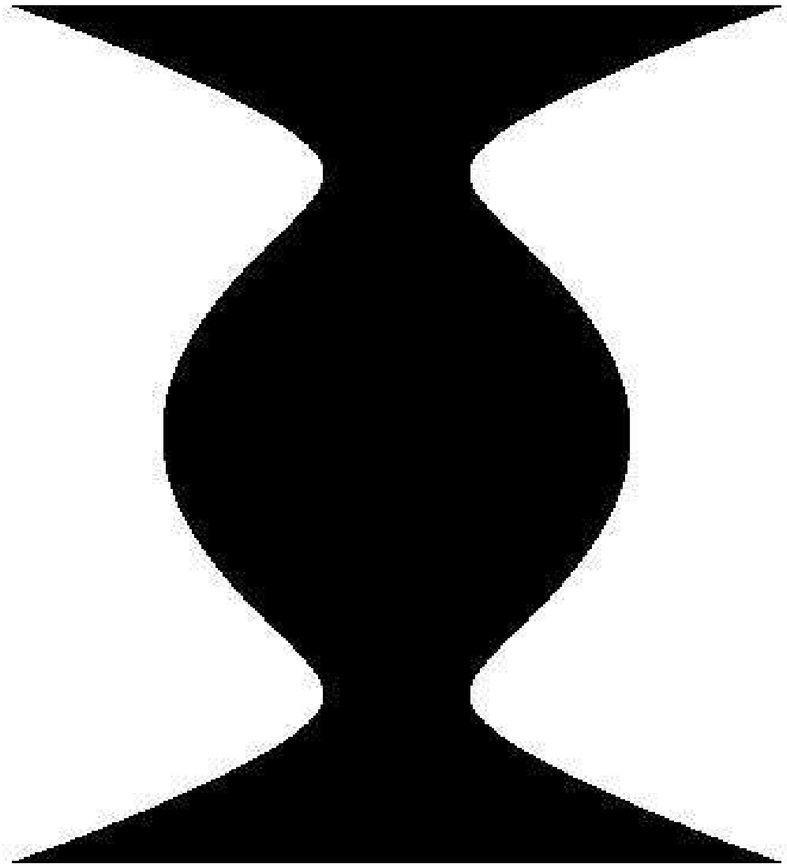


$x^2 = y^4 - 1.9y^2 + 1$

2007 Hisil–Carter–Dawson:
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

2007 Feng–Wu:
$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL
on curves chosen with $a^2 + c^2 = 1$.

More speedups: 2007 Duquesne,
2007 Hisil–Carter–Dawson,
2008.02 Hisil–Wong–Carter–Dawson:
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2XZ)$.
Can combine with Feng–Wu.
Competitive with Edwards!

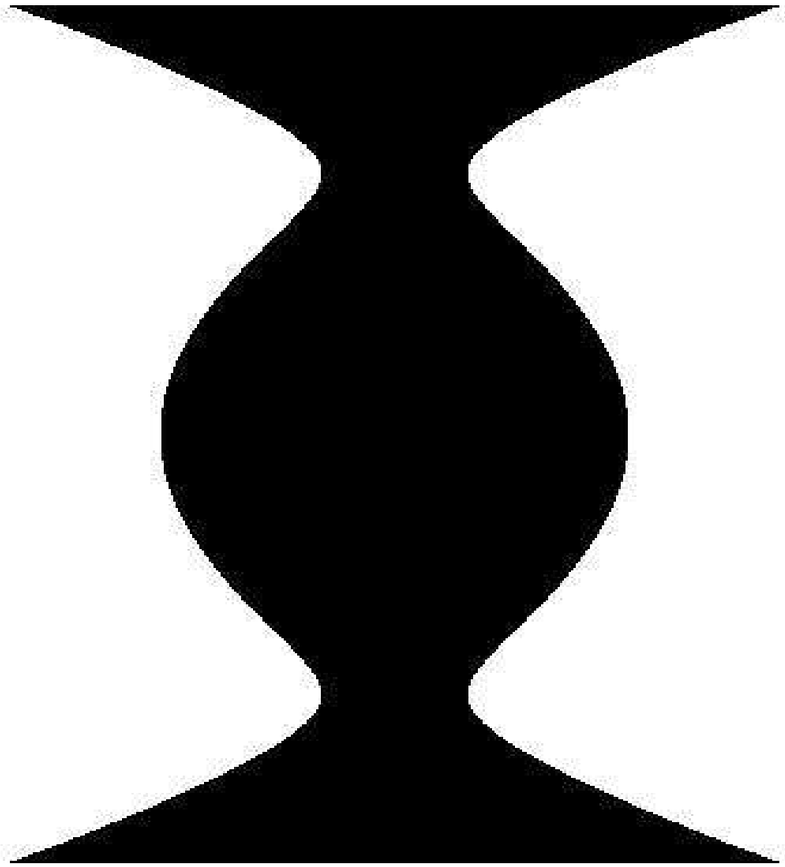


$x^2 = y^4 - 1.9y^2 + 1$

lisil–Carter–Dawson:
$6\mathbf{S} + 2\mathbf{D}$ for DBL.

eng–Wu:
$6\mathbf{S} + 1\mathbf{D}$ for DBL.
$7\mathbf{S} + 3\mathbf{D}$ for DBL
ves chosen with $a^2 + c^2 = 1$.

peedups: 2007 Duquesne,
lisil–Carter–Dawson,
2 Hisil–Wong–Carter–Dawson:
$: Y : Z : X^2 : Z^2)$
$Y : Z : X^2 : Z^2 : 2XZ)$.
mbine with Feng–Wu.
titive with Edwards!

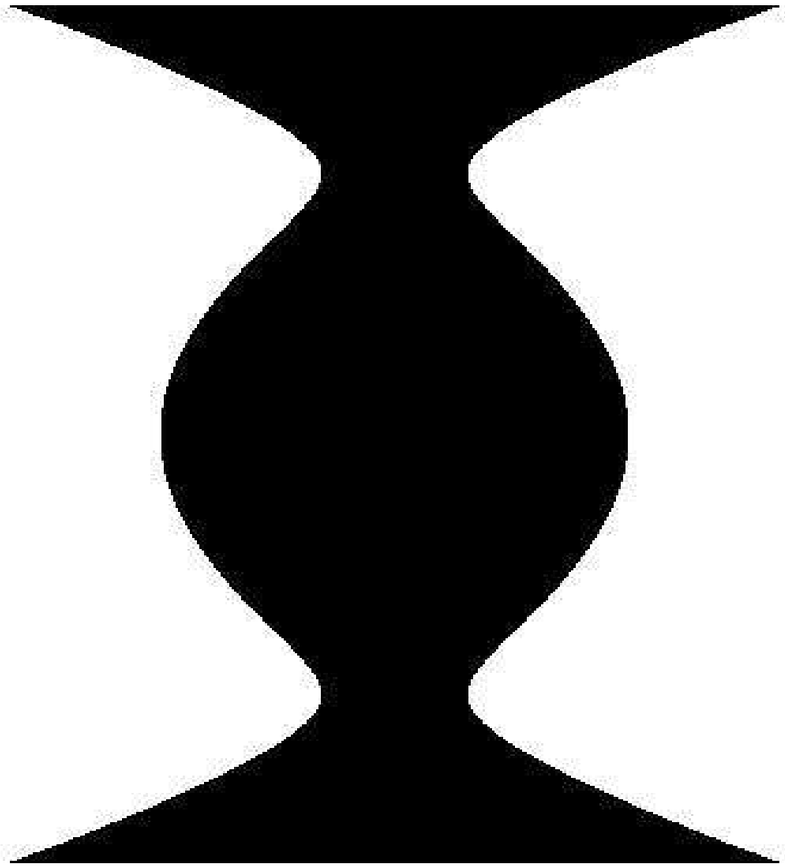


$x^2 = y^4 - 1.9y^2 + 1$

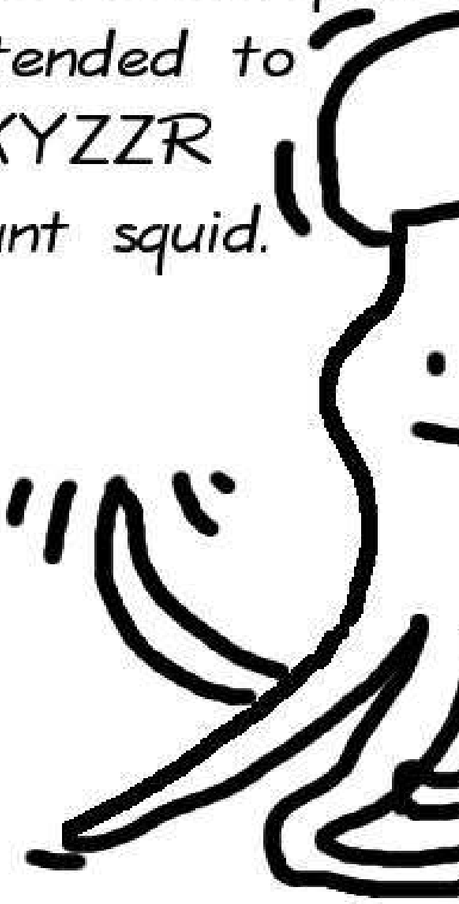
r–Dawson:
or DBL.

or DBL.
or DBL
with $a^2+c^2=1$.

2007 Duquesne,
r–Dawson,
ong–Carter–Dawson:
$X^2 : Z^2)$
$r^2 : Z^2 : 2XZ)$.
h Feng–Wu.
n Edwards!



$x^2 = y^4 - 1.9y^2 + 1$



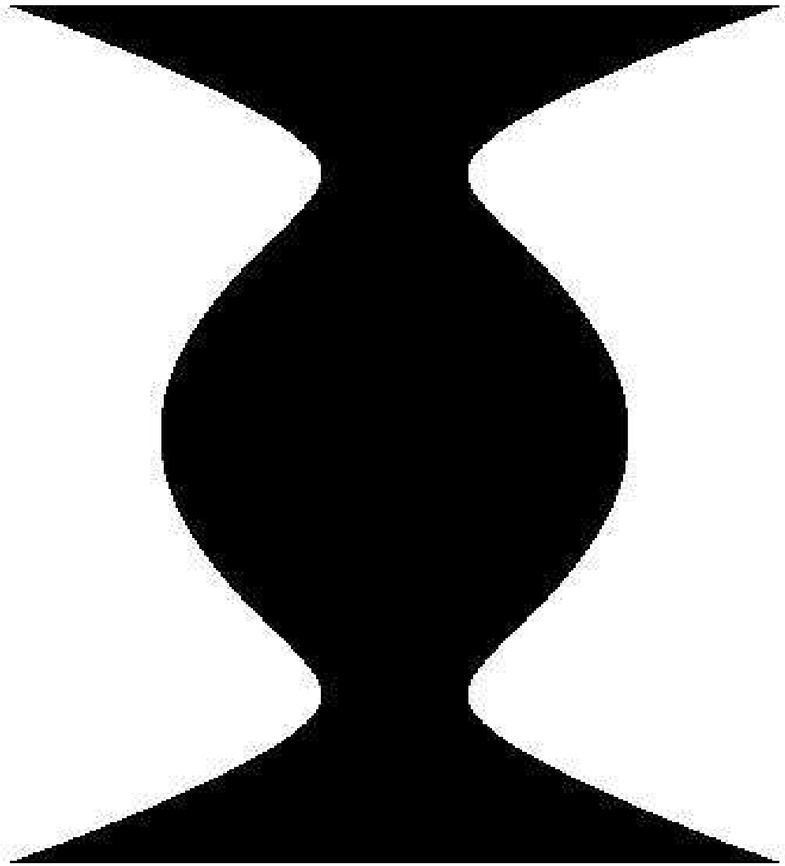
$-c^2 = 1.$

uesne,

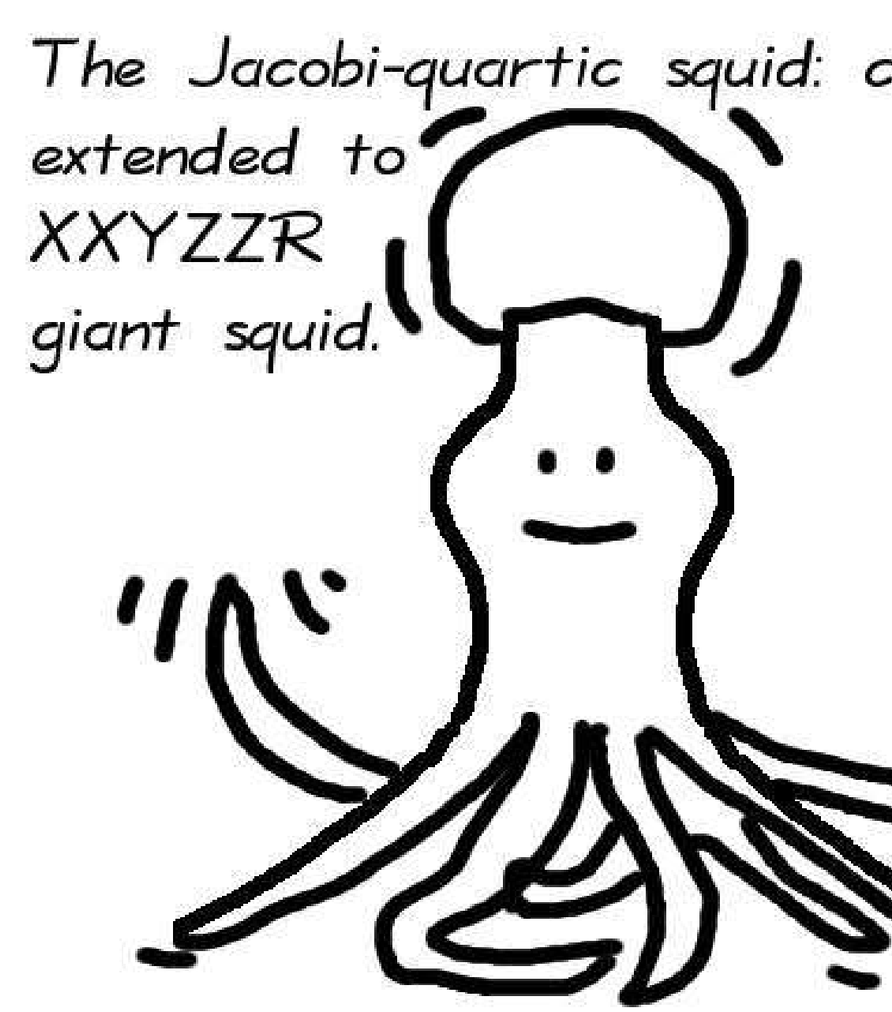
r–Dawson:

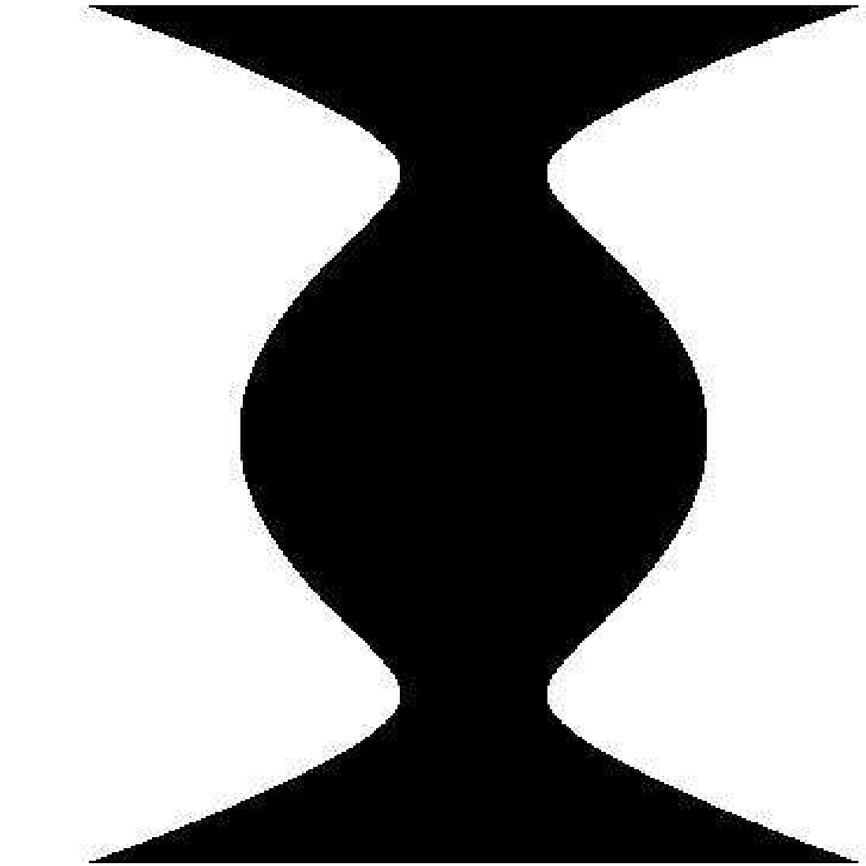
XZ).

/u.



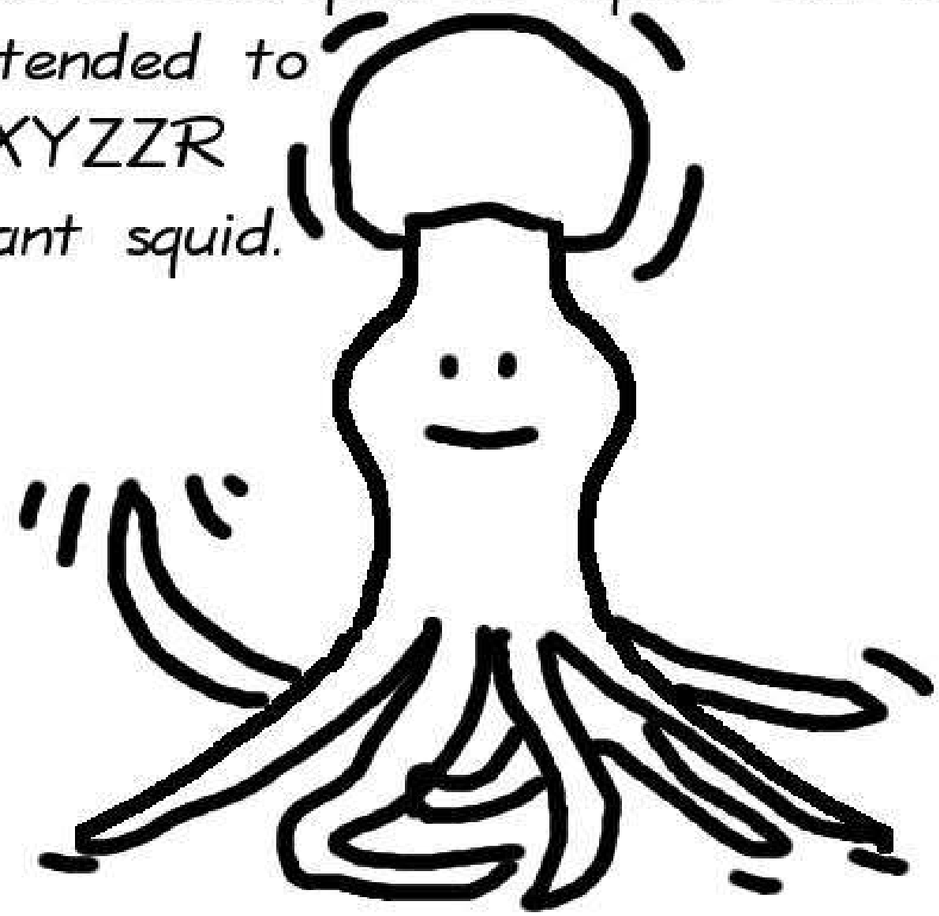
$x^2 = y^4 - 1.9y^2 + 1$

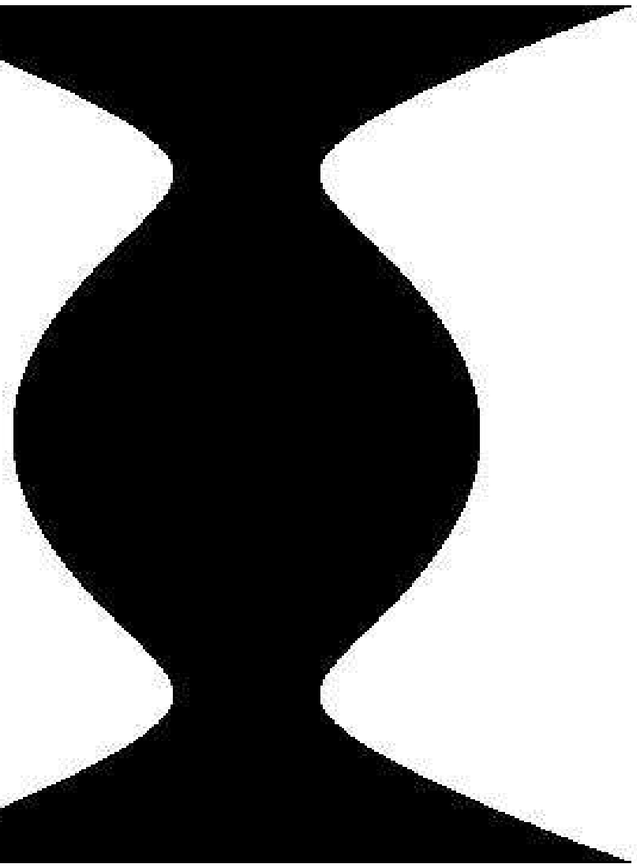




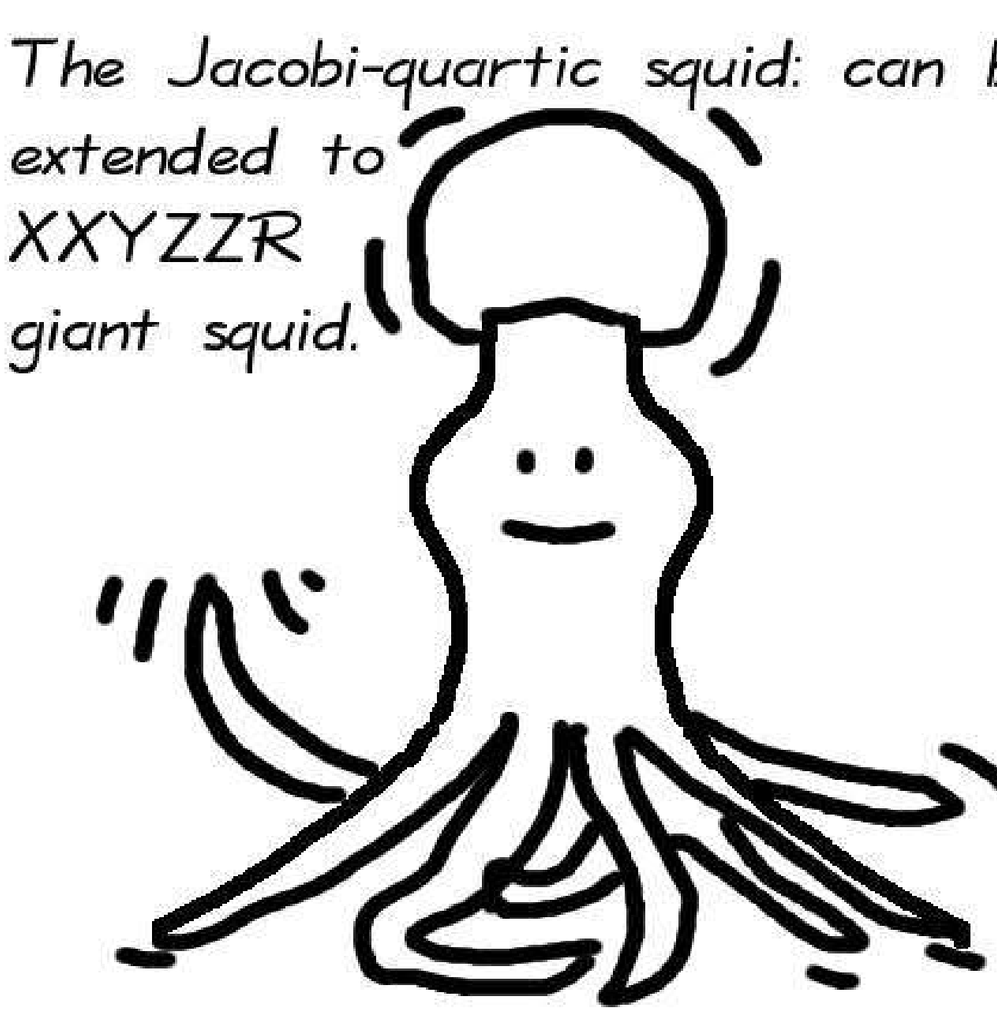
$$x^2 = y^4 - 1.9y^2 + 1$$

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.





$^4 - 1.9y^2 + 1$


The Jacobi-quartic squid: can be extended to XXYZZR giant squid.

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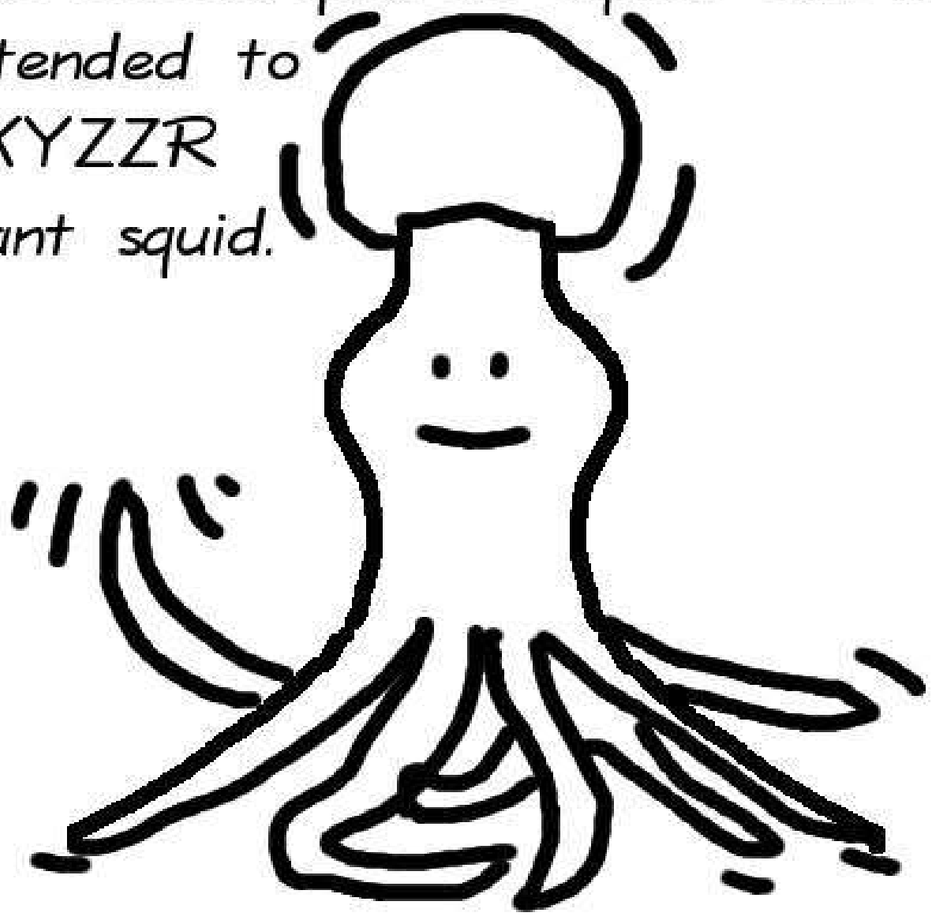
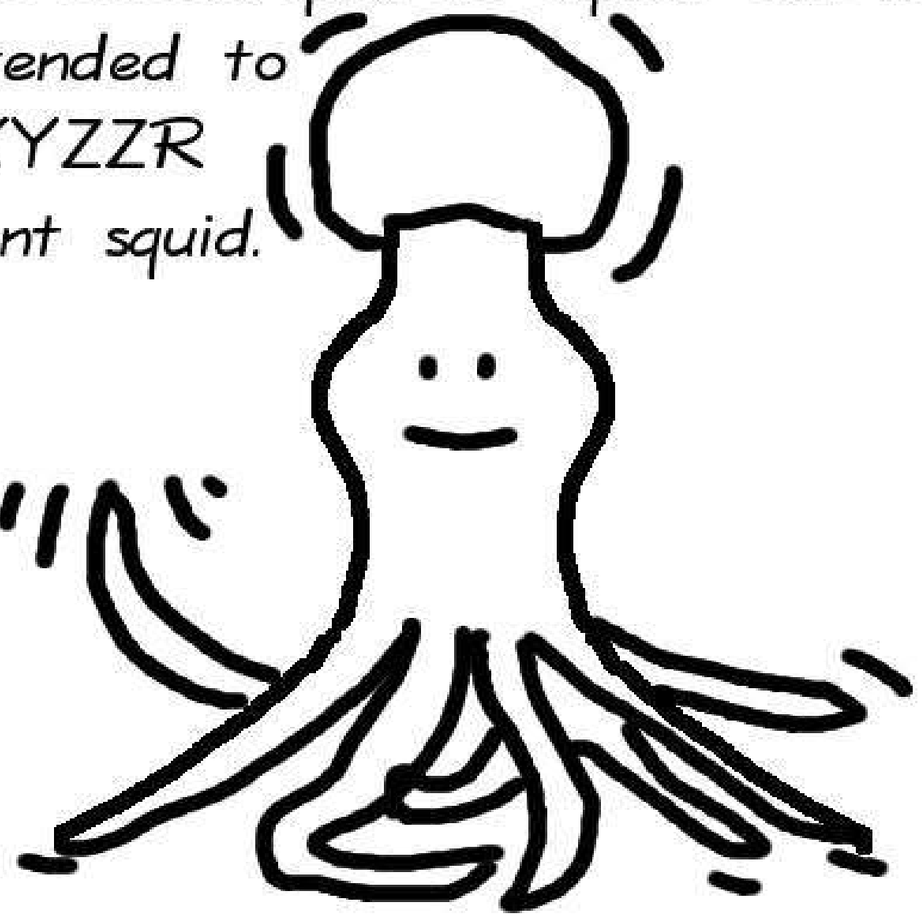
The Jacobi-quartic squid: can be extended to XXYZZR giant squid.



The Jacobi-quartic squid: can be extended to XXYZZR giant squid.



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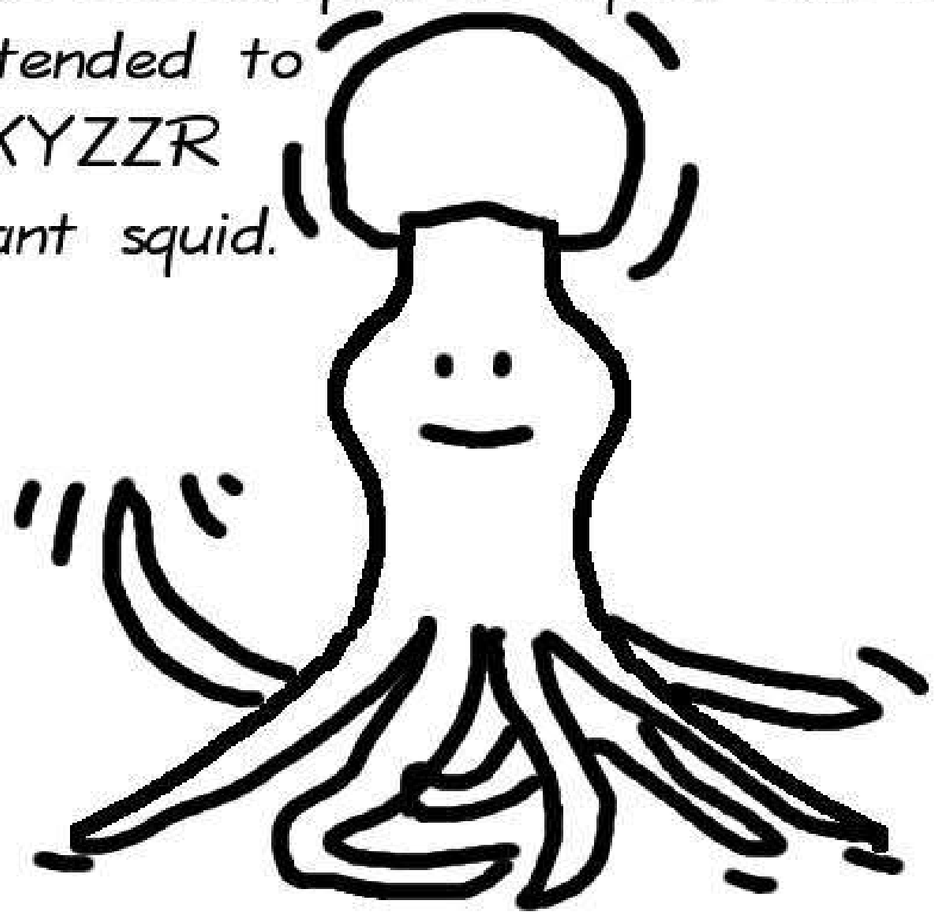

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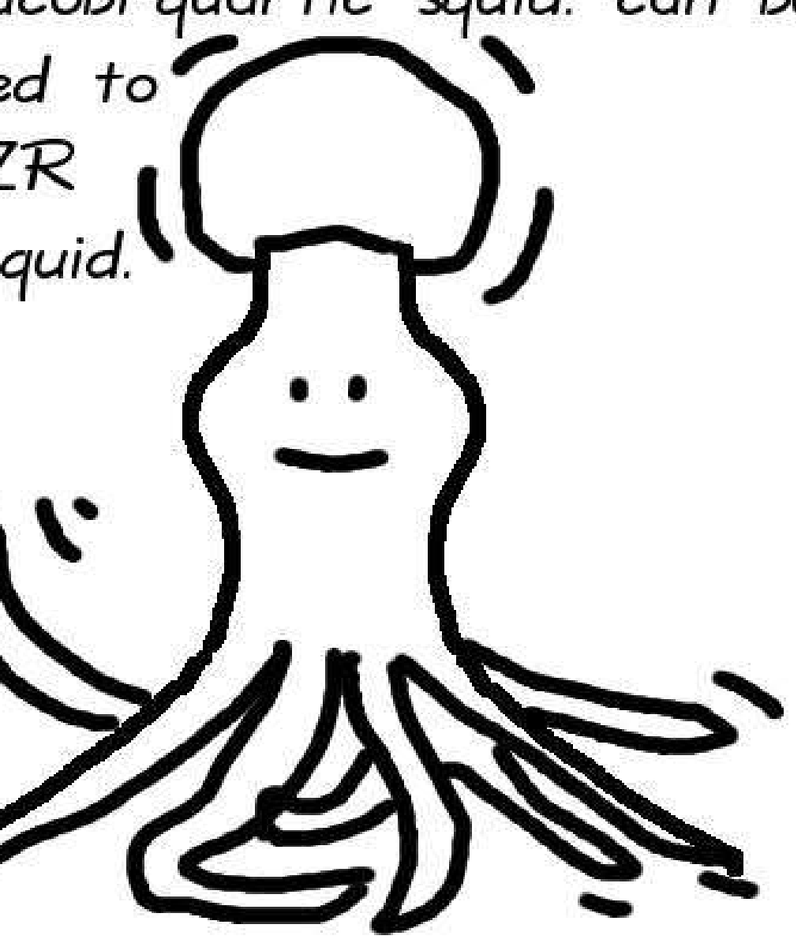

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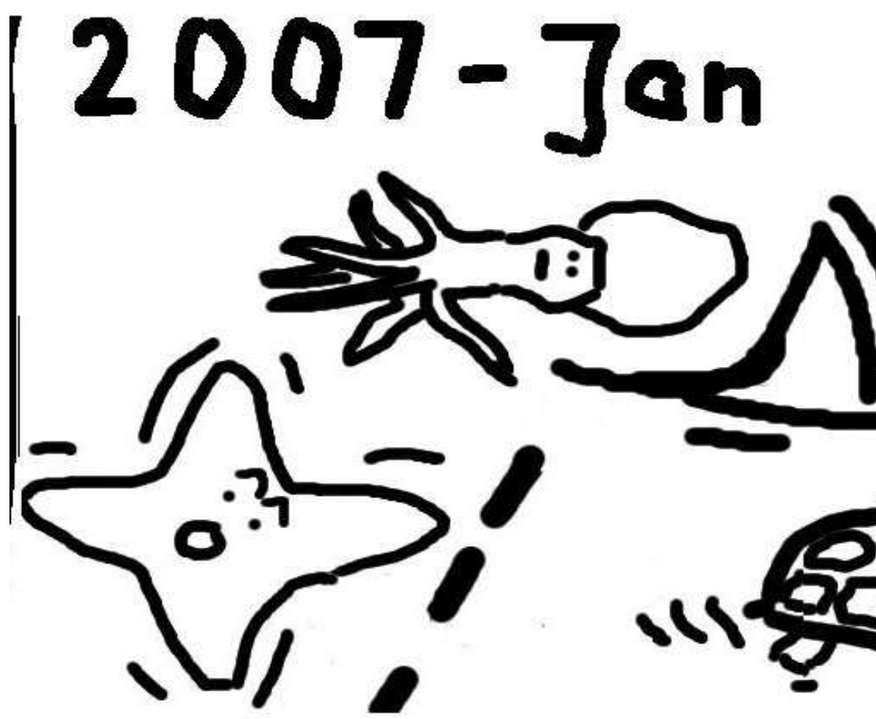
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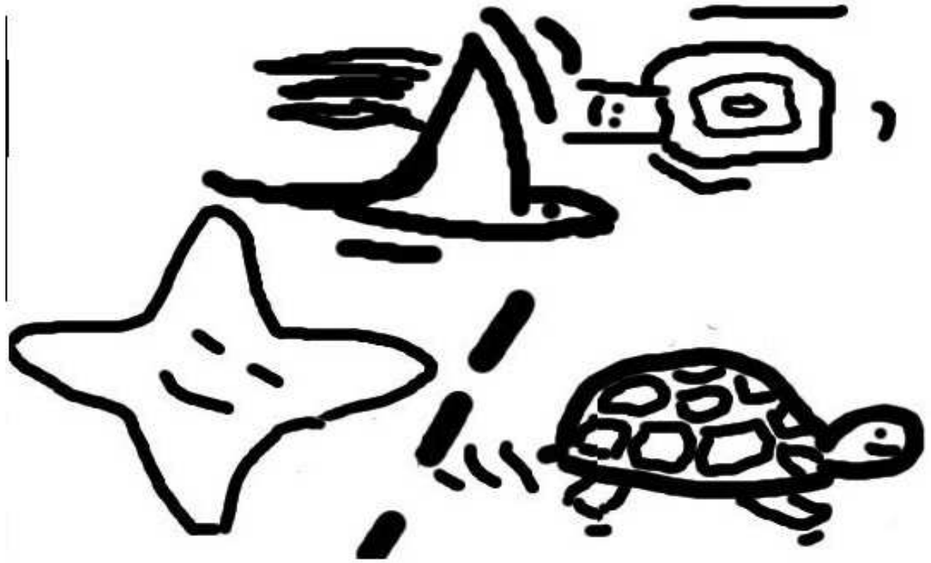

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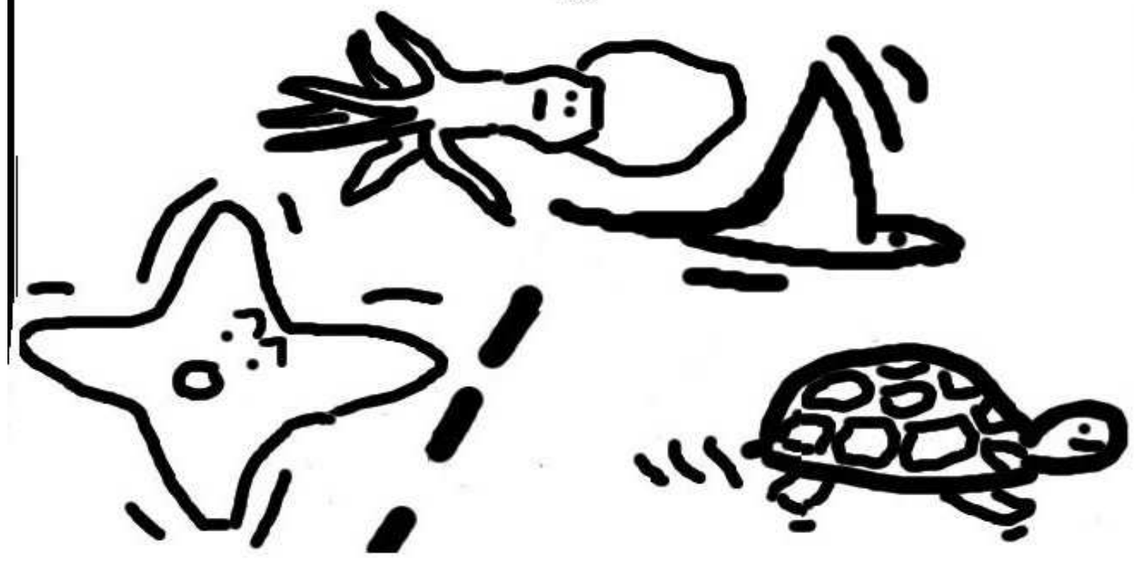

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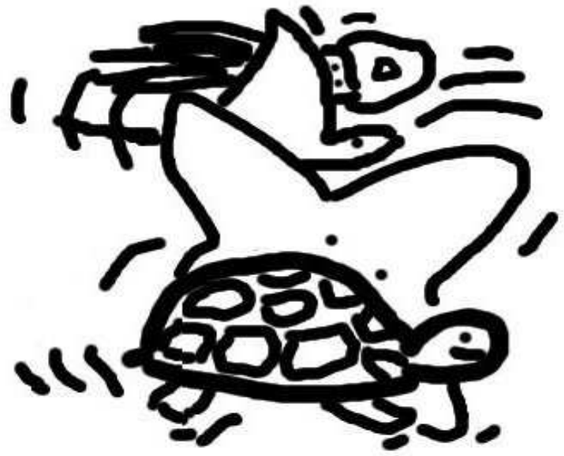

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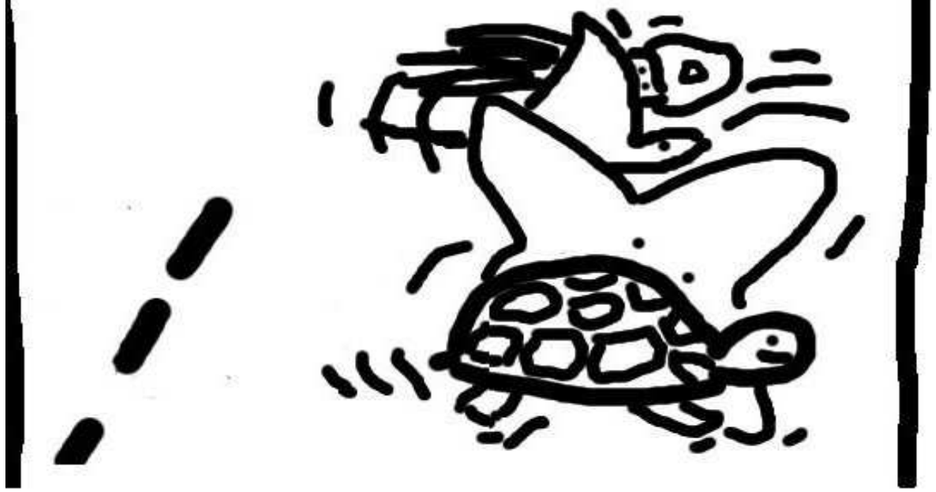

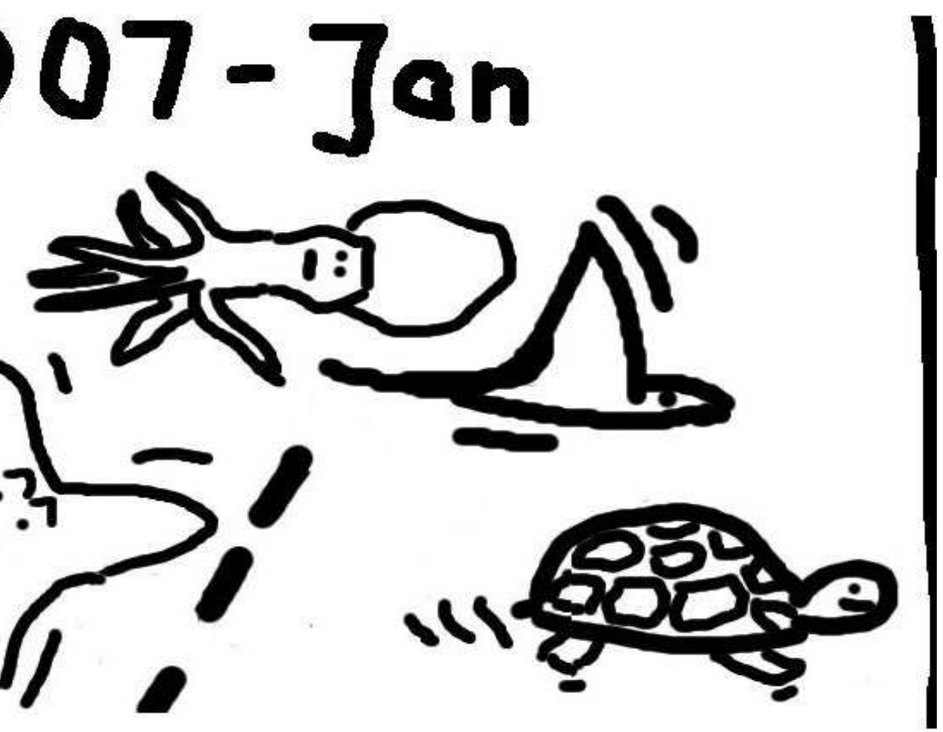
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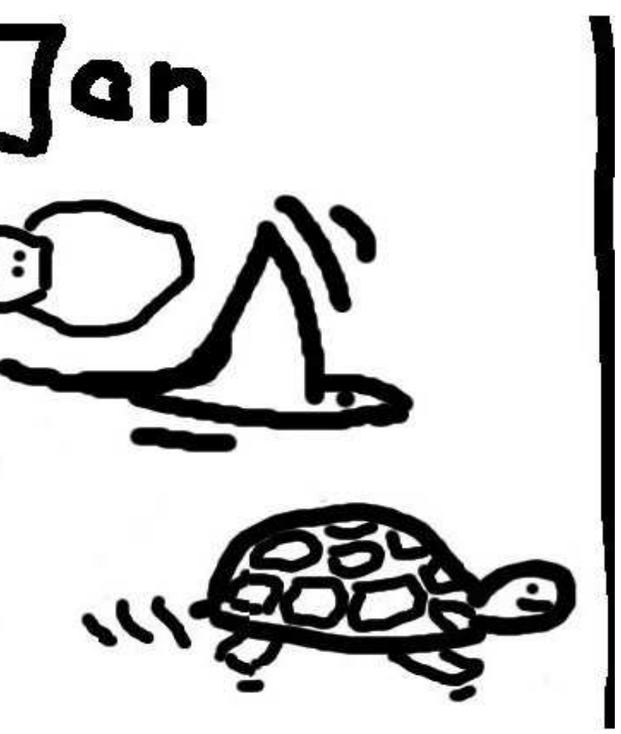
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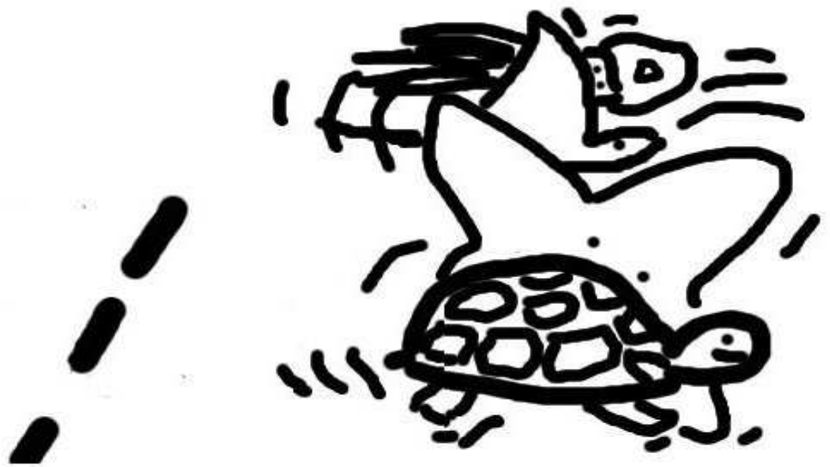

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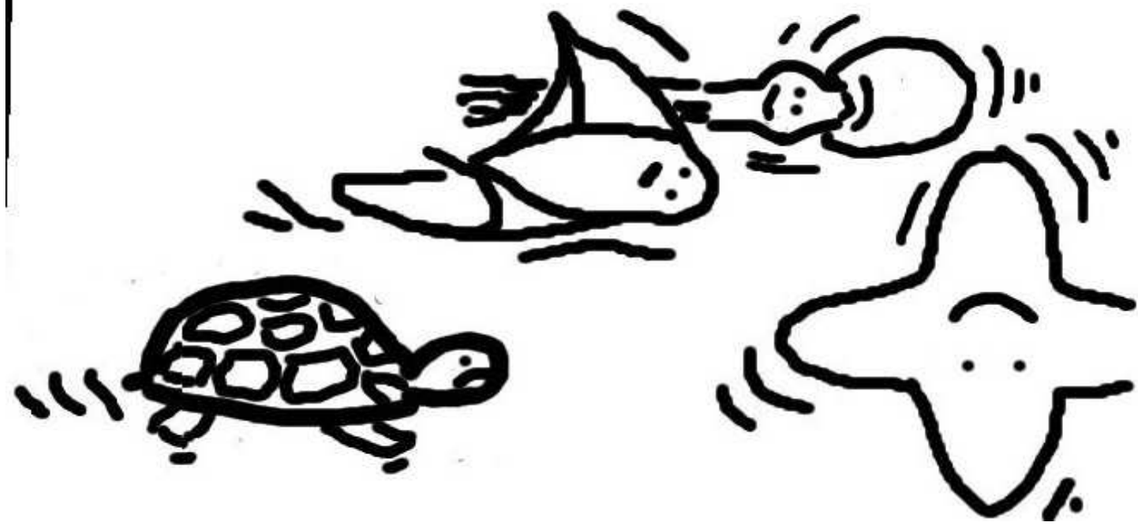

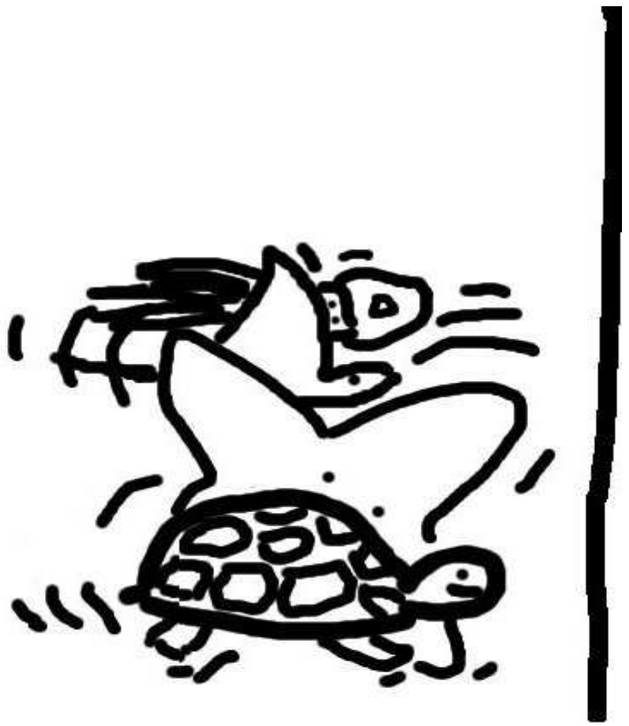

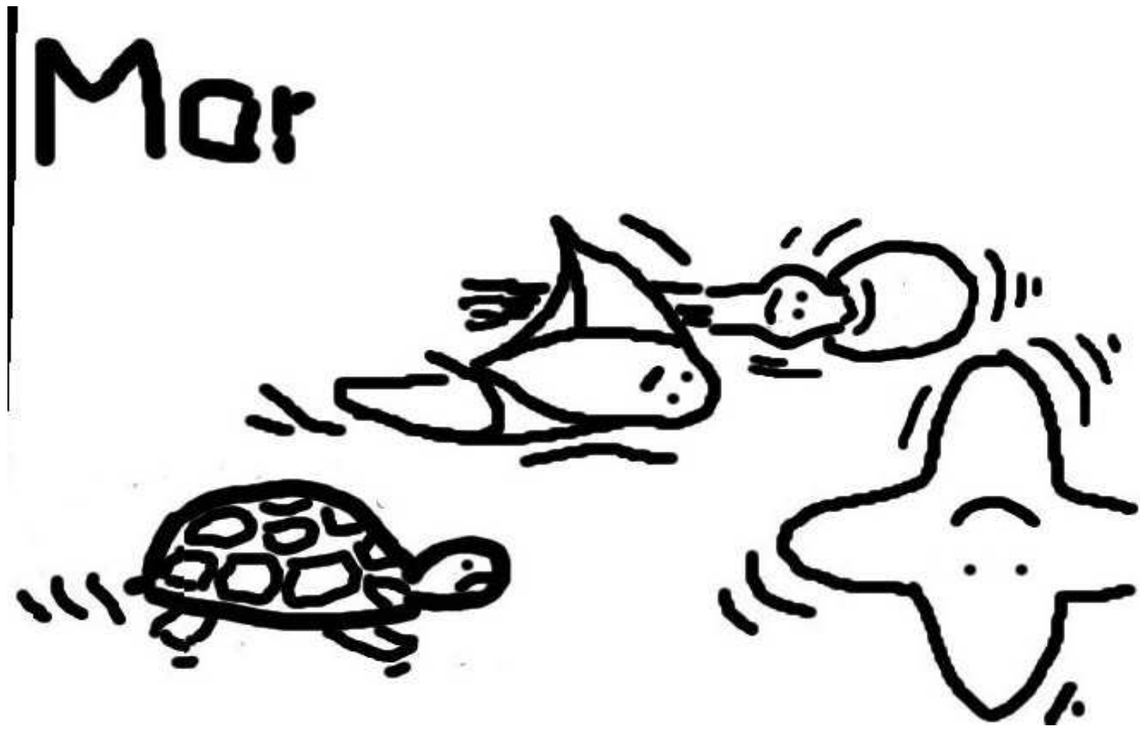

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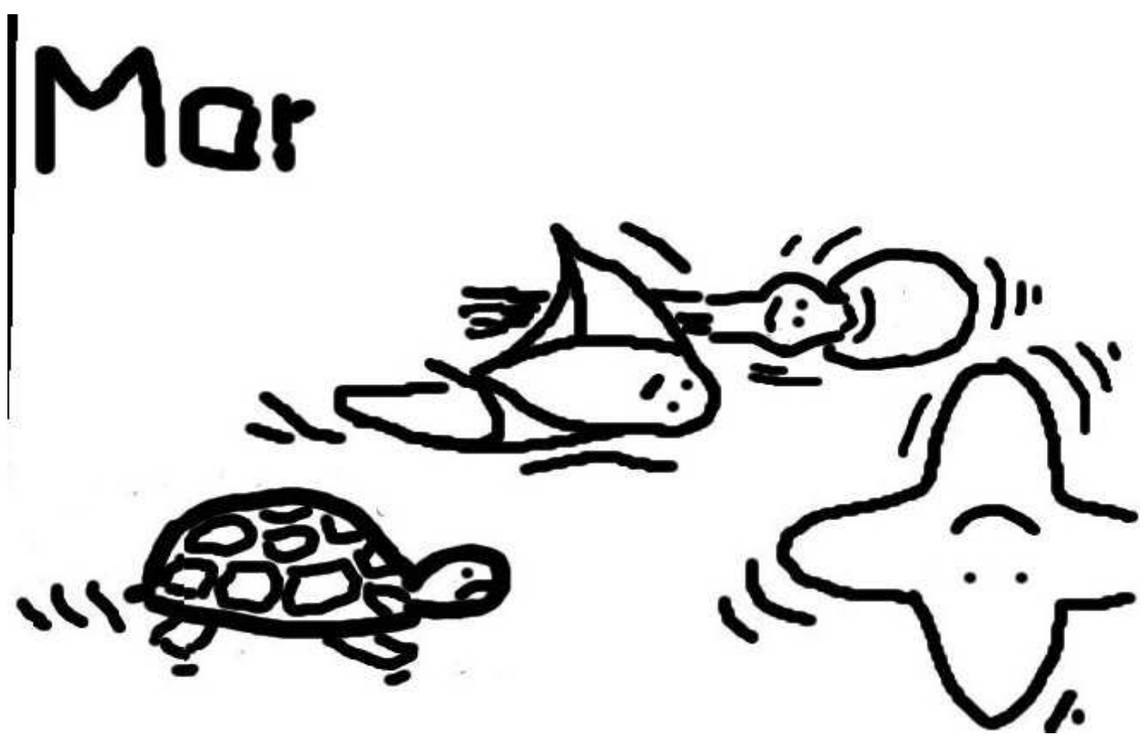

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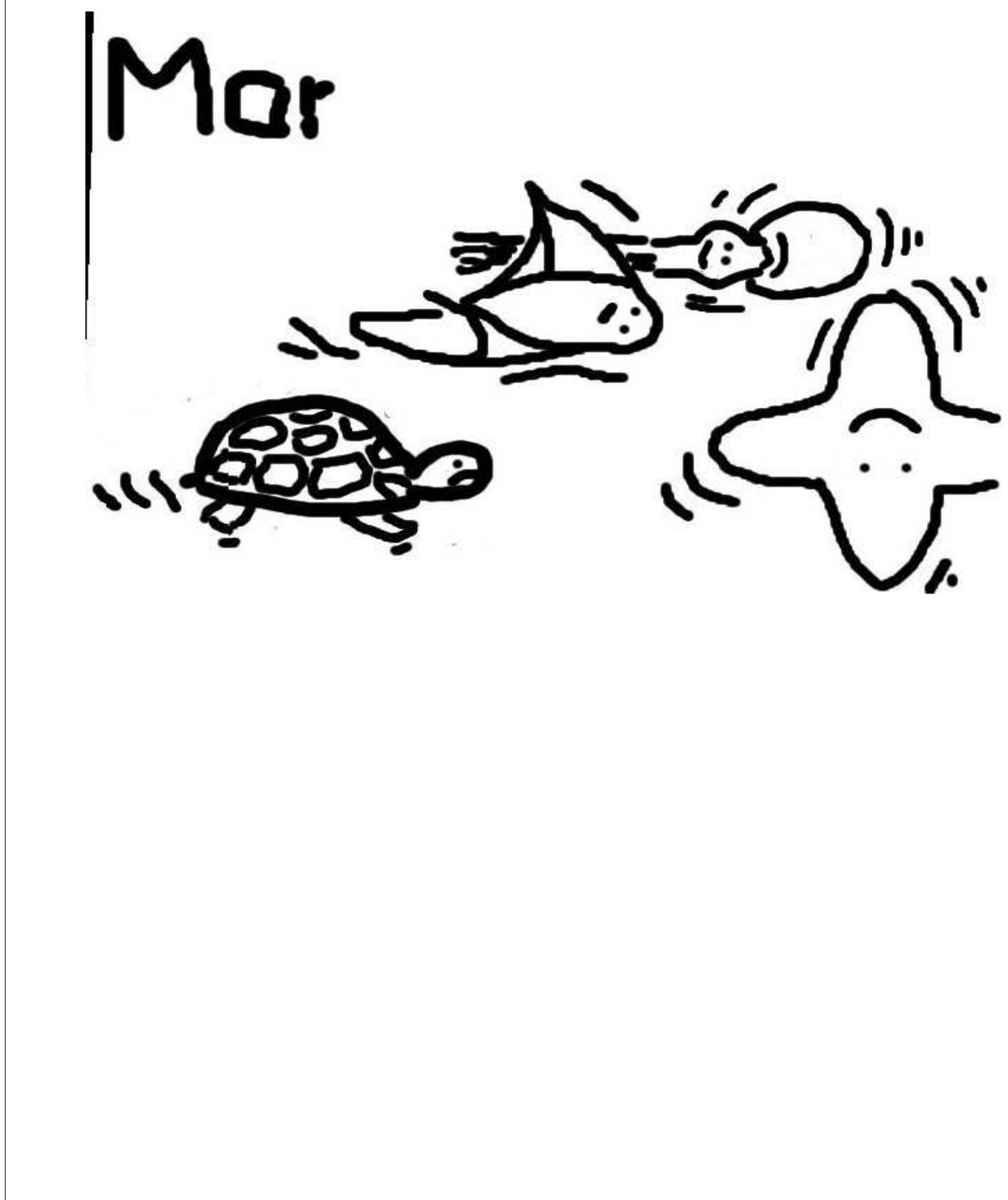

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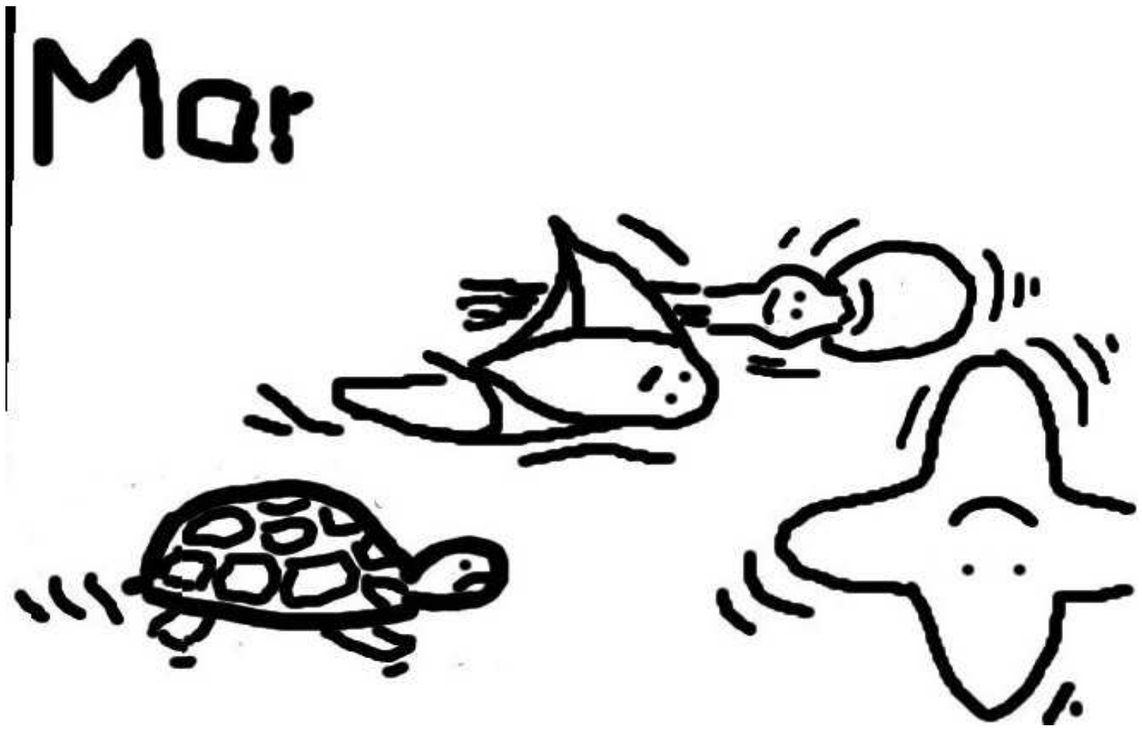

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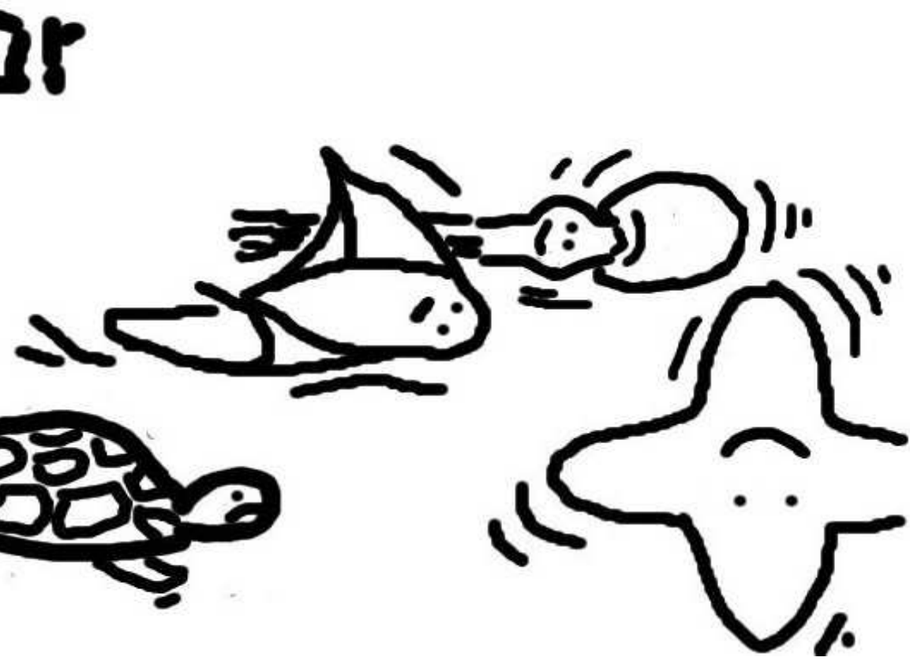

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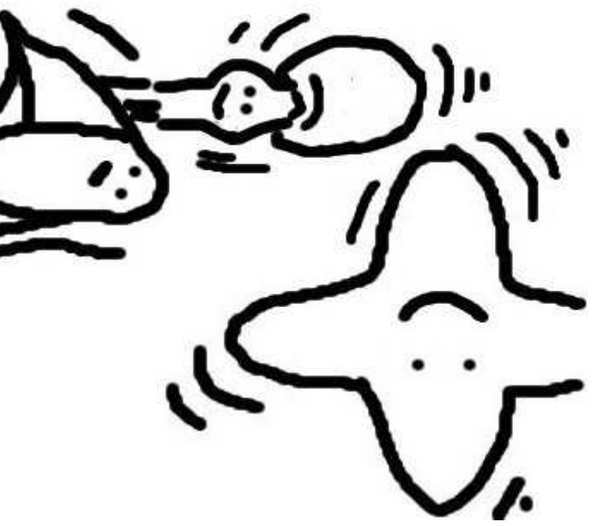

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