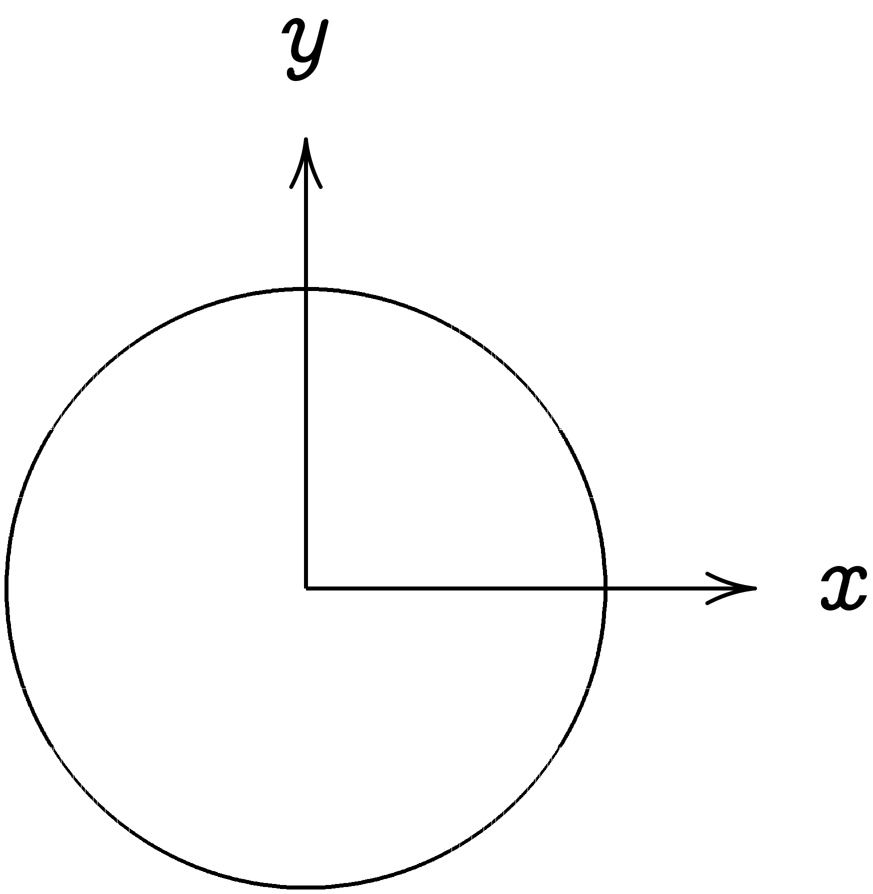
Introduction to elliptic curves

D. J. Bernstein
University of Illinois at Chicago

The clock



This is the curve $x^2 + y^2 = 1$.

Warning:

This is *not* an elliptic curve.

"Elliptic curve" $\neq$ "ellipse."

Examples of points on this curve:

$(0, 1) =$ "12:00".

Examples of points on this curve:

$(0, 1) =$ “12:00”.

$(0, -1) =$

Examples of points on this curve:

$(0, 1) =$ “12:00”.

$(0, -1) =$ “6:00”.

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$

$(-1, 0) =$

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$ "3:00".

$(-1, 0) =$ "9:00".

Examples of points on this curve:

$(0, 1) =$ “12:00”.

$(0, -1) =$ “6:00”.

$(1, 0) =$ “3:00”.

$(-1, 0) =$ “9:00”.

$(\sqrt{3/4}, 1/2) =$

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$ "3:00".

$(-1, 0) =$ "9:00".

$(\sqrt{3/4}, 1/2) =$ "2:00".

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$ "3:00".

$(-1, 0) =$ "9:00".

$(\sqrt{3/4}, 1/2) =$ "2:00".

$(1/2, -\sqrt{3/4}) =$

$(-1/2, -\sqrt{3/4}) =$

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$ "3:00".

$(-1, 0) =$ "9:00".

$(\sqrt{3/4}, 1/2) =$ "2:00".

$(1/2, -\sqrt{3/4}) =$ "5:00".

$(-1/2, -\sqrt{3/4}) =$ "7:00".

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$ "3:00".

$(-1, 0) =$ "9:00".

$(\sqrt{3/4}, 1/2) =$ "2:00".

$(1/2, -\sqrt{3/4}) =$ "5:00".

$(-1/2, -\sqrt{3/4}) =$ "7:00".

$(\sqrt{1/2}, \sqrt{1/2}) =$ "1:30".

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$ "3:00".

$(-1, 0) =$ "9:00".

$(\sqrt{3/4}, 1/2) =$ "2:00".

$(1/2, -\sqrt{3/4}) =$ "5:00".

$(-1/2, -\sqrt{3/4}) =$ "7:00".

$(\sqrt{1/2}, \sqrt{1/2}) =$ "1:30".

$(3/5, 4/5)$. $(-3/5, 4/5)$.

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$ "3:00".

$(-1, 0) =$ "9:00".

$(\sqrt{3/4}, 1/2) =$ "2:00".

$(1/2, -\sqrt{3/4}) =$ "5:00".

$(-1/2, -\sqrt{3/4}) =$ "7:00".

$(\sqrt{1/2}, \sqrt{1/2}) =$ "1:30".

$(3/5, 4/5)$. $(-3/5, 4/5)$.

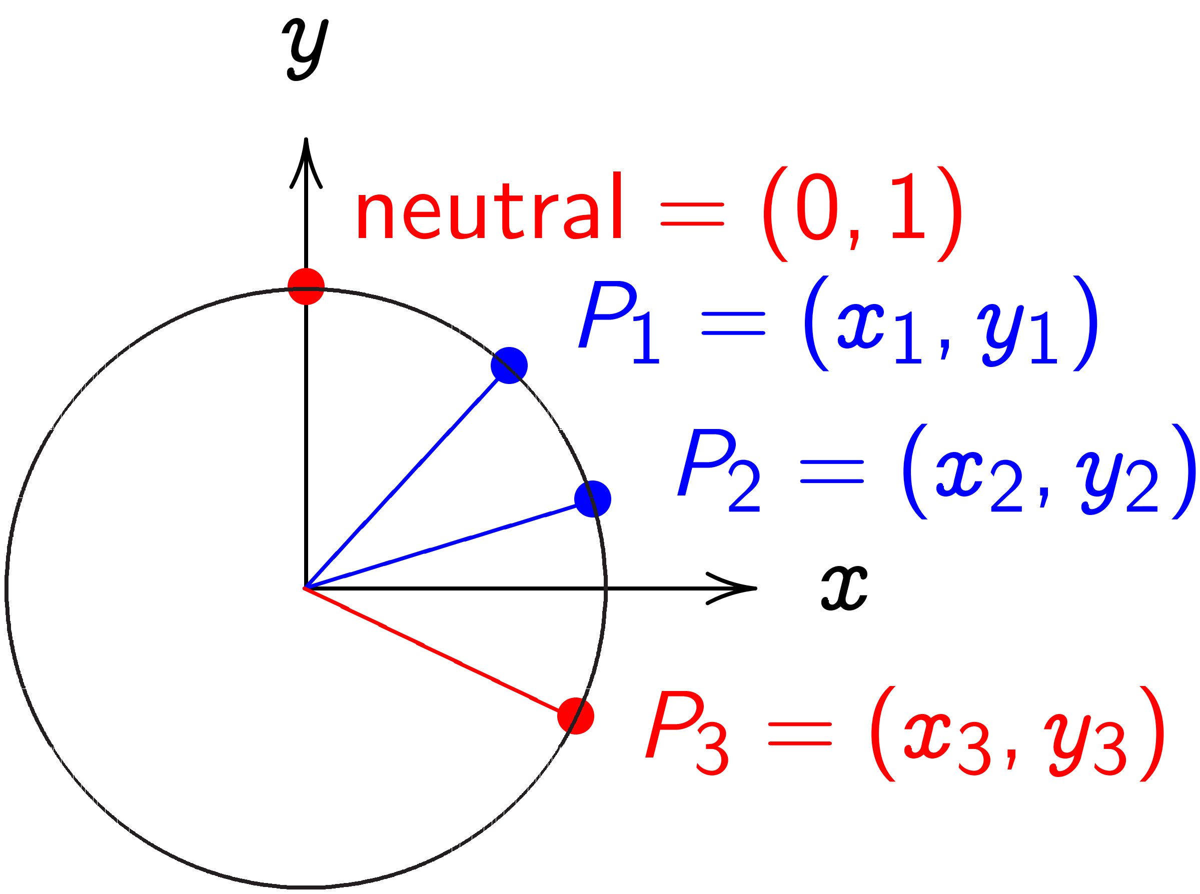
$(3/5, -4/5)$. $(-3/5, -4/5)$.

$(4/5, 3/5)$. $(-4/5, 3/5)$.

$(4/5, -3/5)$. $(-4/5, -3/5)$.

Many more.

Clock addition



Standard addition formula for the clock $x^2 + y^2 = 1$: sum of (x_1, y_1) and (x_2, y_2) is $(x_1y_2 + y_1x_2, y_1y_2 - x_1x_2)$.

Examples of clock addition:

"2:00" + "5:00"

$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$

Examples of clock addition:

"2:00" + "5:00"

$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$

$= (-1/2, -\sqrt{3/4})$

Examples of clock addition:

"2:00" + "5:00"

$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$

$= (-1/2, -\sqrt{3/4}) =$ "7:00".

Examples of clock addition:

"2:00" + "5:00"

$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{"7:00"}.$$

"5:00" + "9:00"

$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{“7:00”}.$$

$$\text{“5:00”} + \text{“9:00”}$$
$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$
$$= (\sqrt{3/4}, 1/2) = \text{“2:00”}.$$

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{“7:00”}.$$

$$\text{“5:00”} + \text{“9:00”}$$
$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$
$$= (\sqrt{3/4}, 1/2) = \text{“2:00”}.$$

$$2\left(\frac{3}{5}, \frac{4}{5}\right) =$$

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{“7:00”}.$$

$$\text{“5:00”} + \text{“9:00”}$$
$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$
$$= (\sqrt{3/4}, 1/2) = \text{“2:00”}.$$

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

Examples of clock addition:

$$\text{"2:00"} + \text{"5:00"}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{"7:00"}.$$

$$\text{"5:00"} + \text{"9:00"}$$
$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$
$$= (\sqrt{3/4}, 1/2) = \text{"2:00"}.$$

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

$$3\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{117}{125}, \frac{-44}{125}\right).$$

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{“7:00”}.$$

$$\text{“5:00”} + \text{“9:00”}$$
$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$
$$= (\sqrt{3/4}, 1/2) = \text{“2:00”}.$$

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

$$3\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{117}{125}, \frac{-44}{125}\right).$$

$$4\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{336}{625}, \frac{-527}{625}\right).$$

Examples of clock addition:

"2:00" + "5:00"

$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{"7:00"}.$$

"5:00" + "9:00"

$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$
$$= (\sqrt{3/4}, 1/2) = \text{"2:00"}.$$

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

$$3\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{117}{125}, \frac{-44}{125}\right).$$

$$4\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{336}{625}, \frac{-527}{625}\right).$$

$$(x_1, y_1) + (0, 1) =$$

Examples of clock addition:

"2:00" + "5:00"

$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$

$$= (-1/2, -\sqrt{3/4}) = \text{"7:00"}.$$

"5:00" + "9:00"

$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$

$$= (\sqrt{3/4}, 1/2) = \text{"2:00"}.$$

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

$$3\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{117}{125}, \frac{-44}{125}\right).$$

$$4\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{336}{625}, \frac{-527}{625}\right).$$

$$(x_1, y_1) + (0, 1) = (x_1, y_1).$$

Examples of clock addition:

$$\text{"2:00"} + \text{"5:00"}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{"7:00"}.$$

$$\text{"5:00"} + \text{"9:00"}$$
$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$
$$= (\sqrt{3/4}, 1/2) = \text{"2:00"}.$$

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

$$3\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{117}{125}, \frac{-44}{125}\right).$$

$$4\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{336}{625}, \frac{-527}{625}\right).$$

$$(x_1, y_1) + (0, 1) = (x_1, y_1).$$

$$(x_1, y_1) + (-x_1, y_1) =$$

Examples of clock addition:

"2:00" + "5:00"

$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{"7:00"}.$$

"5:00" + "9:00"

$$= (1/2, -\sqrt{3/4}) + (-1, 0)$$
$$= (\sqrt{3/4}, 1/2) = \text{"2:00"}.$$

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

$$3\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{117}{125}, \frac{-44}{125}\right).$$

$$4\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{336}{625}, \frac{-527}{625}\right).$$

$$(x_1, y_1) + (0, 1) = (x_1, y_1).$$

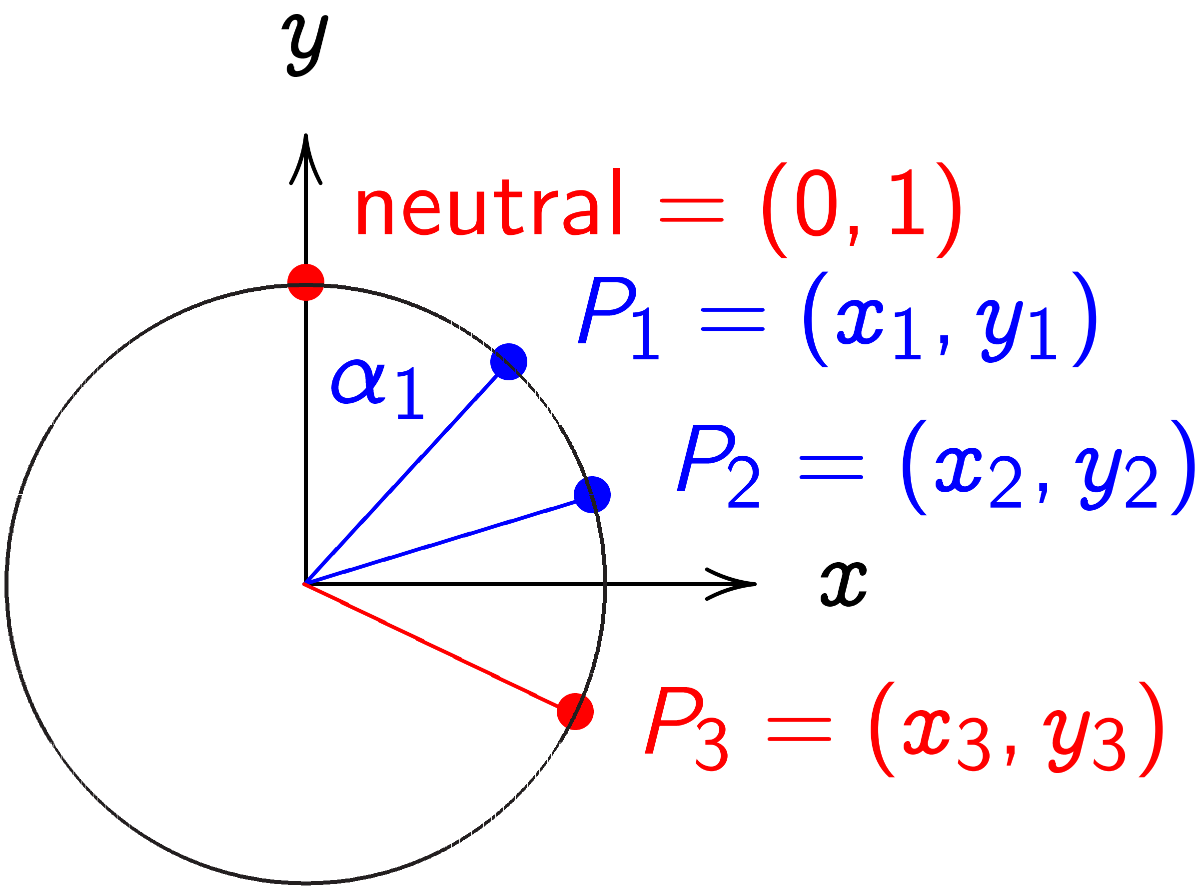
$$(x_1, y_1) + (-x_1, y_1) = (0, 1).$$

Define Clock($\mathbf{R}$) as
$\{(x, y) \in \mathbf{R} \times \mathbf{R} : x^2 + y^2 = 1\}$.
As usual $\mathbf{R} = \{$real numbers$\}$.

Exercise:
Prove that Clock($\mathbf{R}$)
is a commutative group
under clock addition.

In other words:
clock sum is in Clock($\mathbf{R}$);
clock addition is commutative;
clock addition is associative;
there is a neutral element;
each element has a negative.

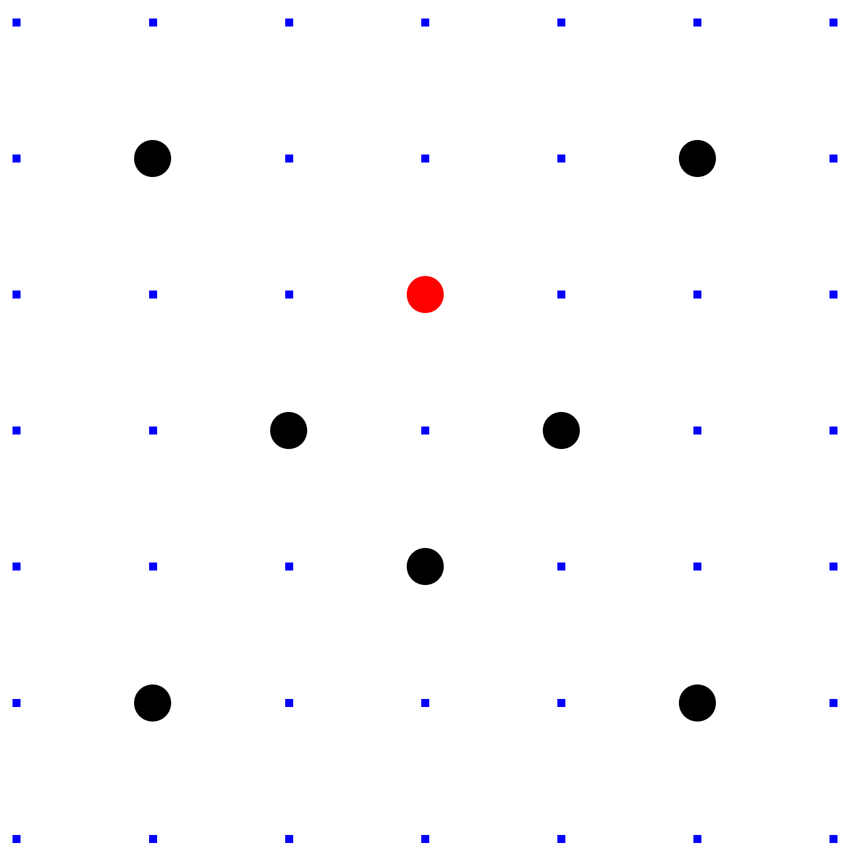
How to remember addition law:



$x^2 + y^2 = 1$, parametrized by $x = \sin\alpha$, $\quad y = \cos\alpha$. Recall

$$(\sin(\alpha_1 + \alpha_2), \cos(\alpha_1 + \alpha_2)) =$$
$$(\sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2,$$
$$\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2).$$

Clocks over finite fields



$\text{Clock}(\mathbf{F}_7) =$
$\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2 = 1\}$.
Here $\mathbf{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$
$= \{0, 1, 2, 3, -3, -2, -1\}$
with $+, -, \times$ modulo 7.

Clock($\mathbf{F}_7$) is a group under the same addition law used for Clock($\mathbf{R}$):
$(x_1, y_1) + (x_2, y_2) =$
$(x_1 y_2 + y_1 x_2, y_1 y_2 - x_1 x_2)$.

Similarly construct a finite group Clock($\mathbf{F}_q$) for each prime power q.

Clock($\mathbf{F}_q$) has $\approx q$ elements. "Index-calculus" attacks find discrete logs in Clock($\mathbf{F}_q$) in time $\exp(O((\log q)^{1/3}(\log\log q)^{2/3}))$.

Can use Clock($\mathbf{F}_q$) for crypto.

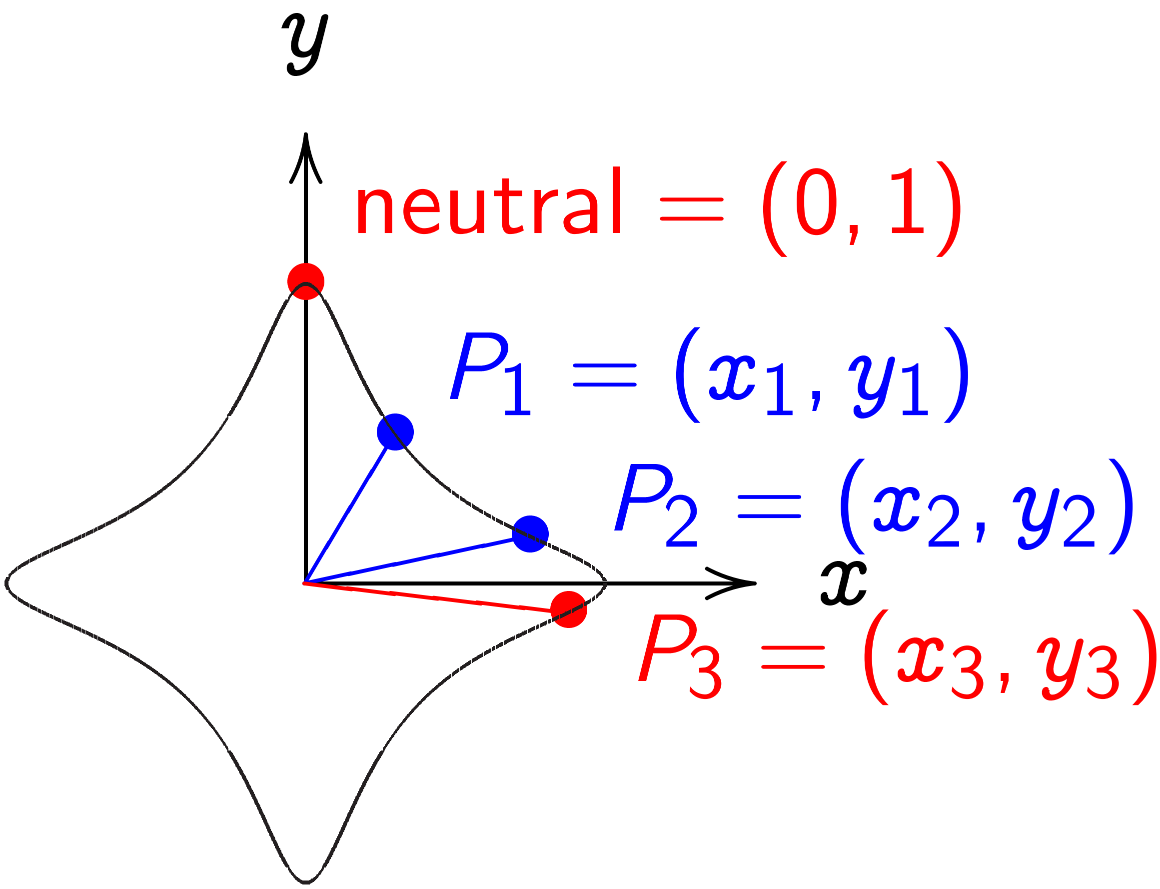
But need hard discrete logs,
so need very slow index calculus,
so need very large q.

This makes arithmetic slow.

Alternative (1985 Miller,
independently 1987 Koblitz):
Switch from $\mathbf{F}_q^*$, Clock($\mathbf{F}_q$), etc.
to an "elliptic curve."

As far as we can tell,
index calculus doesn't work
against most elliptic curves,
so can use much smaller q.

Addition on an Edwards curve



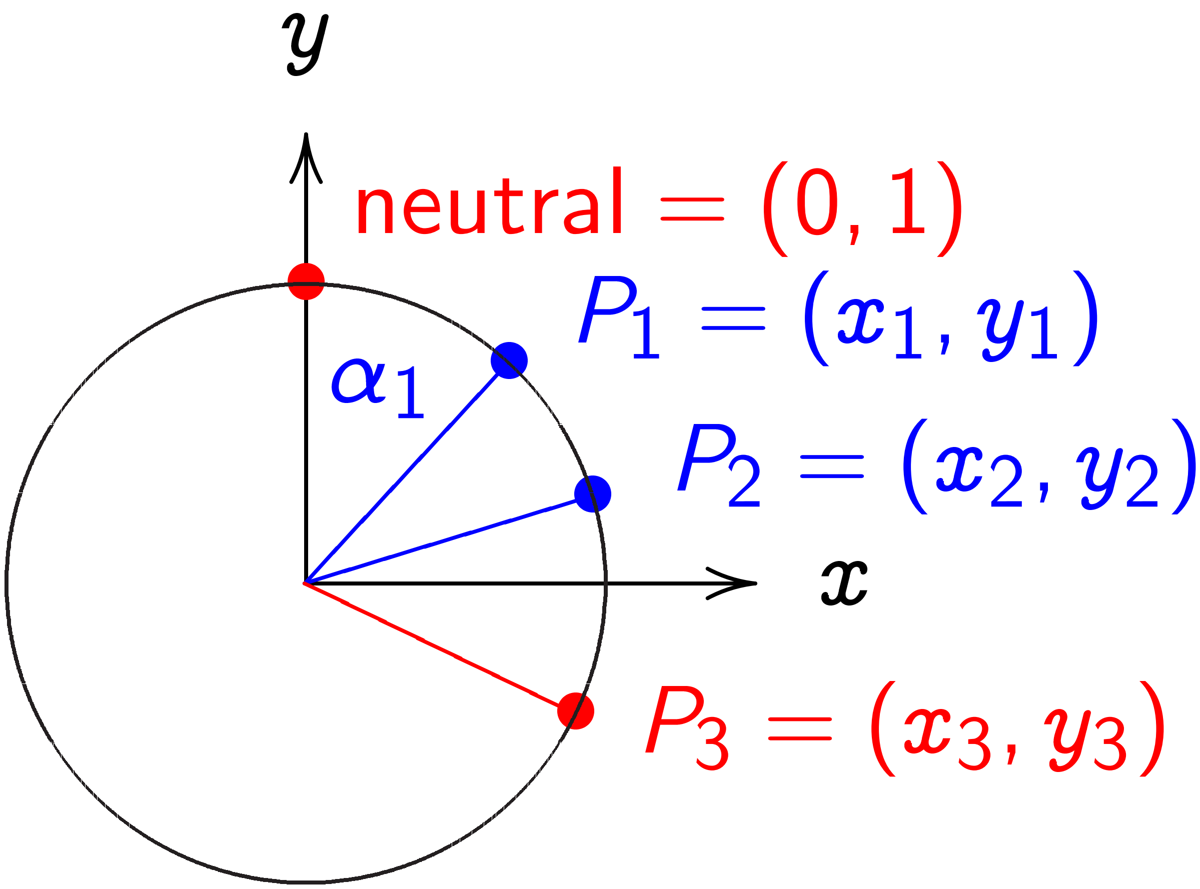
$x^2 + y^2 = 1 - 30x^2y^2$.

Sum of (x_1, y_1) and (x_2, y_2) is

$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$

$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

The clock again, for comparison:



$x^2 + y^2 = 1.$

Sum of (x_1, y_1) and (x_2, y_2) is

$(x_1y_2 + y_1x_2,$

$y_1y_2 - x_1x_2).$

"Hey, there were divisions
in the Edwards addition law!
What if the denominators are 0?"

Answer: They aren't!

If $x^2 + y^2 = 1 - 30x^2y^2$
then $30x^2y^2 < 1$
so $\sqrt{30}\,|xy| < 1$.

“Hey, there were divisions in the Edwards addition law! What if the denominators are 0?”

Answer: They aren’t!

If $x^2 + y^2 = 1 - 30x^2y^2$
then $30x^2y^2 < 1$
so $\sqrt{30}\ |xy| < 1$.

If $x_1^2 + y_1^2 = 1 - 30x_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 - 30x_2^2y_2^2$
then $\sqrt{30}\ |x_1y_1| < 1$
and $\sqrt{30}\ |x_2y_2| < 1$

"Hey, there were divisions
in the Edwards addition law!
What if the denominators are 0?"

Answer: They aren't!

If $x^2 + y^2 = 1 - 30x^2y^2$
then $30x^2y^2 < 1$
so $\sqrt{30}\ |xy| < 1$.

If $x_1^2 + y_1^2 = 1 - 30x_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 - 30x_2^2y_2^2$
then $\sqrt{30}\ |x_1y_1| < 1$
and $\sqrt{30}\ |x_2y_2| < 1$
so $30\,|x_1y_1x_2y_2| < 1$

"Hey, there were divisions
in the Edwards addition law!
What if the denominators are 0?"

Answer: They aren't!

If $x^2 + y^2 = 1 - 30x^2y^2$
then $30x^2y^2 < 1$
so $\sqrt{30}\ |xy| < 1$.

If $x_1^2 + y_1^2 = 1 - 30x_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 - 30x_2^2y_2^2$
then $\sqrt{30}\ |x_1y_1| < 1$
and $\sqrt{30}\ |x_2y_2| < 1$
so $30\,|x_1y_1x_2y_2| < 1$
so $1 \pm 30x_1x_2y_1y_2 > 0$.

The Edwards addition law

$(x_1, y_1) + (x_2, y_2) =$
$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$
is a group law for the curve
$x^2 + y^2 = 1 - 30x^2y^2$.

Some calculation required:
addition result is on curve;
addition law is associative.

Other parts of proof are easy:
addition law is commutative;
$(0, 1)$ is neutral element;
$(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

More Edwards curves

Fix an odd prime power q.
Fix a non-square $d \in \mathbf{F}_q$.

$$\{(x, y) \in \mathbf{F}_q \times \mathbf{F}_q : x^2 + y^2 = 1 + dx^2y^2\}$$

is a commutative group with $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ defined by Edwards addition law:

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

$$y_3 = \frac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

If $x_1^2 + y_1^2 = 1 + dx_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2y_2^2$
and $dx_1x_2y_1y_2 = \pm 1$

Denominators are never 0.

But need different proof;
"$x^2 + y^2 > 0$" doesn't work.

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$
then $dx_1^2 y_1^2 (x_2 + y_2)^2$
$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

If $x_1^2 + y_1^2 = 1 + dx_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2y_2^2$
and $dx_1x_2y_1y_2 = \pm 1$
then $dx_1^2y_1^2(x_2 + y_2)^2$
$= dx_1^2y_1^2(x_2^2 + y_2^2 + 2x_2y_2)$
$= dx_1^2y_1^2(dx_2^2y_2^2 + 1 + 2x_2y_2)$

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$
then $dx_1^2 y_1^2 (x_2 + y_2)^2$
$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$
$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$
$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

If $x_1^2 + y_1^2 = 1 + dx_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2y_2^2$
and $dx_1x_2y_1y_2 = \pm 1$
then $dx_1^2y_1^2(x_2 + y_2)^2$
$= dx_1^2y_1^2(x_2^2 + y_2^2 + 2x_2y_2)$
$= dx_1^2y_1^2(dx_2^2y_2^2 + 1 + 2x_2y_2)$
$= d^2x_1^2y_1^2x_2^2y_2^2 + dx_1^2y_1^2 + 2dx_1^2y_1^2x_2y_2$
$= 1 + dx_1^2y_1^2 \pm 2x_1y_1$

Denominators are never 0.

But need different proof;

“$x^2 + y^2 > 0$” doesn’t work.

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$
then $dx_1^2 y_1^2 (x_2 + y_2)^2$
$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$
$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$
$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$
$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$
$= x_1^2 + y_1^2 \pm 2x_1 y_1$

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

If $x_1^2 + y_1^2 = 1 + dx_1^2 y_1^2$
and $x_2^2 + y_2^2 = 1 + dx_2^2 y_2^2$
and $dx_1 x_2 y_1 y_2 = \pm 1$
then $dx_1^2 y_1^2 (x_2 + y_2)^2$
$= dx_1^2 y_1^2 (x_2^2 + y_2^2 + 2x_2 y_2)$
$= dx_1^2 y_1^2 (dx_2^2 y_2^2 + 1 + 2x_2 y_2)$
$= d^2 x_1^2 y_1^2 x_2^2 y_2^2 + dx_1^2 y_1^2 + 2dx_1^2 y_1^2 x_2 y_2$
$= 1 + dx_1^2 y_1^2 \pm 2x_1 y_1$
$= x_1^2 + y_1^2 \pm 2x_1 y_1$
$= (x_1 \pm y_1)^2$.

Case 1: $x_2 + y_2 \neq 0$. Then

$$d = \left(\frac{x_1 \pm y_1}{x_1 y_1 (x_2 + y_2)} \right)^2,$$

contradiction.

Case 1: $x_2 + y_2 \neq 0$. Then

$$d = \left(\frac{x_1 \pm y_1}{x_1 y_1 (x_2 + y_2)} \right)^2,$$

contradiction.

Case 2: $x_2 - y_2 \neq 0$. Then

$$d = \left(\frac{x_1 \mp y_1}{x_1 y_1 (x_2 - y_2)} \right)^2,$$

contradiction.

Case 1: $x_2 + y_2 \neq 0$. Then

$$d = \left(\frac{x_1 \pm y_1}{x_1 y_1 (x_2 + y_2)} \right)^2,$$

contradiction.

Case 2: $x_2 - y_2 \neq 0$. Then

$$d = \left(\frac{x_1 \mp y_1}{x_1 y_1 (x_2 - y_2)} \right)^2,$$

contradiction.

Case 3: $x_2 + y_2 = x_2 - y_2 = 0$.
Then $x_2 = 0$ and $y_2 = 0$,
contradiction.

This is an elliptic curve (technically, “mod blowups”).

Can use this group in crypto.

... if it’s a “strong” curve.
Need to compute group order.
If no large prime factor in order, must switch to another d; this very often happens.

Also check “twist security,” “embedding degree,” et al.

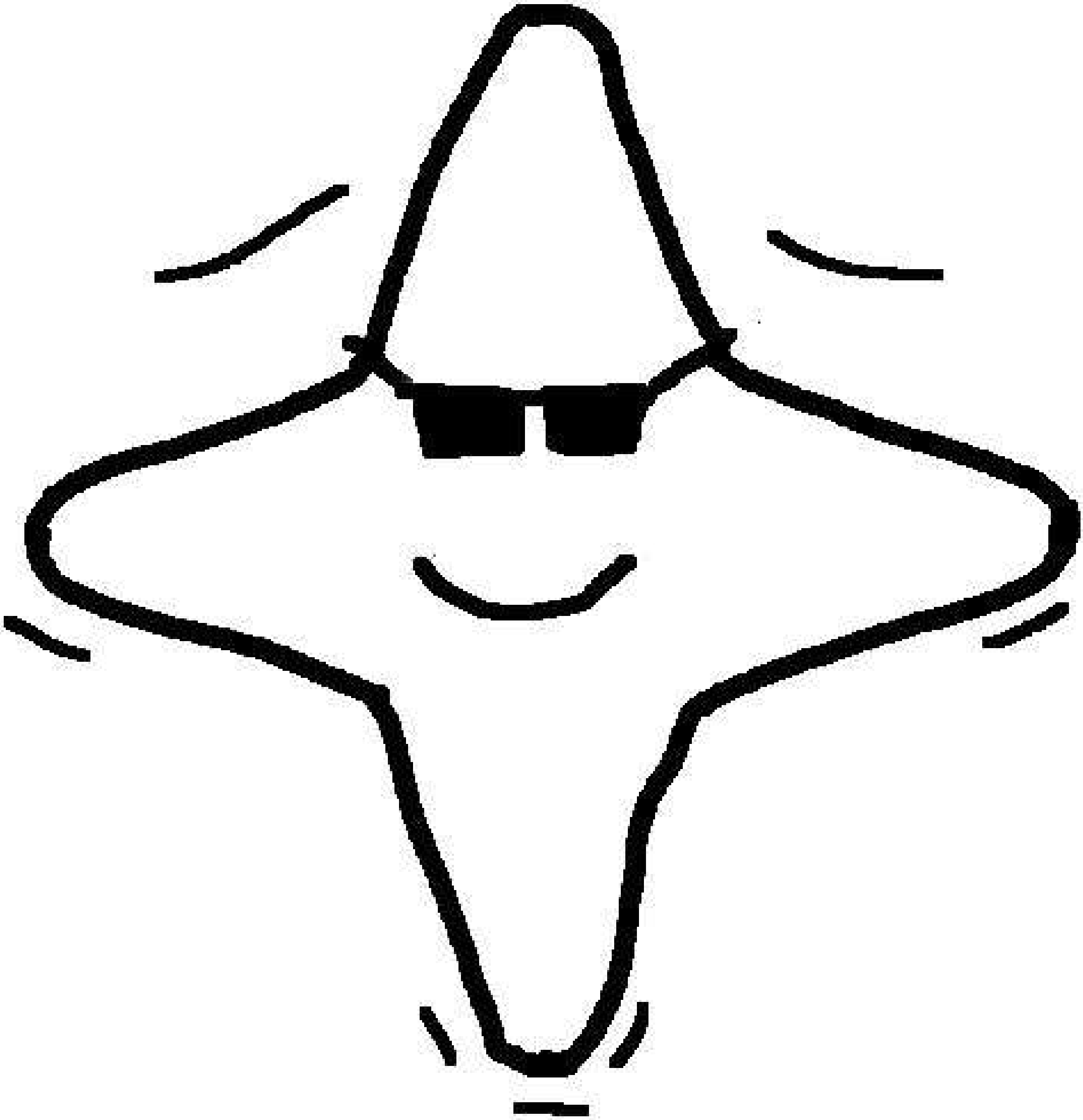
Safe example, “Curve25519”:
$q = 2^{255} - 19$; $d = 1 - 1/121666$.

Historical notes:

1761 Euler, 1866 Gauss
introduced an addition law
for $x^2 + y^2 = 1 - x^2y^2$,
the "lemniscatic elliptic curve."

2007 Edwards generalized to
many curves $x^2 + y^2 = 1 + c^4x^2y^2$.
Theorem: have now obtained
all elliptic curves over $\overline{\mathbf{Q}}$.

2007 Bernstein–Lange:
Edwards addition law is complete
for $x^2 + y^2 = 1 + dx^2y^2$ if $d \neq \square$;
and gives new ECC speed records!



(picture courtesy Tanja Lange)