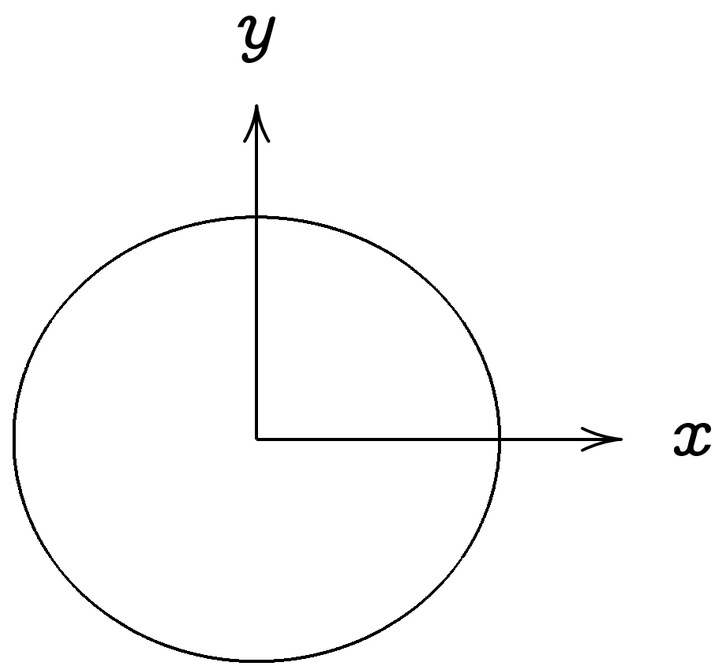
Introduction to elliptic curves

D. J. Bernstein
University of Illinois at Chicago

The clock



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"Elliptic curve" $\neq$ "ellipse."

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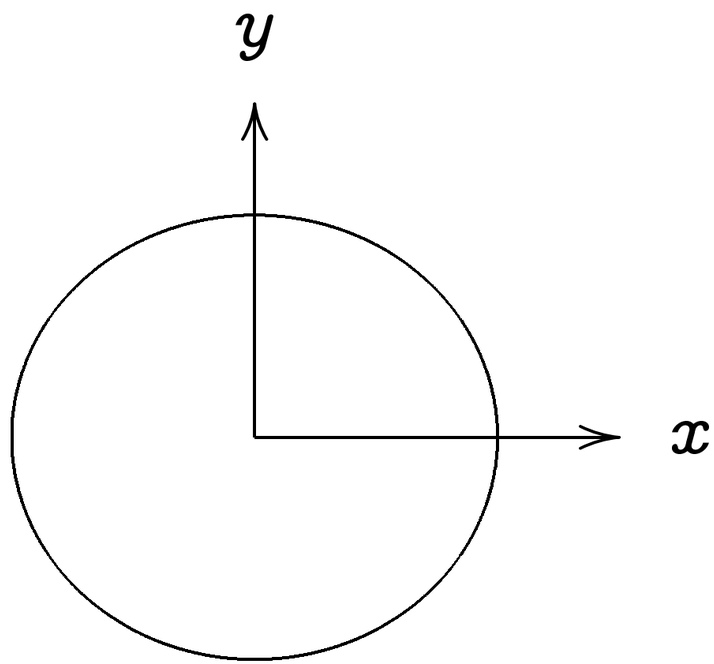

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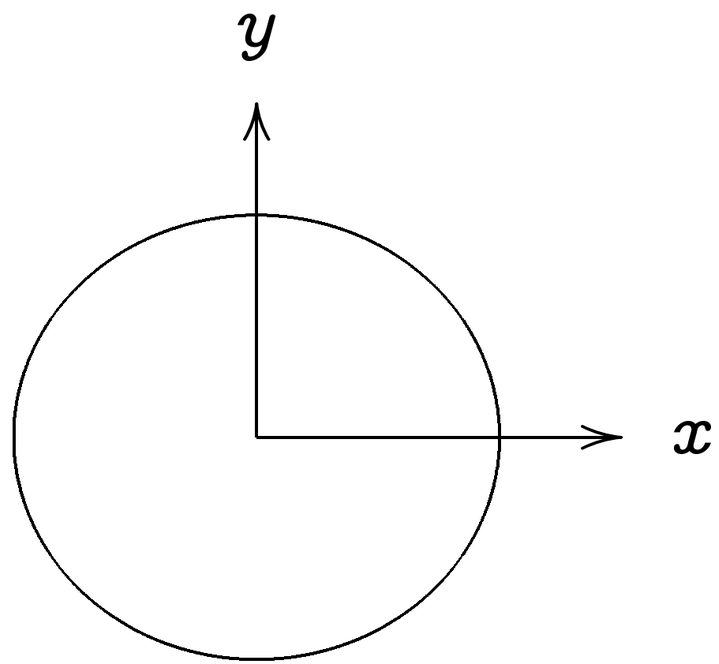

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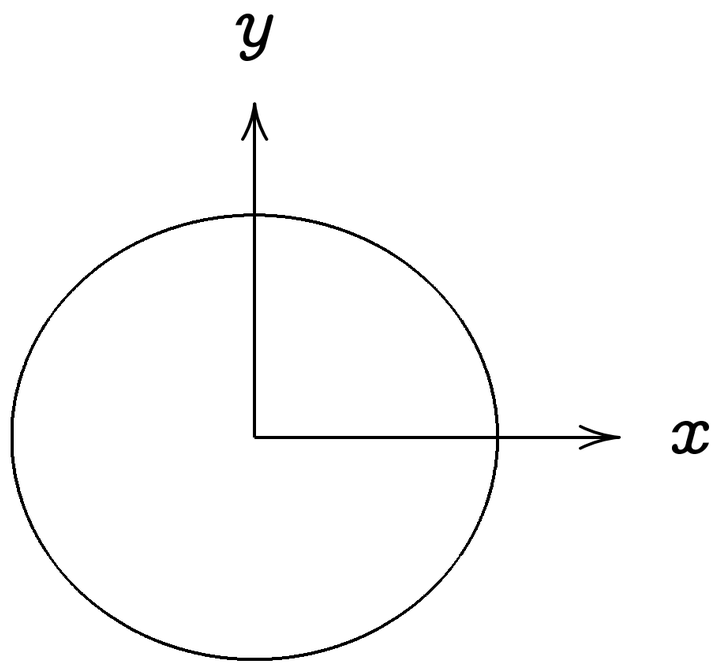

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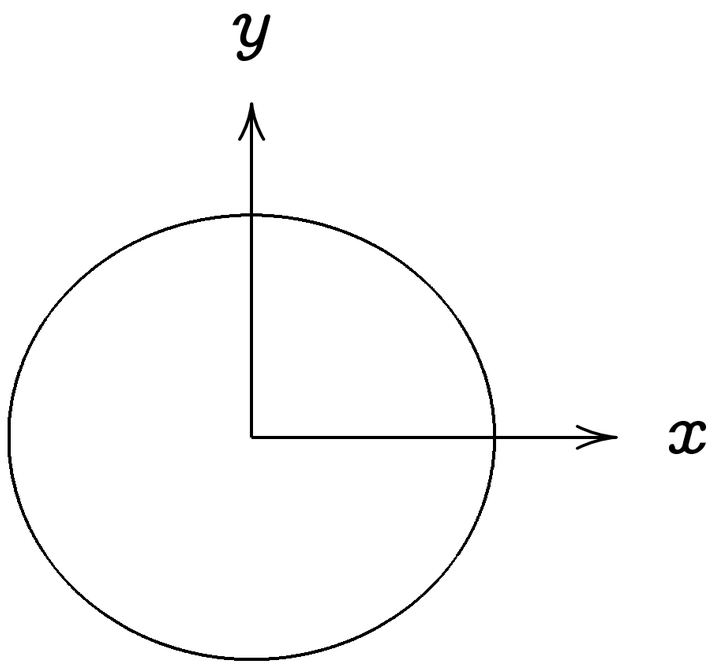

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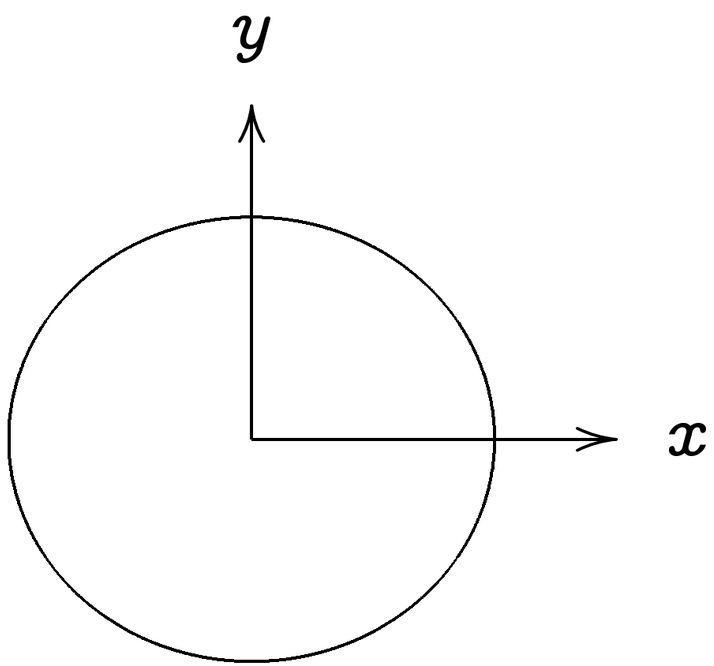

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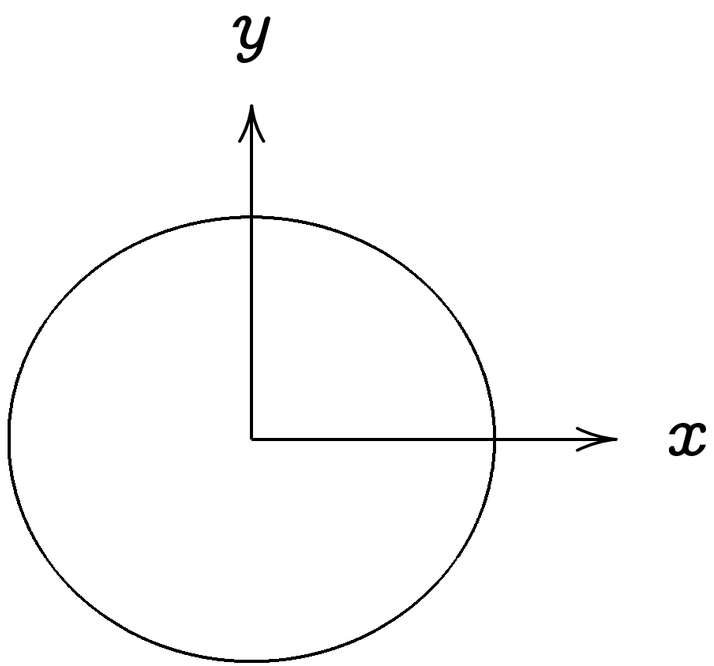

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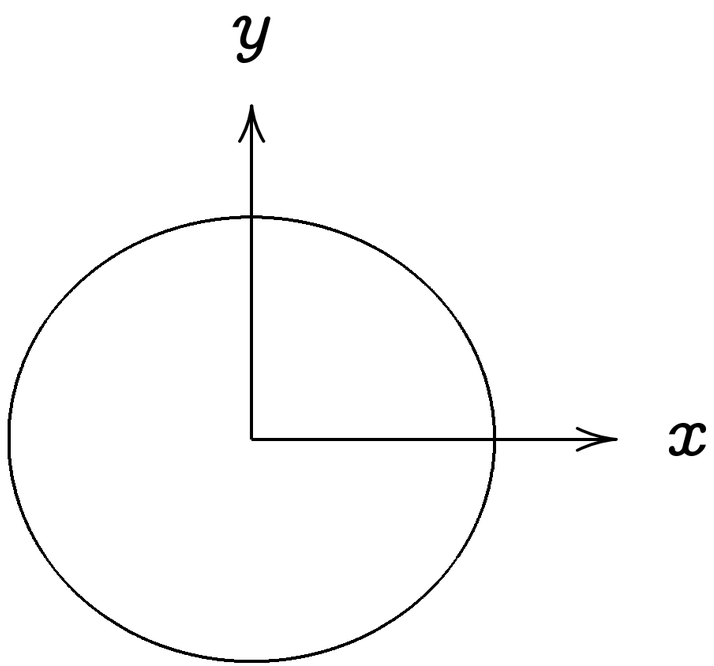

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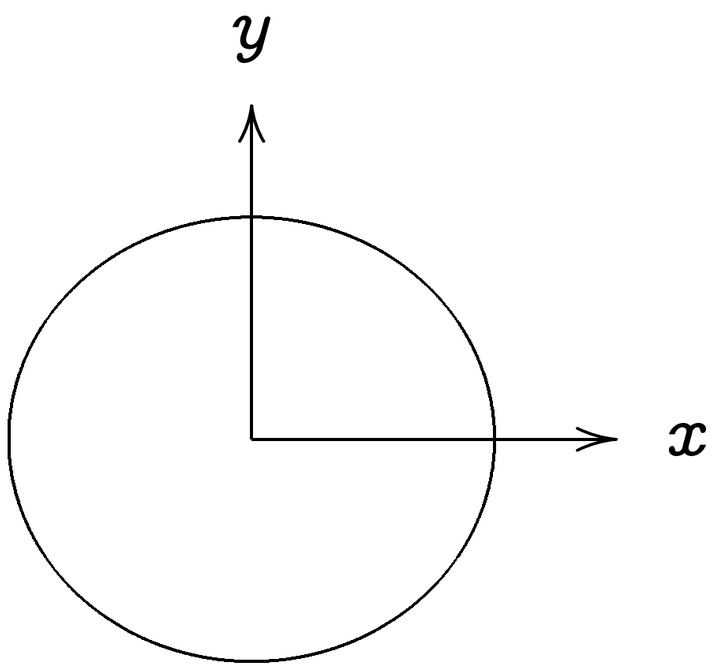

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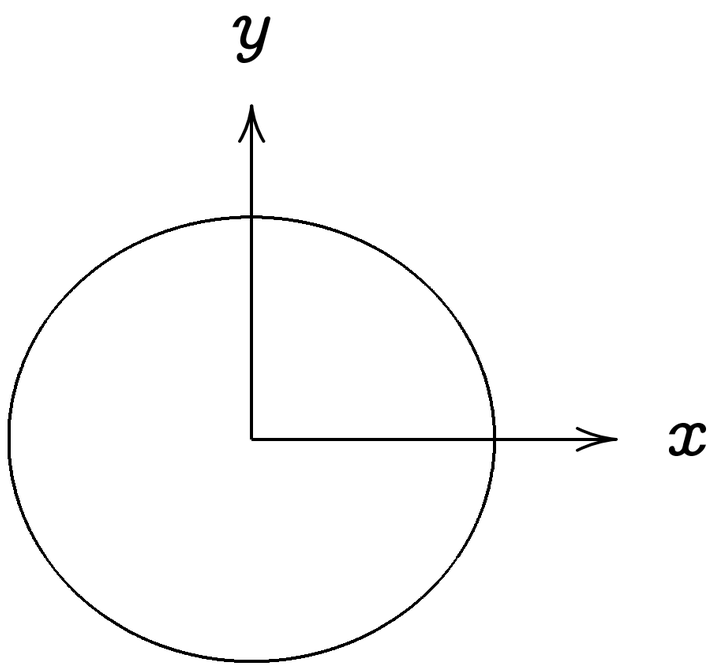

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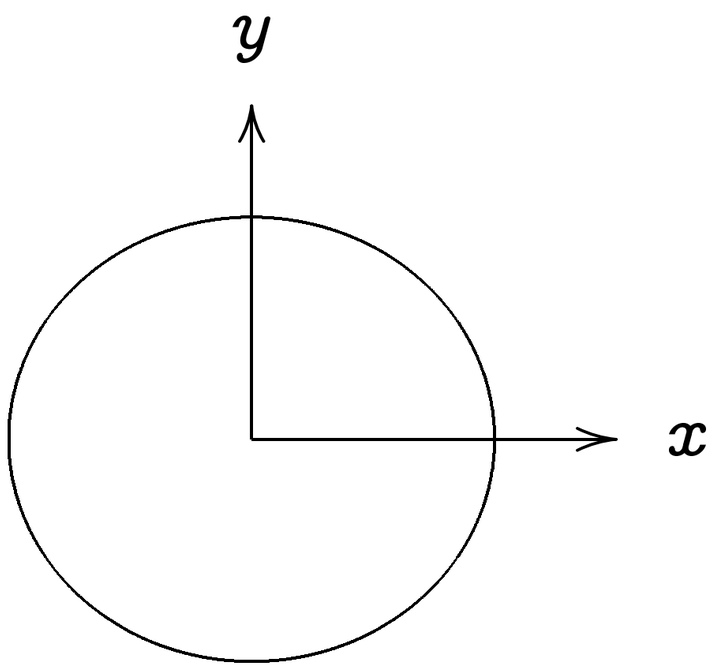

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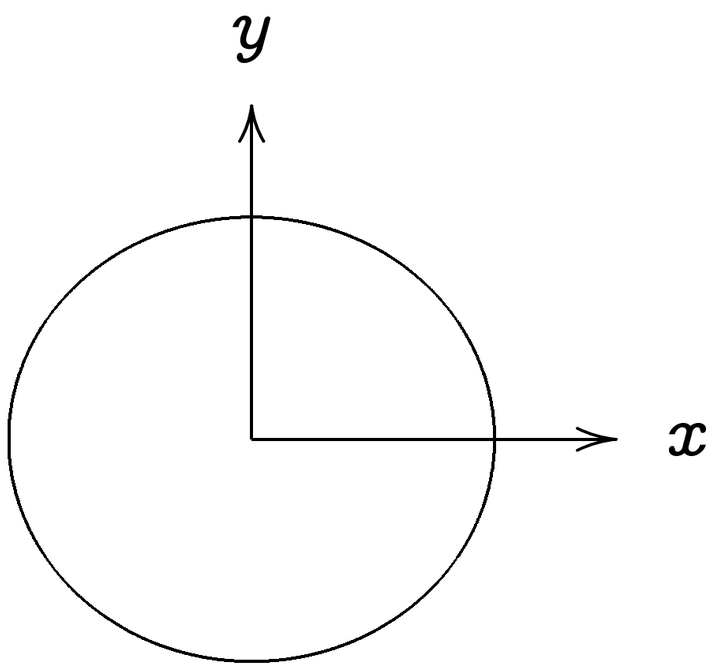

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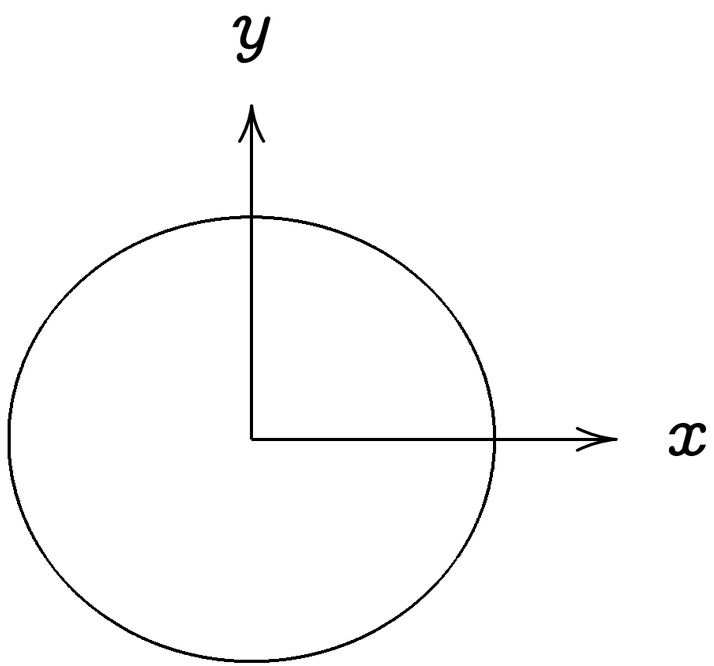

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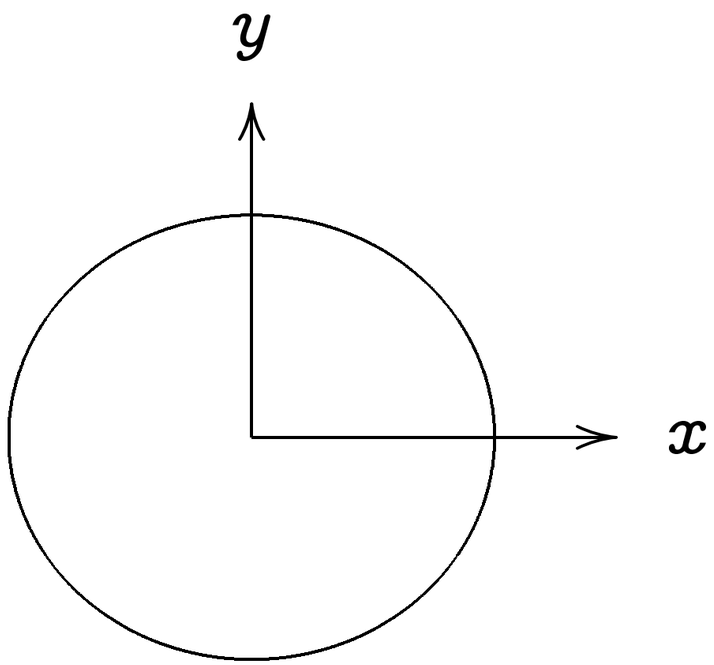

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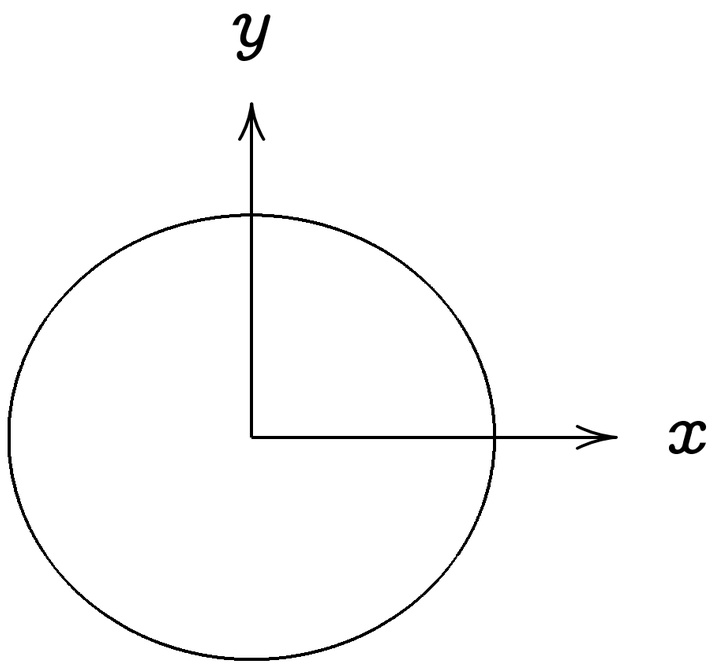

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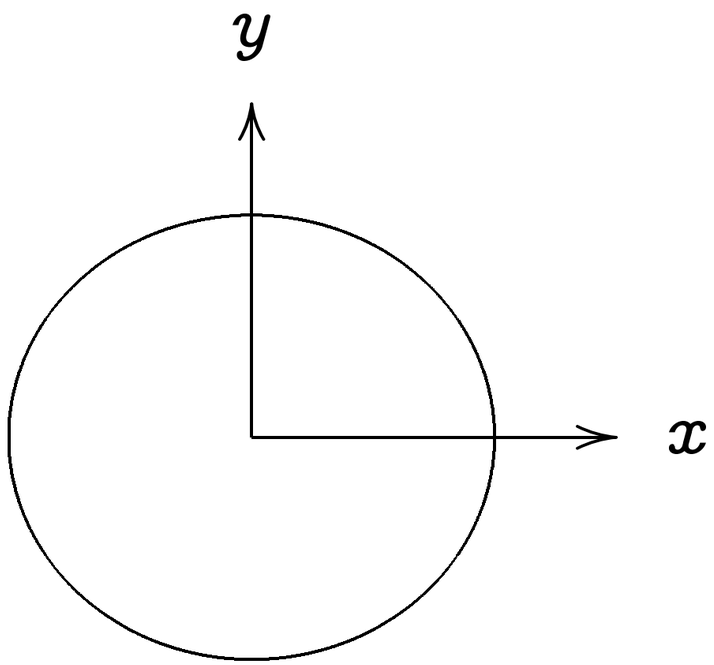

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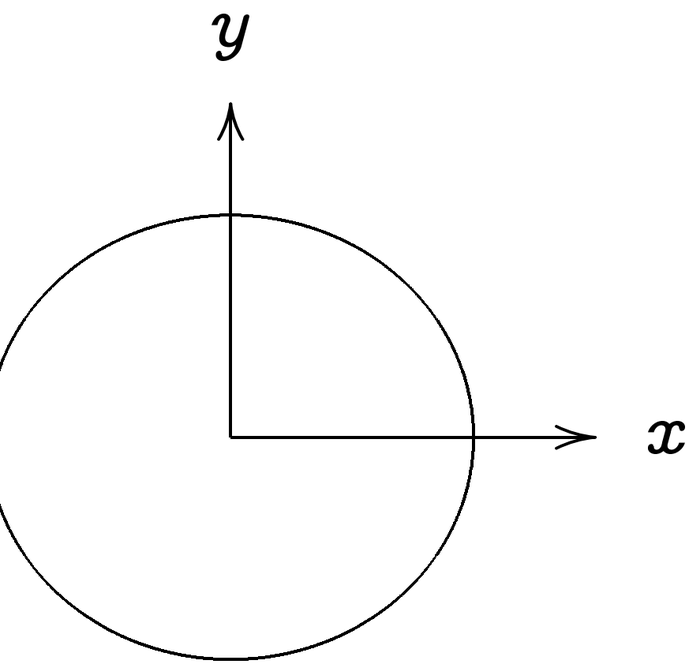

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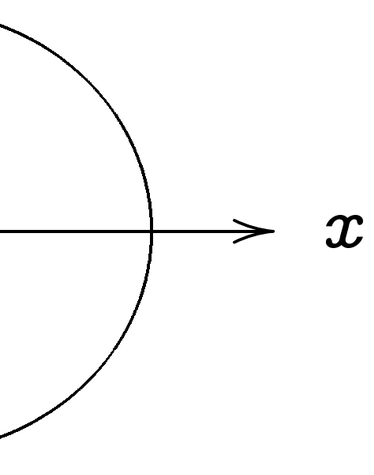

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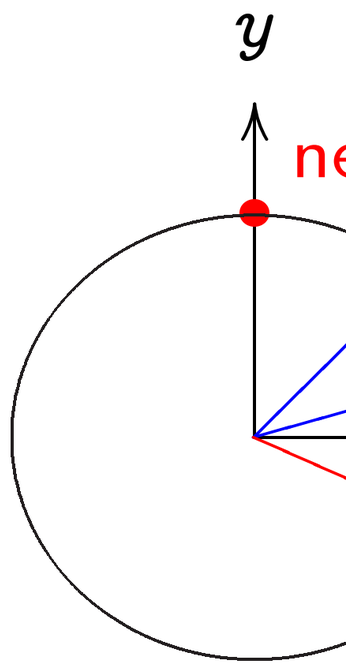

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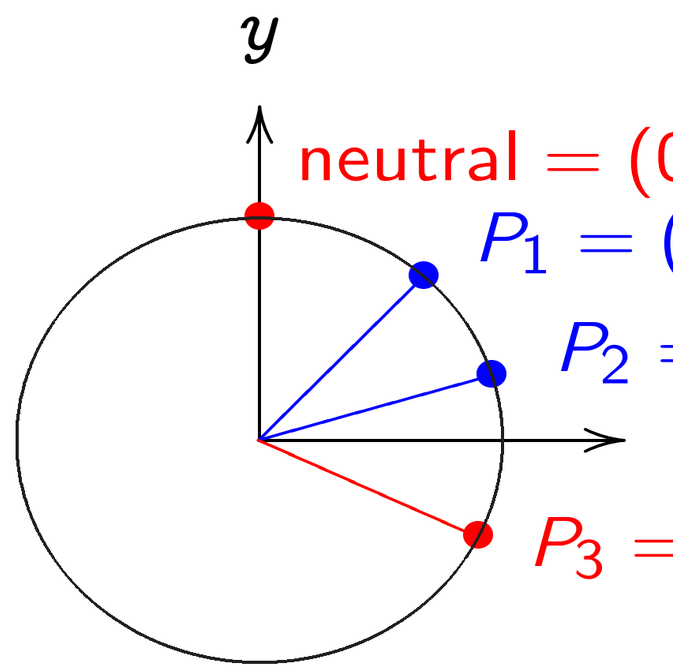

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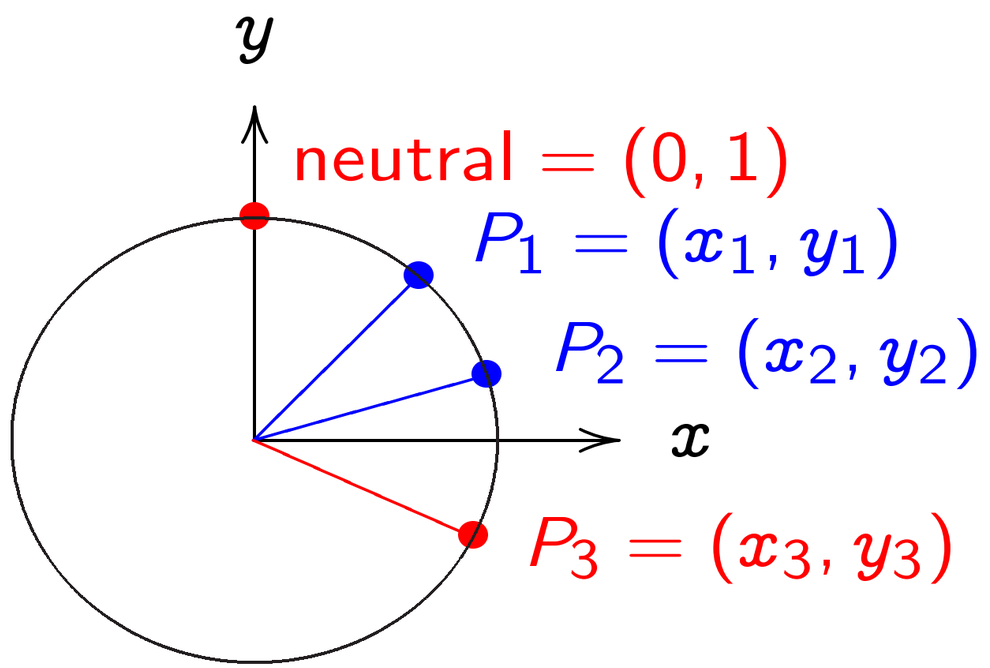

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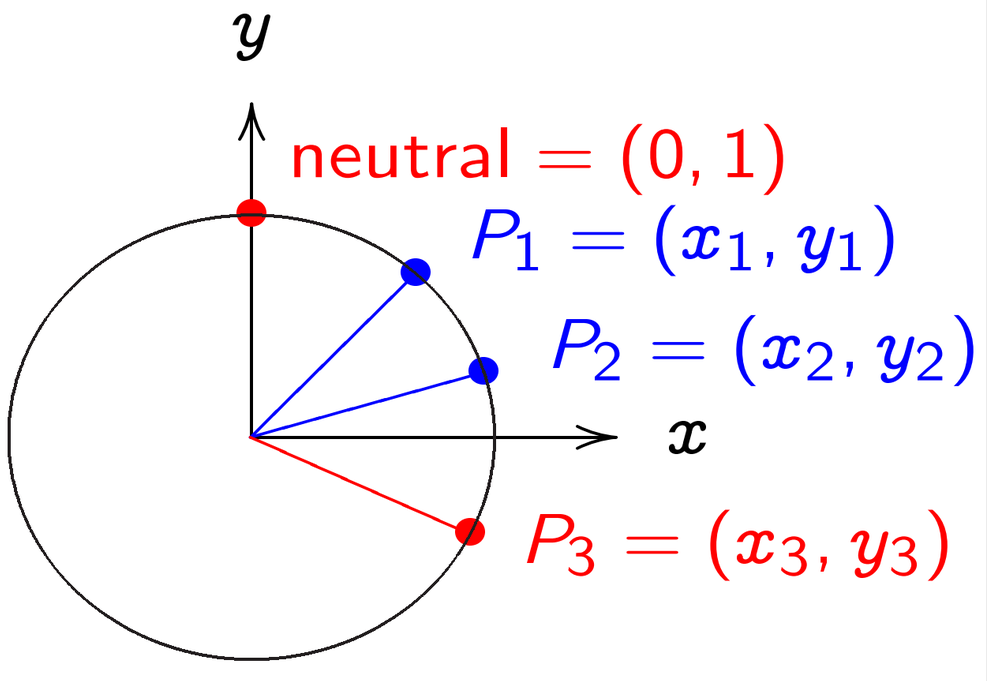

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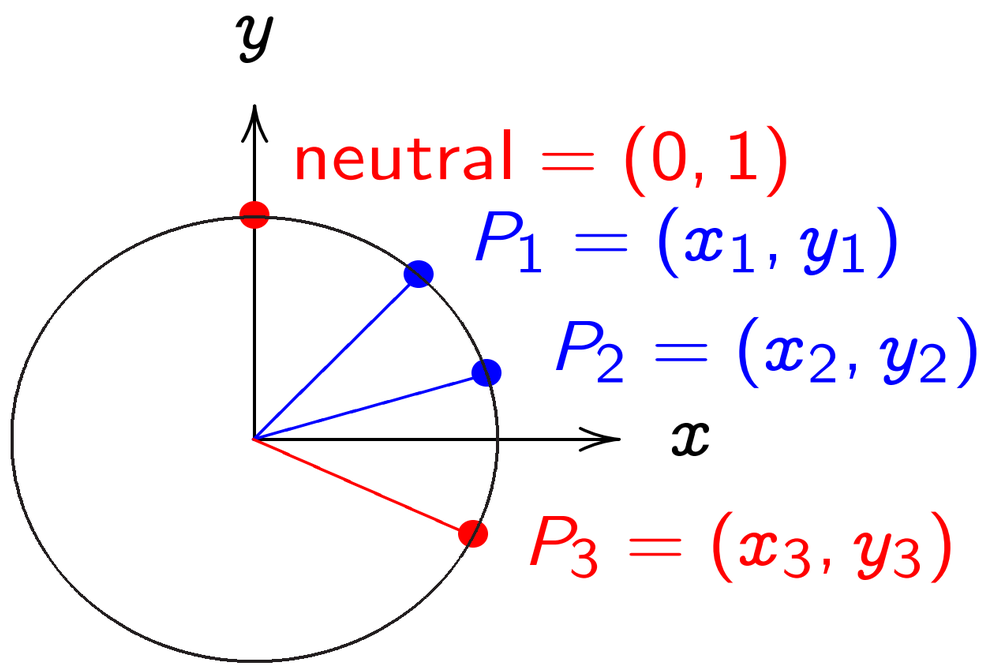

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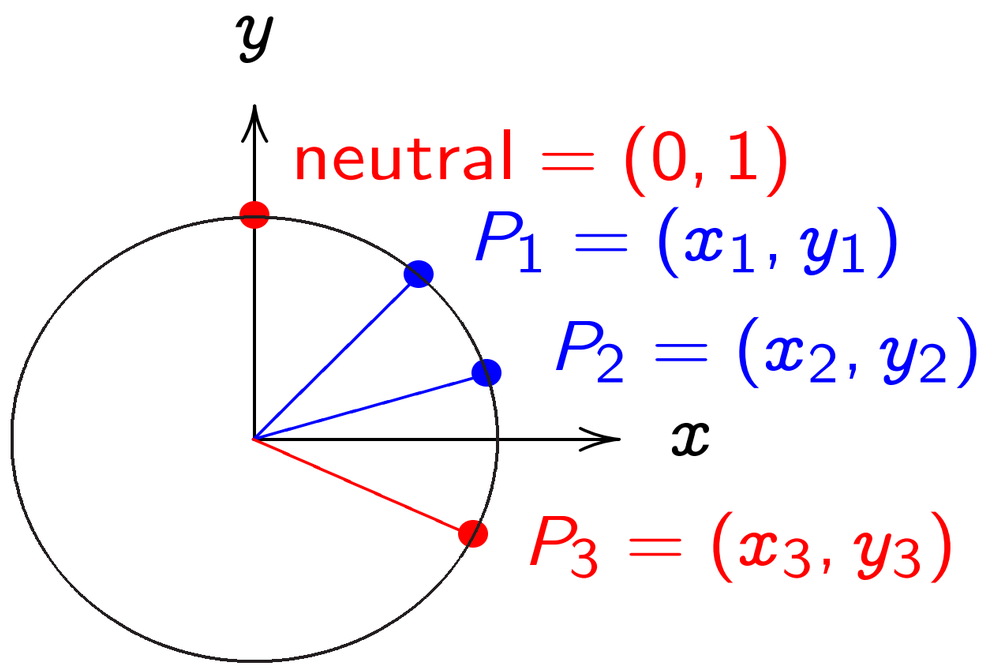

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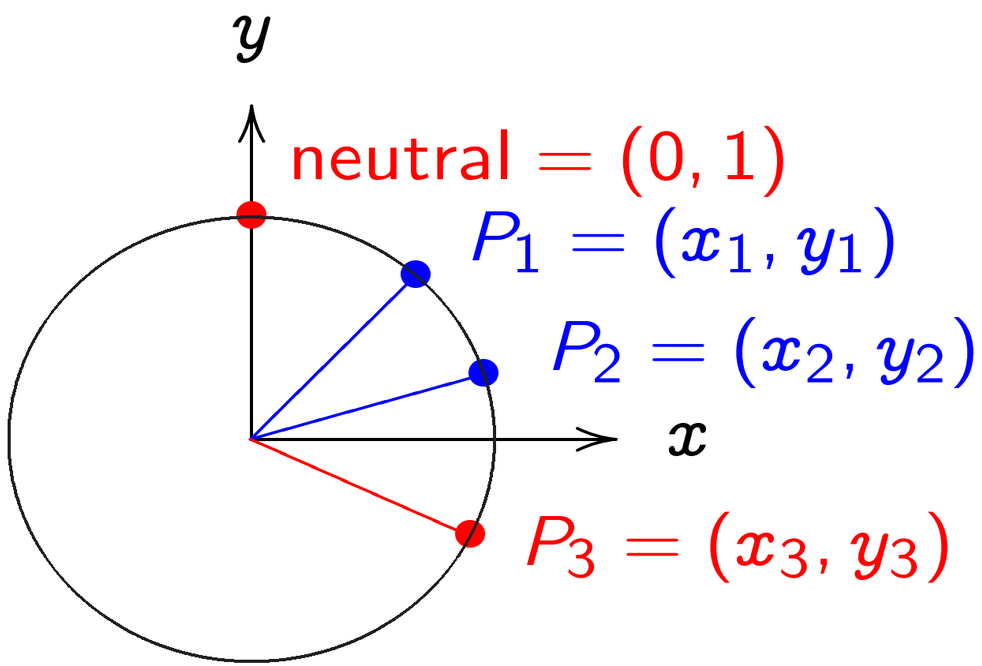

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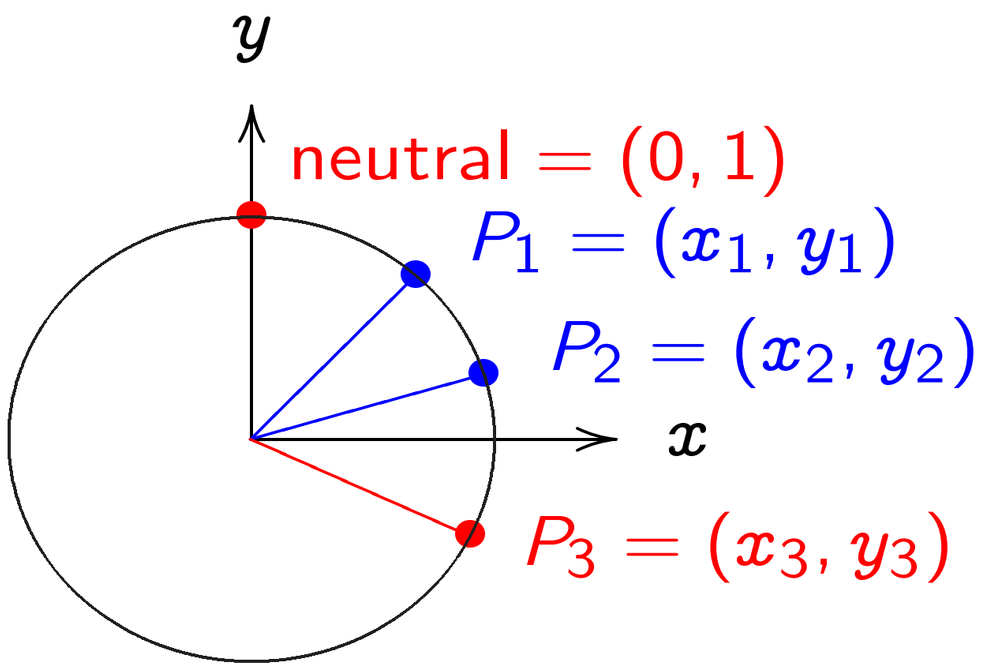

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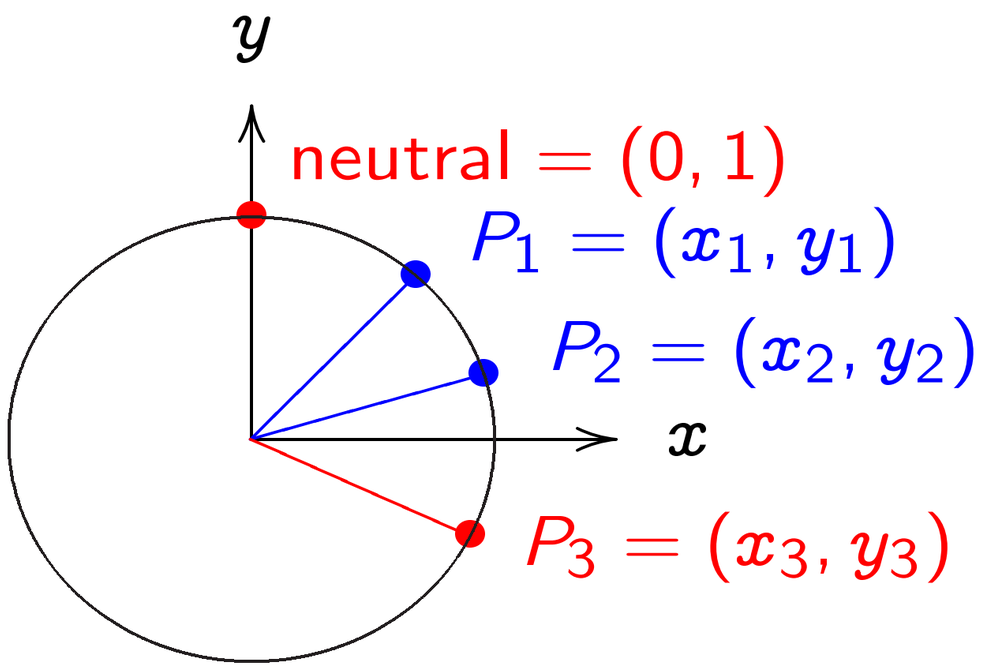

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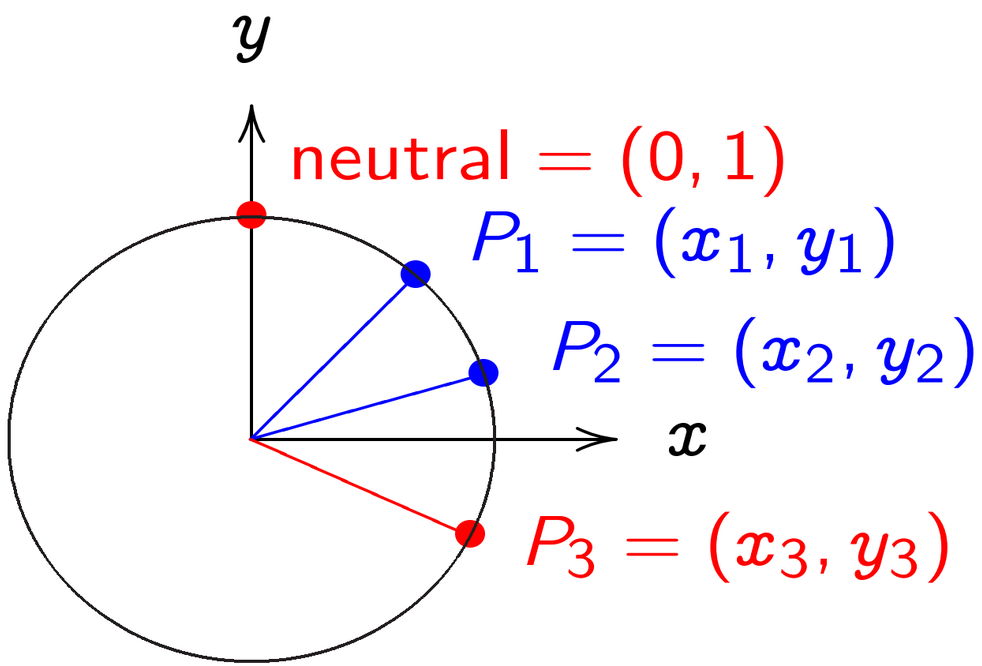

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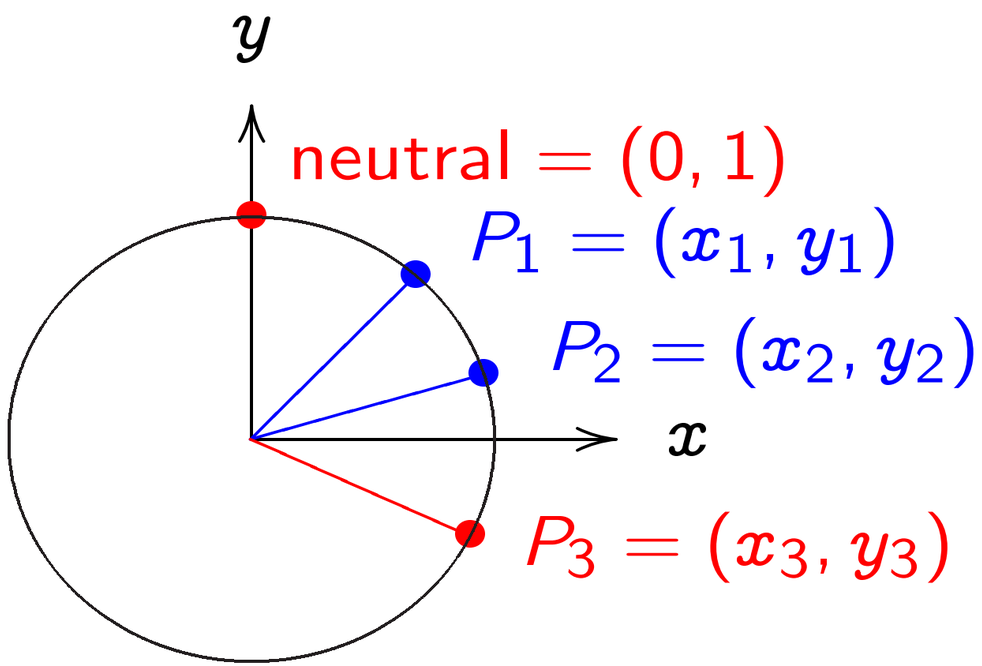

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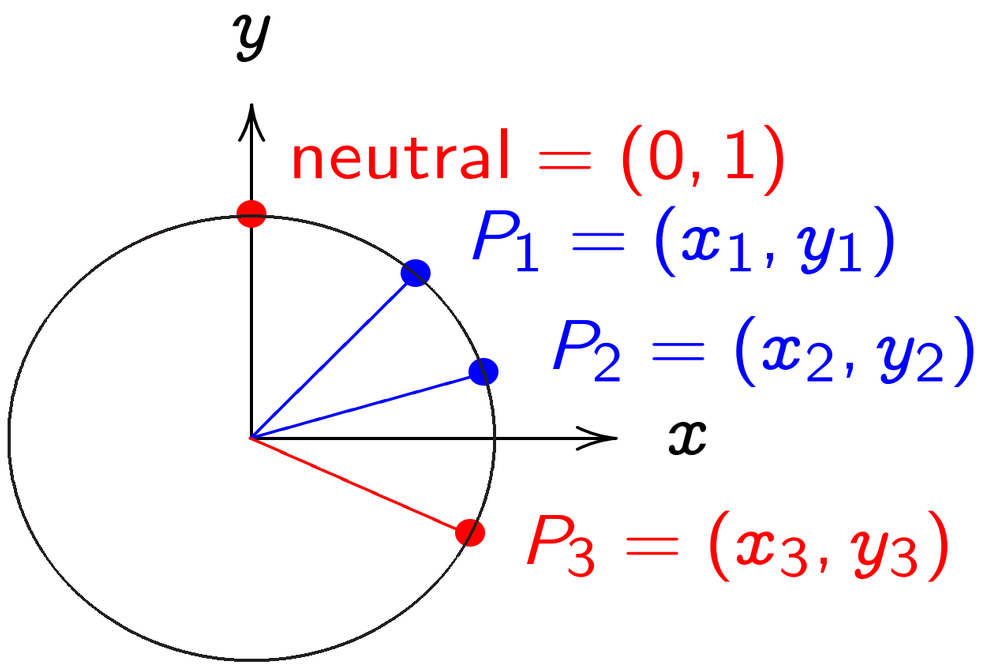

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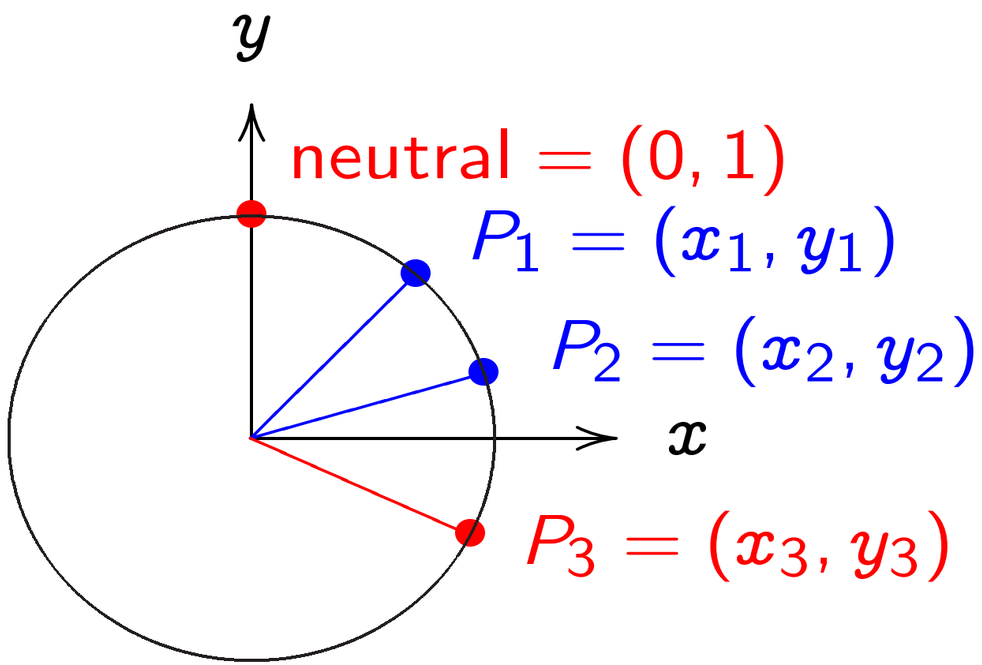

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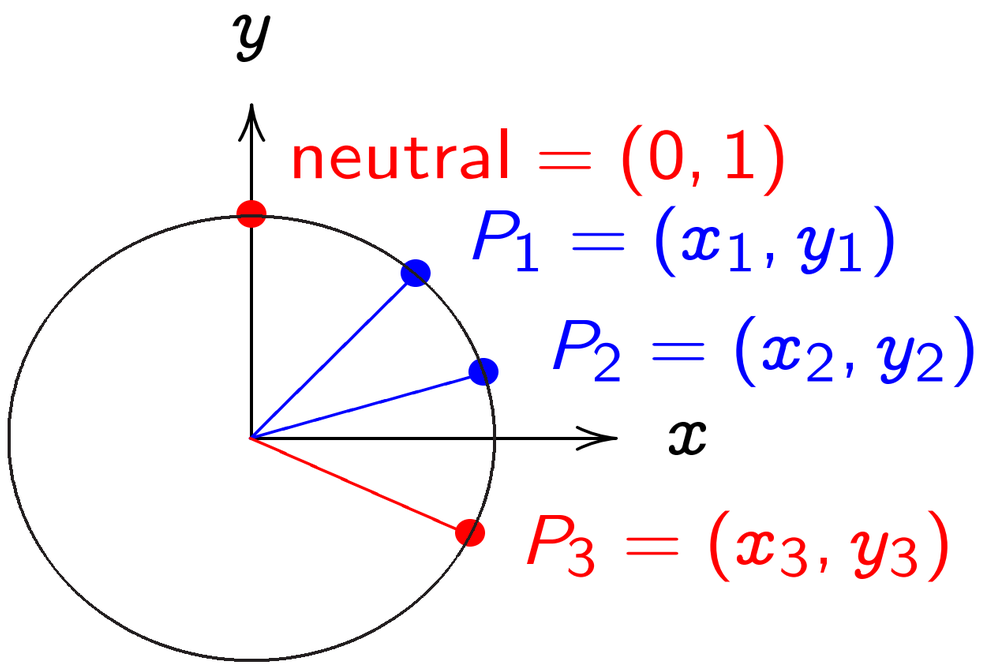

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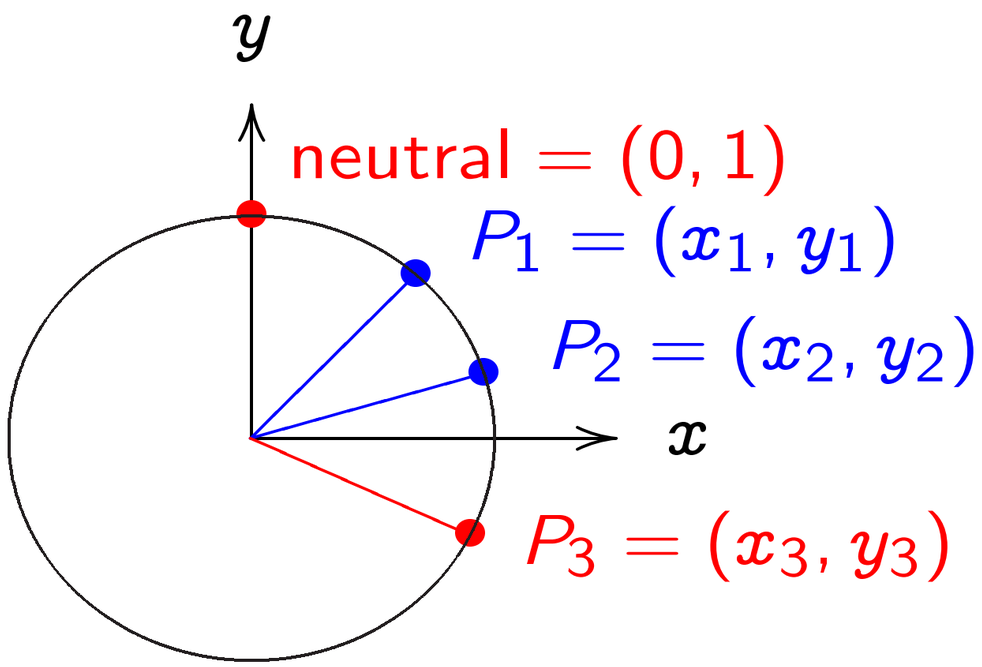

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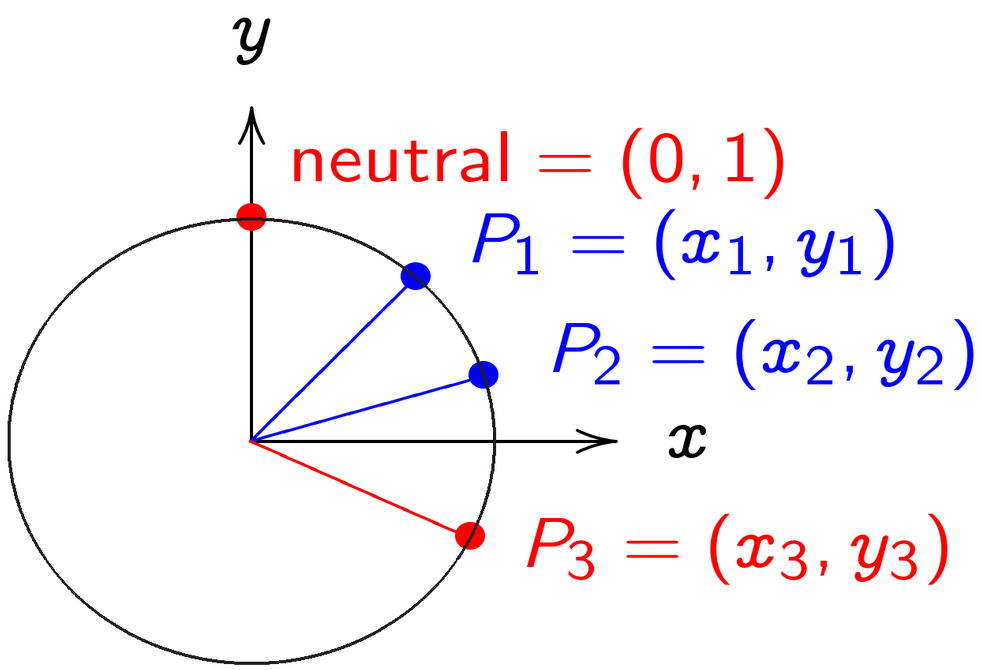

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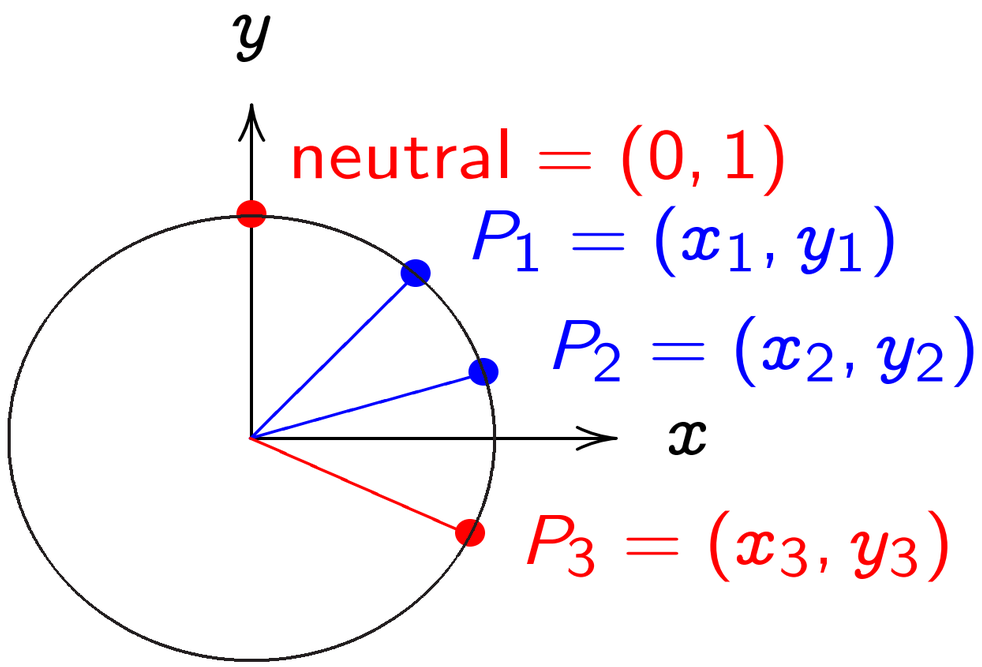

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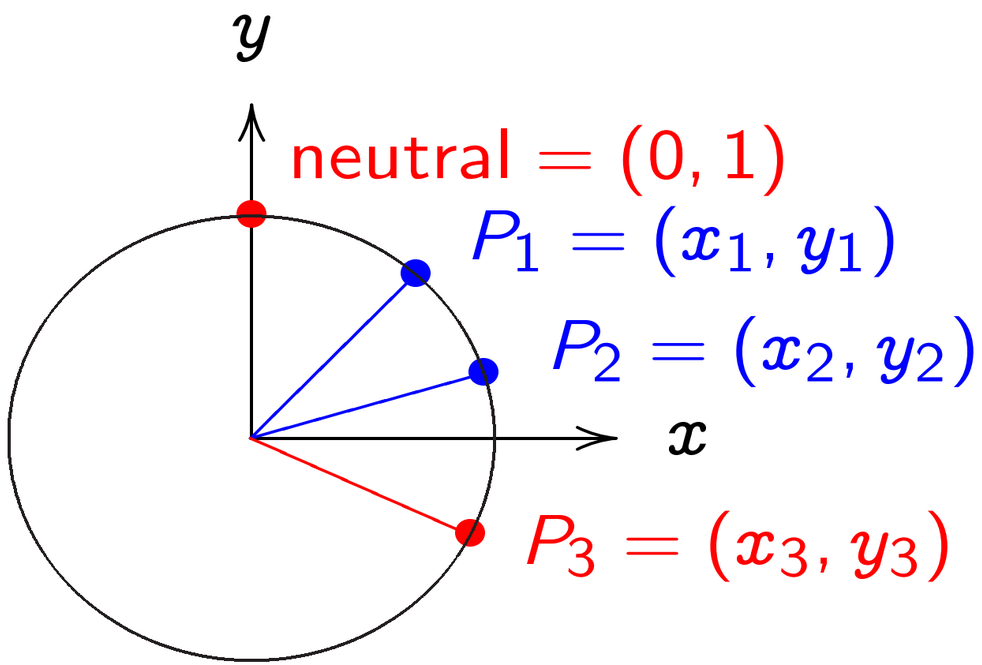

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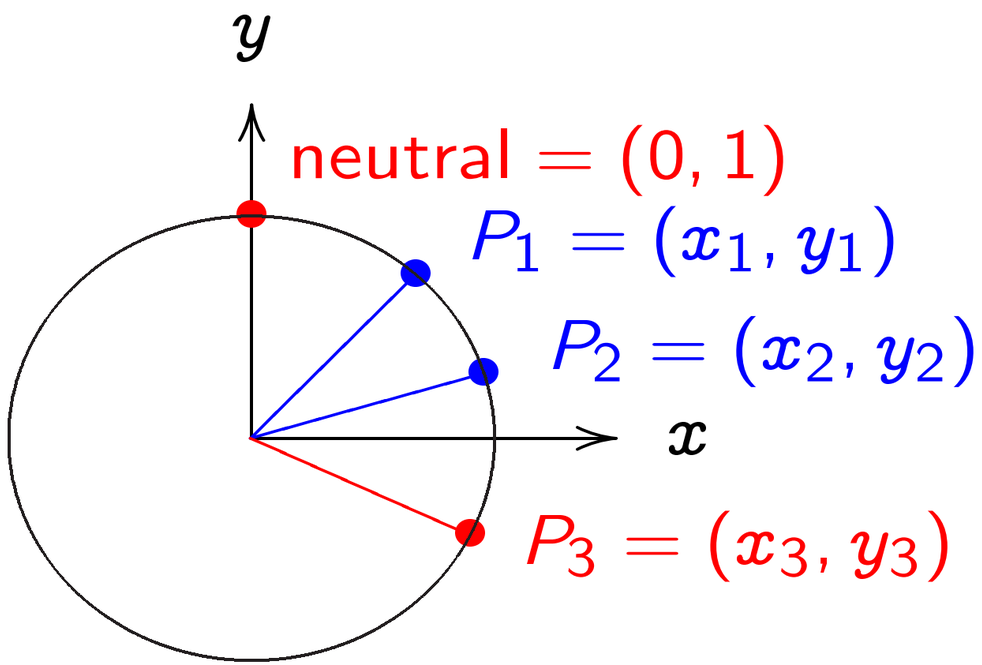

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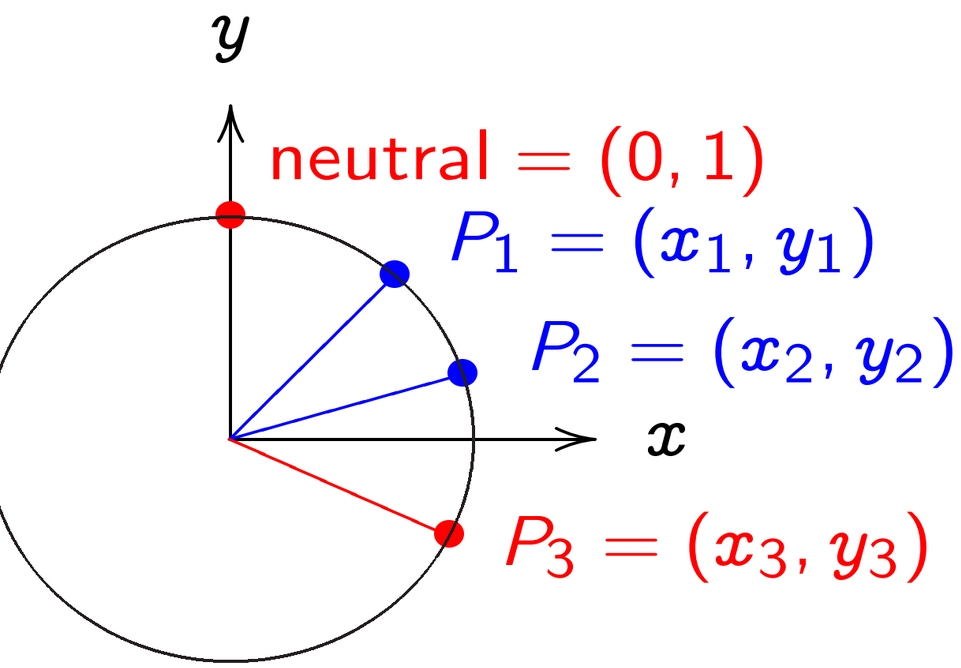

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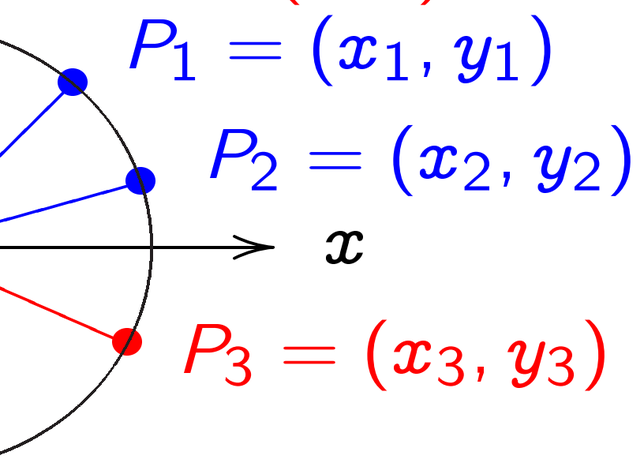

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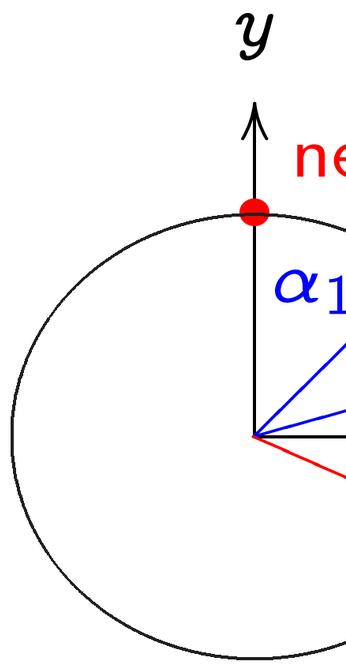

$x^2 + y^2 = 1$, par

$x = \sin\alpha, \quad y =$

$(\sin(\alpha_1 + \alpha_2), \cos$

$(\sin\alpha_1 \cos\alpha_2 +$

$\cos\alpha_1 \cos\alpha_2 -$

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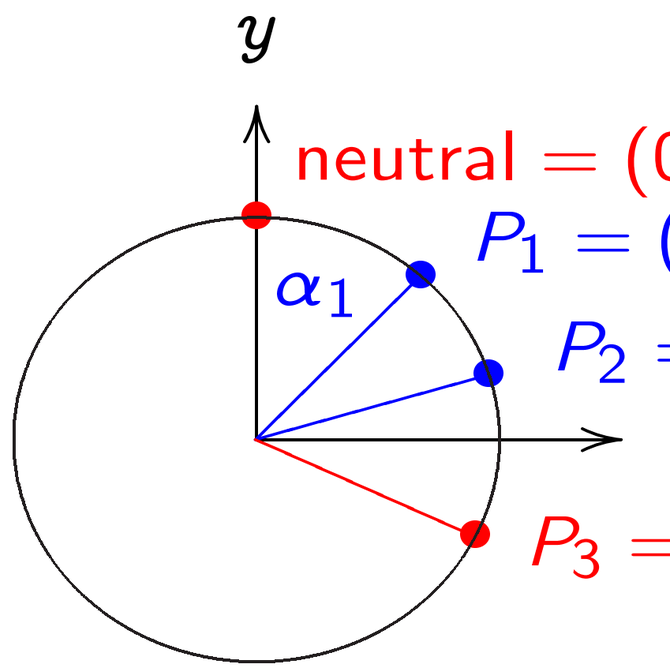

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$(\sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha$

$\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha$

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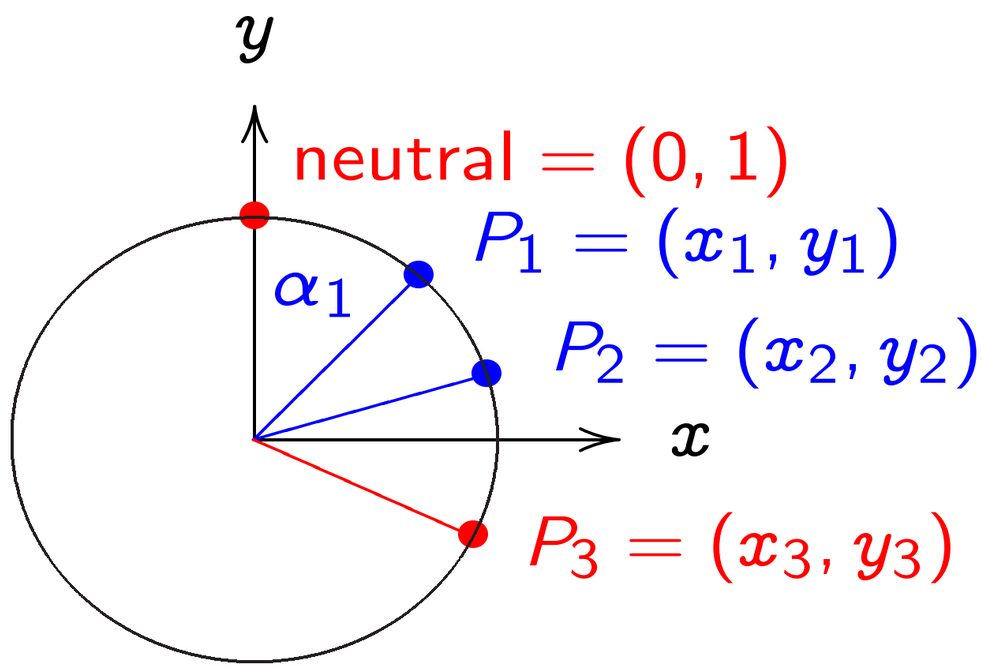

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$x = \sin\alpha, \quad y = \cos\alpha$. Recall
$(\sin(\alpha_1 + \alpha_2), \cos(\alpha_1 + \alpha_2)) =$
$(\sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2,$
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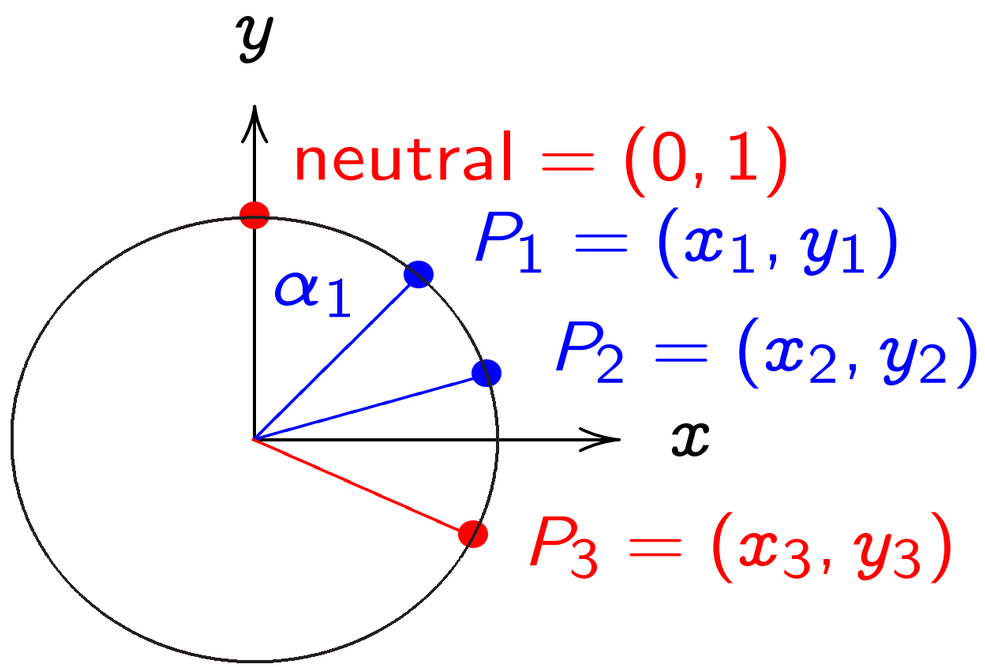
How to remember addition law:



$x^2 + y^2 = 1$, parametrized by

$x = \sin\alpha, \quad y = \cos\alpha$. Recall

$(\sin(\alpha_1 + \alpha_2), \cos(\alpha_1 + \alpha_2)) =$

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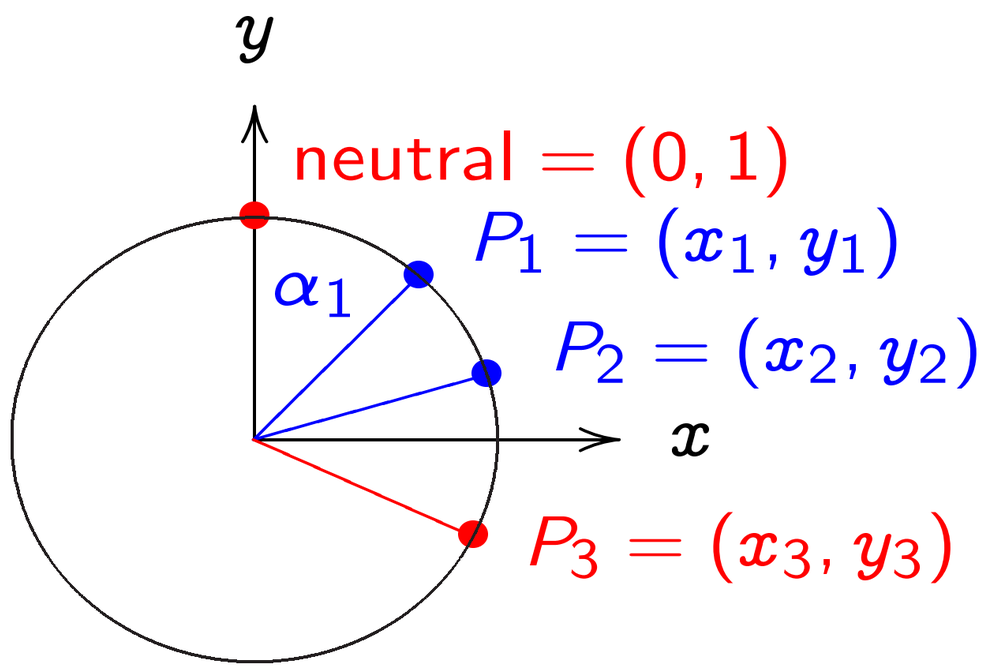

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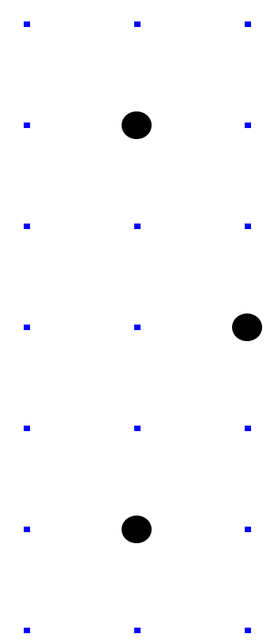
Clocks over finite



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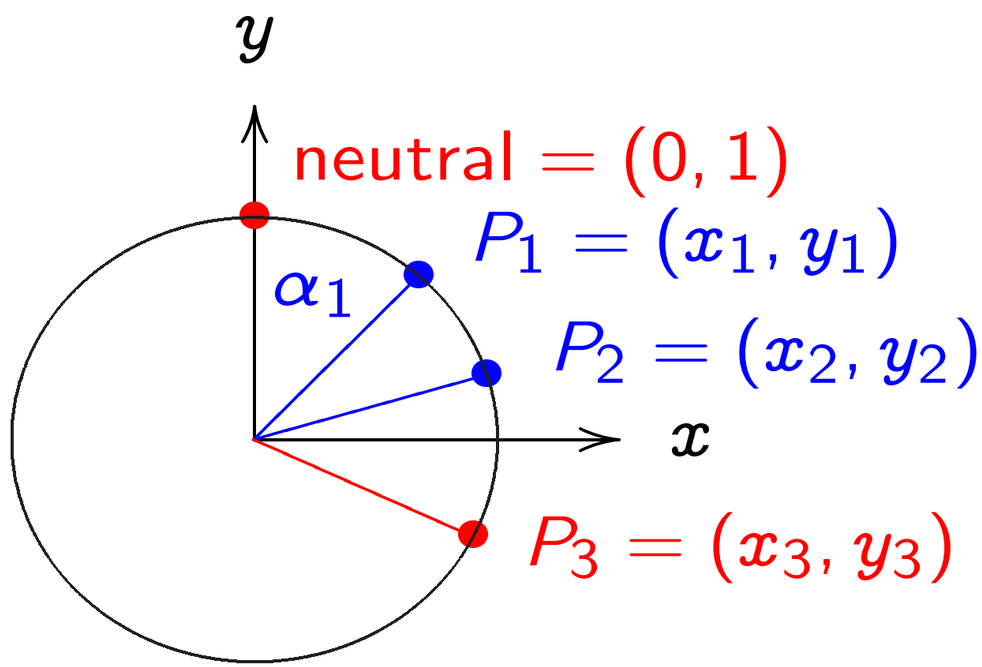

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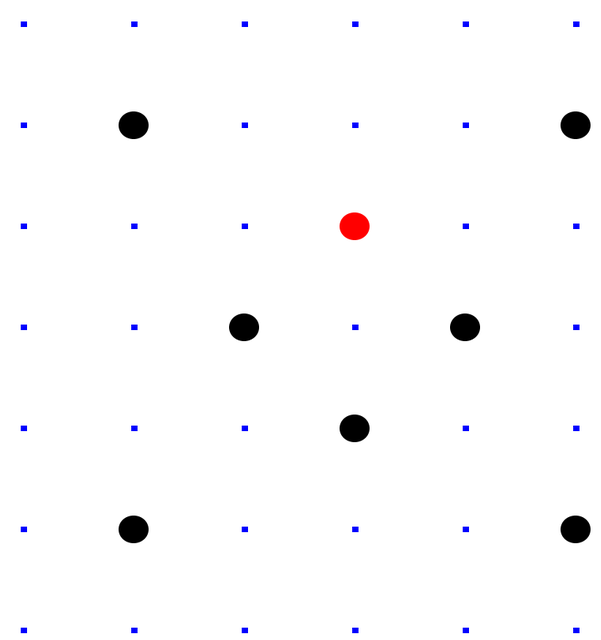
Clocks over finite fields



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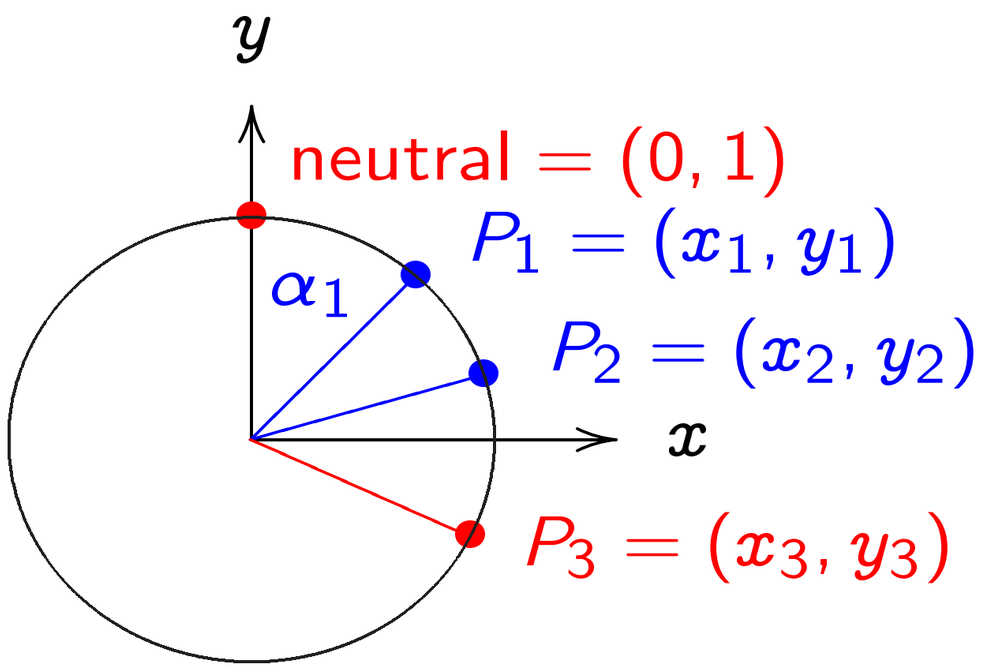

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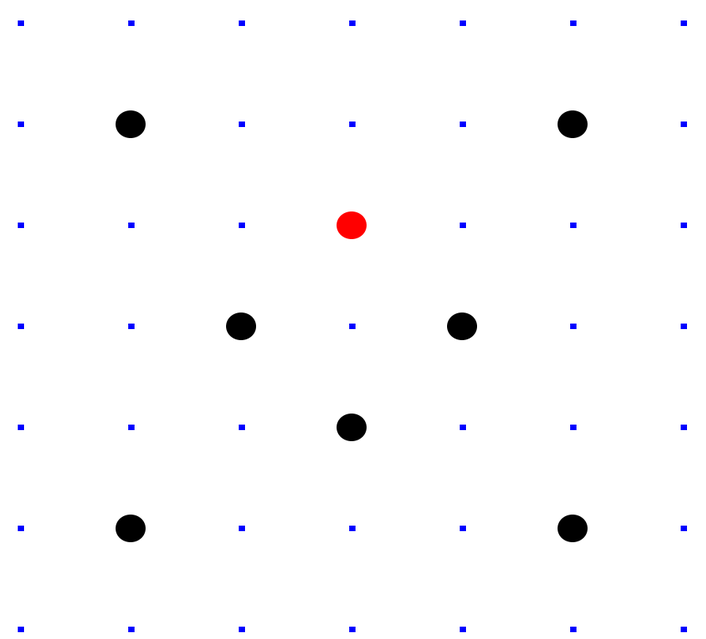

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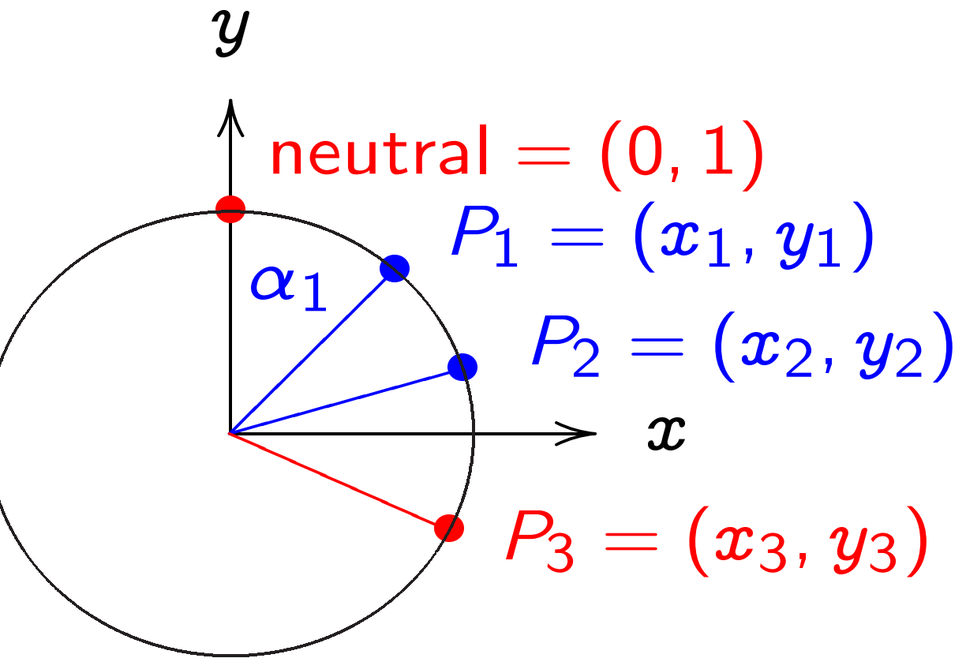

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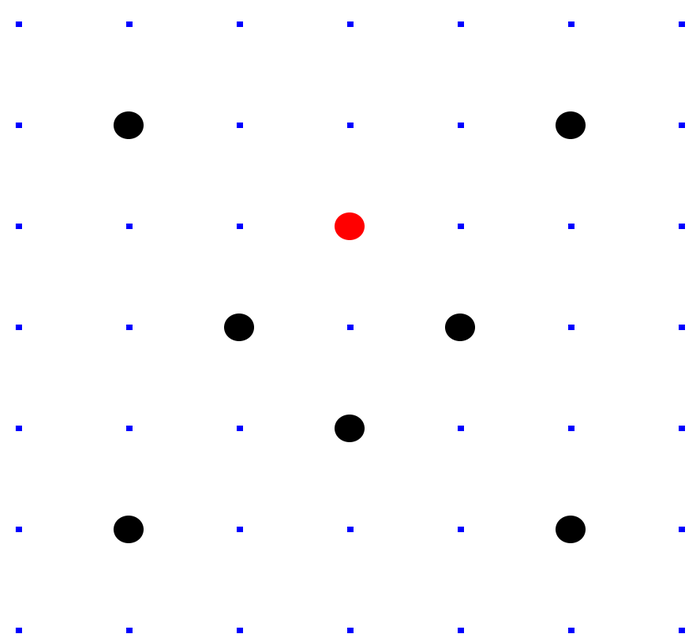

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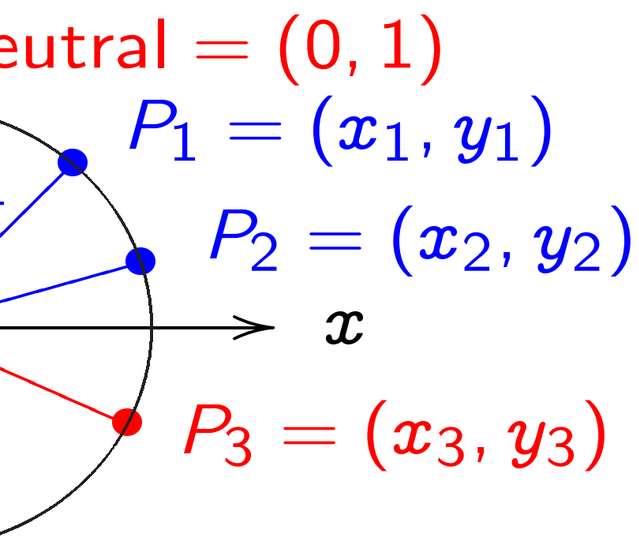

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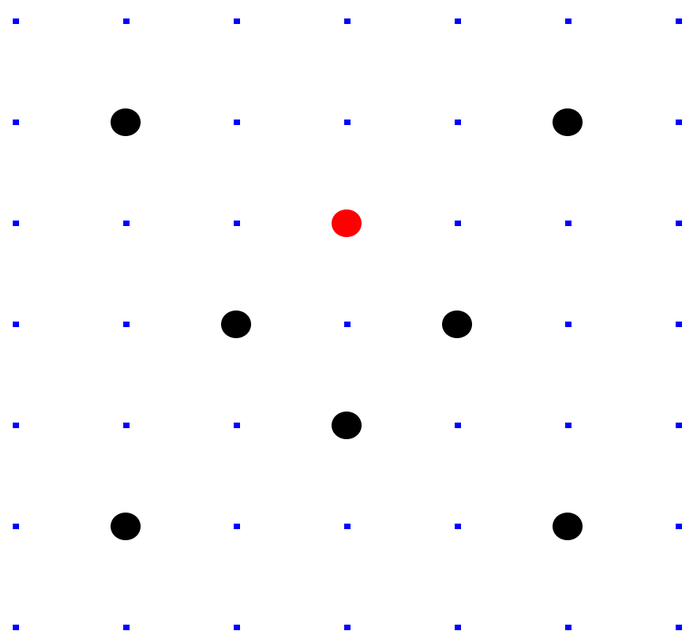

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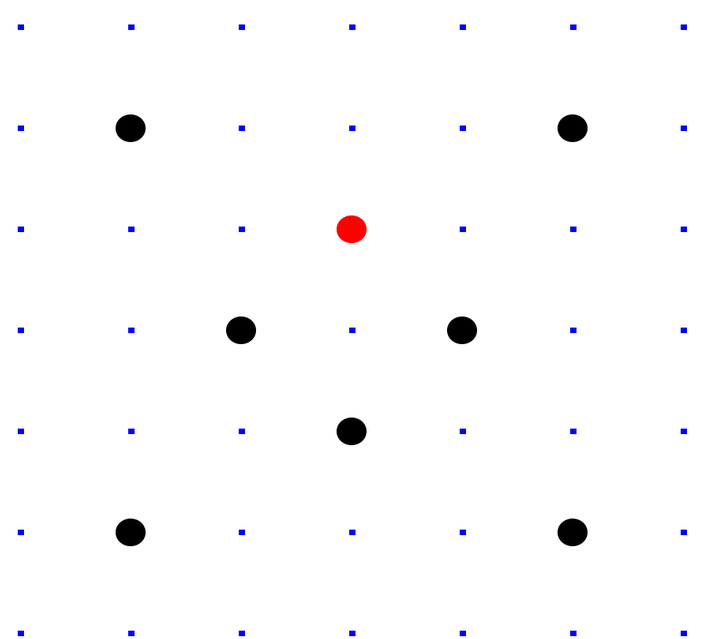

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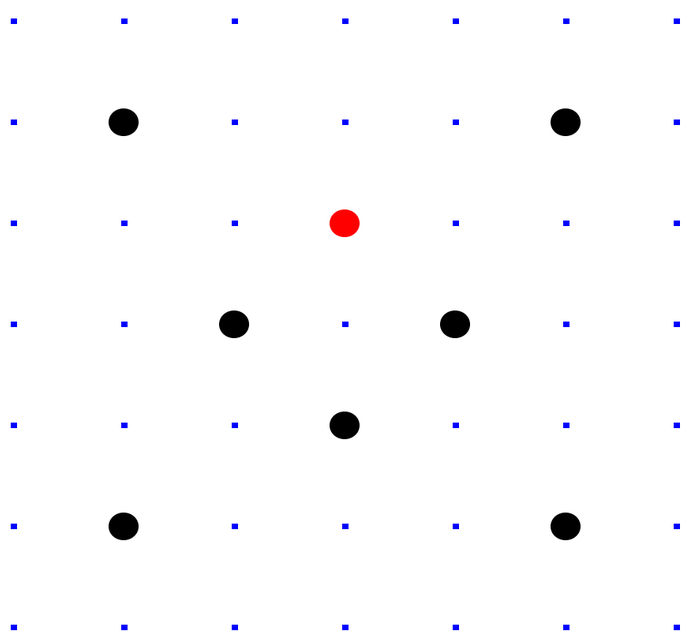

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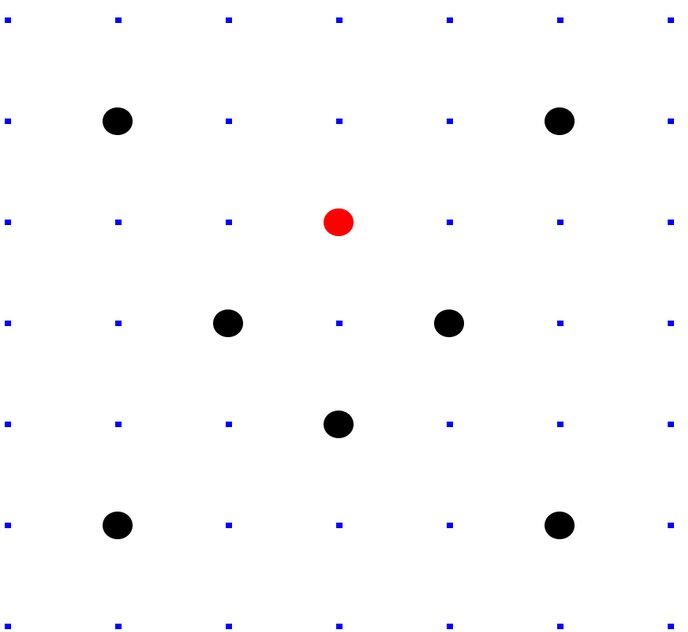

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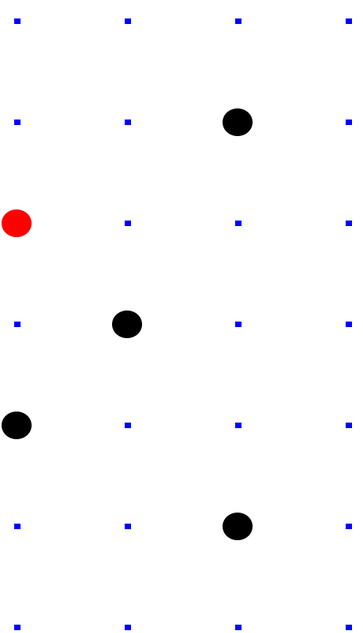

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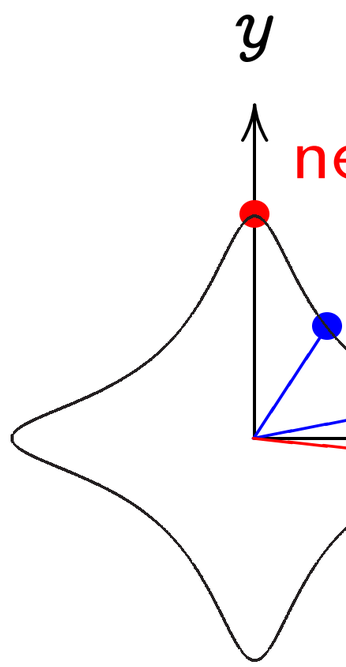

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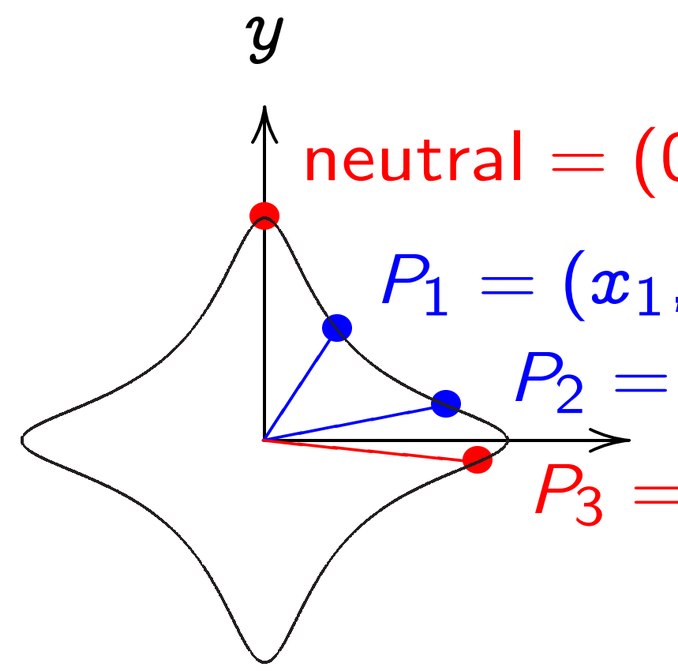

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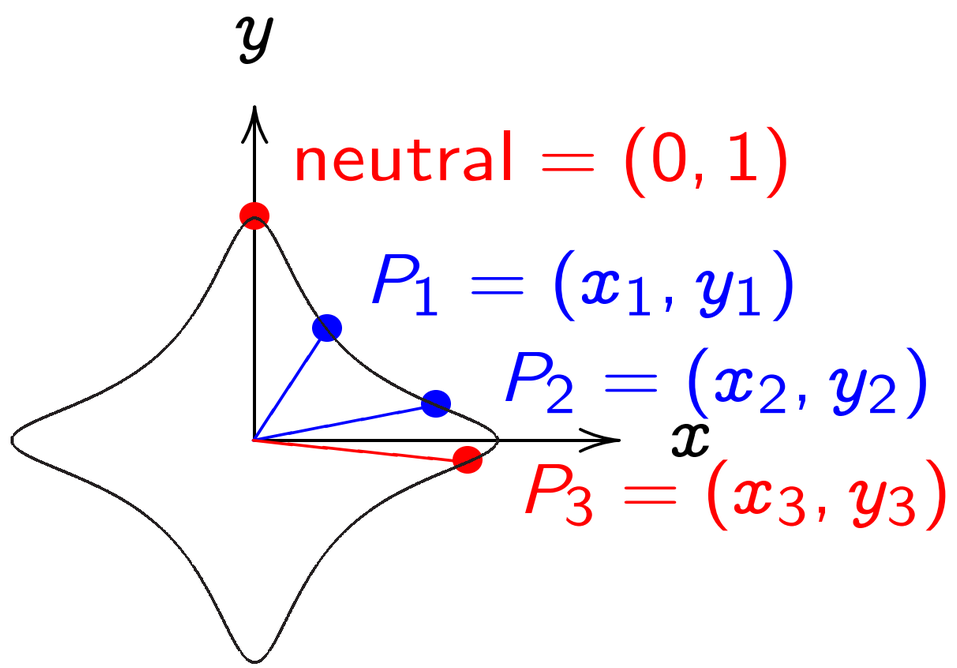

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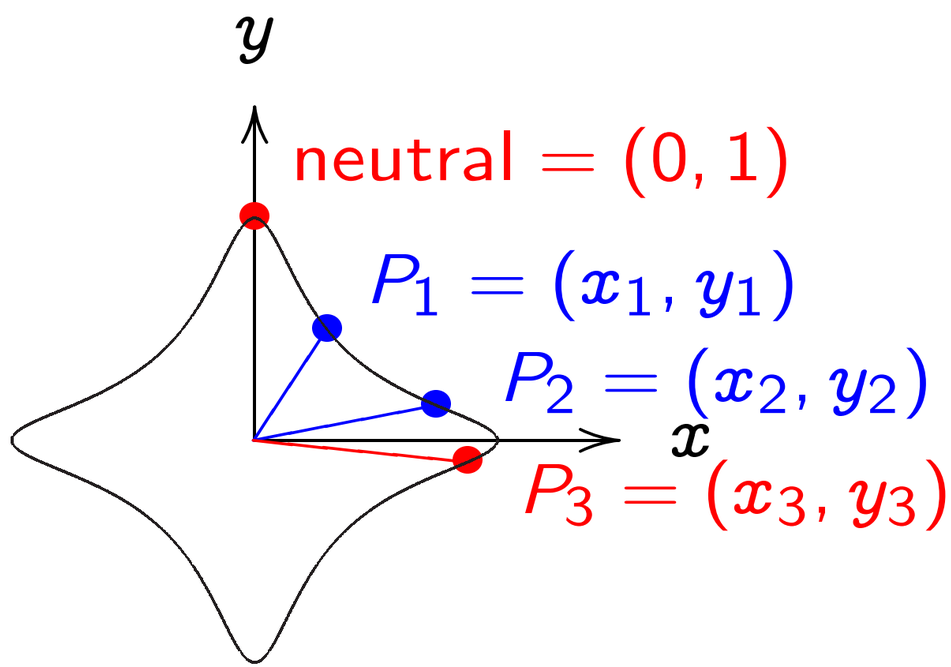

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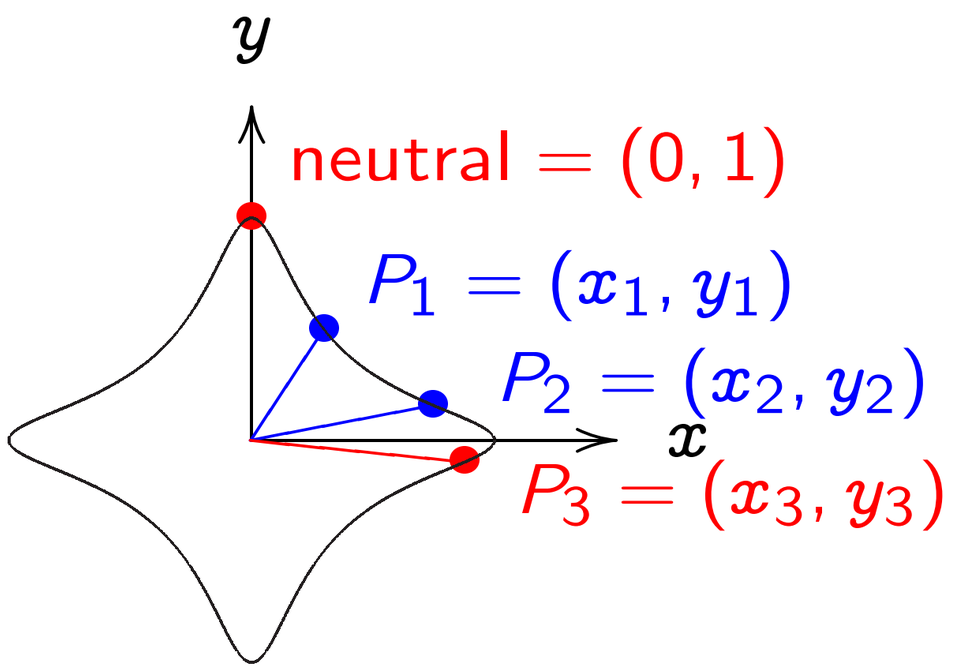

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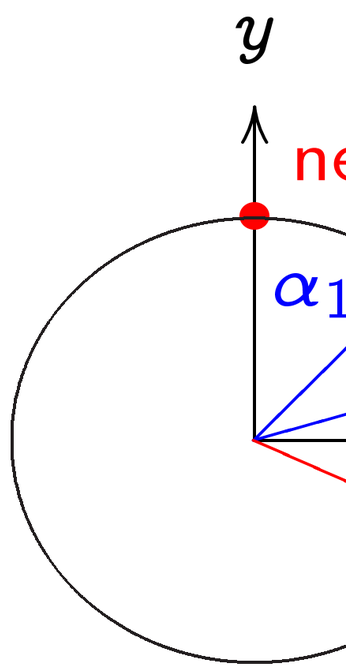

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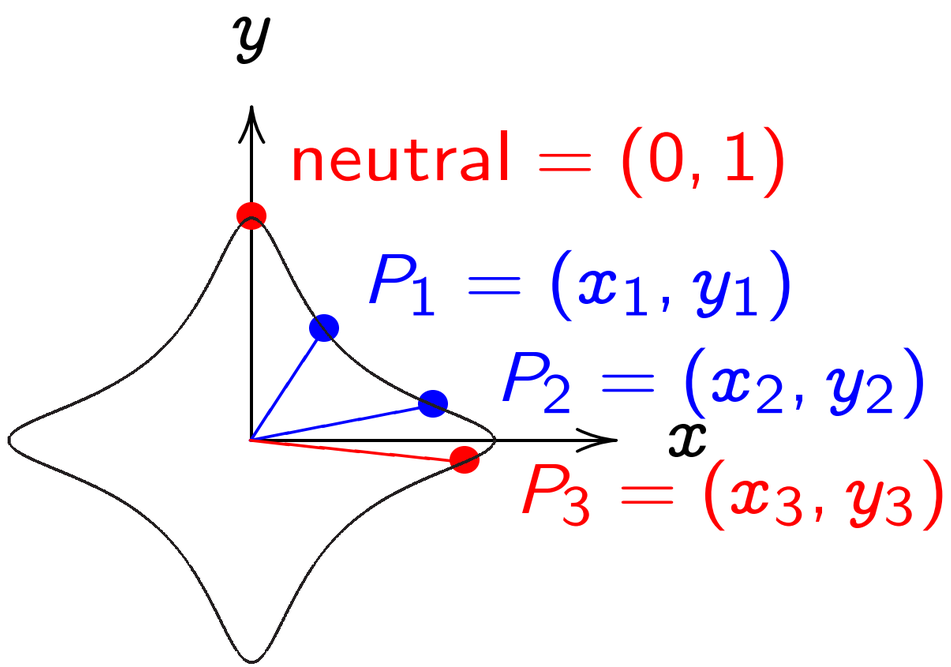

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Sum of (x_1, y_1) and (x_2, y_2) is

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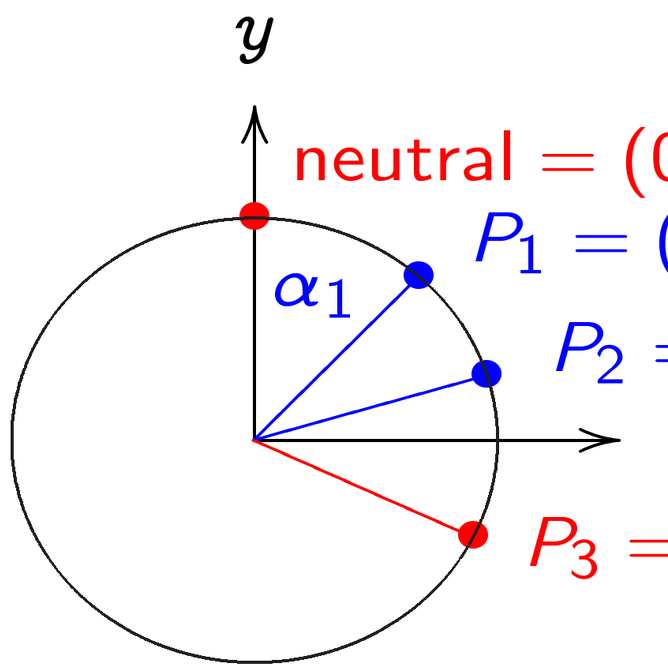

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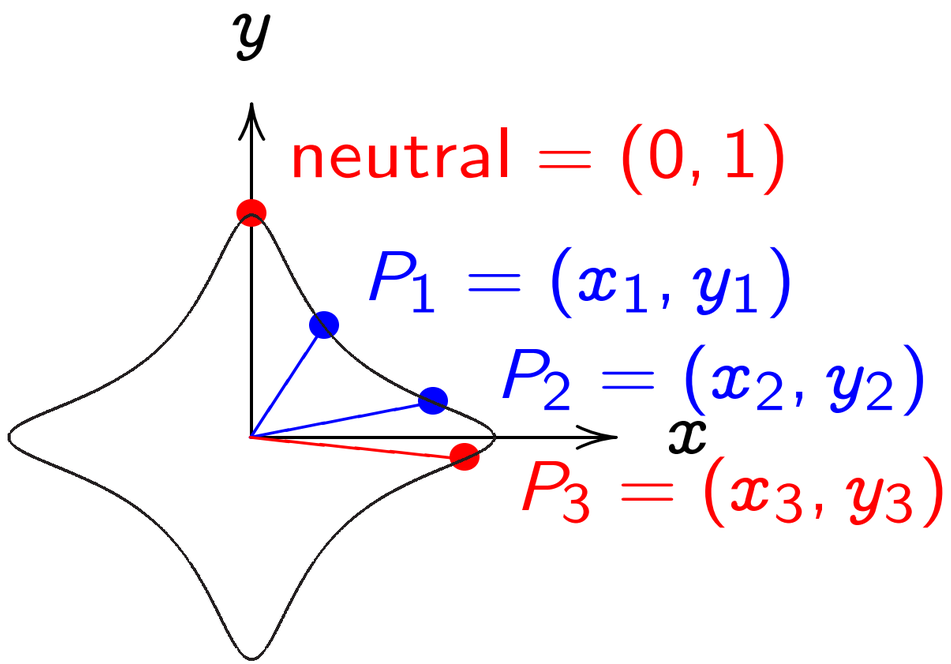
Addition on an Edwards curve



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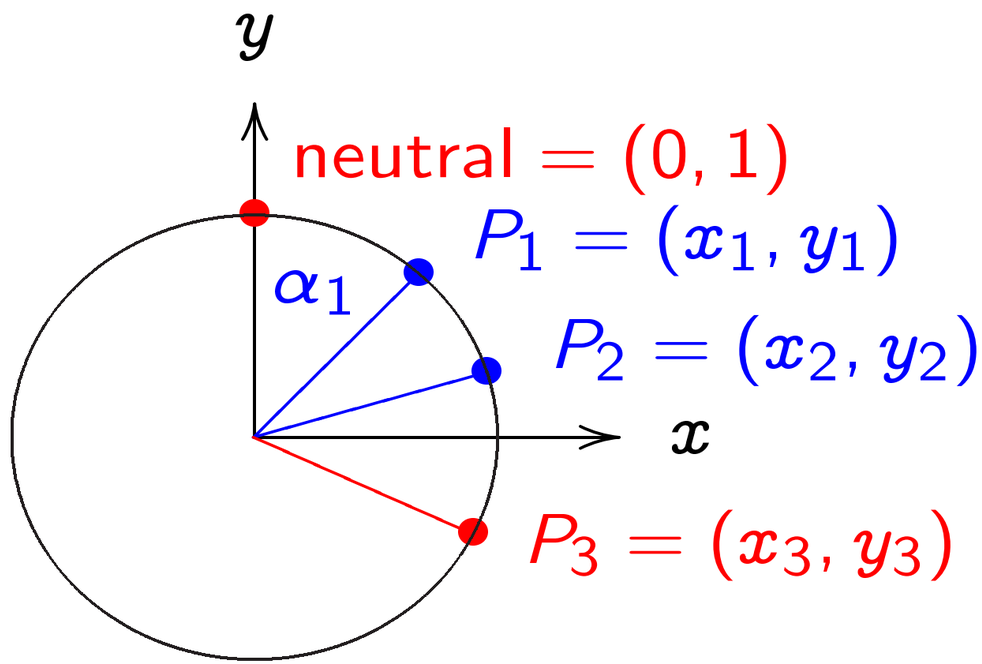

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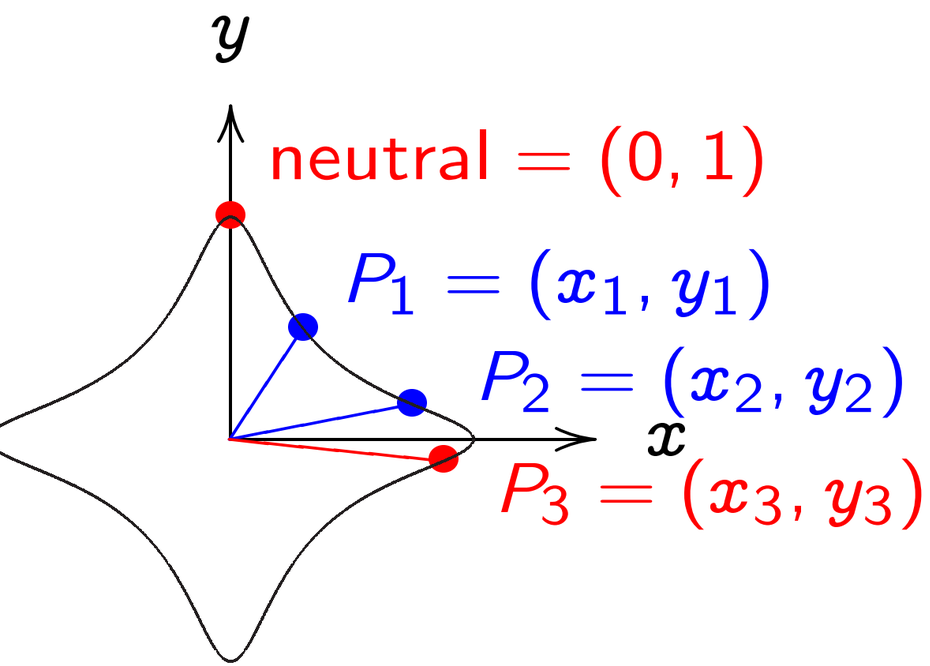

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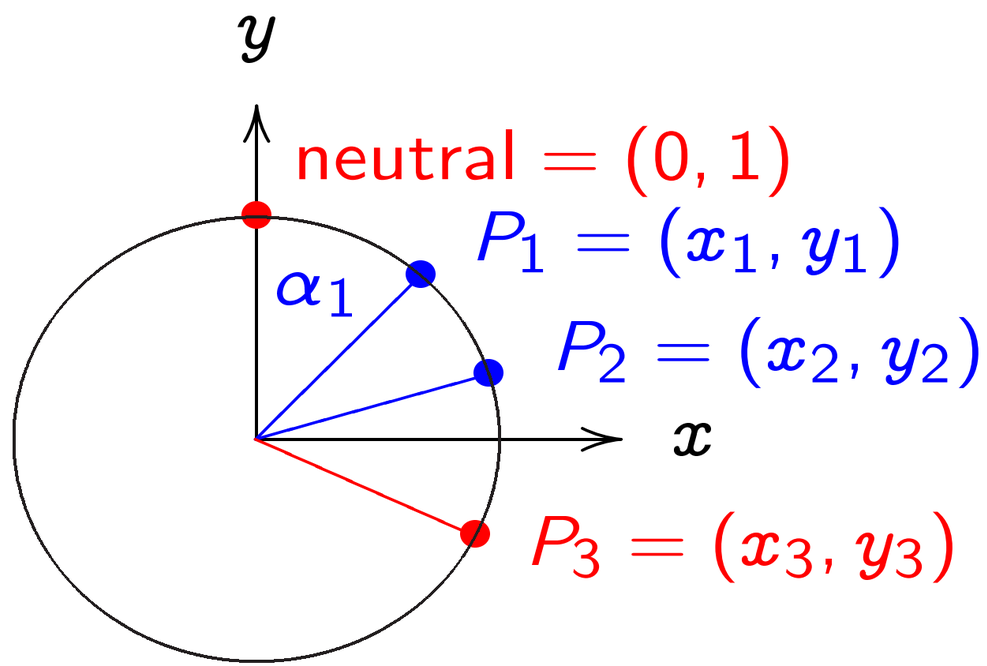

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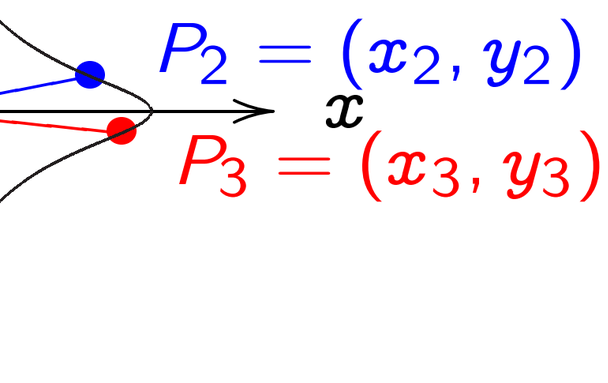

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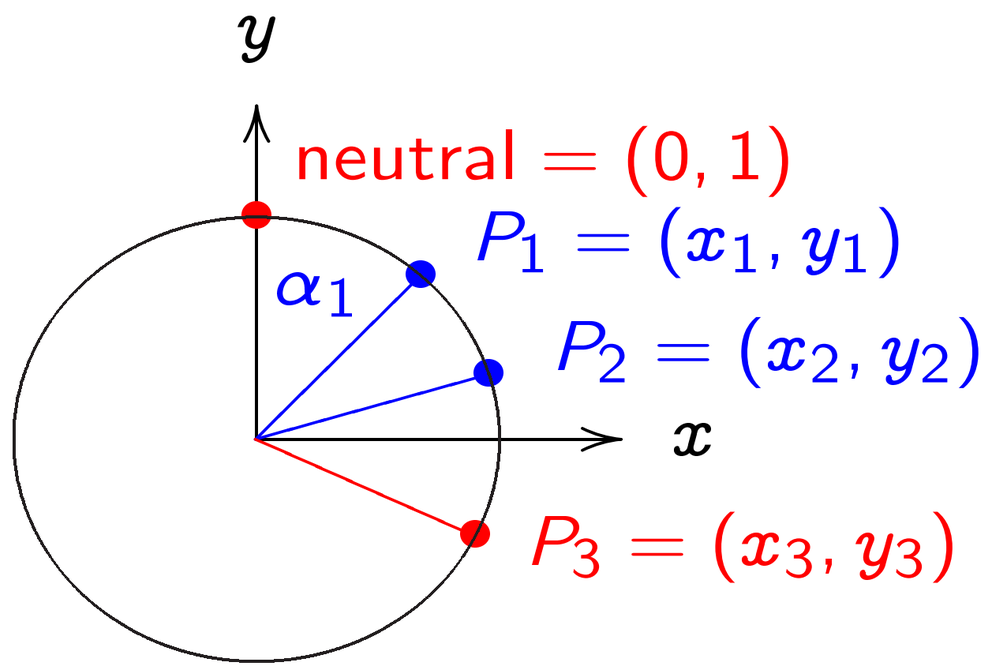

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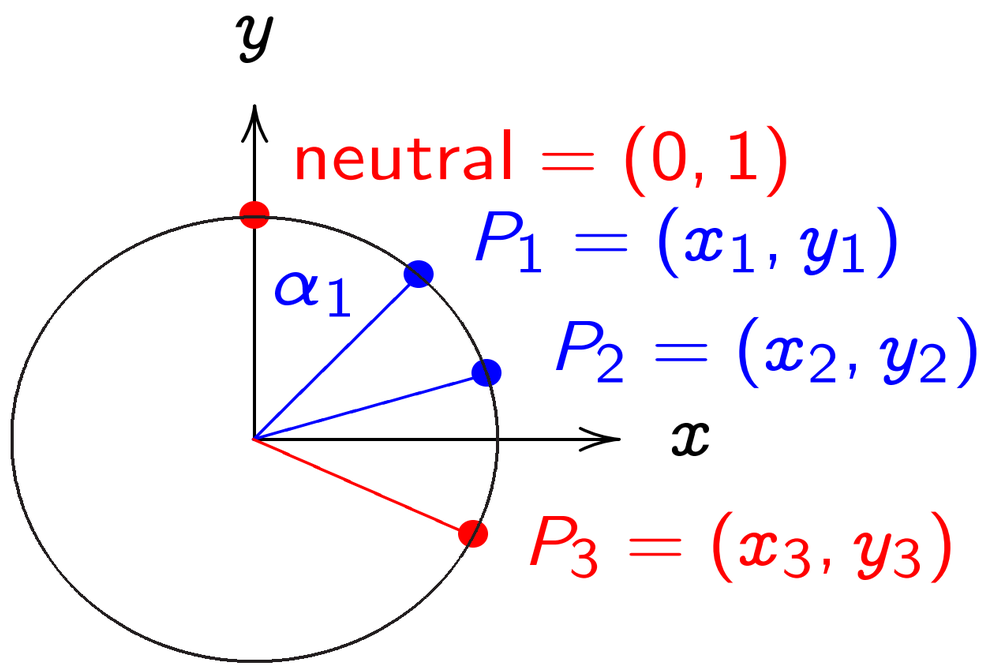

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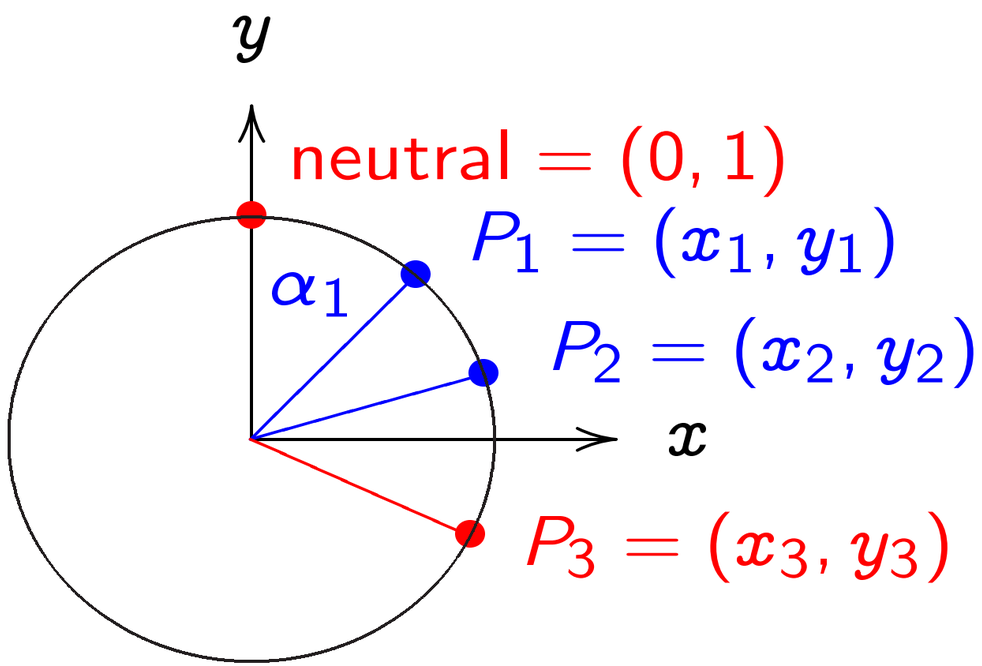

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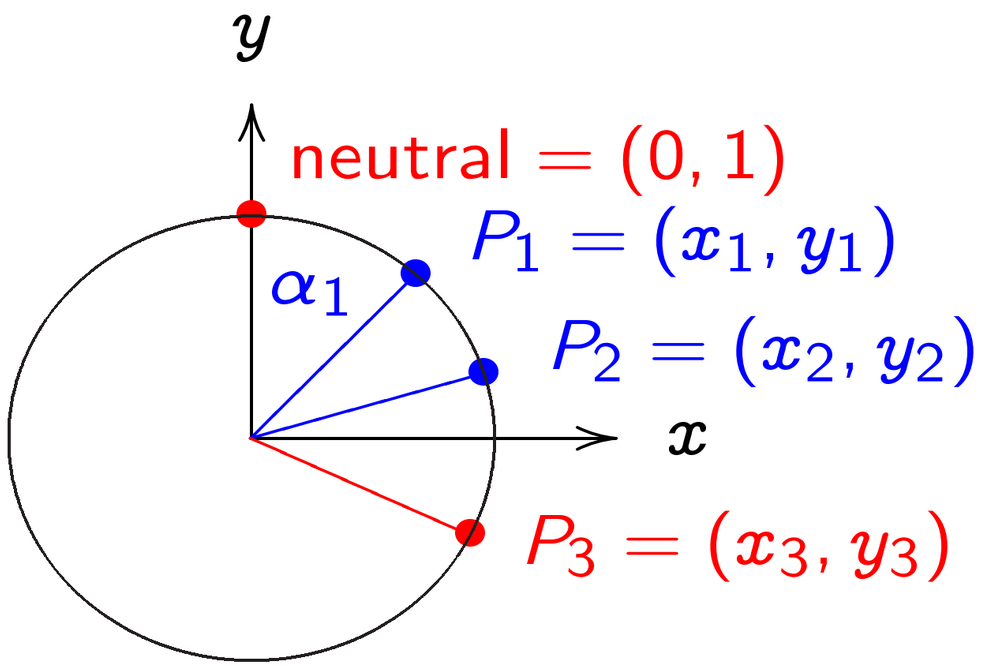

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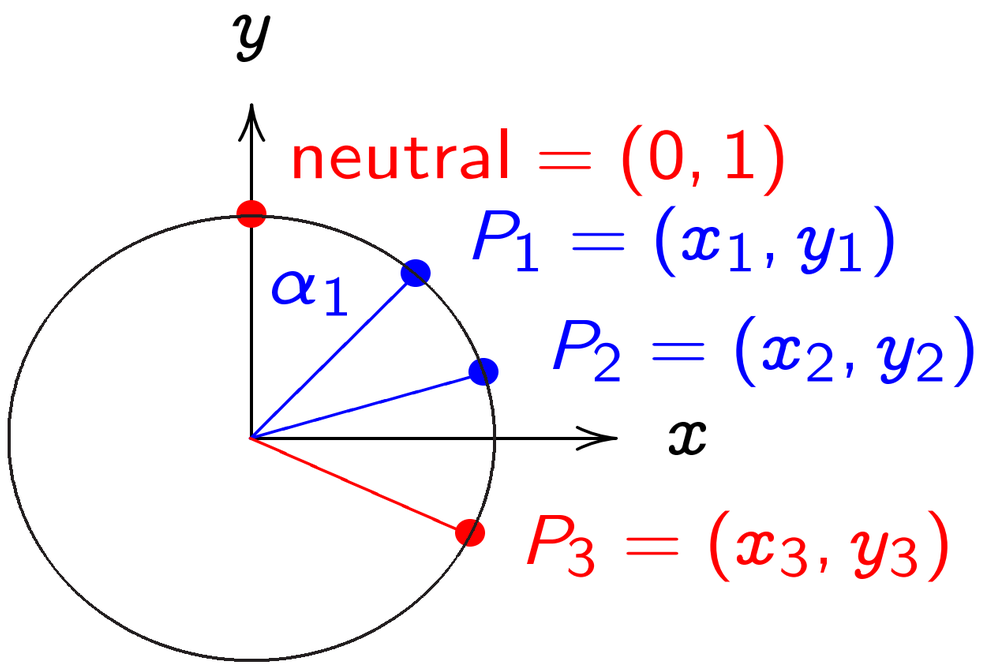

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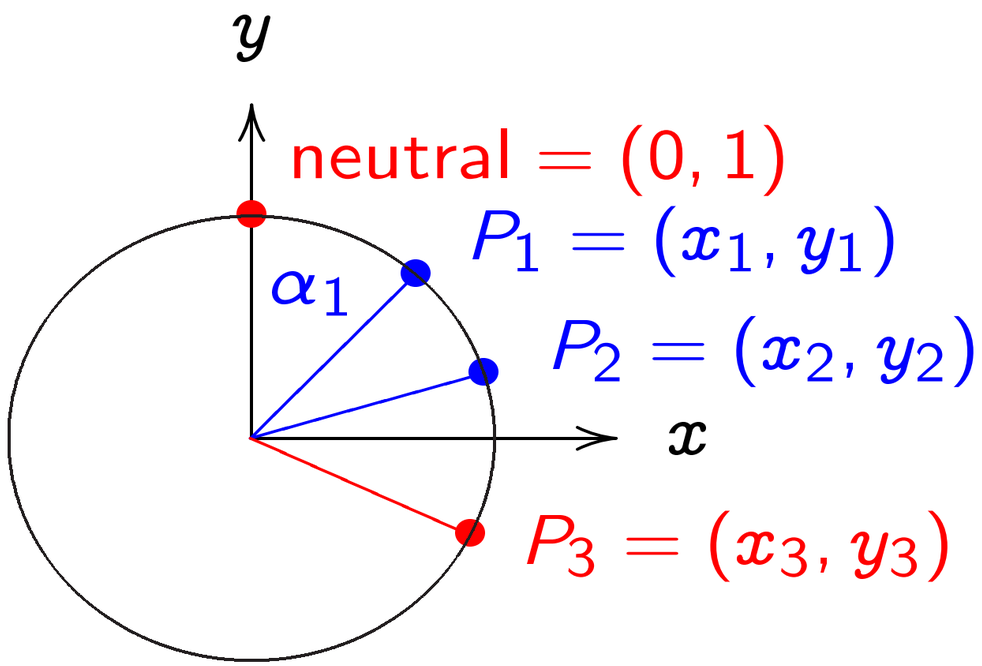

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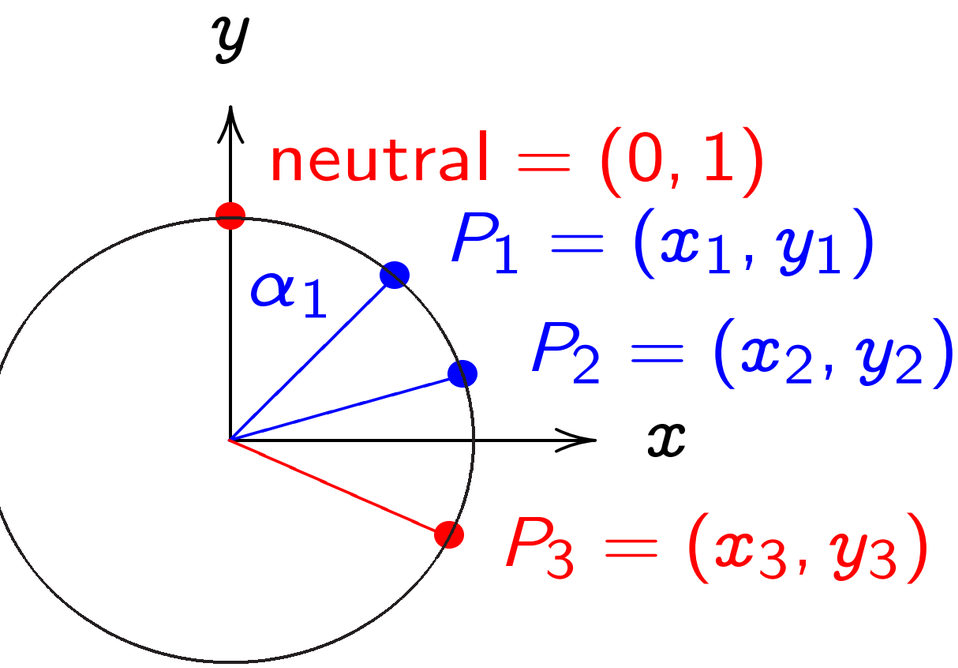

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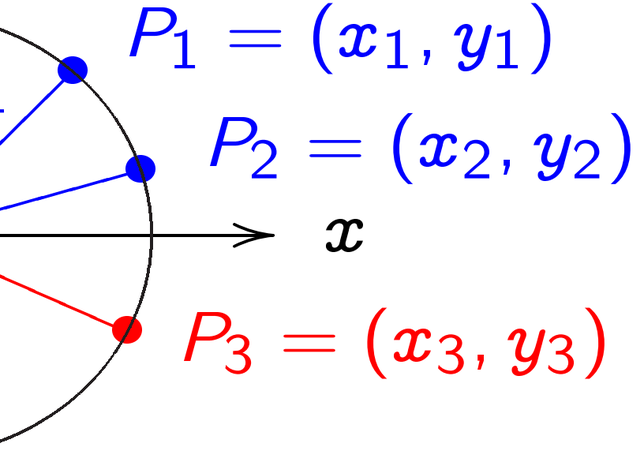

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<u>More E</u>

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More Edwards cu

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Fix a non-square

$\{(x, y) \in \mathbf{F}_q \times \mathbf{F}$
$x^2 + y^2 = 1$

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$(x_1, y_1) + (x_2, y_2$
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$$x_3 = \frac{x_1y_2 + y_1}{1 + dx_1x_2y}$$

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Fix a non-square $d \in \mathbf{F}_q$.

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More Edwards curves

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This is an elliptic curve (technically, "mod blowups").

Can use this group in crypto.

. . . if it's a "strong" curve.
Need to compute group order.
If no large prime factor in order, must switch to another d; this very often happens.

Also check "twist security," "embedding degree," et al.

Safe example, "Curve25519": $q = 2^{255} - 19$; $d = 1 - 1/121666$.

Historical notes:

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Theorem: have n
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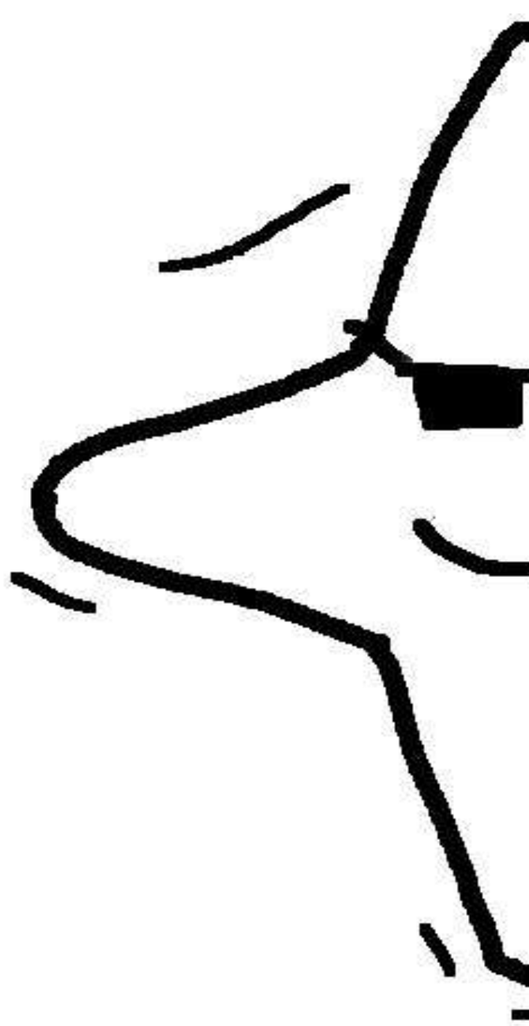

(picture courtesy

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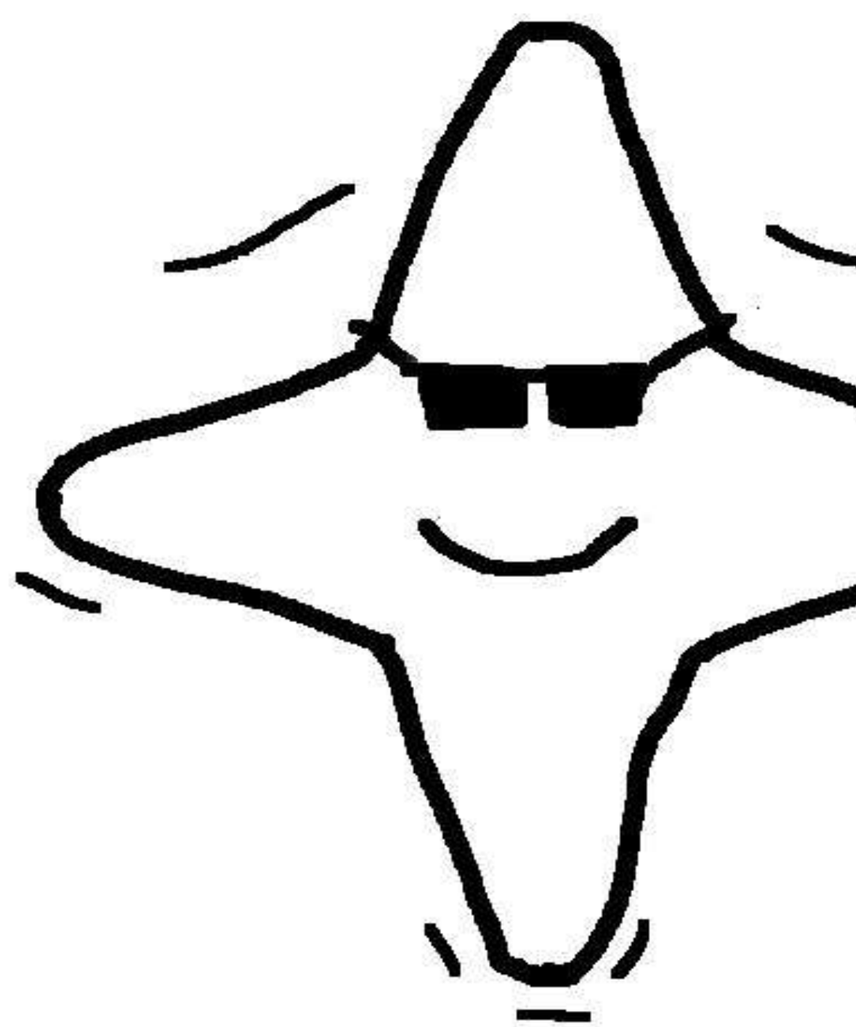

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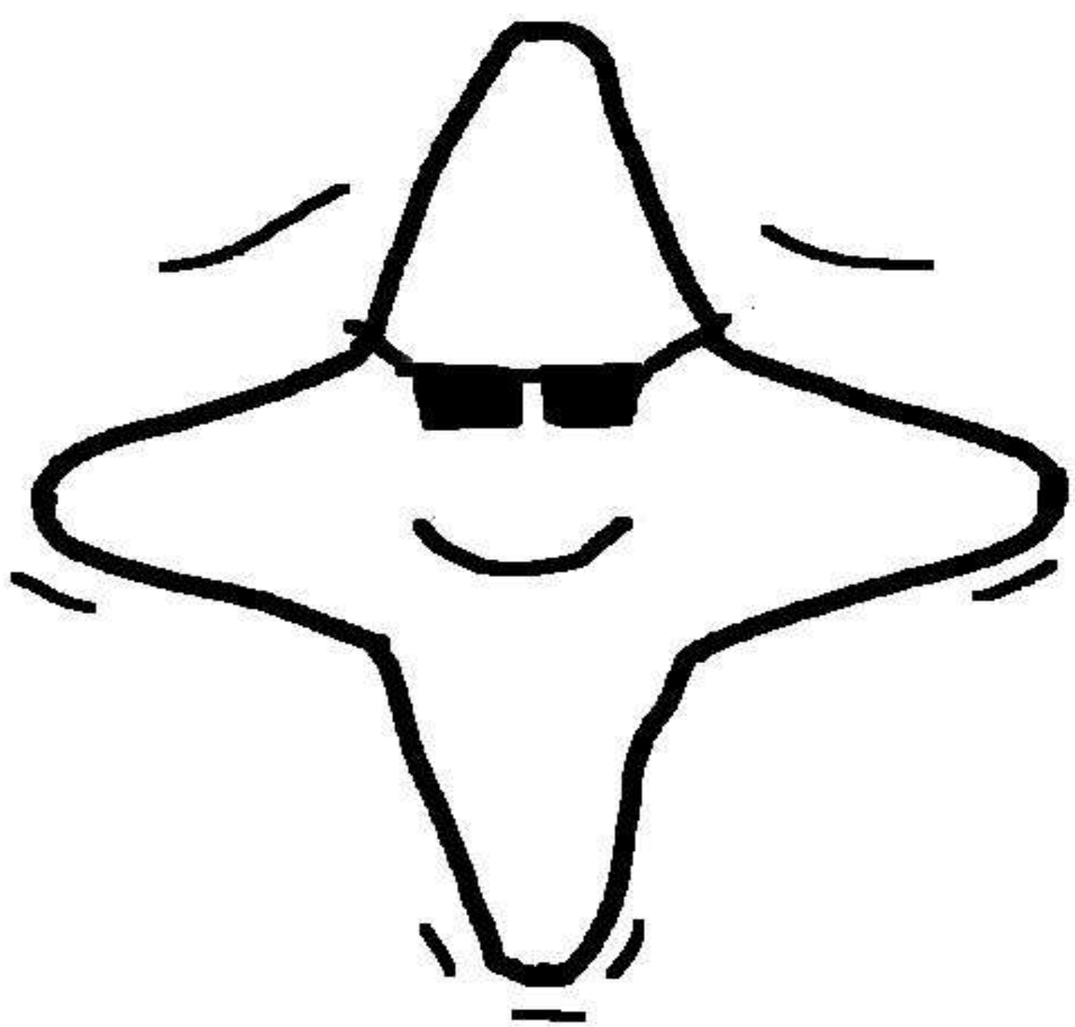

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