# Shapes of Elliptic Curves

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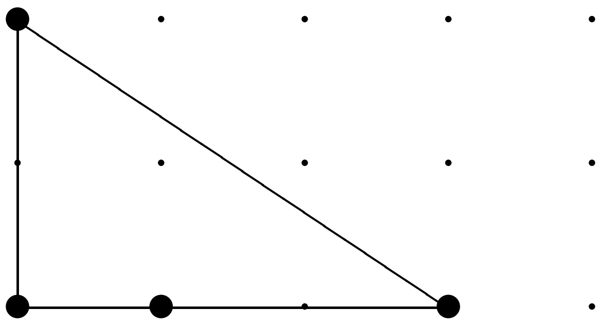
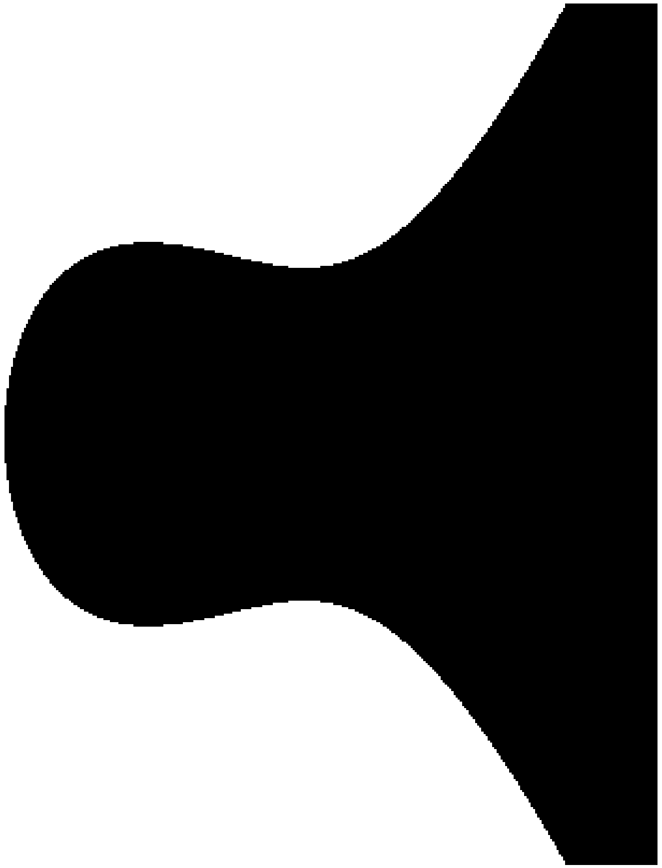
# Starring . . .

# Weierstrass curve

$y^2 = x^3 - 0.4x + 0.7$





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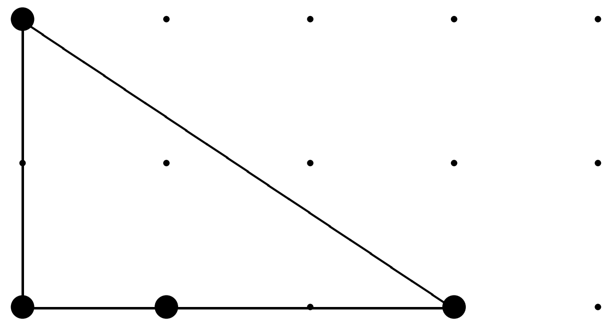
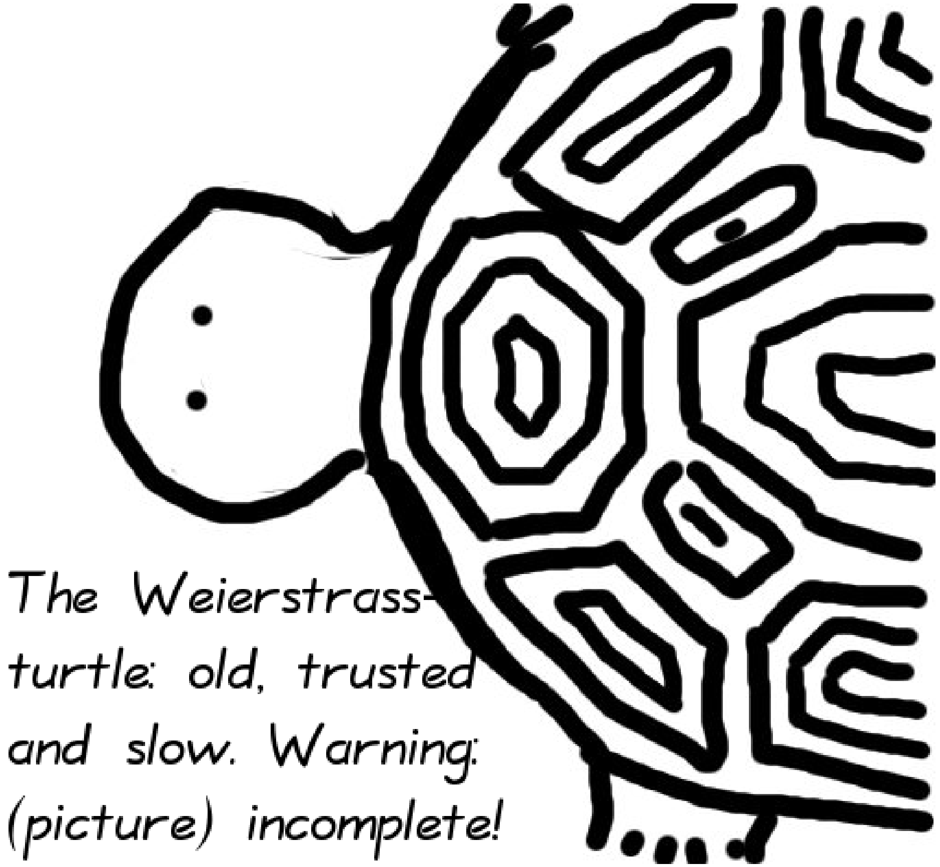
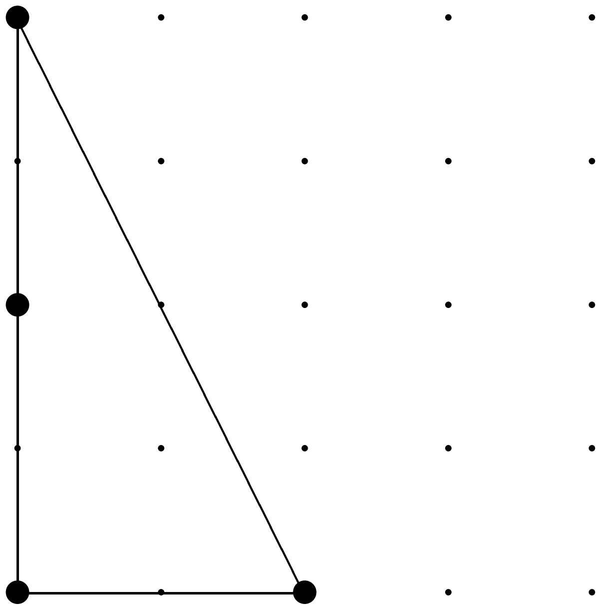
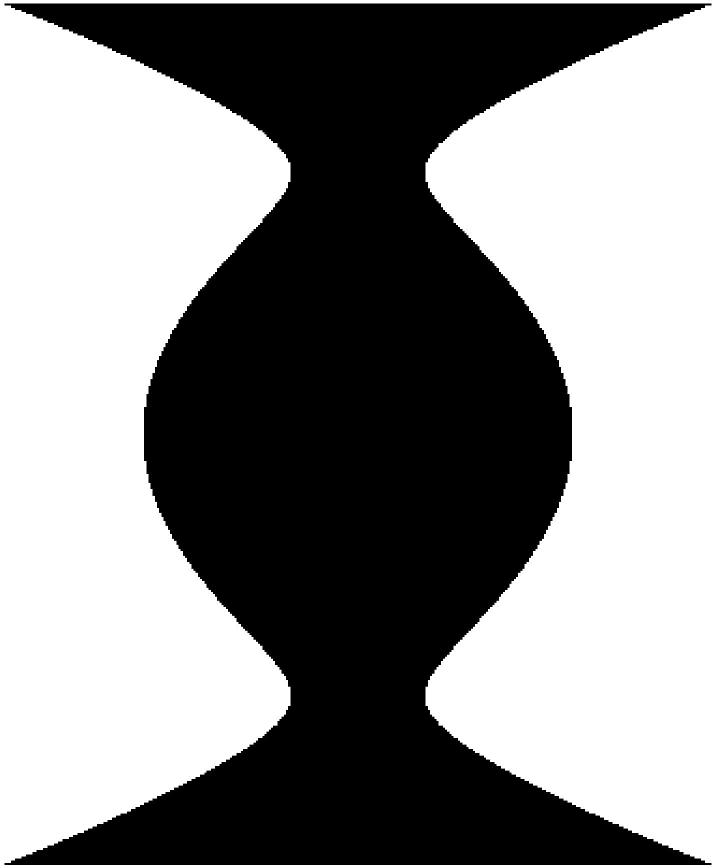




# Jacobi quartic

$$x^2 = y^4 - 1.9y^2 + 1$$





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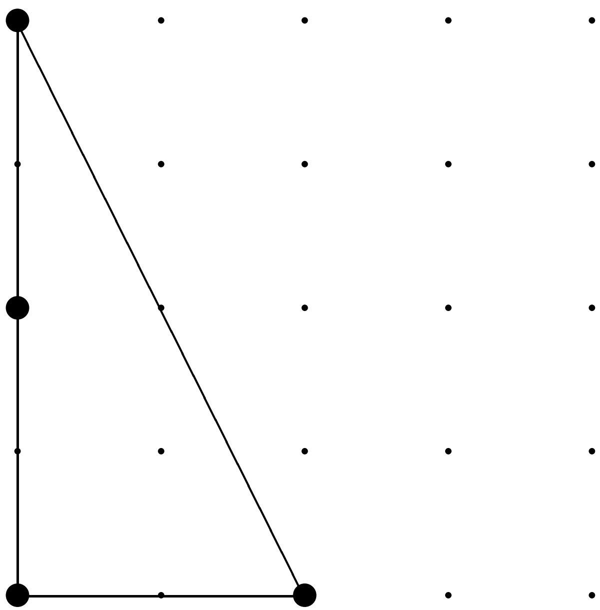
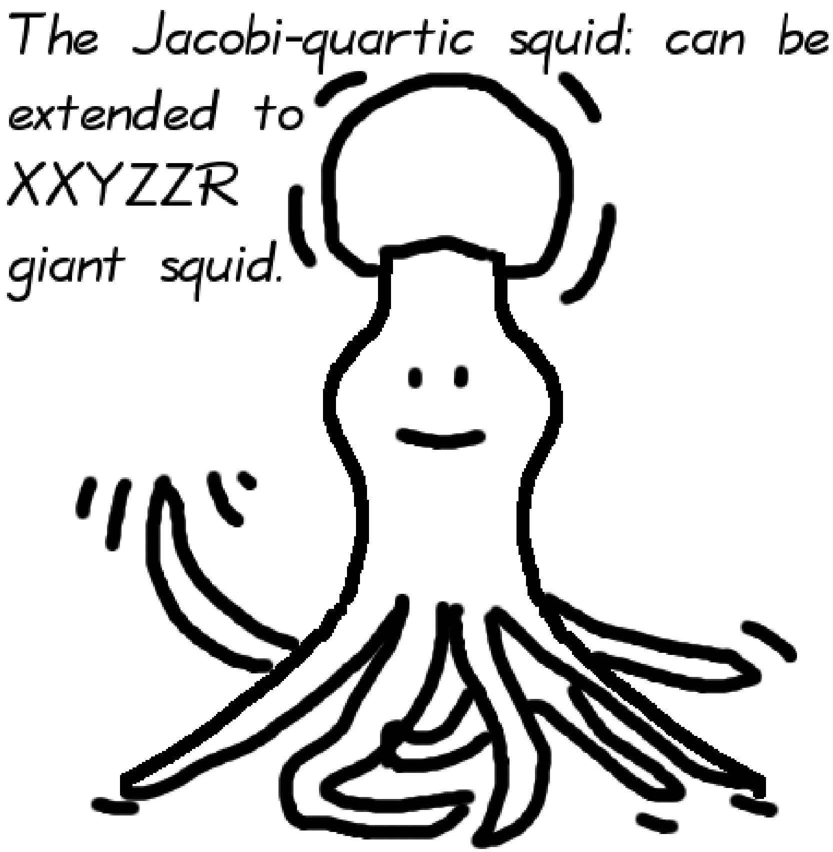
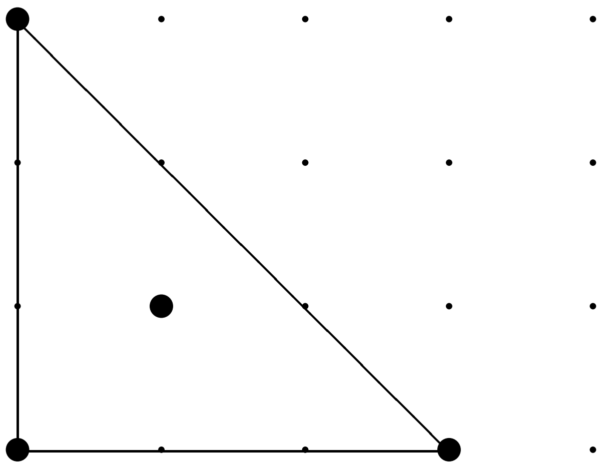
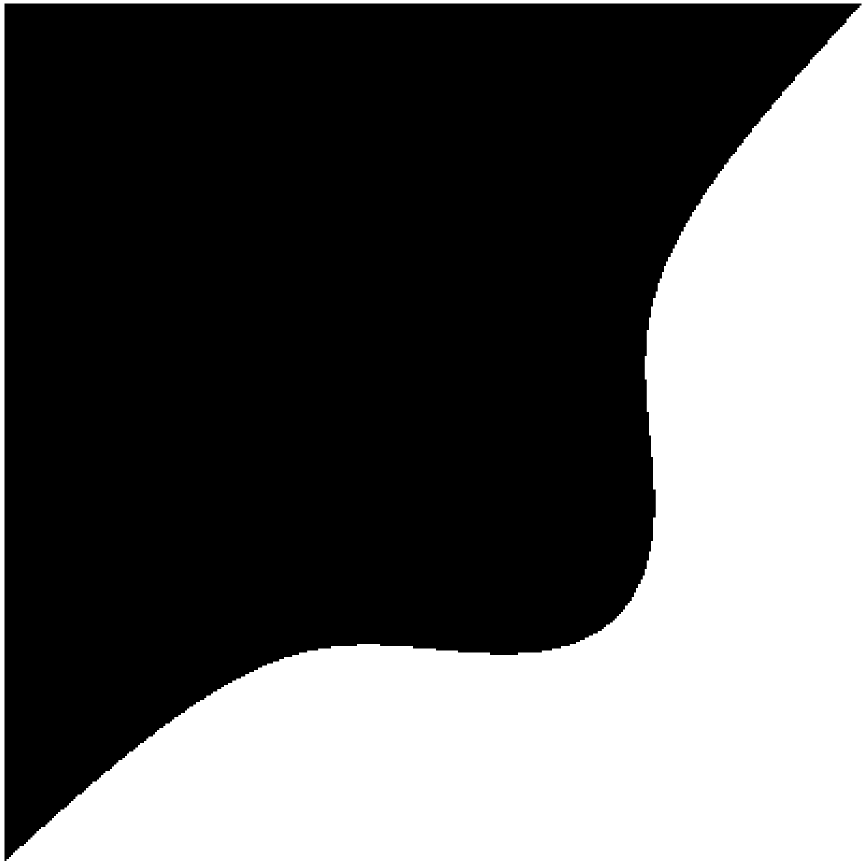




# Hessian curve

$$x^3 - y^3 + 1 = 0.3xy$$





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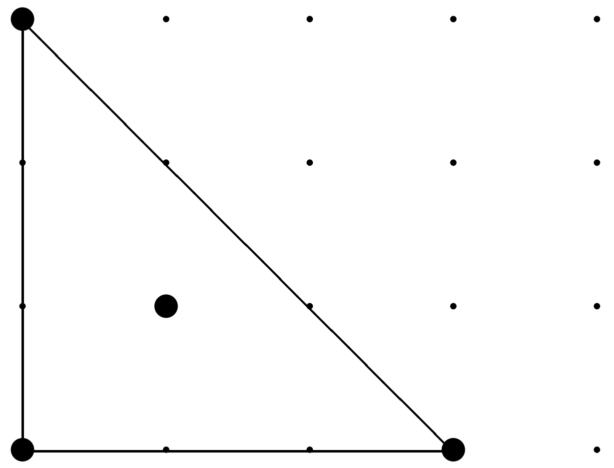
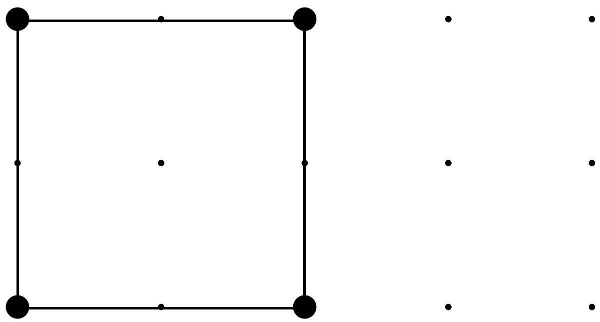
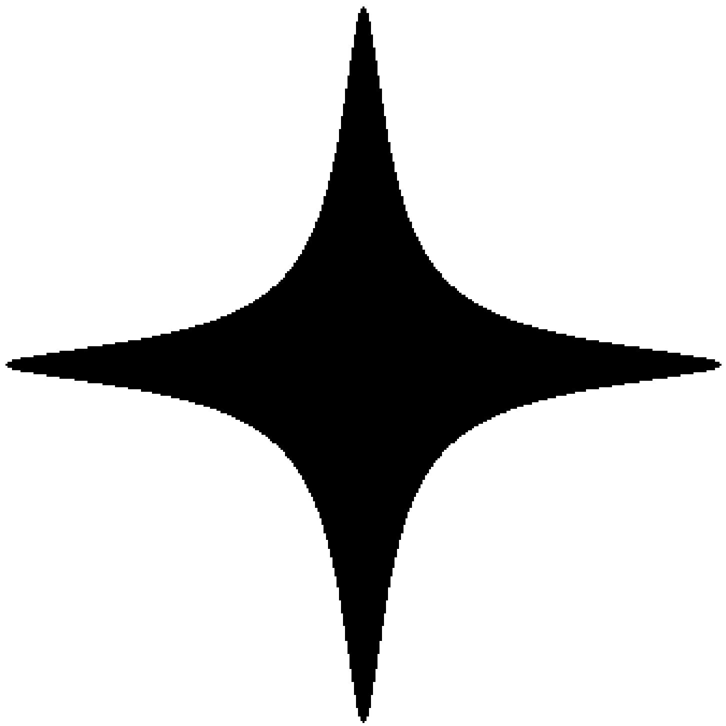


# Edwards curve

$$x^2 + y^2 = 1 - 300x^2y^2$$





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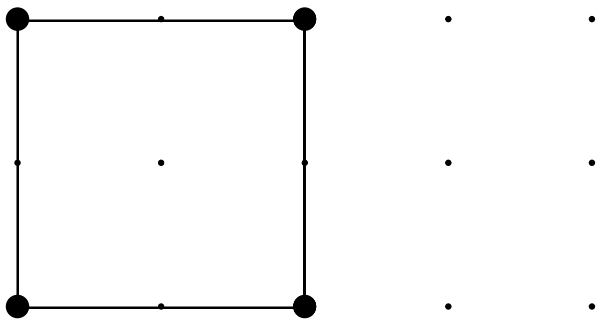
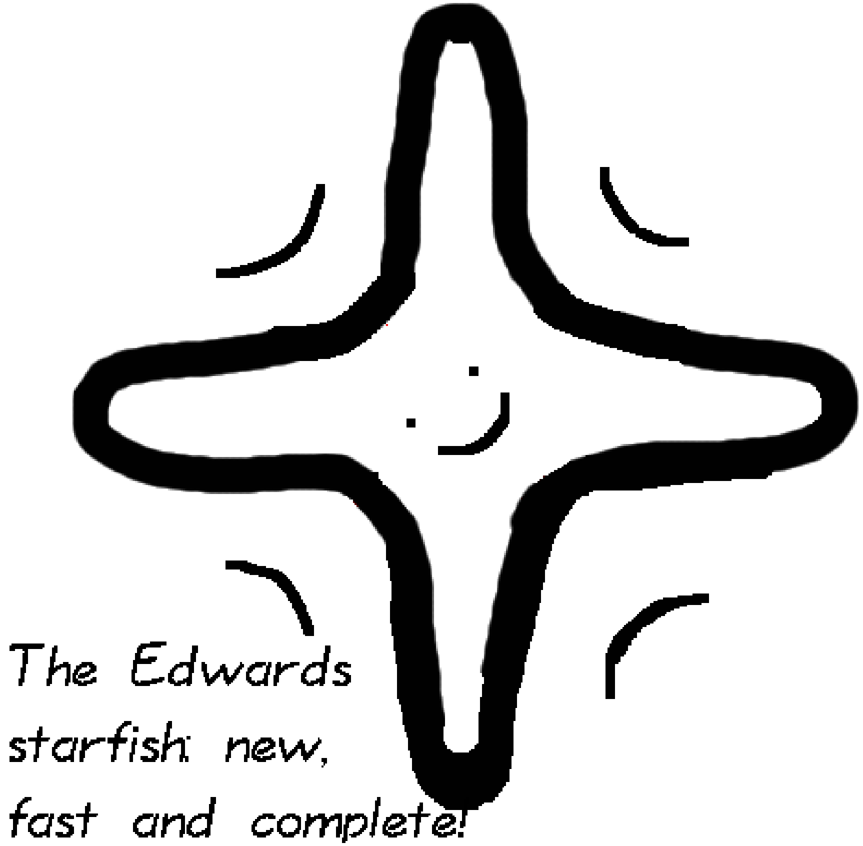
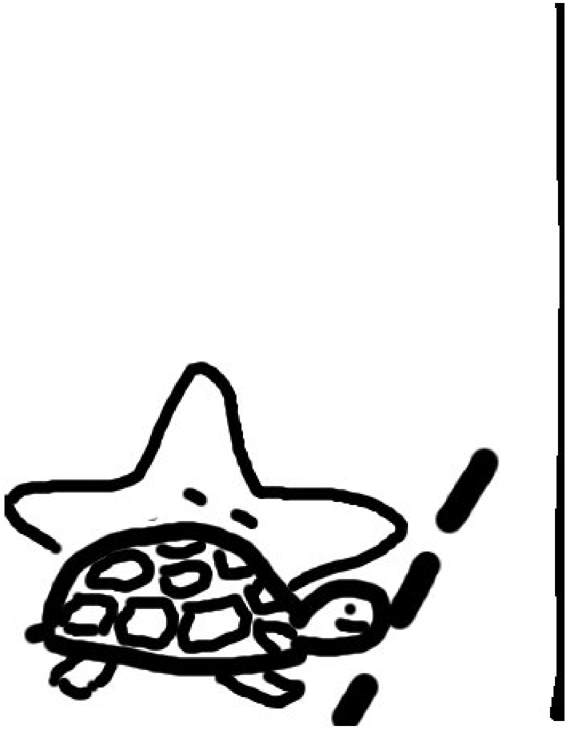




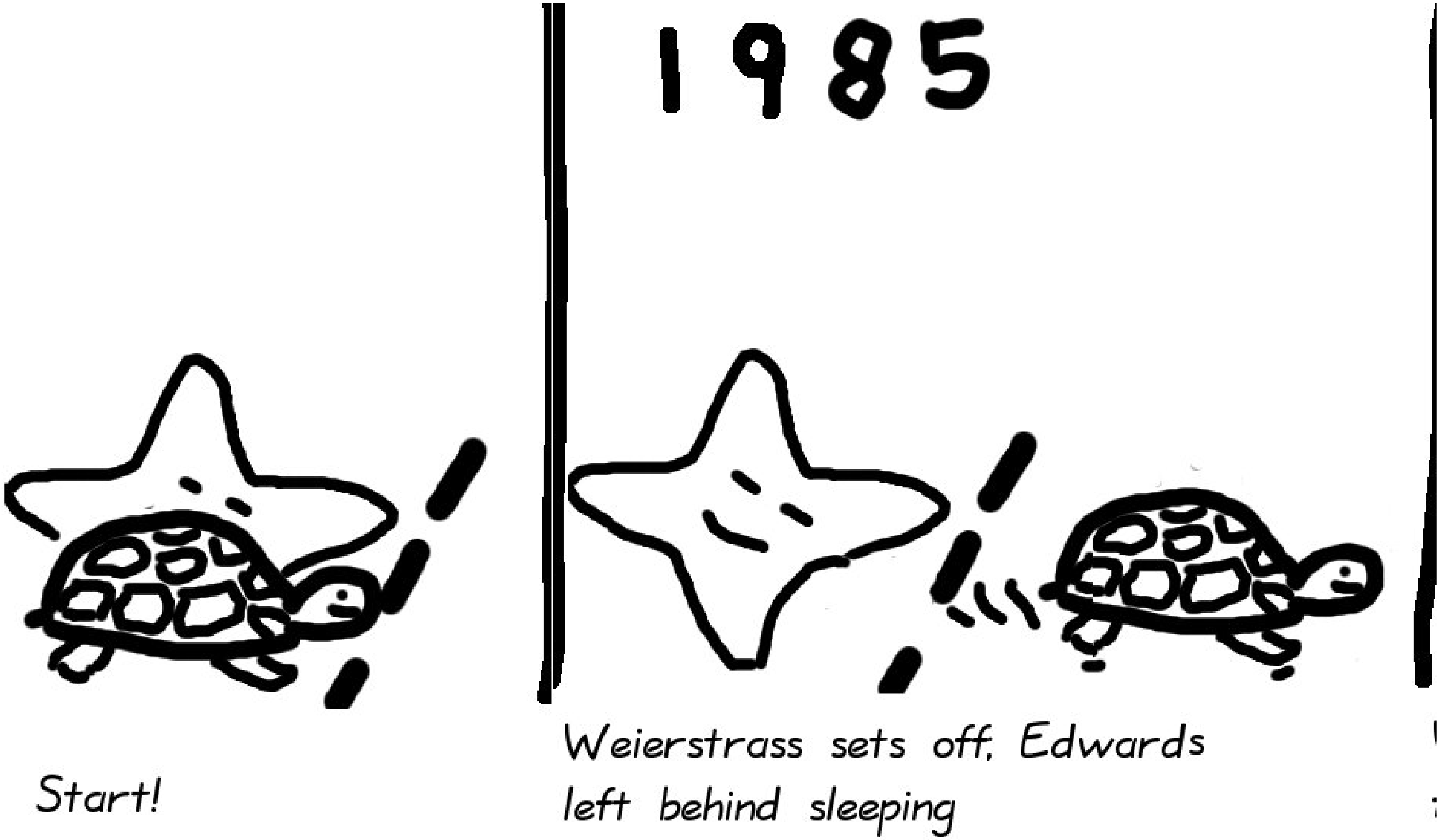
# The race – zoom on Weierstrass and Edwards

# Weierstrass vs. Edwards I

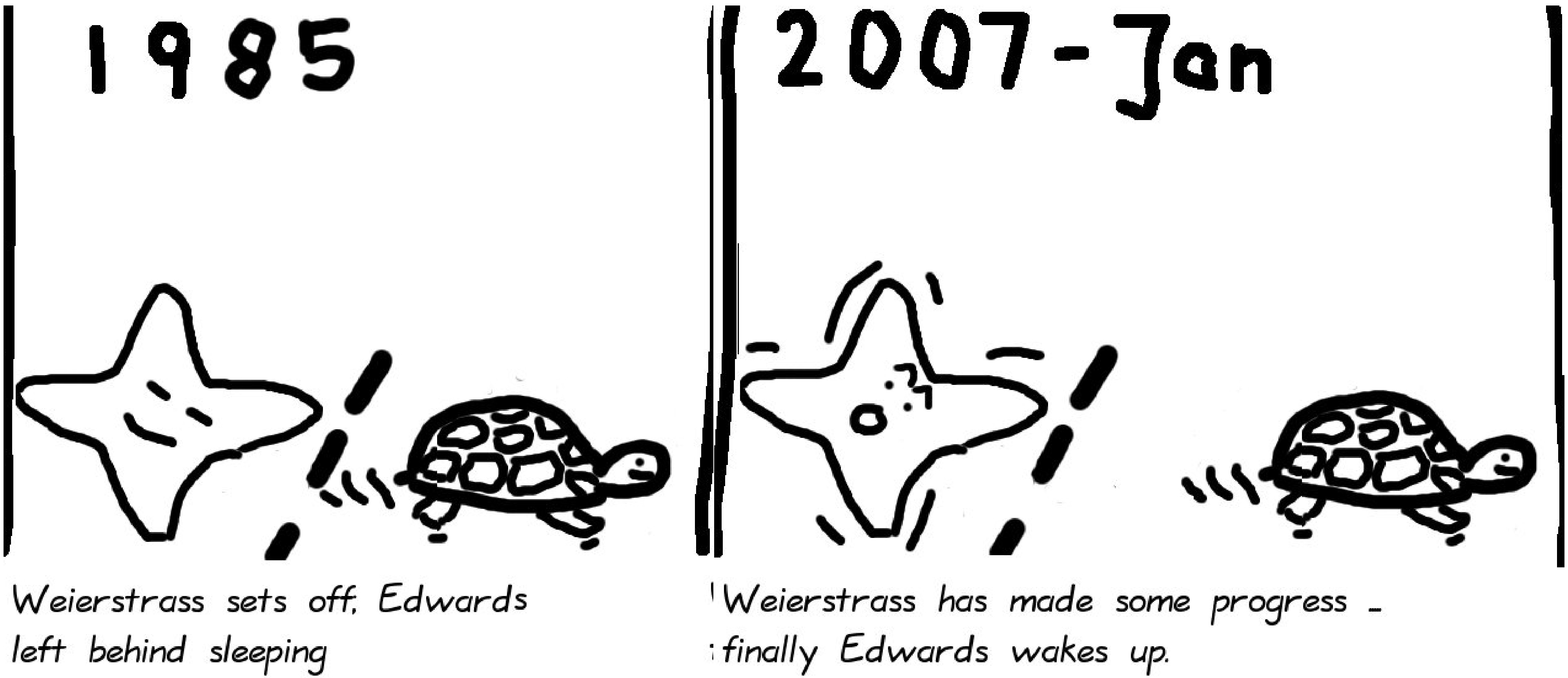


Start!

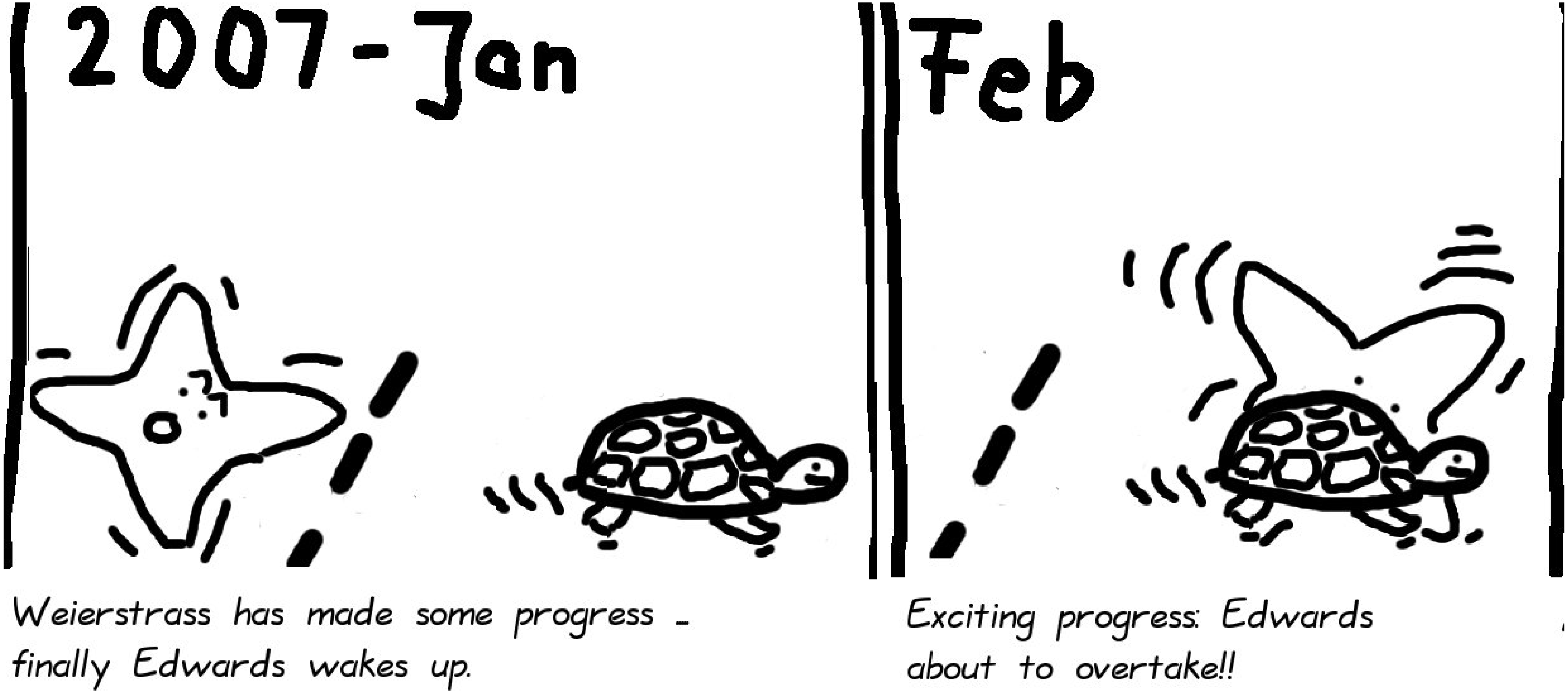
# Weierstrass vs. Edwards II



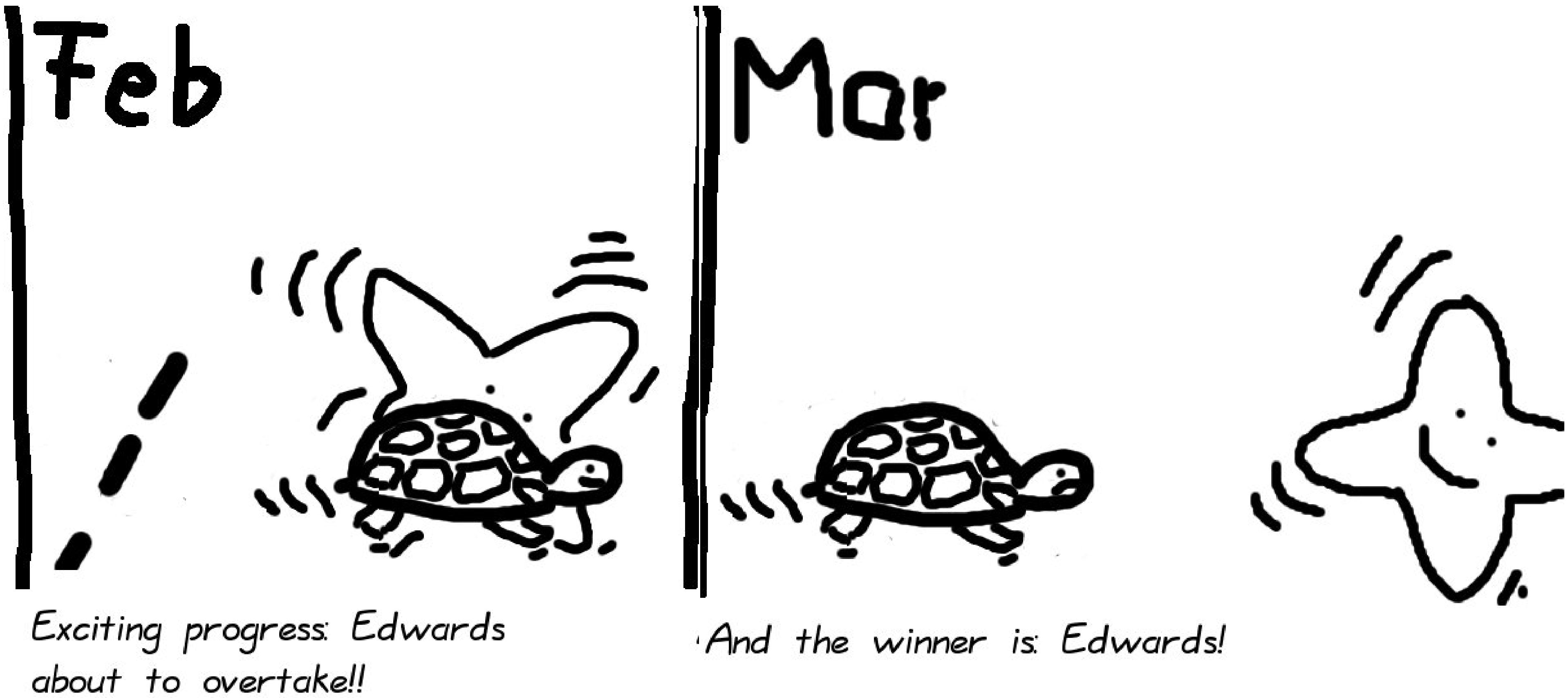
# Weierstrass vs. Edwards III



# Weierstrass vs. Edwards IV



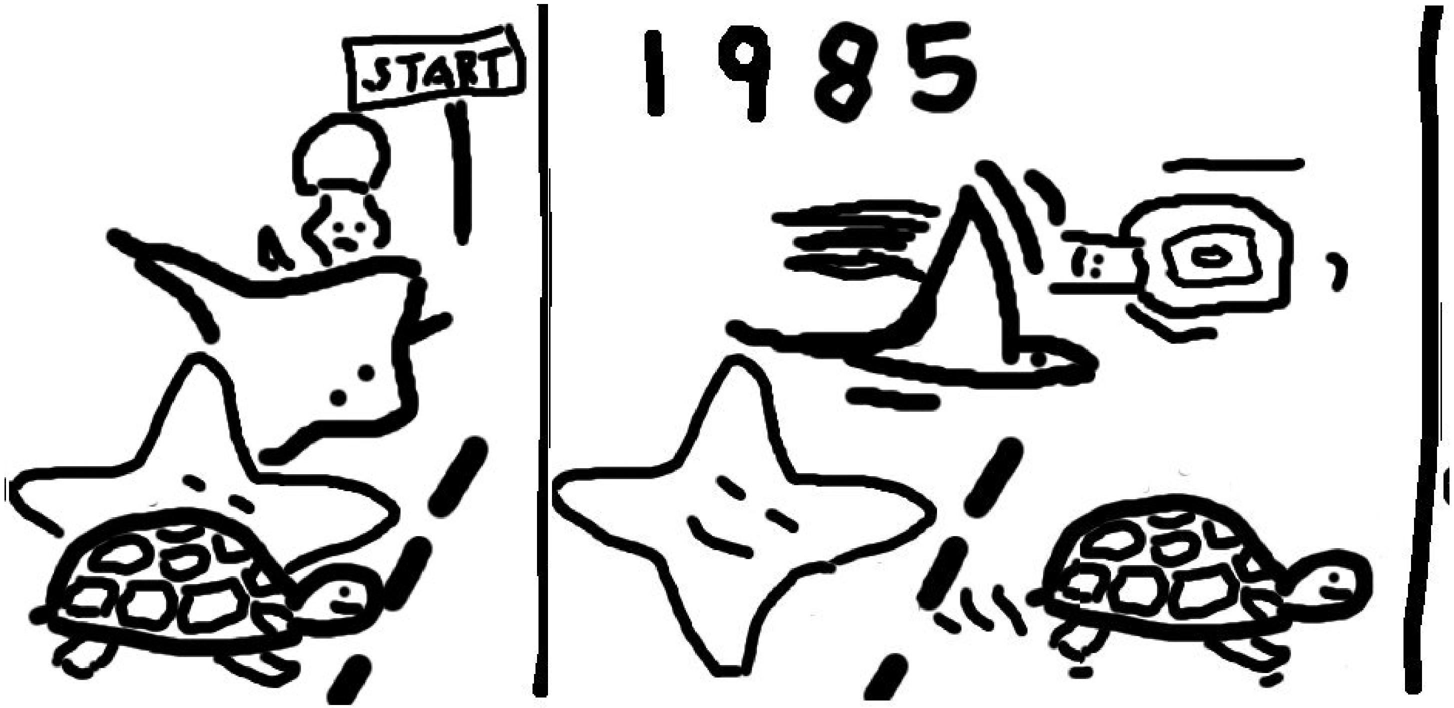
# Weierstrass vs. Edwards V



**all competitors . . .**

# All competitors I

# All competitors II



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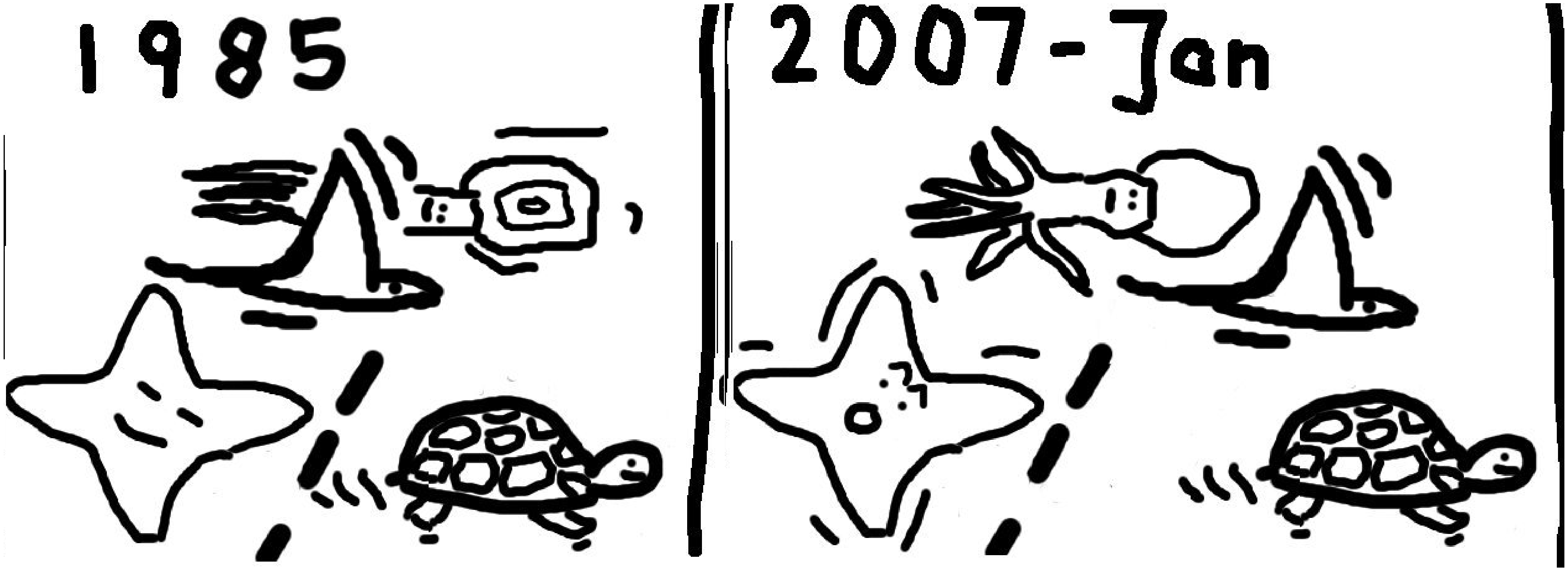

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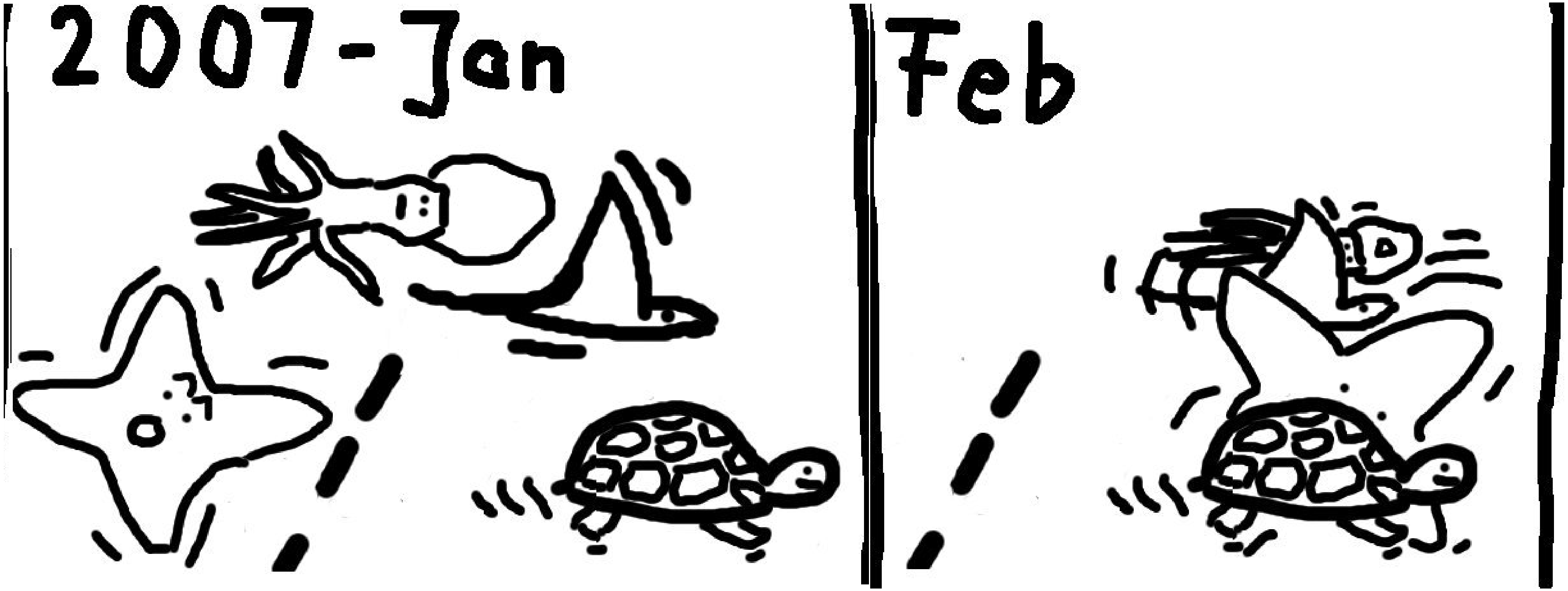
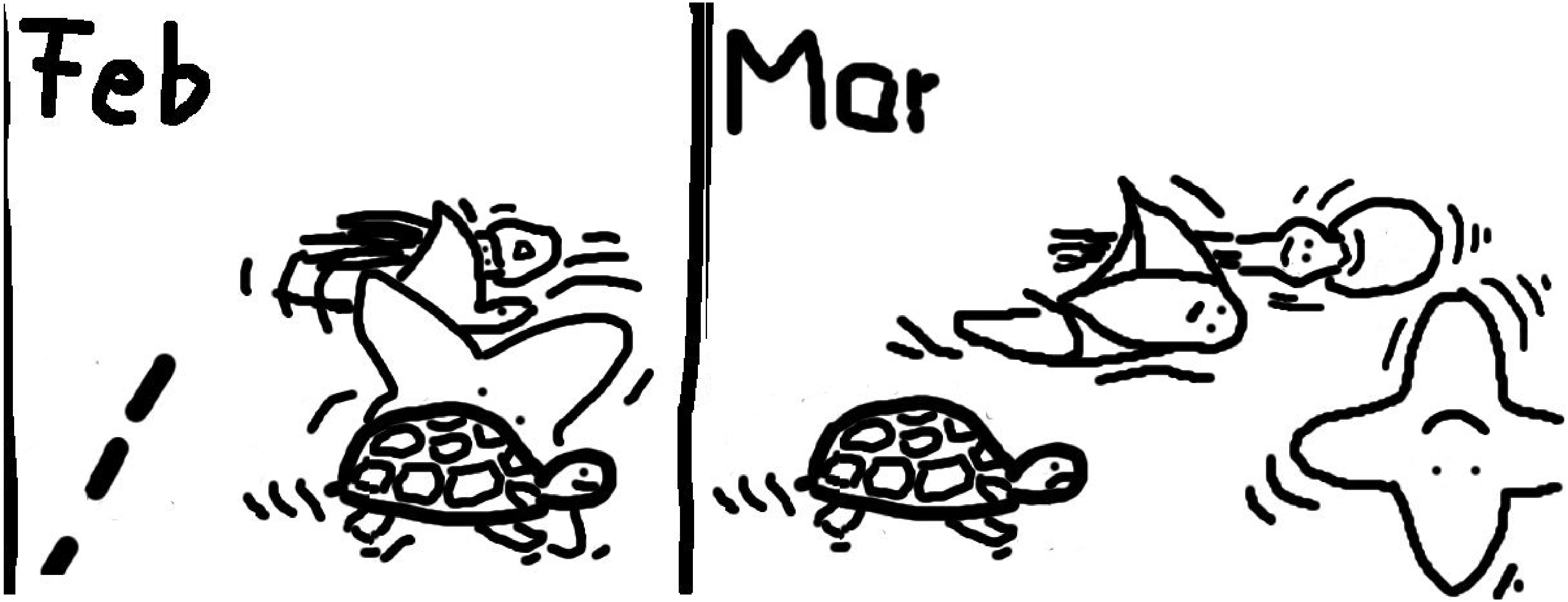


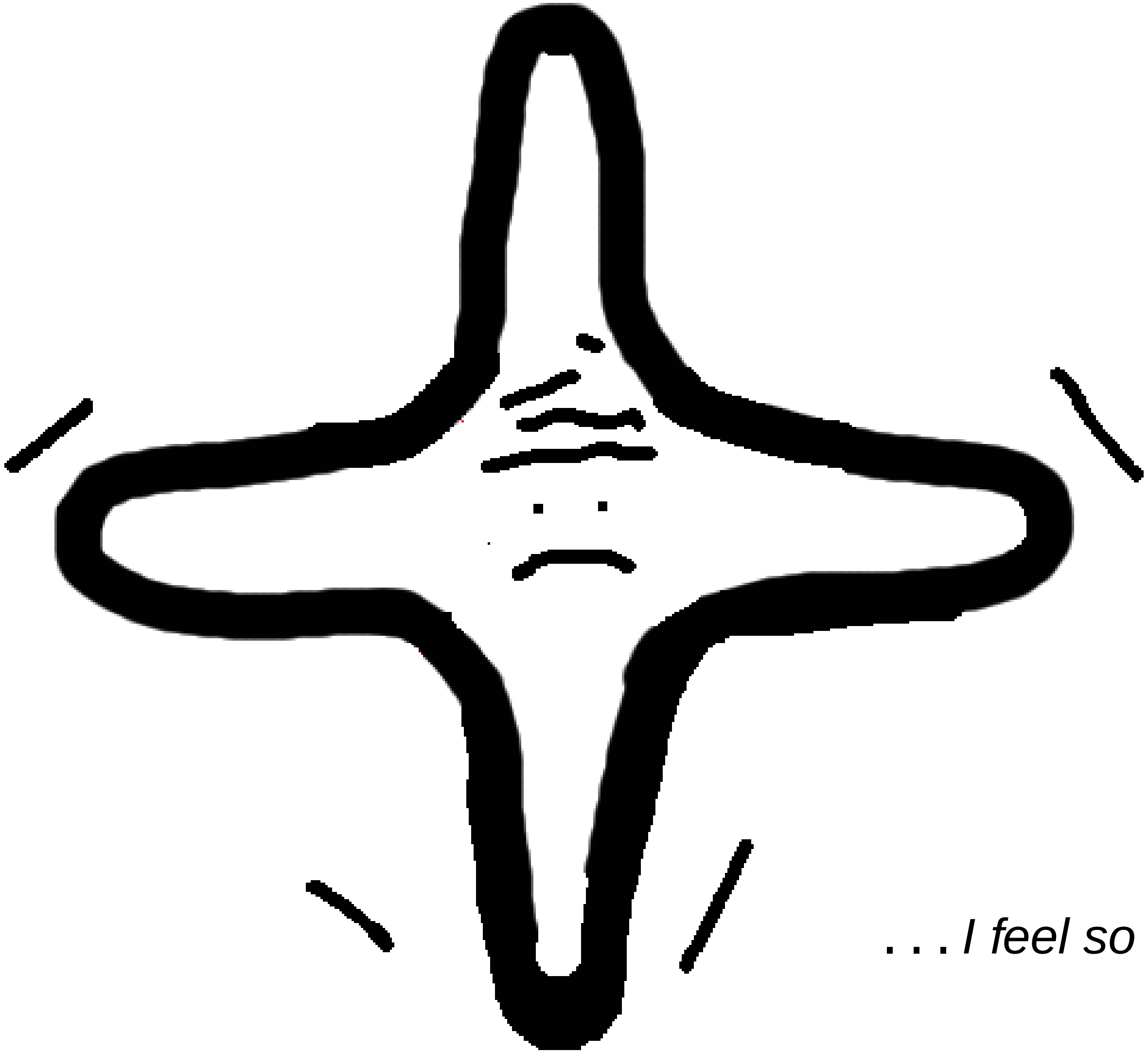
# All competitors V



**Read the full story at:**

`hyperelliptic.org/EFD`

# One year passes . . .



. . . *I feel so odd* . . .

# Exceptions, $2 \neq 0$ . . .

Fix a field $k$ of characteristic different from 2. Fix $c, d \in k$ such that $c \neq 0$, $d \neq 0$, and $dc^4 \neq 1$. Consider the *Edwards addition law*

$$(x_1, y_1), (x_2, y_2) \mapsto \left( \frac{x_1 y_2 + y_1 x_2}{c(1 + d x_1 x_2 y_1 y_2)}, \frac{y_1 y_2 - x_1 x_2}{c(1 - d x_1 x_2 y_1 y_2)} \right)$$

$x^2 + y^2 = a^2(1 + x^2 y^2)$, $a^5 \neq a$ describes an elliptic curve over field $k$ of odd characteristic.

**Theorem 2.1.** *Let $k$ be a field in which $2 \neq 0$. Let $E$ be an elliptic curve over $k$ such that the group $E(k)$ has an element of order 4. Then*

How can there be an incomplete set of complete curves???

# After extensive (finite) field studies . . .

(joint work with Reza Rezaeian Farashahi)

- Assume $d_1, d_2 \in \mathbb{F}_{2^n}$, $d_1 \neq 0$, $\mathrm{Tr}(d_2) = 1$. Then

  $$d_1(x+y) + d_2(x+y)^2 = xy + xy(x+y) + x^2y^2$$

  describes an elliptic curve.
  (Curve shape is symmetric and has highest term $x^2y^2$ like 'classic' Edwards curves $x^2 + y^2 = 1 + dx^2y^2$.)
- Neutral element is $(0, 0)$. Negative of $(x, y)$ is $(y, x)$.
- The addition law on this curve is complete! It works for adding arbitrary points – doubling, adding negatives, adding the neutral element, . . .
- Every ordinary elliptic curve over $\mathbb{F}_{2^n}$ is birationally equivalent to a complete binary Edwards curve.

# Timeline (after some early aborts)

- February 13th, 2008: Binary Edwards curves born
- February 15th, 2008: Binary Edwards takes first steps
- February 15th, 2008: Binary Edwards curves are complete (!) for $d_2 = 1$ and $n$ odd.
- February 16th, 2008: Binary Edwards plays with $d_1, d_2$
- February 20th, 2008: Binary Edwards adds differentially
- February 29th, 2008: Binary Edwards reaches all ordinary curves
- March 31st, 2008: Intel announces support for binary Edwards curves (PCMULQDQ in Westmere)
- April 16th, 2008: Sun announces support for binary Edwards curves in Rock.

# Operation counts

These curves are the first binary curves to offer complete addition laws. They are also surprisingly fast:

- ADD on binary Edwards curves takes 21M+1S+4D, mADD takes 13M+3S+3D.
- Latest results ADD in 18M+2S+7D.
- Differential addition ($P+Q$ given $P, Q$, and $Q-P$) takes 8M+1S+2D; mixed version takes 6M+1S+2D.
- Differential addition+doubling (typical step in Montgomery ladder) takes 8M+4S+2D; mixed version takes 6M+4S+2D.

See our preprint (ePrint 2008/171) or

`cr.yp.to/papers.html#edwards2`

for full details, speedups for $d_1 = d_2$, how to choose small coefficients, affine formulas, . . .

# Comparison with other doubling formulas

Assume curves are chosen with small coefficients.

| System | Cost of doubling |
|---|---|
| Projective | 7M+4S; see HEHCC |
| Jacobian | 4M+5S; see HEHCC |
| Lopez-Dahab | 3M+5S; Lopez-Dahab |
| Edwards | 2M+6S; new, complete |
| Lopez-Dahab $a_2 = 1$ | 2M+5S; Kim-Kim |

Explicit-Formulas Database

`www.hyperelliptic.org/EFD`

for characteristic $2$ is in preparation; our paper already has some speed-ups for Lopez-Dahab coordinates.

# Happy End!