Polynomial evaluation and message authentication

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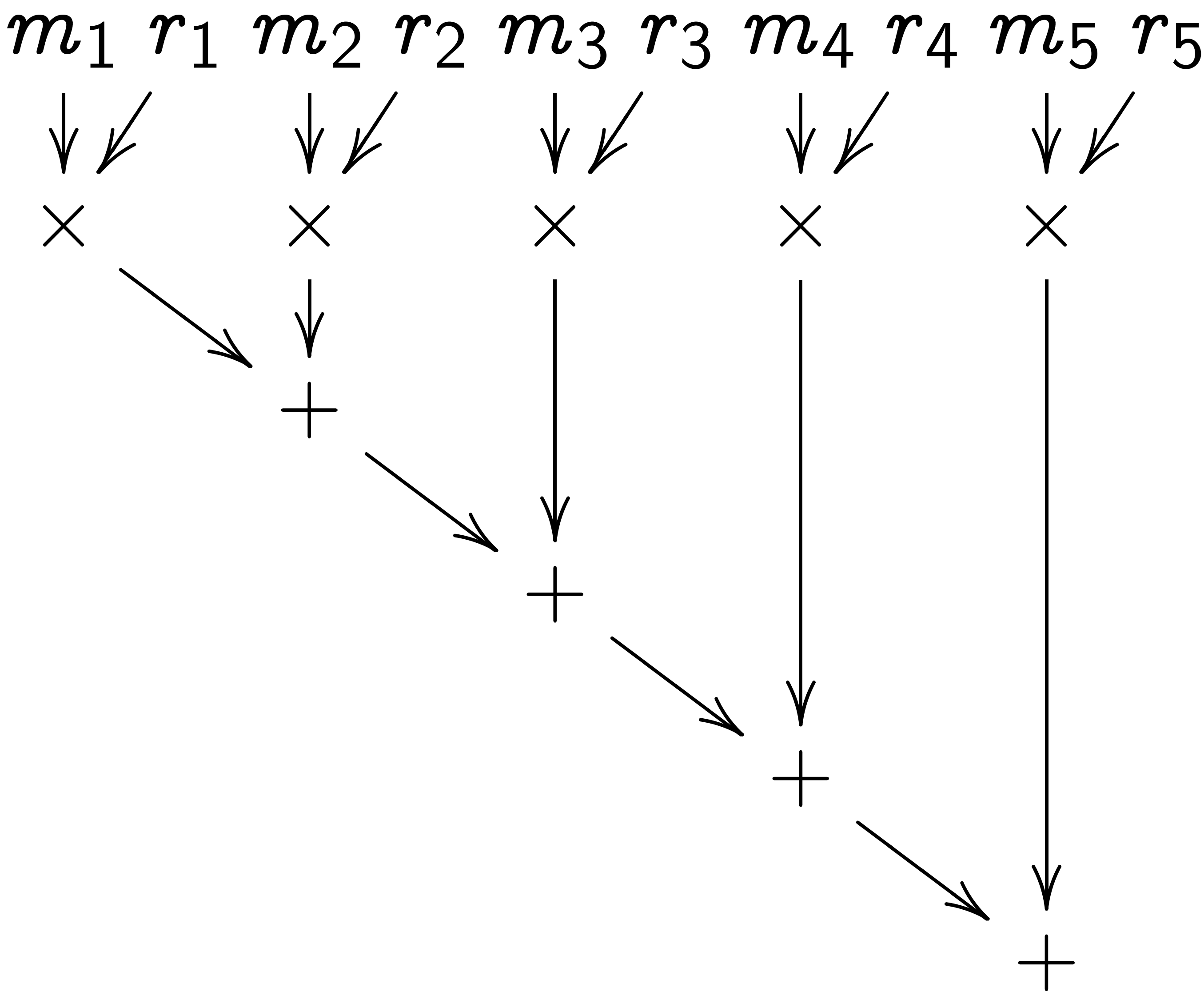
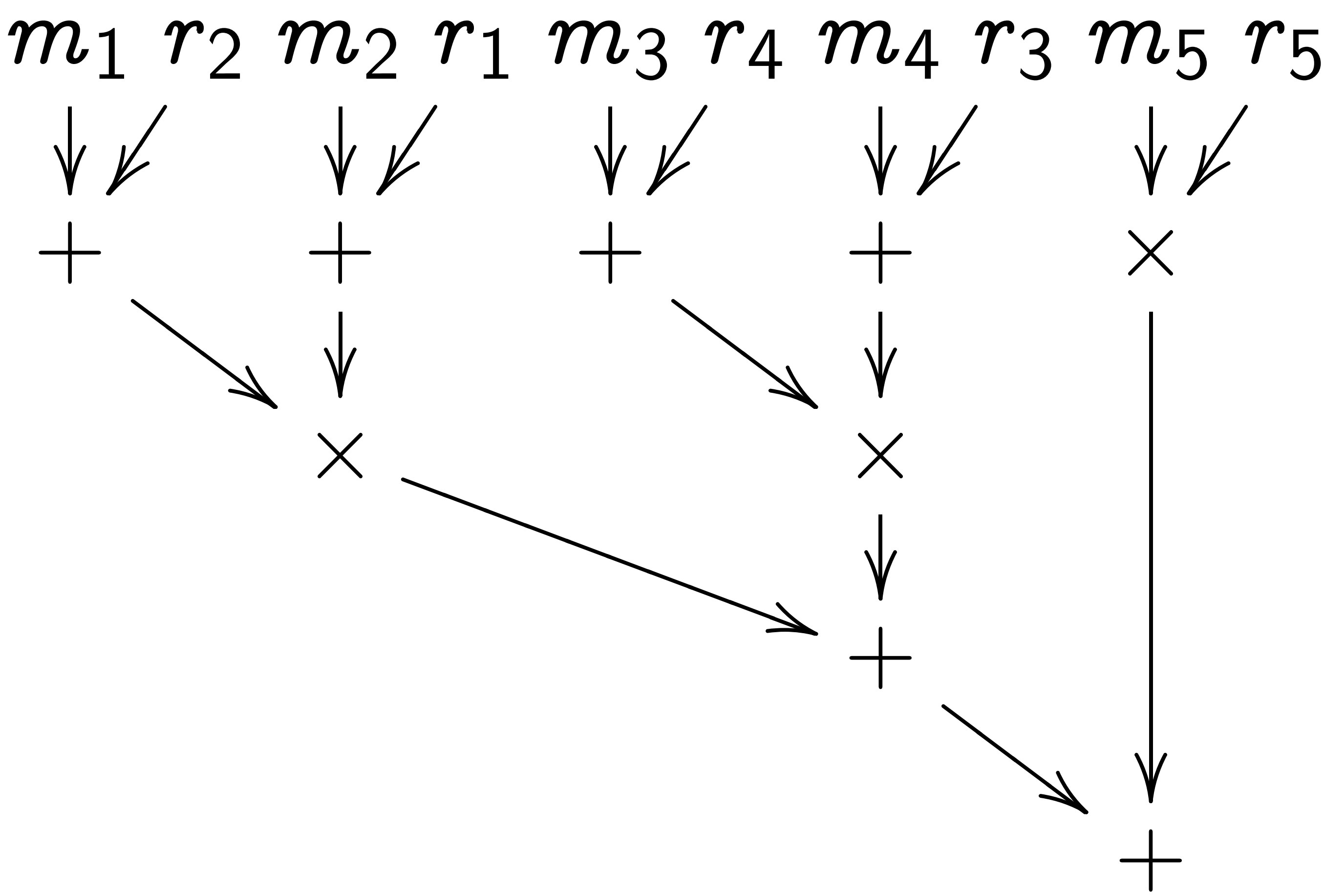


Cost of this algorithm:
5 mults, 4 adds.

Output of this algorithm,
given $m_1, \ldots, r_1, \ldots \in \mathbf{F}_q$:
$m_1 r_1 + \cdots + m_5 r_5$.

Alternative (1968 Winograd), $\approx 2\times$ speedup in matrix mult:



Output in $\mathbf{F}_q[m_1, \ldots, r_1, \ldots]$:
$m_5 r_5 + (m_3 + r_4)(m_4 + r_3) + (m_1 + r_2)(m_2 + r_1) = m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4 + m_5 r_5 + m_1 m_2 + m_3 m_4 + r_1 r_2 + r_3 r_4$.

One good way to recognize forged/corrupted messages:

Standardize a prime $p = 1000003$.

Sender rolls 10-sided die to generate independent uniform random secrets

$r_1 \in \{0, 1, \ldots, 999999\;$,

$r_2 \in \{0, 1, \ldots, 999999\;$,

$\ldots$,

$r_5 \in \{0, 1, \ldots, 999999\;$,

$s_1 \in \{0, 1, \ldots, 999999\;$,

$\ldots$,

$s_{100} \in \{0, 1, \ldots, 999999\;$.

Sender meets receiver in private and tells receiver the same secrets $r_1, r_2, \ldots, r_5, s_1, \ldots, s_{100}$.

Later: Sender wants to send 100 messages $m_1, \ldots, m_{100}$, each m_n having 5 components $m_{n,1}, m_{n,2}, m_{n,3}, m_{n,4}, m_{n,5}$ with $m_{n,i} \in \{0, 1, \ldots, 999999\}$.

Sender transmits 30-digit $m_{n,1}, m_{n,2}, m_{n,3}, m_{n,4}, m_{n,5}$ together with an **authenticator**

$$(m_{n,1}r_1 + \cdots + m_{n,5}r_5 \bmod p) \\ + s_n \bmod 1000000$$

and the message number n.

e.g. $r_1 = 314159$, $r_2 = 265358$,
$r_3 = 979323$, $r_4 = 846264$,
$r_5 = 338327$, $s_{10} = 950288$,
$m_{10} =$ 000006 000007 000000 000000 000000:

Sender computes authenticator

$(6r_1 + 7r_2 \bmod p)$
$\quad + s_{10} \bmod 1000000 =$
$(6 \cdot 314159 + 7 \cdot 265358$
$\quad \bmod 1000003)$
$\quad + 950288 \bmod 1000000 =$
$742451 + 950288 \bmod 1000000 =$
692739.

Sender transmits
10 000006 000007 000000 000000 000000 692739.

Main work is multiplication.
For each 6-digit message chunk, have to do one multiplication by a 6-digit secret r_i.

Scaled up for serious security:
Choose, e.g., $p = 2^{130} - 5$.
For each 128-bit message chunk, have to do one multiplication by a 128-bit secret r_i.
Reduce output mod $2^{130} - 5$.
≈ 5 cycles per message byte, depending on CPU.

Many papers on choosing fields, computing products quickly.

Provably secure authenticators $(m_1 r_1 + m_2 r_2 + \cdots) + s$: 1974 Gilbert/MacWilliams/Sloane.

1999 Black/Halevi/Krawczyk/ Krovetz/Rogaway (crediting unpublished Carter/Wegman, failing to credit Winograd): Replace $m_1 r_1 + m_2 r_2$ with $(m_1 + r_1)(m_2 + r_2)$, replace $m_3 r_3 + m_4 r_4$ with $(m_3 + r_3)(m_4 + r_4)$, etc. Half as many multiplications for each message chunk.

Expand short key k into long secret $r_1, \ldots, s_1, \ldots$ as, e.g., $\text{AES}_k(1), \text{AES}_k(2), \ldots$.

Oops, not uniform random. But easily prove that attack implies attack on AES.

Generate r's, s's on demand? Need $\ell + 1$ AES invocations for $r_1, r_2, \ldots, r_\ell, s_n$.

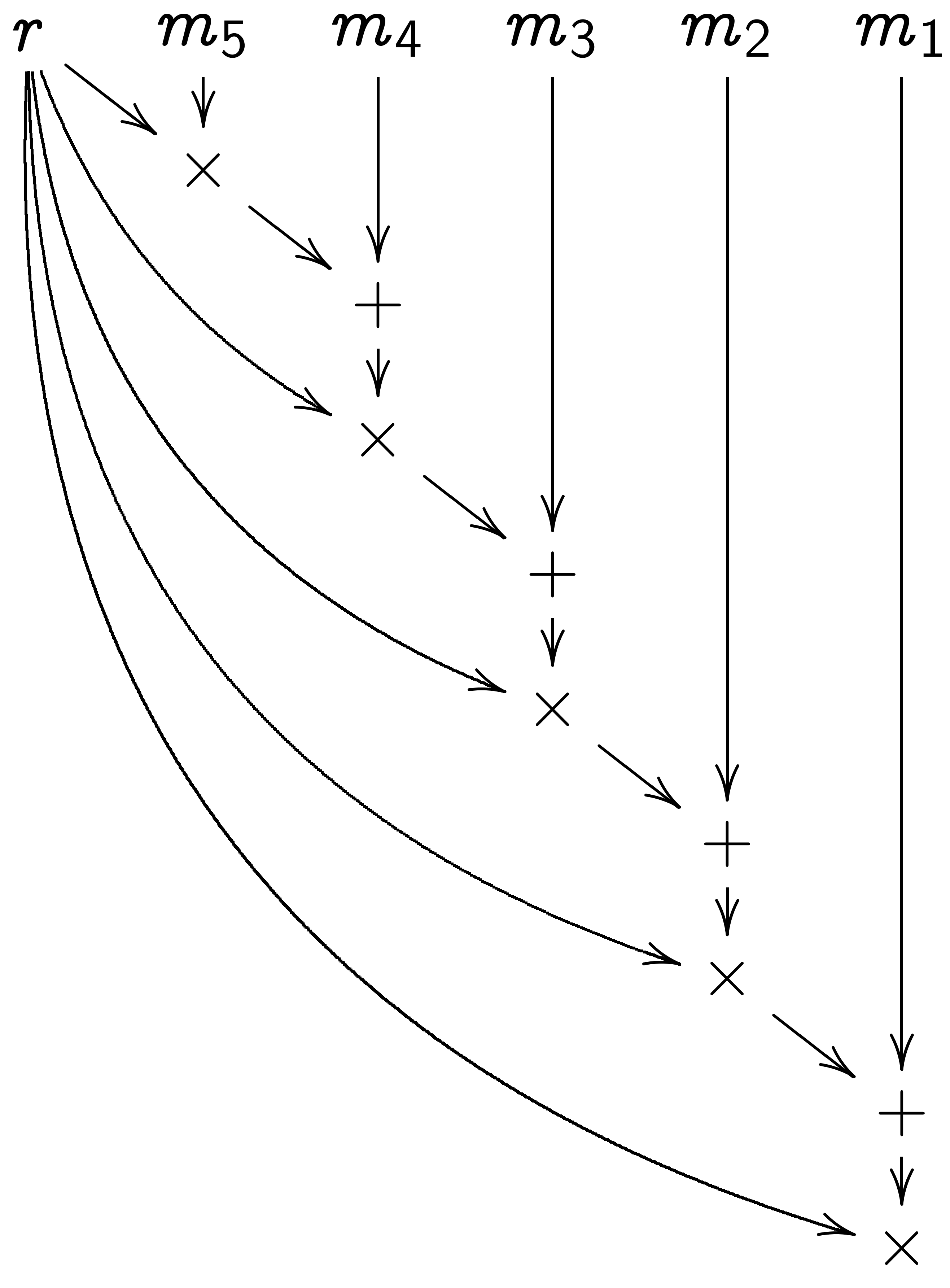
Cache $r_1, r_2, \ldots, r_\ell$? Bad performance for large ℓ: huge initialization cost; many expensive cache misses; too big for low-cost hardware.

1979 Wegman/Carter:
Another authentication function, fewer secrets $r_1, r_2, \ldots$.

1987 Karp/Rabin, 1981 Rabin:
Another authentication function, extremely short secret r, but expensive to generate.

1993 den Boer; independently 1994 Taylor; independently 1994 Johansson/Kabatianskii/Smeets:
Another authentication function, extremely short secret r, trivial to generate.

Horner's rule (const coeff 0):



Cost of this algorithm:
5 mults, 4 adds,
just like dot product.

Output in
$\mathbf{F}_q[m_1, m_2, m_3, m_4, m_5, r]$:
$m_5 r^5 + m_4 r^4 + \cdots + m_1 r$.

Substituting any message
$(m_1, m_2, m_3, m_4, m_5) \in \mathbf{F}_q^5$
produces poly in $\mathbf{F}_q[r]$;
message $\mapsto$ poly is injective.

Secure for authentication:
at most 5 values of r are roots
of any shifted difference
of polys for distinct messages.

1 multiplication per chunk.

Can we do better?

Classic observation (1955 Motzkin, 1958 Belaga, et al.): For each $\varphi \in \mathbf{C}[r]$ there is an algorithm that computes φ using $\approx (\deg \varphi)/2$ multiplications.

Idea: $$\Big(\big(((ar + b)(r^2 + c) + d\big) (r^2 + e) + f\Big)(r^2 + \quad) + h.$$

Doesn't solve the authentication problem. This set of algorithms maps *surjectively* but not *injectively* to $\mathbf{C}[r]$.

1970 Winograd: Can achieve $\approx (\deg \varphi)/2$ multiplications with "rational preparation," i.e., rational map $\varphi \mapsto$ algorithm.

Idea: $\big((r + a)(r^2 + b) + r + c\big)$
$(r^4 + d) + (r + e)(r^2 + f) + r +$.

Adapt idea to non-monic φ and to $\deg \varphi \notin \{1, 3, 7, 15, \ldots$.

"Aha! $\big((r + a)(r^2 + b) + r + c\big)$
$(r^4 + d) + (r + e)(r^2 + f) + r +$
is an authenticator of message $(a, b, c, d, e, f,$)."

Have to be careful. Injective? Not just for fixed degree?

Fix odd prime p. Define
$H : \{0, 2, 4, \ldots, p-3\ * \to \mathbf{F}_p[r]$
by $H() = 0$; $H(m_1) = r + m_1$;
$H(m_1, \ldots, m_\ell) =$
$H(m_{t+1}, \ldots, m_\ell) +$
$(r^t + m_t)H(m_1, \ldots, m_{t-1})$ if
$t \in \{2, 4, 8, 16, \ldots\ , t \le \ell < 2t$.

e.g. $H(m_1, m_2) =$
$(r + m_1)(r^2 + m_2)$;
$H(m_1, m_2, m_3) =$
$(r + m_1)(r^2 + m_2) + (r + m_3)$.

(Could change $H()$ to 1,
avoid special case for $\ell = 1$.
But my H is slightly faster.)

Easy to prove: H is injective.

Use $r H(m) + s_n$ as authenticator of nth message m.

(Good choice of p: $2^{107} - 1$.
Put 13 bytes into each chunk.)

Combines all the advantages of previous authenticators:
extremely short secret r,
trivial to generate;
1/2 multiplications per chunk.