Edwards coordinates for elliptic curves

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Joint work with:

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Weierstrass coordinates

Fix a field k with  $2 \neq 0$ . Well-known fact:  $E: y^2 = x^3 + ax + b$  over k

 $y^2 = x^3 + ax + b$ ?"

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- Fix  $a, b \in k$  with  $4a^3 + 27b^2 \neq 0$ .
- The points of the "elliptic curve"
- form a commutative group E(k).
- "So the group is  $\{(x, y) \in k \times k :$
- Not exactly! It's  $\{(x, y) \in k \times k :$

# Weierstrass coordinates

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The points of the "elliptic curve"  $E: y^2 = x^3 + ax + b$  over kform a commutative group E(k).

"So the group is 
$$\{(x,y)\in k imes k: y^2=x^3+ax+b\}$$
?"

Not exactly! It's  $\{(x, y) \in k \times k :$  $y^2 = x^3 + ax + b\} \cup \{\infty\}.$ 

Define  $x_3 = \lambda^2 - x_1 - x_2$ and  $y_3 = \lambda(x_1 - x_3) - y_1$ Then  $(x_3, y_3) \in E(k)$ .

Geometric interpretation:  $(x_1, y_1), (x_2, y_2), (x_3, -y_3)$  are on the curve  $y^2 = x^3 + ax + b$ and on a line;  $(x_3, y_3), (x_3, -y_3)$  are on a vertical line.

"So that's the group law?"  $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)?$ 

To add  $(x_1, y_1), (x_2, y_2) \in E(k)$ :

where  $\lambda = (y_2 - y_1)/(x_2 - x_1)$ .

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$$x_3=\lambda^2-x_1-x_2$$
  
and  $y_3=\lambda(x_1-x_3)-y_1$   
where  $\lambda=(y_2-y_1)/(x_2-x_1).$   
Then  $(x_3,y_3)\in E(k).$ 

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Not exactly! Definition of  $\lambda$ assumes that  $x_2 \neq x_1$ . Define  $x_3 = \lambda^2 - x_1 - x_2$ and  $y_3 = \lambda(x_1 - x_3) - y_1$ where  $\lambda = (3x_1^2 + a)/2y_1$ . Then  $(x_3, y_3) \in E(k)$ .

Geometric interpretation: The curve's tangent line at

"So that's the group law?"

# One special case for doubling?"

- $(x_1, y_1)$  passes through  $(x_3, -y_3)$ .

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Geometric interpretation: The curve's tangent line at  $(x_1, y_1)$  passes through  $(x_3, -y_3)$ .

"So that's the group law?" One special case for doubling?"

Not exactly! More exceptions: e.g.,  $y_1$  could be 0.  $\infty + (x_2, y_2) = (x_2, y_2);$  $(x_1, y_1) + \infty = (x_1, y_1);$  $(x_1, y_1) + (x_1, -y_1) = \infty;$  $y_3 = \lambda(x_1 - x_3) - y_1$  ,  $\lambda = (3x_1^2 + a)/2y_1;$  $y_3 = \lambda(x_1 - x_3) - y_1$  ,  $\lambda = (y_2 - y_1)/(x_2 - x_1).$ 

Six cases overall:  $\infty + \infty = \infty$ ; for  $y_1 
eq 0$ ,  $(x_1, y_1) + (x_1, y_1) =$  $(x_3, y_3)$  with  $x_3 = \lambda^2 - x_1 - x_2$ , for  $x_1 \neq x_2$ ,  $(x_1, y_1) + (x_2, y_2) =$  $(x_3,y_3)$  with  $x_3=\lambda^2-x_1-x_2$ ,

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for  $y_1 \neq 0$ ,  $(x_1, y_1) + (x_1, y_1) =$   
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E(k) is a commutative group:  $-\infty = \infty; \ -(x,y) = (x,-y).$ Associativity: (P+Q) + R = P + (Q+R).Straightforward but tedious: use a computer-algebra system to check each possible case. Or relate each P + Q case to "ideal-class product." Many other proofs, but can't escape case analysis.

Has neutral element  $\infty$ , and -:

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Projective coordinates Can eliminate some exceptions. Define (X : Y : Z), for  $(X, Y, Z) \in k \times k \times k - \{(0, 0, 0)\},\$ as  $\{(rX, rY, rZ) : r \in k - \{0\}\}.$ Could split into cases: (X : Y : Z) =(X/Z : Y/Z : 1) if  $Z \neq 0$ ; (X : Y : 0) =(X/Y : 1 : 0) if  $Y \neq 0$ ; (X:0:0) = (1:0:0).But scaling unifies all cases.

# Projective coordinates

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Write  $\mathbf{P}^2(k) = \{(X : Y : Z)\}.$ Revised definition: E(k) = $\{(X:Y:Z)\in {\mathbf{P}}^2(k):$ Could split into cases: (X : Y : Z) = (x : y : 1)where x = X/Z, y = Y/Z. Note that  $y^2 = x^3 + ax + b$ . Corresponds to previous (x, y).  $X^3 = 0$  so X = 0 so  $Y \neq 0$ so (X : Y : Z) = (0 : 1 : 0).

Corresponds to previous  $\infty$ .

 $Y^2 Z = X^3 + aXZ^2 + bZ^3$ .

If  $(X : Y : Z) \in E(k)$  and  $Z \neq 0$ : If  $(X : Y : Z) \in E(k)$  and Z = 0:

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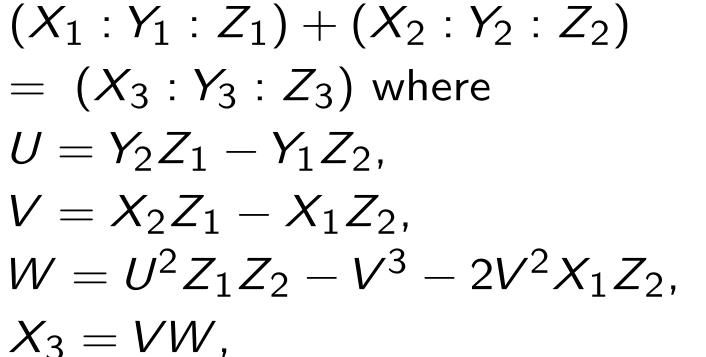
Could split into cases:

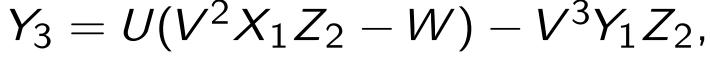
If  $(X : Y : Z) \in E(k)$  and  $Z \neq 0$ : (X : Y : Z) = (x : y : 1)where x = X/Z, y = Y/Z. Note that  $y^2 = x^3 + ax + b$ . Corresponds to previous (x, y).

If 
$$(X : Y : Z) \in E(k)$$
 and  $Z = 0$ :  
 $X^3 = 0$  so  $X = 0$  so  $Y \neq 0$   
so  $(X : Y : Z) = (0 : 1 : 0)$ .  
Corresponds to previous  $\infty$ .

 $(X_1:Y_1:Z_1)+(X_2:Y_2:Z_2)$  $= (X_3 : Y_3 : Z_3)$  where  $U = Y_2 Z_1 - Y_1 Z_2$  $V = X_2 Z_1 - X_1 Z_2$  $X_3 = VW$ ,  $Z_3 = V^3 Z_1 Z_2.$ 

"Aha! No more divisions by 0." Compare to previous formulas:  $x_3 = \lambda^2 - x_1 - x_2$ and  $y_3 = \lambda(x_1 - x_3) - y_1$ where  $\lambda = (y_2 - y_1)/(x_2 - x_1)$ .





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=  $(X_3 : Y_3 : Z_3)$  where  
 $U = Y_2 Z_1 - Y_1 Z_2,$   
 $V = X_2 Z_1 - X_1 Z_2,$   
 $W = U^2 Z_1 Z_2 - V^3 - 2V^2 X_1 Z_2,$   
 $X_3 = VW,$   
 $Y_3 = U(V^2 X_1 Z_2 - W) - V^3 Y_1 Z_2,$   
 $Z_3 = V^3 Z_1 Z_2.$ 

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Compare to previous formulas:  $x_3=\lambda^2-x_1-x_2$ and  $y_3 = \lambda(x_1 - x_3) - y_1$ where  $\lambda = (y_2 - y_1)/(x_2 - x_1)$ .

Oops, still have exceptions! Formulas give bogus  $(X_3, Y_3, Z_3) = (0, 0, 0)$ if  $(X_1 : Y_1 : Z_1) = (0 : 1 : 0)$ . Same problem for doubling. Formulas produce (0 : 1 : 0) for if  $Y_1 \neq 0$  and  $Z_1 \neq 0$ but not if  $Y_1 = 0$ .

To define complete group law, use six cases as before.

 $(X_1:Y_1:Z_1) + (X_1:-Y_1:Z_1)$ 

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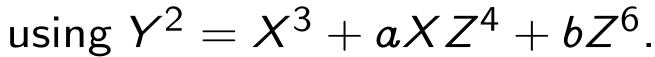
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Jacobian coordinates using weights 2, 3, 1": Redefine (X : Y : Z) as Redefine E(k)Could again split into cases for  $(X : Y : Z) \in E(k)$ : if  $Z \neq 0$  then (X : Y : Z) =

then (X : Y : Z) = (1 : 1 : 0).

- "Weighted projective coordinates
- $\{(r^2X, r^3Y, rZ) : r \in k \{0\}\}.$



 $(X/Z^2 : Y/Z^3 : 1);$  if Z = 0

# Jacobian coordinates

"Weighted projective coordinates using weights 2, 3, 1":

Redefine (X : Y : Z) as  $\{(r^2X, r^3Y, rZ): r \in k - \{0\}\}.$ 

Redefine E(k)using  $Y^2 = X^3 + aXZ^4 + bZ^6$ .

Could again split into cases for  $(X : Y : Z) \in E(k)$ : if  $Z \neq 0$  then (X : Y : Z) = $(X/Z^2 : Y/Z^3 : 1)$ ; if Z = 0then (X : Y : Z) = (1 : 1 : 0).

 $(X_1:Y_1:Z_1)+(X_2:Y_2:Z_2)$  $= (X_3 : Y_3 : Z_3)$  where  $U_1 = X_1 Z_2^2$ ,  $U_2 = X_2 Z_1^2$ ,  $S_1 = Y_1 Z_2^3$ ,  $S_2 = Y_2 Z_1^3$ ,  $H = U_2 - U_1, J = S_2 - S_1,$  $X_3 = -H^3 - 2U_1H^2 + J^2$ .  $Y_3 = -S_1 H^3 + J(U_1 H^2 - X_3),$  $Z_3 = Z_1 Z_2 H.$ 

Streamlined algorithm uses 16 multiplications, of which 4 are squarings.

5 squarings. (2001 Bernstein)

# (1986 Chudnovsky/Chudnovsky)

$$(X_{1}:Y_{1}:Z_{1}) + (X_{2}:Y_{2}:Z_{2})$$

$$= (X_{3}:Y_{3}:Z_{3}) \text{ where}$$

$$U_{1} = X_{1}Z_{2}^{2}, U_{2} = X_{2}Z_{1}^{2},$$

$$S_{1} = Y_{1}Z_{2}^{3}, S_{2} = Y_{2}Z_{1}^{3},$$

$$H = U_{2} - U_{1}, J = S_{2} - S_{1},$$

$$X_{3} = -H^{3} - 2U_{1}H^{2} + J^{2},$$

$$Y_{3} = -S_{1}H^{3} + J(U_{1}H^{2} - X_{3}),$$

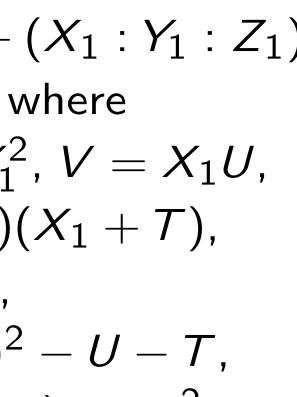
$$Z_{3} = Z_{1}Z_{2}H.$$

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Still need all six cases. Why use Jacobian coordinates? 8 mults (including 5 squarings) if a = -3 (e.g. NIST's curves): If  $Y_1 \neq 0$  then  $(X_1:Y_1:Z_1) + (X_1:Y_1:Z_1)$  $= (X_3, Y_3, Z_3)$  where  $T = Z_1^2, U = Y_1^2, V = X_1 U,$  $W = 3(X_1 - T)(X_1 + T),$  $X_3 = W^2 - 8V$ .  $Z_3 = (Y_1 + Z_1)^2 - U - T$ ,  $Y_3 = W(4V - X_3) - 8U^2$ .

# for Jacobian-coordinate doubling



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# Unified addition laws

Do addition laws have to fail for doublings? Not necessarily!

Example: "Jacobi intersection"  $s^{2} + c^{2} = t^{2}$ .  $as^{2} + d^{2} = t^{2}$ has 17-multiplication addition formula that works for doublings. (1986 Chudnovsky/Chudnovsky) 16. (2001 Liardet/Smart) Many more "unified formulas."

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Do we need 6 cases? No! Can cover  $E(k) \times E(k)$ using 3 addition laws. (1985 H. Lange/Ruppert)

How about just *one* law that covers  $E(k) \times E(k)$ ? One complete addition law?

Bad news: "Theorem 1. The smallest cardinality of a on E equals two." (1995 Bosma/H. Lenstra)

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Interlude: The circle Fix a field k with  $2 \neq 0$ . Fix  $c \in k$  with  $c \neq 0$ . is a commutative group with  $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ where  $x_3 = (x_1y_2 + y_1x_2)/c$ and  $y_3 = (y_1y_2 - x_1x_2)/c$ . Exercise: on curve. Exercise: associative. Look, a complete addition law! But it's not elliptic.

 $\{(x,y)\in k imes k:x^2+y^2=c^2\}$ 

# Interlude: The circle

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# Edwards curves

Fix a field k with  $2 \neq 0$ . and with d not a square.

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is a commutative group with  $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$ defined by Edwards addition law:

$$egin{aligned} x_3 &= rac{x_1y_2 + x_1}{c(1+dx_1)} \ y_3 &= rac{y_1y_2 - x_1}{c(1-dx_1)} \end{aligned}$$

- Fix  $c, d \in k$  with  $cd(1 dc^4) \neq 0$

# $c^{2}(1 + dx^{2}y^{2})$

 $y_1x_2$  $\overline{x_2y_1y_2}$ '

 $x_1 x_2$  $\overline{c(1-dx_1x_2y_1y_2)}.$ 

# Edwards curves

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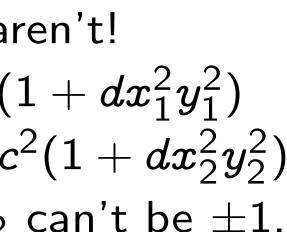
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"What if denominators are 0?"

Answer: They aren't! If  $x_1^2 + y_1^2 = c^2(1 + dx_1^2y_1^2)$ and  $x_2^2 + y_2^2 = c^2(1 + dx_2^2y_2^2)$ then  $dx_1x_2y_1y_2$  can't be  $\pm 1$ .

Outline of proof: If  $(dx_1x_2y_1y_2)^2 = 1$  then curve equation implies  $(x_1 + dx_1x_2y_1y_2y_1)^2 =$  $dx_1^2y_1^2(x_2+y_2)^2$ . Conclude that d is a square. But d is not a square! Q.E.D.



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So 
$$(x_3, y_3)$$
 is all  
 $x_3 = \frac{x_1y_2 + x_1y_2 + x_1y_$ 

Exercise: on curve.

Exercise: associative.

Magma computer-algebra system solves both exercises in 20 secs.

# ways defined:

 $rac{y_1x_2}{x_2y_1y_2)}$ '

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Neutral element (0, c). Commutative. -(x, y) = (-x, y).

So  $(x_3, y_3)$  is always defined:  $x_3 = rac{x_1y_2 + y_1x_2}{c(1 + dx_1x_2y_1y_2)},$  $y_3 = rac{y_1y_2 - x_1x_2}{c(1 - dx_1x_2y_1y_2)}.$ 

Neutral element (0, c). Commutative. -(x, y) = (-x, y).

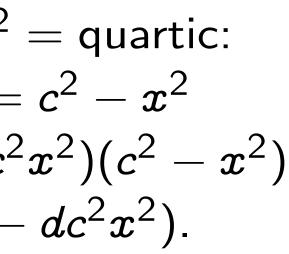
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Is this elliptic (after desingularization)? Yes! Transform to  $z^2 = quartic$ :  $y^2(1-dc^2x^2)=c^2-x^2$ so  $z^2 = (1 - dc^2 x^2)(c^2 - x^2)$ where  $z = y(1 - dc^2 x^2)$ . Or transform to  $v^2 = \text{cubic}$ :  $v^2 = eu^3 + (4 - 2e)u^2 + eu$ where u = (c + y)/(c - y), v=2cu/x,  $e=1-dc^4$ .

Obtain every elliptic curve having a point of order 4 and a unique point of order 2.



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Obtain every elliptic curve having a point of order 4 and a unique point of order 2.

So many elliptic curves have a complete addition law. What about Bosma/Lenstra? Recall "Theorem 1. The smallest cardinality of a on *E* equals two." "Complete" in the theorem means "covers  $E(\overline{k}) \times E(\overline{k})$ "; k is the algebraic closure of k. The Edwards addition law has exceptions defined over  $k(\sqrt{d})$ ,

- complete system of addition laws
- but no exceptions defined over k.

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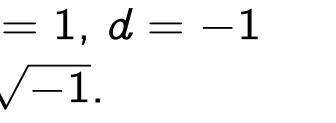
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Historical notes on the addition law:

Euler/Gauss: c = 1, d = -1over field with  $\sqrt{-1}$ .

Theorem: over k, obtain all elliptic curves.

2007 Bernstein/Lange: general d. In particular: complete for non-square d. Also streamlined formulas, coming next!



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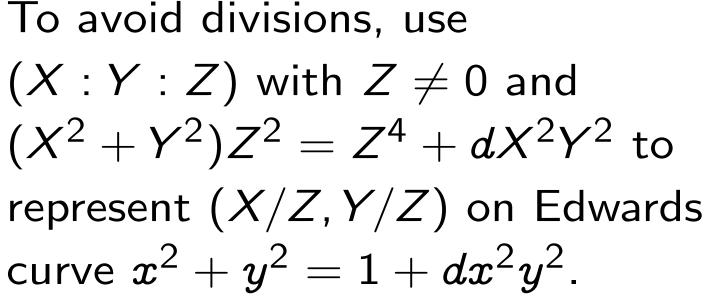
no loss of generality. To avoid divisions, use (X:Y:Z) with  $Z \neq 0$  and curve  $x^2 + y^2 = 1 + dx^2 y^2$ .

$$egin{aligned} x_3 &= rac{x_1y_2 + y_3}{1 + dx_1x_2} \ y_3 &= rac{y_1y_2 - x_3}{1 + dx_1x_2} \end{aligned}$$

 $1 - dx_1 x_2 y_1 y_2$ 

# Computations on Edwards curves

# Take c = 1 for simplicity, speed;



- Edwards addition law (for c = 1):
  - $y_1x_2$
  - $2y_1y_2$
  - $r_1 x_2$

# Computations on Edwards curves

Take c = 1 for simplicity, speed; no loss of generality.

To avoid divisions, use (X:Y:Z) with  $Z \neq 0$  and  $(X^2 + Y^2)Z^2 = Z^4 + dX^2Y^2$  to represent (X/Z, Y/Z) on Edwards curve  $x^2 + y^2 = 1 + dx^2y^2$ .

Edwards addition law (for c = 1):

$$x_3 = rac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2}, \ y_3 = rac{y_1y_2 - x_1x_2}{1 - dx_1x_2y_1y_2}.$$

Clear denominators:

$$X_{3} = Z_{1}Z_{2}(X_{1})$$

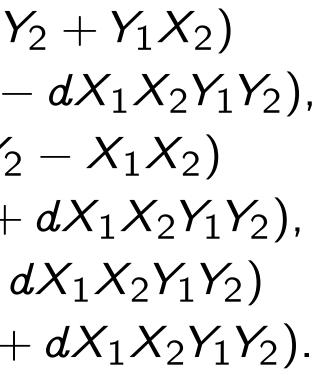
$$\cdot (Z_{1}^{2}Z_{2}^{2} + Z_{3}) = Z_{1}Z_{2}(Y_{1}Y_{2})$$

$$\cdot (Z_{1}^{2}Z_{2}^{2} + Z_{3}) = (Z_{1}^{2}Z_{2}^{2} - Z_{1}^{2})$$

$$\cdot (Z_{1}^{2}Z_{2}^{2} - Z_{1}^{2})$$

Rewrite  $x_1y_2 + x_2y_1$  as  $(x_1+y_1)(x_2+y_2)-x_1x_2-y_1y_2$ , exploit common subexpressions.

12 multiplications (one by d, one a squaring), 7 additions. Still complete.



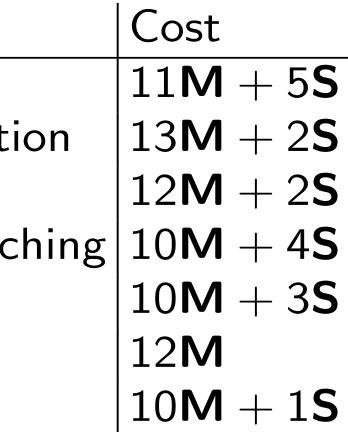
Clear denominators:

$$egin{aligned} X_3 &= Z_1 Z_2 (X_1 Y_2 + Y_1 X_2) \ &\cdot (Z_1^2 Z_2^2 - d X_1 X_2 Y_1 Y_2), \ Y_3 &= Z_1 Z_2 (Y_1 Y_2 - X_1 X_2) \ &\cdot (Z_1^2 Z_2^2 + d X_1 X_2 Y_1 Y_2), \ Z_3 &= (Z_1^2 Z_2^2 - d X_1 X_2 Y_1 Y_2) \ &\cdot (Z_1^2 Z_2^2 + d X_1 X_2 Y_1 Y_2). \end{aligned}$$

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Comparison of addition costs if curve parameters are small: System Jacobian Jacobi intersection Projective Chudnovsky caching Jacobi quartic Hessian Edwards



Comparison of addition costs if curve parameters are small:

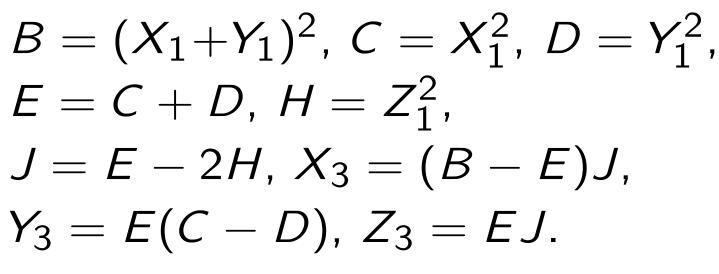
System	Cost
Jacobian	11 <b>M</b> + 5 <b>S</b>
Jacobi intersection	13M + 2S
Projective	12M + 2S
Chudnovsky caching	$10\mathbf{M} + 4\mathbf{S}$
Jacobi quartic	$10\mathbf{M} + 3\mathbf{S}$
Hessian	12 <b>M</b>
Edwards	$10\mathbf{M} + 1\mathbf{S}$

Can save time in doubling: rewrite  $1 + dx_1^2y_1^2$  as  $x_1^2 + y_1^2$ (as suggested by Marc Joye); exploit common subexpressions.

 $E = C + D, H = Z_1^2,$  $J = E - 2H, X_3 = (B - E)J,$  $Y_3 = E(C - D), Z_3 = EJ.$ 

7 multiplications (4 of which are squarings), 6 additions.

# rewrite $1 - dx_1^2 y_1^2$ as $2 - x_1^2 - y_1^2$ ;

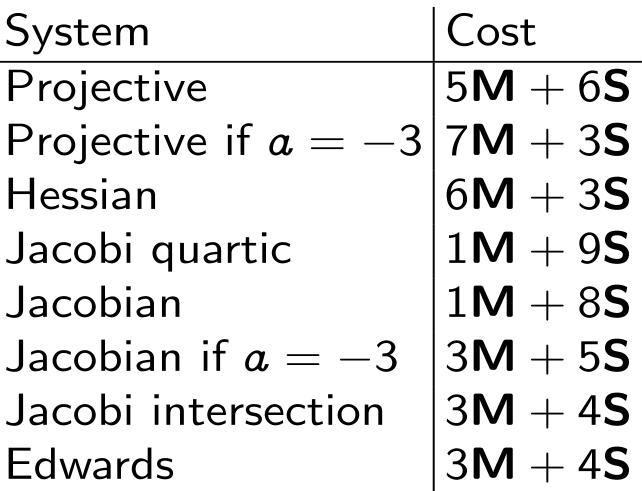


Can save time in doubling: rewrite  $1 + dx_1^2y_1^2$  as  $x_1^2 + y_1^2$ (as suggested by Marc Joye); rewrite  $1 - dx_1^2 y_1^2$  as  $2 - x_1^2 - y_1^2$ ; exploit common subexpressions.

$$B = (X_1 + Y_1)^2$$
,  $C = X_1^2$ ,  $D = Y_1^2$ ,  
 $E = C + D$ ,  $H = Z_1^2$ ,  
 $J = E - 2H$ ,  $X_3 = (B - E)J$ ,  
 $Y_3 = E(C - D)$ ,  $Z_3 = EJ$ .

7 multiplications (4 of which are squarings), 6 additions.

Comparison of doubling costs if curve parameters are small: System Projective Hessian Jacobi quartic Jacobian Jacobian if a = -3Jacobi intersection Edwards



Comparison of doubling costs if curve parameters are small:

System	Cost
Projective	5M + 6S
Projective if $a = -3$	7M + 3S
Hessian	6M + 3S
Jacobi quartic	1M + 9S
Jacobian	1M + 8S
Jacobian if $a = -3$	3 <b>M</b> + 5 <b>S</b>
Jacobi intersection	3 <b>M</b> + 4 <b>S</b>
Edwards	3 <b>M</b> + 4 <b>S</b>

# A cryptographic example

"Curve25519":  $v^2 = u^3 + 486662u^2 + u$ over the field  $k = \mathbf{Z}/(2^{255} - 19)$ . Software speed records for elliptic-curve Diffie-Hellman. (2005 Bernstein)

 $n, P \mapsto nP$  is very fast using Montgomery coordinates. (1987 Montgomery)

 $n_0, n_1, P_0, P_1 \mapsto n_0 P_0 + n_1 P_1?$ Critical for digital signatures. Batch verification: many  $n_i$ 's.

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Multi-scalar multiplication: Montgomery isn't very fast. Jacobian is faster. Edwards is the new winner!

Curve25519 is equivalent over k to the Edwards curve

Transformation is easy:  $x = \sqrt{486664 u/v}$ y = (u - 1)/(u + 1).Map points to Edwards curve. Use Edwards addition law. Map back to Curve25519 or use Edwards everywhere!

 $x^2 + y^2 = 1 + (1 - 1/121666)x^2y^2$ .

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Curve25519 is equivalent over k to the Edwards curve  $x^2 + y^2 = 1 + (1 - 1/121666)x^2y^2$ .

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What about  $n \mapsto nQ$ using standard Q = (9, ...)?Faster than  $n, P \mapsto nP$ ? If  $n = n_0 + 2^{16} n_1 + \cdots$ Precompute  $2^{16}Q$  etc. Use multi-scalar multiplication. Edwards curves work well for all of these applications. Very fast doublings. Very fast additions. Complete addition law helps stop secrets from leaking through side channels.

# then $nQ = n_0Q + 2^{16}n_1Q + \cdots$ .

What about  $n \mapsto nQ$ using standard Q = (9, ...)?Faster than  $n, P \mapsto nP$ ?

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Edwards curves work well for all of these applications. Very fast doublings. Very fast additions. Complete addition law helps stop secrets from leaking through side channels. More on Edwards curves:

http://cr.yp.to /newelliptic.html