Classic question about the Diffie-Hellman system:
How quickly can we compute $n$th powers mod $p$?

“Modular exponentiation.”

Assume standard prime $p$;
e.g. $p = 2^{262} - 5081$.
How quickly can we compute $g^n \mod 2^{262} - 5081$, given integers $g, n$?
This talk asks the analogous question for elliptic-curve Diffie-Hellman: How quickly can we compute $n$th multiples in an elliptic-curve group?

“Elliptic-curve scalar multiplication.”

Assume standard field and standard elliptic curve.
e.g. NIST P-224: the elliptic curve
\[ y^2 = x^3 - 3x + a_6 \] over \( \mathbb{Z}/p \).
Here \( p = 2^{224} - 2^{96} + 1 \)
and \( a_6 = 18958286285566608 
00040866854449392 
64155046809686793 
21075787234672564. \)

e.g. NIST P-256: the elliptic curve
\[ y^2 = x^3 - 3x + \cdots \] over \( \mathbb{Z}/p \) where
\( p = 2^{256} - 2^{224} + 2^{192} + 2^{96} - 1. \)

e.g. Curve25519: the elliptic curve
\[ y^2 = x^3 + 486662x^2 + x \] over \( \mathbb{Z}/p \)
where \( p = 2^{255} - 19. \)
Your task: Given \((x, y)\) on curve, and given integer \(n \geq 0\), compute \(n\)th multiple of \((x, y)\) in the elliptic-curve group.

Warning: Answer is not \((nx, ny)\) unless you’re extremely lucky. Elliptic-curve point addition is not vector addition; \((x, y) + (x', y')\) is almost never \((x + x', y + y')\).

Can emphasize this by changing notation: \(\oplus\), \(\bigoplus\), \([n]\), etc. But this talk uses simplified notation.
Similar tasks are critical for elliptic-curve signatures.

e.g. Schnorr signatures, unfortunately patented:

Signer has secret key $n$, public key $nB$.

To sign $m$: choose random $z$, uniform in $\{0, 1, \ldots, \#\langle B \rangle - 1\}$; compute $r = \text{SHA-256}(zB, m)$; compute $s = z + rn \mod \#\langle B \rangle$; send $(m, r, s)$.

To verify $(m, r, s)$: Check $r = \text{SHA-256}(sB - rnB, m)$. 
Multiples via additions

Typical recursive formulas:

\[ 2P = P + P. \quad 3P = 2P + P. \]
\[ 4P = 2P + 2P. \quad 5P = 3P + 2P. \]
\[ 6P = 3P + 3P. \quad 7P = 5P + 2P. \]
\[ 2nP = 7P + (n - 7)P \text{ if } 4 \leq n < 8. \]
\[ (2n + 1)P = 2nP + P \text{ if } 4 \leq n < 8. \]
\[ (4n + 1)P = 4nP + P \text{ if } 4 \leq n < 8. \]
\[ (4n + 3)P = 4nP + 3P \text{ if } 4 \leq n < 8. \]
\[ 2nP = nP + nP \text{ if } 8 \leq n. \]
\[ (8n + 1)P = 8nP + P \text{ if } 4 \leq n. \]
\[ (8n + 3)P = 8nP + 3P \text{ if } 4 \leq n. \]
\[ (8n + 5)P = 8nP + 5P \text{ if } 4 \leq n. \]
\[ (8n + 7)P = 8nP + 7P \text{ if } 4 \leq n. \]
This “addition chain” ("length-3 sliding windows") uses $\approx \lg n$ doublings and $\approx 0.25 \lg n$ more additions to compute $nP$ for average $n$.

e.g. $\approx 320$ additions for average $n \in \{0, 1, \ldots, 2^{256} - 1\}$.

Some easy improvements from fast negation on elliptic curves: $(16n - 7)P = 16nP - 7P$, etc.

Also use "endomorphisms" for "Koblitz curves," "GLV curves."

More complicated methods replace $0.25$ by $\approx 1/\lg \lg n$. 
Explicit doubling formulas

On curve $y^2 = x^3 - 3x + a_6$:

\[ 2(x, y) = (x'', y'') \] where
\[
\lambda = \frac{(3x^2 - 3)}{2y}, \\
x'' = \lambda^2 - 2x, \\
y'' = \lambda(x - x'') - y.
\]

7 subs etc., 2 squarings, 1 more mult, 1 division.

How do we divide efficiently in a finite field?
\[ \frac{f}{g} = fg^{p-2} \] in prime field \( \mathbb{Z}/p \).
Can compute \( g^{p-2} \) with \( \approx \lg p \) squarings and 
\( \approx (\lg p)/\lg \lg p \) more mults.

e.g. \( p = 2^{224} - 2^{96} + 1: \)
223 squarings, 11 more mults.

More generally, \( \frac{f}{g} = fg^{q-2} \) 
in any field of size \( q \).

There are faster division methods (e.g. “Euclid”—beware timing attacks!); smaller “I/M ratio.” 
Special methods for some fields.
Speedup: delay divisions

Division costs many mults even with fastest division methods.

Save time by delaying divisions.

Naive division-delay method:
Store field elements as fractions until end of computation.
Divide once before output.

Mult fractions with 2 field mults.
Divide fractions with 2 field mults.
Add fractions with 3 field mults.
Speedup: unify denominators

For elliptic-curve doubling, have denominator $2y$
in $\lambda = (3x^2 - 3)/2y$;
denominator $(2y)^2$
in $x'' = \lambda^2 - 2x$;
denominator $(2y)^3$
in $y'' = \lambda(x - x'') - y$.

Subsequent computations will perform separate computations on the denominators $(2y)^2, (2y)^3$ of $x'', y''$.

Save time by manipulating denominators together.
“Jacobian coordinates”:
Store \((x, y, z)\) to represent elliptic-curve point \((x/z^2, y/z^3)\).

\[
2(x/z^2, y/z^3) = (x'', y'') \quad \text{where} \quad \\
\lambda = (3(x/z^2)^2 - 3)/2(y/z^3) = \alpha/2yz \quad \text{with} \quad \alpha = 3x^2 - 3z^4; \\
x'' = \lambda^2 - 2(x/z^2) = (\alpha^2 - 8xy^2)/(2yz)^2; \\
y'' = \lambda((x/z^2) - x'') - (y/z^3) = (12xy^2\alpha - \alpha^3 - 8y^4)/(2yz)^3.
\]
\(2(x/z^2, y/z^3) = (x_2/z_2^2, y_2/z_2^3)\)

where \(z_2 = 2yz\),
\(\alpha = 3x^2 - 3z^4\),
\(x_2 = \alpha^2 - 8xy^2\),
\(y_2 = \alpha(4xy^2 - x_2) - 8y^4\).

Easily compute with 6 squarings, 3 more mults: \(x^2, z^2, x^4, y^2, y^4, yz, xy^2, \alpha^2, \alpha(\cdots)\).

Also some subs, doublings, etc.

Use fast field arithmetic:
e.g., can delay carries and reductions in computing \(y_2\).
Speedup: difference of squares

Can compute $3x^2 - 3z^4$ as $3(x - z^2)(x + z^2)$.

Replace 3 squarings by 1 mult, 1 squaring. Revised total: 4 squarings, 4 more mults.

Note:
$3x^2 - 3z^4$ came from $3x^2 - 3$, derivative of $x^3 - 3x + a_6$.
Wouldn’t have same speedup for, e.g., $x^3 - 5x + a_6$. 
Speedup: $f^2, g^2, 2fg$

After computing $f^2$ and $g^2$ can compute $2fg$
as $(f + g)^2 - f^2 - g^2$.

In particular:
After computing $y^2$ and $z^2$
can compute $2yz$
as $(y + z)^2 - y^2 - z^2$.

Replace 1 mult with 1 squaring.
Revised total: 5 squarings,
3 more mults.
Explicit addition formulas

Similar speedups in formulas for adding distinct points.

5 squarings, 11 more mults.

Again some opportunities to delay carries, etc.
Speedup: cache results

In adding \((x_1/z_1^2, y_1/z_1^3)\) to \((x_2/z_2^2, y_2/z_2^3)\), compute many intermediates, including \(z_1^2, z_1^3\).

Often add same point again to a different point; can reuse \(z_1^2, z_1^3\).

“Chudnovsky coordinates.”
Speedup: delay fewer divisions?

Faster divisions sometimes justify delaying fewer divisions.

e.g. Do we really need fractions for $P, 3P, 5P, 7P$?


Sometimes worthwhile, depending on division speed.
Montgomery coordinates

On elliptic curves with “Montgomery form”

\[ y^2 = x^3 + a_2 x^2 + x, \]

preferably with small \((a_2 - 2)/4:\)

\[ n(x_1, \ldots) = (x_n/z_n, \ldots) \text{ where } \]
\[ z_1 = 1; \quad x_{2m} = (x_m^2 - z_m^2)^2; \]
\[ z_{2m} = 4x_m z_m (x_m^2 + a_2 x_m z_m + z_m^2); \]
\[ x_{2m+1} = 4(x_m x_{m+1} - z_m z_{m+1})^2; \]
\[ z_{2m+1} = 4(x_m z_{m+1} - z_m x_{m+1})^2 x_1. \]

Can also figure out \(y\),
or use cryptographic protocols that ignore \(y\).
Assuming \((a_2 - 2)/4\) small, main operations are 4 squarings, 5 more mults for each bit of \(n\).

Compare to Jacobian coordinates: each bit of \(n\) has 5 squarings, 3 more mults, \textit{and} on occasion 5 more squarings, 11 more mults.

Montgomery form is better if \(n\) is not gigantic.
Choosing curves

Traditional algorithm design: Have a function \( f \).
Want fastest algorithm that computes \( f \).

Cryptographic algorithm design: Have gigantic collection of apparently-safe functions \( f \).
Want fastest algorithm that computes some \( f \).
Elliptic-curve Diffie-Hellman could use any elliptic curve $E$ over any finite field $\mathbb{F}_q$.

Some choices of $E, \mathbb{F}_q$ are better than others. Higher speed: easier to compute $n$th multiples in $E(\mathbb{F}_q)$. Higher security: harder to find $n$ given an $n$th multiple, i.e., to solve ECDLP. Lower bandwidth. Etc.

How do we choose $E, \mathbb{F}_q$? Which curves are best?
Occasionally an application has different criteria for $E, \mathbb{F}_q$.
e.g. Some cryptographic protocols use “pairings” and need specific “embedding degrees.”

For simplicity I’ll focus on traditional protocols:
Diffie-Hellman, ECDSA, etc.

Can also consider, e.g.,
genus-2 hyperelliptic curves.
2006.09: New speed records, faster than elliptic curves.

For simplicity I’ll focus on the elliptic-curve case.
Field size?

The group $E(\mathbb{F}_q)$ has $\approx q$ elements.

“Generic” algorithms such as “Pollard’s rho method” solve ECDLP using $\approx q^{1/2}$ simple operations. Highly parallelizable.

e.g. $\approx 2^{40}$ simple operations to solve ECDLP if $q \approx 2^{80}$. Reject $q$: too small.
$q \approx 2^{256}$ is clearly safe against these ECDLP algorithms. $
 \approx 2^{128}$ simple operations would need massive advances in computer technology.

These algorithms can finish early, but almost never do: e.g., chance
$\approx 2^{-56}$ of finishing after $2^{100}$ simple operations. No serious risk.

Popular today: $q \approx 2^{160}$. Somewhat faster arithmetic.
I don’t recommend this; I can imagine $2^{80}$ simple operations.
Field degree?

Field size $q$ is a power of field characteristic $p$. Many possibilities for field degree $(\log q)/(\log p)$.

e.g. $q = 2^{255} - 19$; prime;
$p = 2^{255} - 19$; degree 1.

e.g. $q = (2^{61} - 1)^5$;
$p = 2^{61} - 1$; degree 5.

e.g. $q = 2^{255}$;
$p = 2$; degree 255.

What’s the best degree?
Degree $> 1$ has a possible security problem: “Weil descent.”

e.g. Degree divisible by 4 allows ECDLP to be solved with only about $q^{0.375}$ simple operations. Need to increase $q$, outweighing all known benefits.

Other degrees are at risk too. Exactly which curves are broken by Weil descent? Very complicated answer; active research area.

Maybe we can be comfortable with degree $> 1$ despite Weil descent.
Standard argument for using small characteristic, large degree:

Arithmetic on polynomials mod 2 is just like integer arithmetic but faster: skip the carries.

Also have fast squarings.
Use fast curve endomorphisms.

Fewer bit operations for scalar multiplication in characteristic 2, compared to large characteristic.
Speculation: $\approx 4$ times fewer?
Counterargument:

Typical CPU includes circuits for integer multiplication, not for poly mult mod 2.

Large char is slower in hardware than char 2, but char 2 is slower in software than large char. Hard for char-2 standards to survive.

For simplicity I’ll assume that the counterargument wins: we won’t use char 2.
Medium char? Similar problems.

e.g. \( q = (2^{31} - 1)^8, \ p = 2^{31} - 1 \),
degree 8, polys with coefficients in \( \{0, 1, \ldots, 2^{31} - 2\} \):

Coefficient products fit comfortably into 64 bits.
Also have fast inversion.

But hard to take advantage of 128-bit products; and hard to fit into 53-bit floating-point products.
Big speed loss on many CPUs, outweighing all known benefits.
Prime shape?

Assume prime field from now on; $\mathbb{F}_q = \mathbb{F}_p = \mathbb{Z}/p$.

How to choose prime $p$? Three common choices in literature.

“Binomial”:
e.g., $2^{255} - 19$.

“Radix $2^{32}$”:
e.g., NIST prime $2^{224} - 2^{96} + 1$.

“Random”:
no special shape for $p$. 
Classic Diffie-Hellman had an argument for random primes.

Here’s the argument:
Best attack so far, namely modern “NFS” index calculus, is faster for special primes, requiring larger primes, outweighing any possible speedup.

Argument disappears for elliptic curves over prime fields. Attacker doesn’t seem to benefit from special primes; don’t have anything like NFS.
So choose prime
very close to power of 2,
saving time in field operations.

Binomial primes allow very fast
reduction, as we’ve seen.

Radix-$2^{32}$ primes also allow
very fast reduction *if*
integer arithmetic uses radix $2^{32}$.
Otherwise not quite as fast.
Different CPUs want
different choices of radix,
so binomial primes are better.
Which power of 2?

Primes not far below $2^{32}$ allow field elements to fit in $4w$ bytes, minimal waste.

Comfortable security, $w = 8$: $2^{253} + 39, 2^{253} + 51, 2^{254} + 79, 2^{255} - 31, 2^{255} - 19, 2^{255} + 95$. I recommend $2^{255} - 19$. 
Subgroup shape?

Elliptic-curve Diffie-Hellman uses standard base point $B$. Bob’s secret key is $n$; Bob’s public key is $nB$.

Order of $B$ in group should be a prime $\ell \approx q$. Otherwise ECDLP is accelerated by “Pohlig-Hellman algorithm.”

This constrains curve choice: number of elements of $E(\mathbb{F}_q)$ must have large prime divisor $\ell$. 
Quickly compute $\#E(F_q)$, number of elements of $E(F_q)$, using “Schoof’s algorithm.” Then can check for $\ell$.

Also enforce other constraints:
$\gcd\{\#E(F_q), q\} = 1$ to stop “anomalous curve attack”;
large prime divisor of “twist order” $2q + 2 - \#E(F_q)$ to stop “twist attacks”;
large embedding degree to eliminate “pairings.”
Curve shape?

How to choose $a_1, a_2, a_3, a_4, a_6$ defining elliptic curve

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6? \]

See some coefficients in explicit formulas for curve operations.

e.g. Derivative $3x^2 + 2a_2 x + a_4$ usually creates mult by $a_2$.

But formulas vary: e.g., mult by $(a_2 - 2)/4$ in Montgomery’s formulas.
Save time in these formulas by specializing coefficients.

e.g. \( y^2 = x^3 - 3x + a_6 \).

e.g. \( y^2 = x^3 + a_2 x^2 + x \).

Many other interesting choices.

Warning: some specializations can force low embedding degree or otherwise create security problems. Remember to check all the security conditions.
Note on comparing curves and comparing explicit formulas: Count CPU cycles, not field ops! Otherwise you make bad choices.

Reality: mult by small constant is as expensive as several adds.

Reality: square-to-multiply ratio is 2/3 for a typical field, not the often-presumed 4/5.

Reality: $a^2 + b^2 + c^2$ is faster than $(a^2, b^2, c^2)$. 
Current speed records use curve \( y^2 = x^3 + a_2 x^2 + x \) with small \((a_2 - 2)/4\).

Additional advantages: easily resist timing attacks; easily eliminate \( y \).

\( a_2 = 486662 \) has near-prime curve order and twist order.

“Curve25519”:

http://cr.yp.to/ecdh.html
How fast is this curve?

Let’s focus on Pentium M.

Each Pentium M cycle does \( \leq 1 \) floating-point operation: fp add or fp sub or fp mult.

Current scalar-multiplication software for Curve25519:
640838 Pentium M cycles.
589825 fp ops; \( \approx 0.92 \) per cycle.

Understand cycle counts fairly well by simply counting fp ops.
Main loop: 545700 fp ops.
2140 times 255 iterations.

Reciprocal: 43821 fp ops.
$41148 = 254 \cdot 162$ for 254 squares;
$2673 = 11 \cdot 243$ for 11 more mults.

Additional work: 304 fp ops.

Inside one main-loop iteration:
$80 = 8 \cdot 10$ for 8 adds/subs;
$55$ for mult by 121665;
$648 = 4 \cdot 162$ for 4 squarings;
$1215 = 5 \cdot 243$ for 5 more mults;
$142$ for $bx[1] + (1 - b)x[0]$ etc.
An integer mod $2^{255} - 19$ is represented in radix $2^{25.5}$ as a sum of 10 fp numbers in specified ranges.

Add/sub: 10 fp adds/subs. Delay reductions and carries!

Mult: poly mult using $10^2$ fp mults, $9^2$ fp adds; reduce using 9 fp mults, 9 fp adds; carry 11 times, each 4 fp adds; overall $2 \cdot 10^2 + 4 \cdot 10 + 3$ fp ops.

Squaring: first do 9 fp doublings; then eliminate $9^2 + 9$ fp ops; overall $1 \cdot 10^2 + 6 \cdot 10 + 2$ fp ops.