# Differential addition chains

D. J. Bernstein

Thanks to:
University of Illinois at Chicago
Danmarks Tekniske Universitet
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Motivating problem:
Given elliptic curve $E$,
integer $n$, and point $P$ on $E$,
compute $nP$ on $E$
as quickly as possible.

Many variations of problem.
Some applications reuse one $n$
for many $P$'s.
Some applications don't.
Some applications use secret $n$;
must not leak $n$ through timing.
Some applications use public $n$.
Etc.

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Focus on large-ch
curves $y^2 = x^3 +$
with small $a \in \{$

Use pair $(x, z)$ to
$P = (x/z, \ldots)$.

Computing $Q$, $R$,
takes 6 mults.
Only 5 mults if $Q$
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1987 Montgomery:

Focus on large-characteristic
curves $y^2 = x^3 + ax^2 + x$
with small $a \in \{6, 10, 14, \ldots\}$.

Use pair $(x, z)$ to represent point
$P = (x/z, \ldots)$.

Computing $Q, R, Q - R \mapsto Q + R$
takes 6 mults.
Only 5 mults if $Q - R$ has small
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Only 4 mults if $Q - R$ has small
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Only 4 mults if $Q = R$.

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Given $n$, write $P$

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$Q, R, Q - R \mapsto Q$

e.g. $n = 10$: cor

$P, \; P, \; 0 \mapsto 2P$

$2P, \; P, P \mapsto 3P$

$3P, 2P, P \mapsto 5P$

$5P, 5P, \; 0 \mapsto 10P$

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Only 5 mults if $Q - R$ has small denominator.
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Only 4 mults if $Q = R$.

Given $n$, write $P \mapsto nP$ as composition of additions $Q, R, Q - R \mapsto Q + R$.

e.g. $n = 10$: compute
$P, \; P, \; 0 \mapsto 2P$ with 4 mults;
$2P, \; P, P \mapsto 3P$ with 6 mults;
$3P, 2P, P \mapsto 5P$ with 6 mults;
$5P, 5P, \; 0 \mapsto 10P$ with 4 mults.
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$0, P, 2P, 3P, 5P, 10P$ is a **differential addition chain** starting from $0, P$: each subsequent term is $Q+R$ for some $Q, R, Q-R$ already in chain.

$0, 1, 2, 3, 5, 10$ is a differential addition chain starting from $0, 1$.

Question: Given $n$, how to find short differential addition chain starting from $0, 1$ and ending $n$? Variations: measure shortness by mults, CPU cycles, etc.

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The binary method: obtain $n, n + 1$ from $\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1$ using one addition with difference 1, one addition with difference 0.

e.g. $13P, 13P, 0 \mapsto 26P$ with 4 mults; $14P, 13P, P \mapsto 27P$ with 5 mults, if $P$ has small denominator.

Overall 9 mults for each bit of $n$, if $P$ has small denominator.

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Experiments for average 128-bit $n$
find length $\approx 1.533$ per bit,
instead of 2 per bit.
Lower bound $\approx 1.440$ per bit.

Count mults instead of length:
$\approx 8.885$ per bit,
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Disadvantages: harder to find;
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Two-dimensional question:
Given $m, n$, how to find short differential addition chain starting from the vectors $(0,0), (1,0), (0,1), (1,-1)$ and ending $(m, n)$?

Motivating problem:
Given elliptic curve $E$, integers $m, n$, and points $P, Q, P - Q$, compute $mP + nQ$ on $E$ as quickly as possible.

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For average 128-
small $P, Q, P-Q$

| dim | method |
|---|---|
| 2 | easy binary |
| 2 | Schoenmake |
| 2 | Akishita |
| 2 | new binary |
| 2 | Montgomer |
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For average 128-bit exponents,
small $P, Q, P-Q$ denominators:

| dim | method | mults per bit | unif |
|---|---|---|---|
| 2 | easy binary | 19.000 | yes |
| 2 | Schoenmakers | 17.250 | no |
| 2 | Akishita | 14.250 | no |
| 2 | new binary | 14.000 | yes |
| 2 | Montgomery | 10.261 | no |
| 2 | new ext gcd | 9.918 | no |
| 1 | easy binary | 9.000 | yes |
| 1 | standard | 8.885 | no |
| | Fibonacci case | 8.643 | |

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addition chain
e vectors
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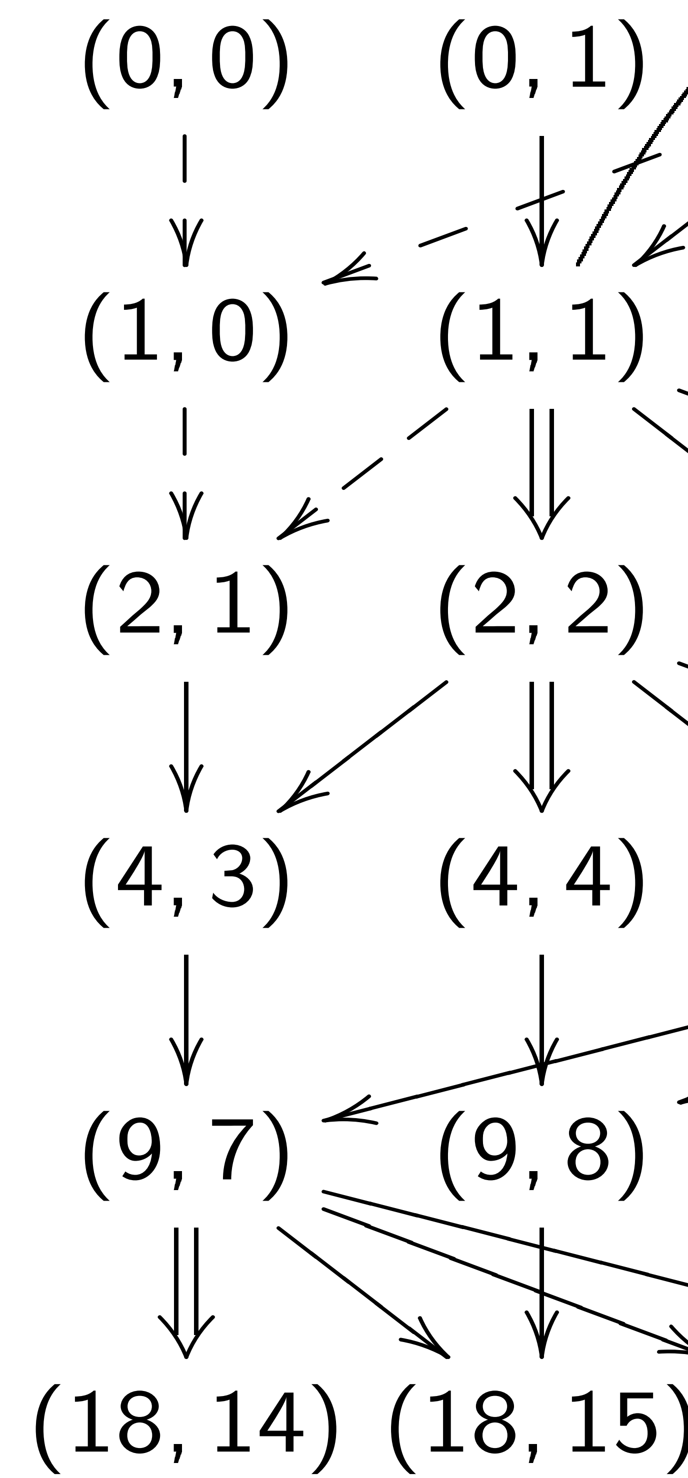

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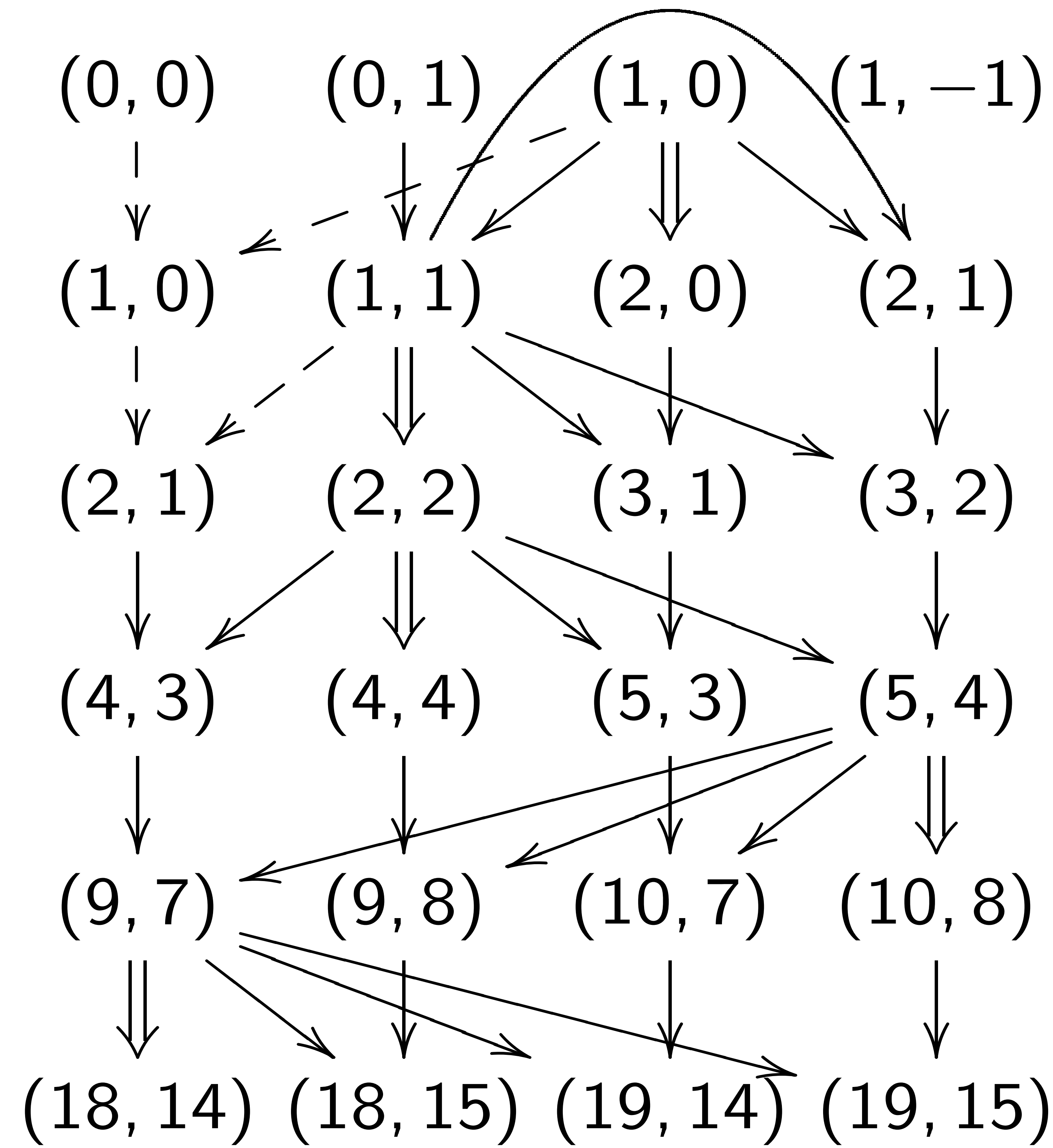

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Q denominators:

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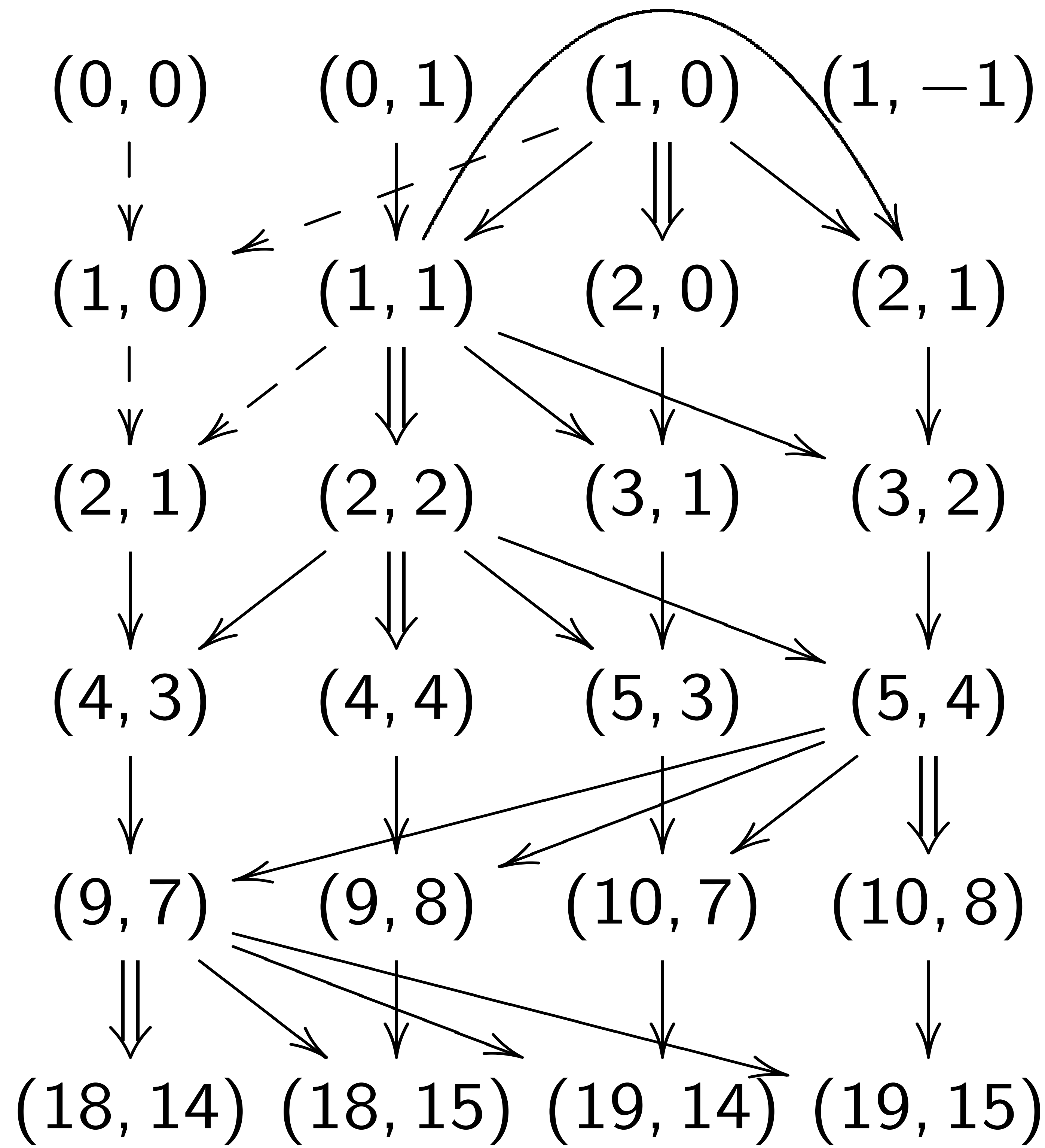
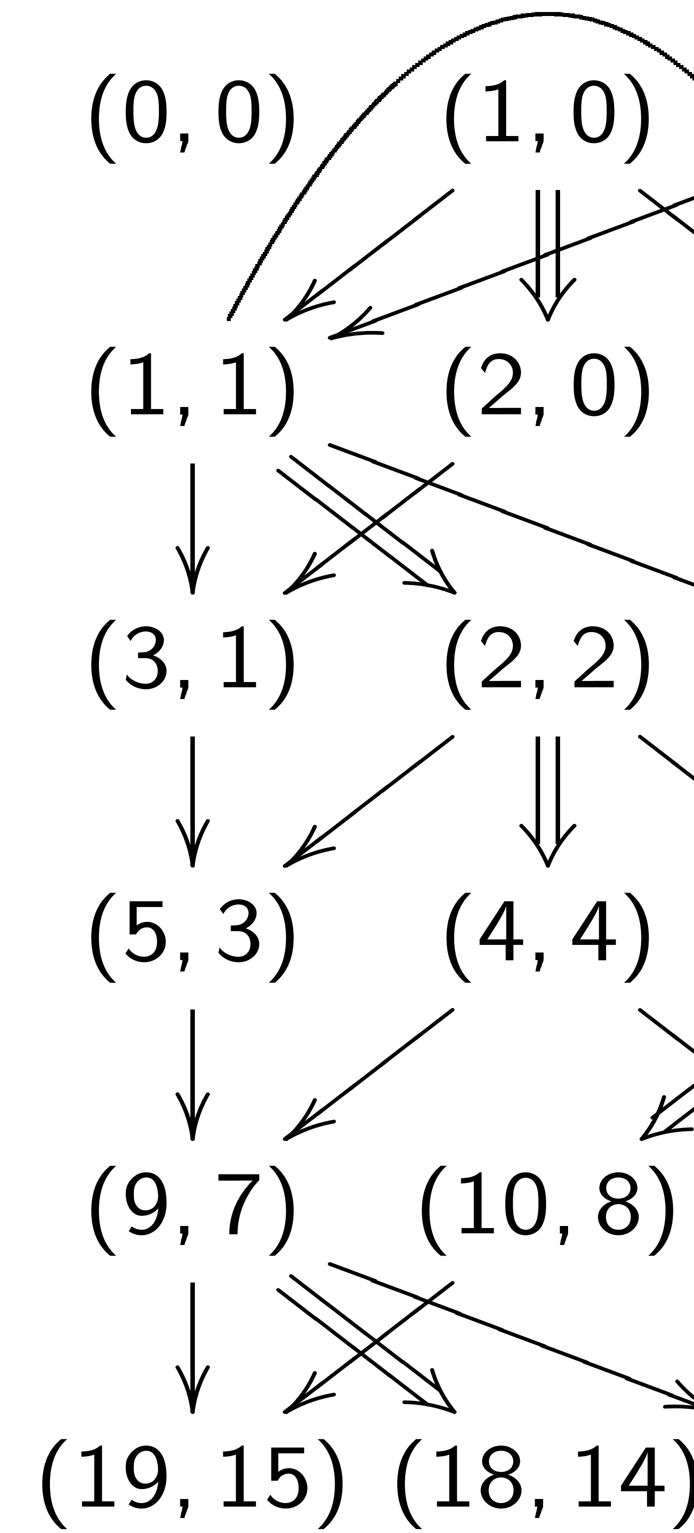



New dim-2 binar



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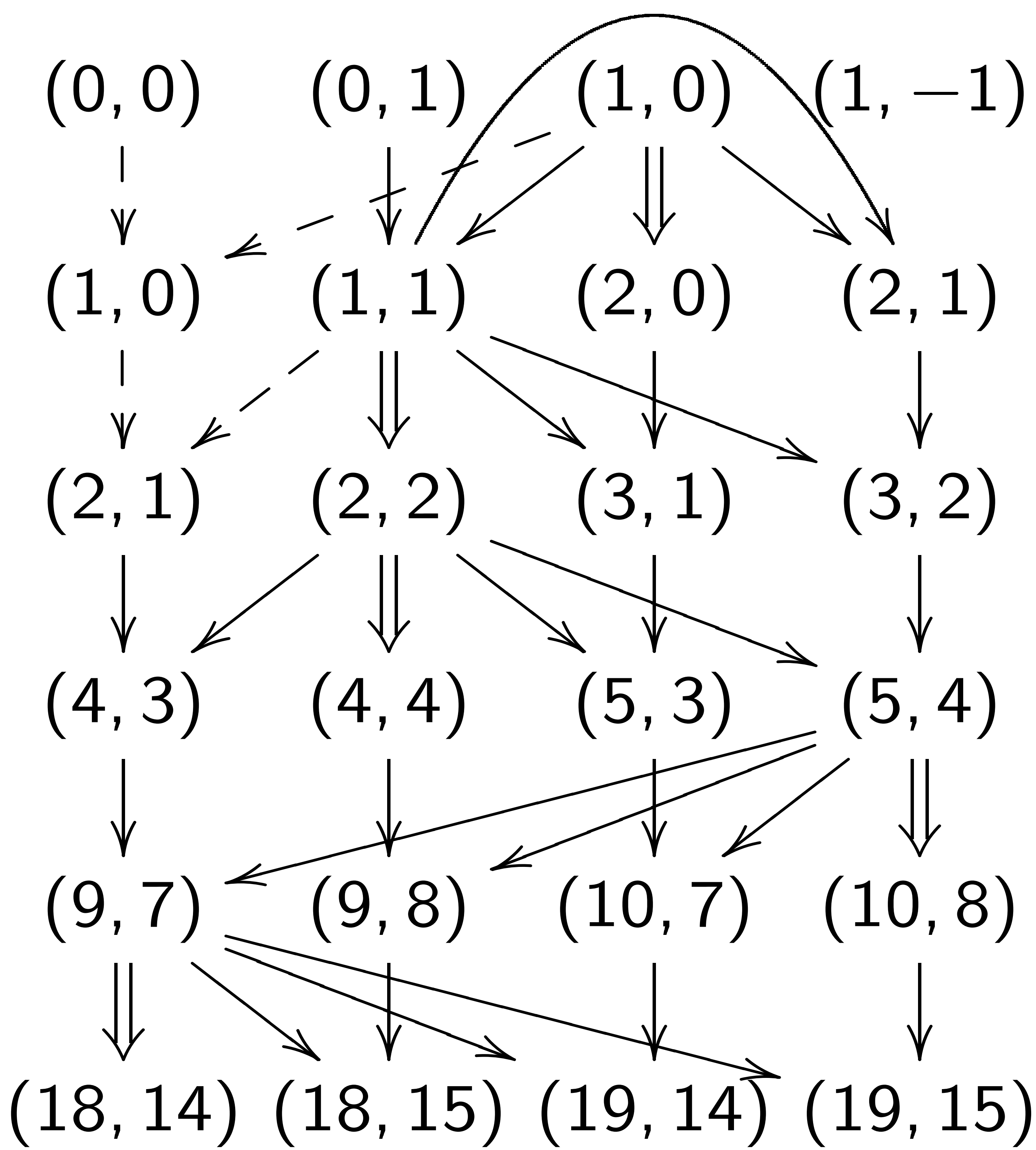


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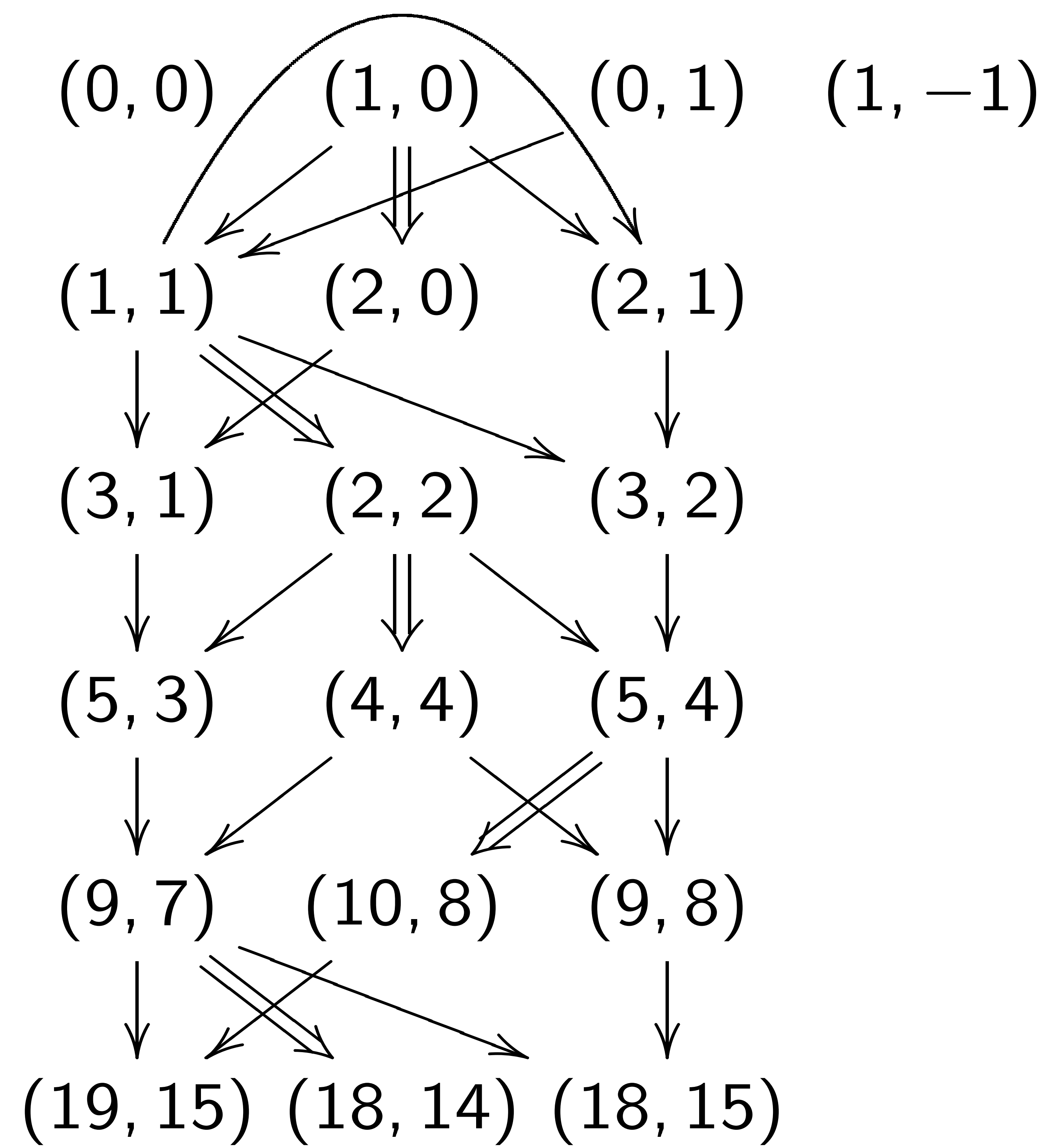

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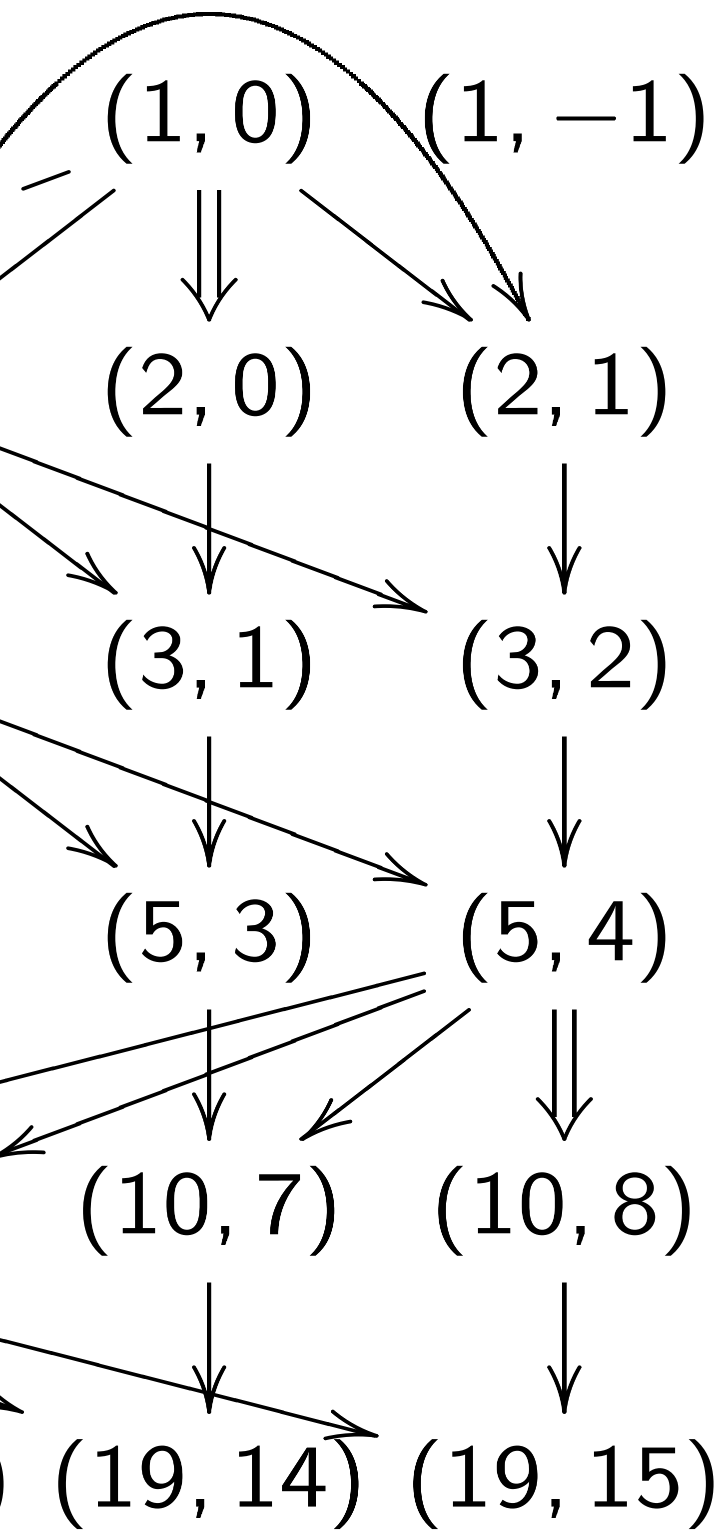


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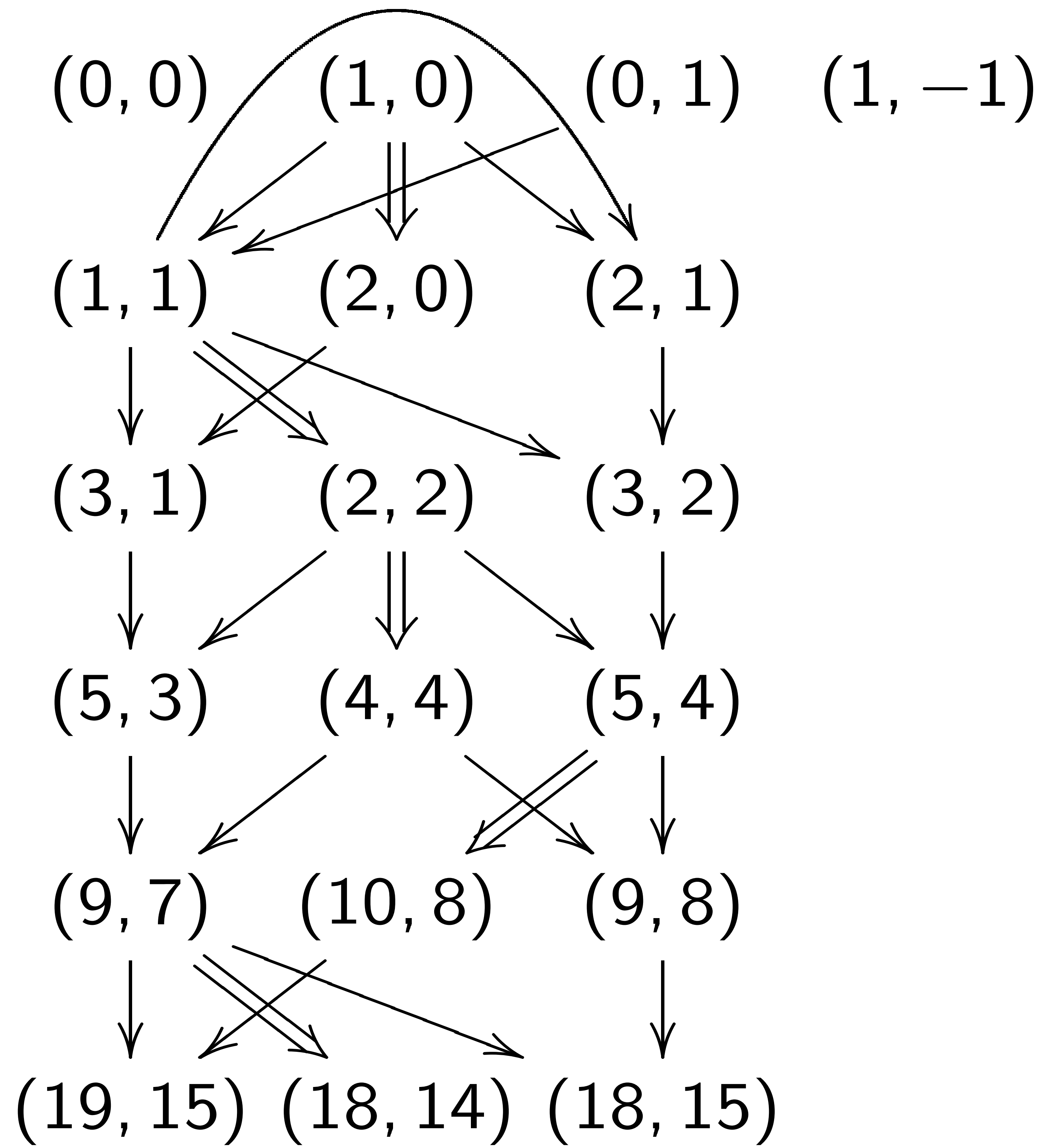


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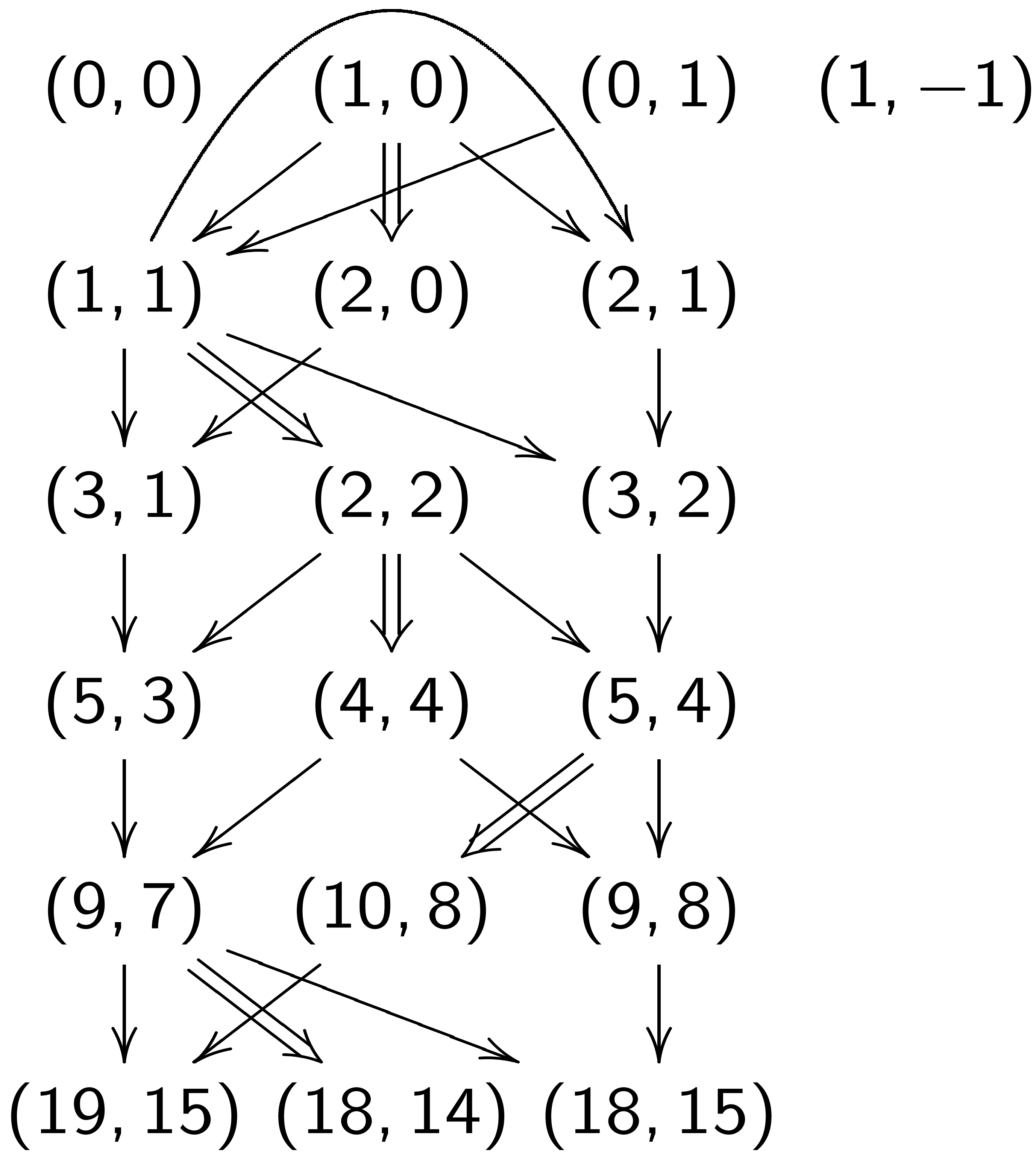


Line in easy binary chain has $(a, b)$, $(a, b+1)$, $(a+1, b)$, $(a+1, b+1)$. Obtain next line by double-add-add-add.

New observation: can omit (even, odd) or (odd, even), chosen recursively so that next line can be obtained by double-add-add. 14 mults if $P, Q, P-Q$ have small denominators.

Intermediate results: 2000 Schoenmakers, 2001 Akishita.

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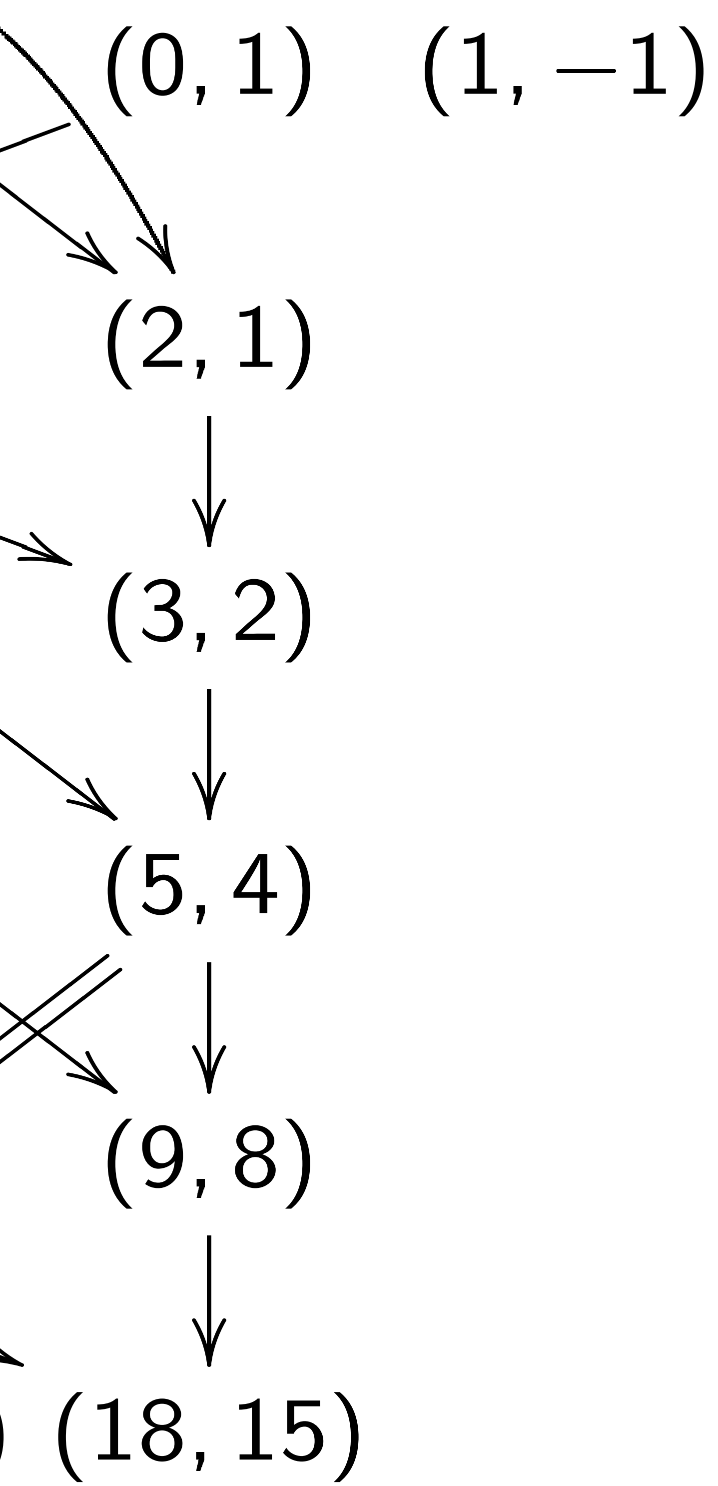


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Intermediate results: 2000 Schoenmakers, 2001 Akishita.

How to do better than binary? Don't worry about uniformity.

Critical idea for dim 1: Build chain $0, 1, \ldots, n$ by choosing $r \approx n(\sqrt{5} - 1)/2$ and building chain $0, 1, \ldots, r, n - r, n$. Try many $r$'s, keep best.

Some further choices here: could build $\{r, n - r, n\}$ from $\{r, n - 2r, n - r\}$ or from $\{n - r, 2r - n, r\}$ or from $\{r, n/2 - r, n/2\}$ or $\ldots$.

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e.g. $n = 100$, $r =$
Build chain
$0, 1, 2, 3, 5, 7, 12,$
by building $\{39, 6$
from $\{22, 39, 61\}$

What about dim
Obvious adaptati
Build chain $\ldots, ($
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e.g. $n = 100$, $r = 39$:
Build chain
$0, 1, 2, 3, 5, 7, 12, 17, 22, 39, 61, 100$
by building $\{39, 61, 100\}$
from $\{22, 39, 61\}$ etc.

What about dim 2?
Obvious adaptation of idea:
Build chain $\ldots, (m, n)$
by choosing $(q, r)$
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$\ldots, (q, r), (m - q, n - r), (m, n)$.

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Build chain $\ldots, (m, n)$
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e.g. Work backw
$(314, 271)$ and (1
$(120, 104)$, then
$(46, 41)$, then (28
$(18, 19)$, then (10
$(8, 16)$.

Hmmm, what's t
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Several plausible
but all of them s
Normally this con
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e.g. $n = 100$, $r = 39$:
Build chain
$0, 1, 2, 3, 5, 7, 12, 17, 22, 39, 61, 100$
by building $\{39, 61, 100\}$
from $\{22, 39, 61\}$ etc.

What about dim 2?
Obvious adaptation of idea:
Build chain $\ldots, (m, n)$
by choosing $(q, r)$
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$\ldots, (q, r), (m - q, n - r), (m, n)$.

e.g. Work backwards from
$(314, 271)$ and $(194, 167)$ to
$(120, 104)$, then $(74, 63)$, then
$(46, 41)$, then $(28, 22)$, then
$(18, 19)$, then $(10, 3)$, then
$(8, 16)$.

Hmmm, what's the endgame?
How to build short chain with
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Several plausible approaches,
but all of them scale badly.
Normally this construction
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17, 22, 39, 61, 100

61, 100}

etc.

2?

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$(m, n)$

)

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e.g. Work backwards from (314, 271) and (194, 167) to (120, 104), then (74, 63), then (46, 41), then (28, 22), then (18, 19), then (10, 3), then (8, 16).

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Several plausible approaches, but all of them scale badly. Normally this construction is abandoned.

New observation

Simple endgames

if $rm - qn = \Delta$

with, e.g., $\Delta =$

Often find very g

Easy to find $(q, r$

given $(m, n, \Delta)$:

standard ext-gcd

What if $(m, n)$ n

Great! Exploit fa

Try many good c

for $(\Delta, q, r)$, kee

e.g. Work backwards from $(314, 271)$ and $(194, 167)$ to $(120, 104)$, then $(74, 63)$, then $(46, 41)$, then $(28, 22)$, then $(18, 19)$, then $(10, 3)$, then $(8, 16)$.

Hmmm, what's the endgame? How to build short chain with $\{(8, 16), (10, 3), (18, 19)\}$?

Several plausible approaches, but all of them scale badly. Normally this construction is abandoned.

New observation: Simple endgames work well if $rm - qn = \Delta$ with, e.g., $\Delta = \pm 2^a 3^b$. Often find very good chains.

Easy to find $(q, r)$ given $(m, n, \Delta)$: standard ext-gcd computation.

What if $(m, n)$ not coprime? Great! Exploit factor.

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Example of new
$(0, 0)$, $(1, 0)$, $(0,$
$(1, 1)$, $(1, 2)$, $(2,$
$(4, 7)$, $(5, 9)$, $(9,$
$(19, 34)$, $(33, 59)$
$(66, 118)$, $(71, 12$
$(132, 236)$, $(203,$
$(325, 581)$, $(528,$
$(731, 1307)$, $(125$
$(1787, 3195)$, $(25$
$(3249, 5809)$, $(50$
$(6823, 12199)$, $(1$
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Example of new chain:
$(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, -1)$,
$(1, 1)$, $(1, 2)$, $(2, 3)$, $(3, 5)$,
$(4, 7)$, $(5, 9)$, $(9, 16)$, $(14, 25)$,
$(19, 34)$, $(33, 59)$, $(38, 68)$,
$(66, 118)$, $(71, 127)$, $(61, 109)$,
$(132, 236)$, $(203, 363)$, $(264, 472)$,
$(325, 581)$, $(528, 944)$,
$(731, 1307)$, $(1259, 2251)$,
$(1787, 3195)$, $(2518, 4502)$,
$(3249, 5809)$, $(5036, 9004)$,
$(6823, 12199)$, $(10072, 18008)$,
$(16895, 30207)$, $(26967, 48215)$.