Problem A1
Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another.
(For example, $23 = 9 + 8 + 6$.)

Problem A2
Let $S = \{(a,b) \mid a = 1,2,\ldots,n, \ b = 1,2,3\}$. A rook tour of $S$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (i) $p_i \in S$, (ii) $p_i$ and $p_{i+1}$ are a unit distance apart, for $1 \leq i < 3n$, (iii) for each $p \in S$ there is a unique $i$ such that $p_i = p$. How many rook tours are there that begin at $(1,1)$ and end at $(n,1)$?
(An example of such a rook tour for $n = 5$ is depicted below.)

Problem A3
Let $p(z)$ be a polynomial of degree $n$, all of whose zeros have absolute value 1 in the complex plane. Put $g(z) = p(z)/z^{n/2}$. Show that all zeros of $g'(z) = 0$ have absolute value 1.

Problem A4
Let $H$ be an $n \times n$ matrix all of whose entries are $\pm 1$ and whose rows are mutually orthogonal. Suppose $H$ has an $a \times b$ submatrix whose entries are all 1. Show that $ab \leq n$.

Problem A5
Evaluate $\int_0^1 \frac{\ln(x + 1)}{x^2 + 1} \, dx$.

Problem A6
Let $n$ be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are $n$ randomly, independently and uniformly, chosen points on a circle. Consider the convex $n$-gon whose vertices are the $P_i$. What is the probability that at least one of the vertex angles of this polygon is acute?
Problem B1

Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers $a$. (Note: $\lfloor v \rfloor$ is the greatest integer less than or equal to $v$.)

Problem B2

Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1.$$

Problem B3

Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number $a$ such that

$$f'(\frac{a}{x}) = \frac{x}{f(x)}$$

for all $x > 0$.

Problem B4

For positive integers $m$ and $n$, let $f(m, n)$ denote the number of $n$-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

Problem B5

Let $P(x_1, \ldots, x_n)$ denote a polynomial with real coefficients in the variables $x_1, \ldots, x_n$, and suppose that

(a) \( \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \ldots, x_n) = 0 \) \hspace{1cm} \text{(identically)}

and that

(b) \( x_1^2 + \cdots + x_n^2 \) divides $P(x_1, \ldots, x_n)$.

Show that $P = 0$ identically.

Problem B6

Let $S_n$ denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if $\pi$ is an even permutation and $\sigma(\pi) = -1$ if $\pi$ is an odd permutation. Also, let $v(\pi)$ denote the number of fixed points of $\pi$. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{v(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$
Problem A1

Show that every positive integer is a sum of one or more numbers of the form $2^r3^s$, where $r$ and $s$ are nonnegative integers and no summand divides another.

(For example, $23 = 9 + 8 + 6$.)

**Solution:** For each $n \geq 0$ define a sequence $E(n)$ of elements of $2^N3^N$ as follows:

- if $n = 0$ then $E(n)$ is the empty sequence ($\emptyset$);
- if $n > 0$ and $n$ is even then $E(n)$ is $2E(n/2)$, the sequence obtained by doubling each component of $E(n/2)$;
- if $n > 0$ and $n$ is odd then $E(n)$ is $(E(n-3^k), 3^k)$, the sequence obtained by appending $3^k$ to $E(n-3^k)$, where $k$ is the largest integer such that $3^k \leq n$.

I claim that the sum of $E(n)$ is $n$; that each component of $E(n)$ is even if $n$ is even; and that no component of $E(n)$ divides another component. Proof:

- $n = 0$: $E(n)$ is empty so it has sum 0.
- $n > 0$ and $n$ is even: Assume inductively that $E(n/2)$ has sum $n/2$ and that no component of $E(n/2)$ divides another component. Then $E(n) = 2E(n/2)$ has sum $2(n/2) = n$; each component of $E(n)$ is even; and no component divides another component.
- $n > 0$ and $n$ is odd: Find the largest integer $k$ such that $3^k \leq n$. Note that $n - 3^k$ is even. Assume inductively that $E(n-3^k)$ has sum $n - 3^k$; that each component of $E(n-3^k)$ is even; and that no component of $E(n-3^k)$ divides another component. Then $E(n) = (E(n-3^k), 3^k)$ has sum $(n - 3^k) + 3^k = n$; each component of $E(n-3^k)$, being even, does not divide $3^k$; and each component of $E(n-3^k)/2$, being at most $(n - 3^k)/2 < (3^{k+1} - 3^k)/2 = 3^k$, is not divisible by $3^k$, so each component of $E(n-3^k)$ is not divisible by $3^k$.

In particular, for $n \geq 1$, the components of $E(n)$ are one or more elements of $2^N3^N$, adding up to $n$, none dividing the others.

Problem A2

Let $S = \{(a,b) \mid a = 1, 2, \ldots, n, \ b = 1, 2, 3\}$. A rook tour of $S$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (i) $p_i \in S$, (ii) $p_i$ and $p_{i+1}$ are a unit distance apart, for $1 \leq i < 3n$, (iii) for each $p \in S$ there is a
unique $i$ such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?
(An example of such a rook tour for $n = 5$ is depicted below.)

\[ \text{Solution: } \text{The answer is } 0 \text{ for } n = 1 \text{ and } 2^{n-2} \text{ for } n \geq 2. \]

For each $n \geq 1$, define $r_n$ as the number of $n \times 3$ rook tours beginning at $(1, 1)$ and ending at $(n, 1)$, and define $s_n$ as the number of $n \times 3$ rook tours beginning at $(1, 1)$ and ending at $(n, 3)$.

One can obtain an $n \times 3$ rook tour beginning at $(1, 1)$ and ending at $(n, 1)$ as follows. Choose $k \in \{1, 2, \ldots, n-1\}$. Take a $k \times 3$ rook tour beginning at $(1, 1)$ and ending at $(k, 3)$. Step to the right $n-k$ times to $(n, 3)$; then down once to $(n, 2)$; then left $n-k-1$ times to $(k+1, 2)$; then down once to $(k+1, 1)$; then right $n-k-1$ times to $(n, 1)$.

Every $n \times 3$ rook tour beginning at $(1, 1)$ and ending at $(n, 1)$ can be obtained uniquely in this way. Indeed, the final step down on the tour must be from $(k+1, 2)$ to $(k+1, 1)$ for a unique $k \in \{1, 2, \ldots, n-1\}$; it must be followed by $n-k-1$ steps right to $(n, 1)$; it must be preceded by $n-k-1$ steps left from $(n, 2)$, since any earlier step up (or left) would prevent the tour from reaching $(n, 2)$; $(n, 2)$ must be preceded by a step down from $(n, 1)$; and $(n, 1)$ must be preceded by $n-k$ steps right from $(k, 1)$.

Consequently $r_n = s_1 + s_2 + \cdots + s_{n-1}$ for all $n \geq 1$. In particular, $r_1 = 0$.

Similarly, one can obtain an $n \times 3$ rook tour beginning at $(1, 1)$ and ending at $(n, 3)$ by choosing $k \in \{1, 2, \ldots, n-1\}$, taking a $k \times 3$ rook tour beginning at $(1, 1)$ and ending at $(k, 1)$, stepping to the right $n-k$ times, stepping up once, stepping left $n-k-1$ times, stepping up once, and stepping right $n-k-1$ times; or by simply starting from $(1, 1)$, stepping to the right $n-1$ times, stepping up once, stepping left $n-1$ times, stepping up once, and stepping right $n-1$ times. Every $n \times 3$ rook tour beginning at $(1, 1)$ and ending at $(n, 3)$ can be obtained uniquely in this way.

Consequently $s_n = r_1 + r_2 + \cdots + r_{n-1} + 1$ for all $n \geq 1$. In particular, $s_1 = 1$.

Now $r_n = s_n = 2^{n-2}$ for all $n \geq 2$. Indeed, assume inductively that $r_k = s_k = 2^{k-2}$ for $2 \leq k < n$. Then $r_n = s_1 + s_2 + \cdots + s_{n-1} = 1 + 2^0 + \cdots + 2^{n-3} = 2^{n-2}$ and $s_n = r_1 + r_2 + \cdots + r_{n-1} + 1 = 0 + 2^0 + \cdots + 2^{n-3} + 1 = 2^{n-2}$.

**Problem A3**
Let \( p(z) \) be a polynomial of degree \( n \), all of whose zeros have absolute value 1 in the complex plane. Put \( g(z) = p(z)/z^{n/2} \). Show that all zeros of \( g'(z) = 0 \) have absolute value 1.

**Solution:** I’m annoyed by this problem, for two reasons.

First, the definition of \( g(z) \) is ambiguous when \( n \) is odd. Does \( z^{n/2} \) mean the principal branch of the \( n/2 \) power, applied to \( z \)? Or does it mean the principal branch of the square root, applied to \( z^n \)? Or is \( z \) not actually a complex number, but an element of a Riemann sheet chosen so that the square root does not need a branch cut? My proof works for any of these choices of \( g \), but I can imagine proofs that work with all the roots of \( g \) and that occasionally break down for the first two choices of \( g \). The problem should have said “Show that all zeros of \( zp'(z) - (n/2)p(z) \) have absolute value 1.”

Second, the statement is false for \( n = 0 \). Consider, for example, \( p(z) = 1 \). This is a polynomial of degree 0, and it has no zeros, so all of its zeros have absolute value 1. The function \( g(z) = p(z)/z^{n/2} \) is then constant, so its derivative is 0, so its derivative has every complex number (in the ambiguous domain) as a root, not just complex numbers of absolute value 1.

Assume from now on that \( n \geq 1 \). Factor \( p(z) \) as \( p_n(z - r_1)(z - r_2)\cdots(z - r_n) \). By hypothesis each \( r_i \) has absolute value 1. If \( |z| > 1 \) then, by Lemma 2, \( (z + r_j)/(z - r_j) \) has positive real part for each \( j \), so—since \( n \geq 1 \)—\( \sum_j(z + r_j)/(z - r_j) \) has positive real part. Similarly, if \( |z| < 1 \) then \( \sum_j(z + r_j)/(z - r_j) \) has negative real part. Either way \( \sum_j(z + r_j)/(z - r_j) \) is nonzero; i.e., \( \sum_j 2z/(z - r_j) \neq \sum_j(z - r_j)/(z - r_j) = n \); i.e., \( 2zp'(z)/p(z) \neq n \); i.e., \( g'(z) \neq 0 \).

**Lemma 1:** If \( z \in \mathbb{C} \) then \((z + 1)/(z - 1)\) has positive real part when \(|z| > 1\) and negative real part when \(|z| < 1\).

**Proof:** Write \( z \) in polar coordinates as \( re^{i\theta} \). Then \((z + 1)/(z - 1) = (re^{i\theta} + 1)/(re^{i\theta} - 1)\) has real part \((r^2 - 1)/((r \cos \theta - 1)^2 + (r \sin \theta)^2)\), which is positive if \( r > 1 \) and negative if \( 0 \leq r < 1 \).

**Lemma 2:** If \( r \in \mathbb{C}, \ |r| = 1, \) and \(|z| \in \mathbb{C}\), then \((z + r)/(z - r)\) has positive real part when \(|z| > 1\) and negative real part when \(|z| < 1\).

**Proof:** Apply Lemma 1 to \( z/r \).

**Problem A4**

Let \( H \) be an \( n \times n \) matrix all of whose entries are \( \pm 1 \) and whose rows are mutually orthogonal. Suppose \( H \) has an \( a \times b \) submatrix whose entries are all 1. Show that \( ab \leq n \).

**Solution:** Write \( v_1, v_2, \ldots, v_b \) for the length-\( n \) row vectors covering \( H \). By assumption
each entry of $v_i$ is $\pm 1$, so $v_i$ has squared length $n$. By assumption $v_1, v_2, \ldots, v_b$ are pairwise orthogonal, so $v_1 + v_2 + \cdots + v_b$ has squared length $nb$. On the other hand, by assumption $v_1 + v_2 + \cdots + v_b$ has $a$ entries equal to $b$, so $v_1 + v_2 + \cdots + v_b$ has squared length at least $ab^2$. Thus $ab^2 \leq nb$; consequently $ab \leq n$, whether or not $b = 0$.

**Problem A5**

Evaluate $\int_0^1 \frac{\ln(x + 1)}{x^2 + 1} \, dx$.

**Solution:** The answer is $(\pi/8) \log 2$. Fast proof by Bhargava, Kedlaya, and Ng: The integral is $\int_0^{\pi/4} \log(\tan \theta + 1) \, d\theta = \int_0^{\pi/4} \left((1/2) \log 2 + \log \cos(\pi/4 - \theta) - \log \cos \theta\right) \, d\theta = \int_0^{\pi/4} ((1/2) \log 2) \, d\theta = (\pi/8) \log 2$ since $\int_0^{\pi/4} \log \cos(\pi/4 - \theta) \, d\theta = \int_0^{\pi/4} \log \cos \theta \, d\theta$.

**Problem A6**

Let $n$ be given, $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are $n$ randomly, independently and uniformly, chosen points on a circle. Consider the convex $n$-gon whose vertices are the $P_i$. What is the probability that at least one of the vertex angles of this polygon is acute?

**Solution:** The answer is $(n^2 - 2n)/2^{n-1}$.

Define $\theta(p, q) \in [0, 2\pi)$, where $p$ and $q$ are points on the circle, as the angle from $p$ to $q$ in the clockwise direction.

Define a vertex as “happy” if it is immediately before an acute-angled vertex. In other words, if $v, w, x$ are consecutive vertices in clockwise order, then $v$ is happy if and only if the angle at vertex $w$ is acute. Equivalently: $v$ is happy if and only if $\theta(v, x) > \pi$. Equivalently: $v$ is happy if and only if at most one other vertex $P$ has $\theta(v, P) \leq \pi$.

Critical fact 1: $P_1$ is happy with probability $n/2^{n-1}$.

Indeed, here is a partition of $P_1$’s happiness into $n$ disjoint possibilities, each occurring with probability $1/2^{n-1}$:

- $\theta(P_1, P_i) > \pi$ for all $i \neq 1$;
- $\theta(P_1, P_i) > \pi$ for all $i \notin \{1, 2\}$ while $\theta(P_1, P_2) \leq \pi$;
- $\theta(P_1, P_i) > \pi$ for all $i \notin \{1, 3\}$ while $\theta(P_1, P_3) \leq \pi$;
- $\ldots$;
- $\theta(P_1, P_i) > \pi$ for all $i \notin \{1, n\}$ while $\theta(P_1, P_n) \leq \pi$.

Critical fact 2: $P_1$ and $P_2$ are simultaneously happy with probability $1/2^{n-3}(n - 1)$.

This probability is the average, over $P_1, P_2$, of the conditional probability given $P_1, P_2$. I
Putting it all together: Permute indices to see that \( P \) are distinct; and that

\[
\frac{1}{\pi} \int_0^\pi \frac{(\pi - \alpha)^{n-2} + (n-2)\alpha(\pi - \alpha)^{n-3}}{(2\pi)^{n-2}} \, d\alpha
\]

Proof of the claim: Assume without loss of generality that \( \theta(P_1, P_2) \leq \pi \), i.e., that \( \alpha = \theta(P_1, P_2) \). Now \( P_1 \) is happy if and only if \( \theta(P_1, P_3), \ldots, \theta(P_1, P_n) \) are all \( > \pi \); and \( P_2 \) is happy if and only if at most one of \( \theta(P_2, P_3), \ldots, \theta(P_2, P_n) \) is \( \leq \pi \). Thus \( P_1 \) and \( P_2 \) are simultaneously happy if and only if one of the following disjoint events occurs:

- \( \pi + \alpha < \theta(P_1, P_i) < 2\pi \) for each \( i \notin \{1, 2\} \)—which, given \( P_1 \) and \( P_2 \), occurs with conditional probability \( (\pi - \alpha)^{n-2}/(2\pi)^{n-2} \);
- \( \pi + \alpha < \theta(P_1, P_i) < 2\pi \) for each \( i \notin \{1, 2, 3\} \) while \( \pi < \theta(P_1, P_3) \leq \pi + \alpha \)—which, given \( P_1 \) and \( P_2 \), occurs with conditional probability \( \alpha(\pi - \alpha)^{n-3}/(2\pi)^{n-2} \);
- \( \pi + \alpha < \theta(P_1, P_i) < 2\pi \) for each \( i \notin \{1, 2, 4\} \) while \( \pi < \theta(P_1, P_4) \leq \pi + \alpha \)—which, given \( P_1 \) and \( P_2 \), occurs with conditional probability \( \alpha(\pi - \alpha)^{n-3}/(2\pi)^{n-2} \);
- \( \ldots \);
- \( \pi + \alpha < \theta(P_1, P_i) < 2\pi \) for each \( i \notin \{1, 2, n\} \) while \( \pi < \theta(P_1, P_n) \leq \pi + \alpha \)—which, given \( P_1 \) and \( P_2 \), occurs with conditional probability \( \alpha(\pi - \alpha)^{n-3}/(2\pi)^{n-2} \).

Add to obtain \((\pi - \alpha)^{n-2} + (n-2)\alpha(\pi - \alpha)^{n-3})/(2\pi)^{n-2}\) as claimed.

Critical fact 3: \( P_1, P_2, P_3 \) are all happy with probability 0.

Indeed, in each of the above ways for \( P_1, P_2 \) to be happy, \( P_3 \) is visibly unhappy: either \( \pi + \alpha < \theta(P_1, P_3) \leq 2\pi \), in which case both \( \theta(P_3, P_1) \) and \( \theta(P_3, P_2) \) are below \( \pi \), or \( \pi < \theta(P_1, P_3) \leq \pi + \alpha \) while \( \pi + \alpha < \theta(P_1, P_4) \leq 2\pi \), in which case both \( \theta(P_3, P_1) \) and \( \theta(P_3, P_4) \) are below \( \pi \). This is where the proof uses the hypothesis that \( n \geq 4 \).

Putting it all together: Permute indices to see that \( P_i \) is happy with probability \( n/2^{n-1} \); that \( P_i, P_j \) are simultaneously happy with probability \( 1/2^{n-3}(n-1) \); if the indices \( i, j \) are distinct; and that \( P_i, P_j, P_k \) are simultaneously happy with probability 0, if \( i, j, k \) are distinct. By inclusion-exclusion, the probability of at least one happy vertex is

\[
n(n/2^{n-1}) - \binom{n}{2}(1/2^{n-3}(n-1)) = (n^2 - 2n)/2^{n-1}.
\]
Problem B1

Find a nonzero polynomial $P(x, y)$ such that $P([a], [2a]) = 0$ for all real numbers $a$.  
(Not: $[v]$ is the greatest integer less than or equal to $v$.)

Solution: One answer is the nonzero polynomial $P(x, y) = (y - 2x)(y - 2x - 1)$.

Define $i = [a]$. Then $i \leq a < i + 1$. If $i \leq a < i + 0.5$ then $2i \leq 2a < 2i + 1$ so $[2a] = 2i = 2[i] \Rightarrow [2a] = 2i - 2[i] = 0$. Otherwise $i + 0.5 \leq a < i + 1$ so $2i + 1 \leq 2a < 2i + 2$ so $[2a] = 2i + 1 = 2[i] + 1$ so $[2a] - 2[i] - 1 = 0$. Either way $P([a], [2a]) = 0$.

Problem B2

Find all positive integers $n, k_1, \ldots, k_n$ such that $k_1 + \cdots + k_n = 5n - 4$ and

$$\frac{1}{k_1} + \cdots + \frac{1}{k_n} = 1. $$

Solution: 1, 1; 3, 2, 3, 6; 3, 2, 6, 3; 3, 3, 2, 6; 3, 3, 6, 2; 3, 6, 2, 3; 3, 6, 3, 2; 4, 4, 4, 4.

By inspection each of these possibilities works. Conversely, assume that $k_1 + \cdots + k_n = 5n - 4$ and $1/k_1 + \cdots + 1/k_n = 1$; I will show that $n, k_1, \ldots, k_n$ is one of these possibilities.

If $k_1, \ldots, k_n$ are all equal then $1 = 1/k_1 + \cdots + 1/k_n = n/k_1$ so $k_1 = n$ and $5n - 4 = k_1 + \cdots + k_n = nk_1 = n^2$. Hence $(n - 4)(n - 1) = n^2 - 5n + 4 = 0$. Either $n = 1$, in which case $(n, k_1, \ldots, k_n) = (1, 1)$; or $n = 4$, in which case $(n, k_1, \ldots, k_n) = (4, 4, 4, 4, 4)$.

Assume from now on that $k_1, \ldots, k_n$ are not all equal. The average of $k_1, \ldots, k_n$ is $(5n - 4)/n$ so the geometric average of $k_1, \ldots, k_n$ is below $(5n - 4)/n$. The average of $1/k_1, \ldots, 1/k_n$ is $1/n$ so the geometric average of $1/k_1, \ldots, 1/k_n$ is below $1/n$. Thus the geometric average of $k_1, \ldots, k_n, 1/k_1, \ldots, 1/k_n$ is below $\sqrt{((5n - 4)/n)(1/n)}$; but this geometric average is equal to 1. Therefore $1 < (5n - 4)/n^2$; so $(n - 1)(n - 4) < 0$; so $1 < n < 4$; so $n = 2$ or $n = 3$.

If $n = 2$ then $k_1 + k_2 = 5n - 4 = 6$ so $1/k_1 + 1/k_2$ is one of $1/1 + 1/5, 1/2 + 1/4, 1/3 + 1/3, 1/4 + 1/4$, none of which equal 1.

If $n = 3$ then $k_1 + k_2 + k_3 = 5n - 4 = 11$ so $1/k_1 + 1/k_2 + 1/k_3$ is one of $1/1 + \cdots, 1/2 + 1/2 + \cdots, 1/2 + 1/3 + 1/6, 1/2 + 1/4 + 1/5, 1/3 + 1/3 + 1/5, 1/3 + 1/4 + 1/4$. By inspection none of these are 1 except $1/2 + 1/3 + 1/6$. Thus $(k_1, k_2, k_3)$ is a permutation of $(2, 3, 6)$.

Problem B3

Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number $a$ such that

$$f'(\frac{a}{x}) = \frac{x}{f(x)}.$$
for all $x > 0$.

**Solution:** Here are two classes of qualifying functions $f$:

- Define $f(x) = x$. Then $f'(x) = 1$ so $f'(1/x) = 1 = x/f(x)$.

- Choose positive real numbers $\alpha, \beta$ with $\beta \neq 1$, and define $f(x) = \alpha x^\beta$. Then $f'(a/x) f(x) = \alpha \beta (a/x)^{\beta - 1} \alpha x^\beta = \alpha^2 \beta a^{\beta - 1} x = x$ where $a = (1/\alpha^2 \beta)^{1/(\beta - 1)}$.

I claim that there are no other possibilities: if $f'(a/x) = x/f(x)$ then $f$ is one of the above functions. Indeed, substitute $a/x$ for $x$: $f'(x) = a/x f(a/x)$. The right side is differentiable, so the left side is too:

$$f''(x) = \frac{-a}{(x f(a/x))^2} \left( x f'(a/x) \frac{-a}{x^2} + f(a/x) \right).$$

Substitute $f(a/x) = a/x f'(x)$ and $f'(a/x) = x/f(x)$:

$$f''(x) = \frac{-a}{(a/f'(x))^2} \left( \frac{x^2}{f(x)} \frac{-a}{x^2} + \frac{a}{xf'(x)} \right)$$

$$= \frac{-f'(x)^2}{a} \left( \frac{-a}{f(x)} + \frac{a}{xf'(x)} \right) = \frac{f'(x)^2}{f(x)} - \frac{f'(x)}{x}.$$

Define $g(x) = \log f(x)$. Then $g'(x) = f'(x)/f(x)$; note that $f'(x) > 0$ so $g'(x) > 0$. Differentiate again:

$$g''(x) = \frac{f(x) f''(x) - f'(x)^2}{f(x)^2} = \frac{-f(x) f'(x)/x}{f(x)^2} = -\frac{f'(x)}{x} = \frac{-g'(x)}{x}.$$

Define $h(x) = \log g'(x)$. Then $h'(x) = g''(x)/g'(x) = -1/x$. Integrate: there is a real number $d$ such that $h(x) = d - \log x$. Exponentiate: $g'(x) = \beta/x$ where $\beta = \exp d$. Integrate again: there is a real number $c$ such that $g(x) = c + \beta \log x$. Exponentiate: $f(x) = \alpha x^\beta$ where $\alpha = \exp c$. If $\beta = 1$ then $f(x) = \alpha x$ so $\alpha = f'(a/x) = x/f(x) = 1/\alpha$ so $\alpha = 1$ so $f(x) = x$ as claimed. Otherwise $\alpha, \beta$ are positive real numbers, $\beta \neq 1$, and $f(x) = \alpha x^\beta$ as claimed.

**Problem B4**

For positive integers $m$ and $n$, let $f(m, n)$ denote the number of $n$-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

**Solution:** Extend the same definition to all nonnegative integers $m, n$.

If $n = 0$ then there is exactly one $n$-tuple, and its sum of absolute values is $0 \leq m$. Thus $f(m, 0) = 1$. 
If \( m = 0 \) then the only qualifying \( n \)-tuple is \((0, 0, \ldots, 0)\). Thus \( f(0, n) = 1 \).

If \( n \geq 1 \) and \( m \geq 0 \) then one can construct a qualifying \( n \)-tuple as follows: choose \( x_n \) in \([-m, -m + 1, \ldots, m - 1, m]\); choose an \((n-1)\)-tuple \((x_1, x_2, \ldots, x_{n-1})\) satisfying 
\[
|x_1| + |x_2| + \cdots + |x_{n-1}| \leq m - |x_n|.
\]
Every qualifying \( n \)-tuple arises uniquely in this way. Thus \( f(m, n) = f(m, n-1) + 2f(m-1, n-1) + 2f(m-2, n-1) + \cdots + 2f(0, n-1) \).

Consequently, \( f(m + 1, n + 1) = f(m, n + 1) + f(m + 1, n) + f(m, n) \) if \( n \geq 0 \) and \( m \geq 0 \). Indeed, \( f(m, n + 1) = f(m, n) + 2f(m-1, n) + 2f(m-2, n) + \cdots + 2f(0, n) \). and \( f(m + 1, n + 1) = f(m + 1, n) + 2f(m, n) + 2f(m - 1, n) + 2f(m - 2, n) + \cdots + 2f(0, n) \); subtract.

Theorem: \( f(m, n) = f(n, m) \) for all nonnegative integers \( m, n \). Proof: If \( m = 0 \) then \( f(m, n) = f(0, n) = 1 = f(n, 0) = f(n, m) \) as claimed. If \( n = 0 \) then \( f(m, n) = f(m, 0) = 1 = f(0, m) = f(n, m) \) as claimed. So assume that \( m \geq 1 \) and \( n \geq 1 \). Then \( f(m, n) = f(m - 1, n) + f(m, n - 1) + f(m - 1, n - 1) \) and \( f(n, m) = f(n - 1, m) + f(n, m - 1) + f(n - 1, m - 1) \). Induct on \( m + n \).

Alternate approaches: One can, with marginally more work, prove the symmetric formula 
\( f(m, n) = 1 + 2(m + n - 1)!/(m - 1)!(n - 1)! \). One can use other bijections; partitioning by choices of \( x_n \) is straightforward but might not produce the shortest proof.

**Problem B5**

Let \( P(x_1, \ldots, x_n) \) denote a polynomial with real coefficients in the variables \( x_1, \ldots, x_n \), and suppose that

\[
(a) \quad \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \right) P(x_1, \ldots, x_n) = 0 \quad \text{(identically)}
\]

and that \( x_1^2 + \cdots + x_n^2 \) divides \( P(x_1, \ldots, x_n) \).

Show that \( P = 0 \) identically.

**Solution:** Assume that \( n \geq 1 \). Define \( X = x_1^2 + \cdots + x_n^2 \) and \( D = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2 \). The problem is to show that if \( X \) divides \( P \) and \( D(P) = 0 \) then \( P = 0 \).

Suppose that \( P \neq 0 \). Find the maximum positive integer \( e \) such that \( X^e \) divides \( P \). Write \( P/X^e \) as \( \sum_{i \geq 0} H_i \) where \( H_i \) is homogeneous of degree \( i \).

Then \( 0 = D(P) = D(\sum_i X^e H_i) = \sum_i D(X^e H_i) \). The terms \( D(X^e H_i) \) are homogeneous of different degrees, namely \( 2e - 2 + i \), so \( D(X^e H_i) = 0 \) for each \( i \). Thus \( XD(H_i) + e(4(e - 1) + 4 \deg H_i + 2n)H_i = 0 \) by Lemma 2. The coefficient \( e(4(e - 1) + 4 \deg H_i + 2n) \) is positive since \( n \geq 1 \) and \( e \geq 1 \); thus \( H_i \) is a multiple of \( X \). This is true for every \( i \), so \( P/X^e \) is a multiple of \( X \), contradicting the definition of \( e \).
Lemma 1: If $H$ is homogeneous then $D(XH) = XD(H) + (4 \deg H + 2n)H$.

Proof: For $e = 0$: $D(X^eH) = D(H) = X^eD(H) + e(\cdots)$. For $e \geq 1$: $D(X^eH) = XD(X^{e-1}H) + (4 \deg X^{e-1}H + 2n)X^{e-1}H$ by Lemma 1. Assume inductively that $D(X^{e-1}H) = X^{e-1}D(H) + (e-1)(4(e-2) + 4 \deg H + 2n)X^{e-2}H$. Then

$$D(X^eH) = X^eD(H) + (e-1)(4(e-2) + 4 \deg H + 2n)X^{e-1}H + (4 \deg X^{e-1}H + 2n)X^{e-1}H = X^eD(H) + (4(e-1)e-2) + 4 \deg H + 2n)X^{e-1}H$$

since $\deg X = 2$.

Problem B6

Let $S_n$ denote the set of all permutations of the numbers $1, 2, \ldots, n$. For $\pi \in S_n$, let $\sigma(\pi) = 1$ if $\pi$ is an even permutation and $\sigma(\pi) = -1$ if $\pi$ is an odd permutation. Also, let $v(\pi)$ denote the number of fixed points of $\pi$. Show that

$$\sum_{\pi \in S_n} \frac{\sigma(\pi)}{v(\pi) + 1} = (-1)^{n+1} \frac{n}{n+1}.$$

Solution: Define $e_n$ as the number of even permutations of $\{1, 2, \ldots, n\}$. Recall that $e_n = 1$ if $n = 0$; $e_n = 1$ if $n = 1$; and $e_n = n!/2$ if $n \geq 2$.

Define $f_k$ as the number of even derangements of $\{1, 2, \ldots, k\}$, i.e., the number of even permutations with no fixed points. Define $g_k$ as the number of odd derangements of $\{1, 2, \ldots, k\}$, i.e., the number of odd permutations with no fixed points.

By choosing $k$ elements of $\{1, \ldots, n\}$, choosing an even derangement of those $k$ elements, and fixing the other $n-k$ elements, one obtains an even permutation of $\{1, \ldots, n\}$ with exactly $n-k$ fixed points. Every such permutation arises in this way. Thus there are exactly $\binom{n}{k} f_k$ even permutations of $\{1, \ldots, n\}$ with exactly $n-k$ fixed points. Sum over $k$ to see that $\sum_{0 \leq k \leq n} \binom{n}{k} f_k = e_n$. 
Similarly, there are exactly \( \binom{n}{k} g_k \) odd permutations of \( \{1, \ldots, n\} \) with exactly \( n-k \) fixed points, and \( \sum_{0 \leq k \leq n} \binom{n}{k} g_k = n! - e_n \).

I claim that \( f_n - g_n = (-1)^{n-1}(n-1) \) for all \( n \geq 0 \). Proof: The point is that \( f_n - g_n \) is determined recursively by the equation \( \sum_k \binom{n}{k} (f_k - g_k) = 2e_n - n! \); so one simply has to check that \( \sum_k \binom{n}{k} (-1)^{k-1}(k-1) = 2e_n - n! \). For \( n = 0 \) the latter sum is \( (-1)^{-1}(-1) = 1 = 2e_0 - 0! \) as desired. For \( n = 1 \) the sum is \( (-1)^{-1}(-1) + (-1)^0(0) = 1 = 2e_1 - 1! \) as desired. For \( n \geq 2 \) one has \( \sum_k \binom{n}{k} (-1)^k = (1 - 1)^n = 0 \) and \( \sum_k \binom{n}{k} (-1)^{k-1}k = \sum_k \binom{n-1}{k-1} (-1)^{k-1} = (1 - 1)^{n-1} = 0 \) so \( \sum_k \binom{n}{k} (-1)^{k-1}(k-1) = 0 = 2e_n - n! \) as desired.

Now if \( n \geq 1 \) then \( \sum_{0 \leq k \leq n} \binom{n+1}{k} (f_k - g_k) = (-1)^{n+1}n \). Proof: \( \binom{n+1}{n+1} (f_{n+1} - g_{n+1}) = (-1)^n n \) and \( \sum_{0 \leq k \leq n+1} \binom{n+1}{k} (f_k - g_k) = 2e_{n+1} - (n+1)! = 0 \).

The problem asks for the sum of \( \sigma(\pi)/(1 + v(\pi)) \) over all permutations \( \pi \) of \( \{1, \ldots, n\} \). There are \( \binom{n}{k} f_k \) even permutations \( \pi \) with \( v(\pi) = n-k \), contributing \( \binom{n}{k} f_k / (1 + n-k) = \binom{n+1}{k} f_k / (n+1) \) to the sum. There are also \( \binom{n}{k} g_k \) odd permutations \( \pi \) with \( v(\pi) = n-k \), contributing \( -\binom{n}{k} g_k / (1 + n-k) = -\binom{n+1}{k} g_k / (n+1) \) to the sum. Overall the sum is \( \sum_{0 \leq k \leq n} \binom{n+1}{k} (f_k - g_k) / (n+1) = (-1)^{n+1} n / (n+1) \) if \( n \geq 1 \).

Beware that this formula is wrong for \( n = 0 \). The problem should have said that \( n \) is a positive integer.