Problem A1

Basketball star Shanille O’Keal’s team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

Problem A2

For $i = 1, 2$, let $T_i$ be a triangle with side lengths $a_i, b_i, c_i$, and area $A_i$. Suppose that $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$, and that $T_2$ is an acute triangle. Does it follow that $A_1 \leq A_2$?

Problem A3

Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that $u_n$ is an integer for all $n$. (By convention, $0! = 1$.)

Problem A4

Show that for any positive integer $n$ there is an integer $N$ such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the $c_i$ are rational numbers and each $a_{ij}$ is one of the numbers, $-1, 0, 1$.

Problem A5

An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability 1/2. We say that two squares, $p$ and $q$, are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at $p$ and ending at $q$, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$. 
Problem A6

Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Show that

$$
\int_0^1 \left( \int_0^1 f(x, y) \, dx \right)^2 \, dy + \int_0^1 \left( \int_0^1 f(x, y) \, dy \right)^2 \, dx 
\leq \left( \int_0^1 \int_0^1 f(x, y) \, dx \, dy \right)^2 + \int_0^1 \int_0^1 [f(x, y)]^2 \, dx \, dy.
$$
Problem B1
Let \( P(x) = c_nx^n + c_{n-1}x^{n-1} + \cdots + c_0 \) be a polynomial with integer coefficients. Suppose that \( r \) is a rational number such that \( P(r) = 0 \). Show that the \( n \) numbers
\[
c_n r, \quad c_n r^2 + c_{n-1} r, \quad c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \quad \ldots, \quad c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r
\]
are integers.

Problem B2
Let \( m \) and \( n \) be positive integers. Show that
\[
\frac{(m + n)!}{(m + n)^{m+n}} < \frac{m!}{m^m} \cdot \frac{n!}{n^n}.
\]

Problem B3
Determine all real numbers \( a > 0 \) for which there exists a nonnegative continuous function \( f(x) \) defined on \([0, a]\) with the property that the region
\[
R = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq f(x)\}
\]
has perimeter \( k \) units and area \( k \) square units for some real number \( k \).

Problem B4
Let \( n \) be a positive integer, \( n \geq 2 \), and put \( \theta = 2\pi/n \). Define points \( P_k = (k, 0) \) in the \( xy \)-plane, for \( k = 1, 2, \ldots, n \). Let \( R_k \) be the map that rotates the plane counterclockwise by the angle \( \theta \) about the point \( P_k \). Let \( R \) denote the map obtained by applying, in order, \( R_1, \) then \( R_2, \ldots, \) then \( R_n \). For an arbitrary point \( (x, y) \), find, and simplify, the coordinates of \( R(x, y) \).

Problem B5
Evaluate
\[
\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left( \frac{1 + x^{n+1}}{1 + x^n} \right)^x.
\]

Problem B6
Let \( \mathcal{A} \) be a non-empty set of positive integers, and let \( N(x) \) denote the number of elements of \( \mathcal{A} \) not exceeding \( x \). Let \( \mathcal{B} \) denote the set of positive integers \( b \) that can be written in the form \( b = a - a' \) with \( a \in \mathcal{A} \) and \( a' \in \mathcal{A} \). Let \( b_1 < b_2 < \cdots \) be the members of \( \mathcal{B} \), listed in increasing order. Show that if the sequence \( b_{i+1} - b_i \) is unbounded, then \( \lim_{x \to \infty} N(x)/x = 0 \).
Solutions
D. J. Bernstein, 6 December 2004

Problem A1
Basketball star Shanille O’Keal’s team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first $N$ attempts of the season. Early in the season, $S(N)$ was less than 80% of $N$, but by the end of the season, $S(N)$ was more than 80% of $N$. Was there necessarily a moment in between when $S(N)$ was exactly 80% of $N$?

Solution: Yes.
By hypothesis $S(N_1) < 0.8N_1$ for some $N_1$ but $S(N_2) > 0.8N_2$ for some $N_2 > N_1$. Find the smallest $N \geq N_1$ such that $S(N) \geq 0.8N$. Then $N \neq N_1$, so $S(N-1) < 0.8(N-1)$.
If she does not make her $N$th free throw then $S(N) = S(N-1) < 0.8(N-1) < 0.8N \leq S(N)$, contradiction. If she makes her $N$th free throw then $S(N) = S(N-1) + 1$ so

$$0 \leq 5S(N) - 4N = 5S(N-1) + 5 - 4N < 4(N-1) + 5 - 4N = 1.$$  

The quantity $5S(N) - 4N$ is an integer, so it must be 0; i.e., $S(N) = 0.8N$ as claimed.

Problem A2
For $i = 1, 2$, let $T_i$ be a triangle with side lengths $a_i, b_i, c_i$, and area $A_i$. Suppose that $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, and that $T_2$ is an acute triangle. Does it follow that $A_1 \leq A_2$?

Solution: Yes.
Recall Heron’s formula $(4A)^2 = 4a^2b^2 - (a^2 + b^2 - c^2)^2$ for the area $A$ of a triangle with side lengths $a, b, c$. The derivative of $(4A)^2$ with respect to $c$ is $4c(a^2 + b^2 - c^2)$, which is positive if $c^2 < a^2 + b^2$, i.e., if the angle opposite $c$ is acute. By symmetry, the derivative of $(4A)^2$ with respect to $a$ is positive if the angle opposite $a$ is acute, and the derivative of $(4A)^2$ with respect to $b$ is positive if the angle opposite $b$ is acute.

Find $(a, b, c)$ in the compact set $[a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$ to maximize $A$. It is not possible to have exactly one of $a, b, c$ smaller than $a_2, b_2, c_2$ respectively: for example, if $a < a_2$ and $b = b_2$ and $c = c_2$, then $a^2 < a_2^2 < b_2^2 + c_2^2 = b^2 + c^2$ since $T_2$ is acute, so the angle opposite $a$ is acute, so increasing $a$ increases $A$, contradiction. Similarly, it is not possible to have two or three of $a, b, c$ smaller than $a_2, b_2, c_2$ respectively: for example, if $a < a_2$ and $b < b_2$, then at least one angle opposite $a$ or $b$ must be acute, so increasing $a$ or $b$ increases $A$, contradiction.

Thus $(a, b, c) = (a_2, b_2, c_2)$. In particular, $A_1 \leq A_2$. 
Problem A3  
Define a sequence \( \{u_n\}_{n=0}^{\infty} \) by \( u_0 = u_1 = u_2 = 1 \), and thereafter by the condition that 
\[
\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!
\]
for all \( n \geq 0 \). Show that \( u_n \) is an integer for all \( n \). (By convention, \( 0! = 1 \).) 

**Solution:** Define \( v_0 = 1, \ v_1 = 1, \) and \( v_n = (n - 1)v_{n-2} \) for all \( n \geq 2 \). Then \( v_n \) is an integer.

I claim that \( v_nv_{n+1} = n! \) for all \( n \geq 0 \). Proof: If \( n = 0 \) then \( v_nv_{n+1} = v_0v_1 = 1 = 0! = n! \) as claimed. If \( n \geq 1 \) then assume inductively that \( v_{n-1}v_n = (n - 1)! \). By definition \( v_{n+1} = nv_{n-1} \), so \( v_nv_{n+1} = nv_{n-1}v_n = n(n - 1)! = n! \) as claimed.

I now claim that \( u_n = v_n \) for all \( n \geq 0 \). Proof: If \( n = 0 \), or \( n = 1 \), or \( n = 2 \), then \( u_n = 1 = v_n \) as claimed. For \( n \geq 3 \), assume inductively that \( u_{n-3} = v_{n-3} \), that \( u_{n-2} = v_{n-2} \), and that \( u_{n-1} = v_{n-1} \). By hypothesis \( (n-3)! = u_{n-3}u_{n} - u_{n-1}u_{n-2} \), so

\[
u_n = \frac{(n-3)! + u_{n-1}u_{n-2}}{u_{n-3}} = \frac{(n-3)! + v_{n-1}v_{n-2}}{v_{n-3}} = \frac{v_{n-3}v_{n-2} + (n-2)v_{n-3}v_{n-2}}{v_{n-3}} = (n-1)v_{n-2} = v_n
\]
as claimed.

Hence \( u_n \) is an integer.

Problem A4  
Show that for any positive integer \( n \) there is an integer \( N \) such that the product \( x_1x_2\cdots x_n \) can be expressed identically in the form

\[
x_1x_2\cdots x_n = \sum_{i=1}^{N} c_i(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n
\]

where the \( c_i \) are rational numbers and each \( a_{ij} \) is one of the numbers, \(-1, 0, 1\).

**Solution:** One can take \( N = 2^n \). Specifically, I claim that \( x_1x_2\cdots x_n \) is the sum of \((a_1a_2\cdots a_n/2^nn!)(a_1x_1 + a_2x_2 + \cdots + a_n)x_n)^n\) over all \((a_1, a_2, \ldots, a_n) \in \{1, -1\}^n\).

Define \( P_0(x_1, x_2, \ldots, x_n) = (x_1 + x_2 + \cdots + x_n)^n \). Note for future reference that the coefficient of \( x_1x_2\cdots x_n \) in \( P_0 \) is \( n! \).

Define \( P_1(x_1, x_2, \ldots, x_n) = P_0(x_1, x_2, \ldots, x_n) - P_0(-x_1, x_2, \ldots, x_n) \). The coefficient of \( x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n} \) in \( P_1 \) is \( 1 - (-1)^{e_1} \) times the corresponding coefficient in \( P_0 \).
Define $P_2(x_1, x_2, \ldots, x_n) = P_1(x_1, x_2, \ldots, x_n) - P_1(x_1, -x_2, \ldots, x_n)$. The coefficient of $x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$ in $P_2$ is $1 - (-1)^{e_2}$ times the corresponding coefficient in $P_1$; in other words, $(1 - (-1)^{e_1})(1 - (-1)^{e_2})$ times the corresponding coefficient in $P_0$.

Define $P_3, P_4, \ldots, P_n$ similarly. Then the coefficient of $x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$ in $P_n$ is exactly $(1 - (-1)^{e_1})(1 - (-1)^{e_2})\cdots (1 - (-1)^{e_n})$ times the corresponding coefficient in $P_0$. The factor $(1 - (-1)^{e_1})(1 - (-1)^{e_2})\cdots (1 - (-1)^{e_n})$ is $2^n$ if $e_1, e_2, \ldots, e_n$ are all odd, otherwise 0. The coefficient in $P_0$ is 0 unless $e_1 + e_2 + \cdots + e_n = n$. The only way for odd numbers $e_1, e_2, \ldots, e_n$ to have sum $n$ is for all of them to be 1. Hence $P_n(x_1, x_2, \ldots, x_n) = 2^n n! x_1 x_2 \cdots x_n$.

**Problem A5**

An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, $p$ and $q$, are in the same connected monochromatic region if there is a sequence of squares, all of the same color, starting at $p$ and ending at $q$, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

**Solution:** Sorry, I haven’t solved this one yet.

**Problem A6**

Suppose that $f(x, y)$ is a continuous real-valued function on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Show that

$$
\int_0^1 \left( \int_0^1 f(x, y) \, dx \right)^2 \, dy + \int_0^1 \left( \int_0^1 f(x, y) \, dy \right)^2 \, dx \\
\leq \left( \int_0^1 \int_0^1 f(x, y) \, dx \, dy \right)^2 + \int_0^1 \int_0^1 \left[ f(x, y) \right]^2 \, dx \, dy.
$$

**Solution:** Dave Rusin writes: “Let $F(x, y, z, w) = f(x, y) + f(z, w) - f(x, w) - f(z, y)$; then integrate $F^2$ over the box $[0, 1]^4$. Done!”
**Problem B1**

Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that $r$ is a rational number such that $P(r) = 0$. Show that the $n$ numbers

$$c_n r, \ c_n r^2 + c_{n-1} r, \ c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \ \ldots, \ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

**Solution:** Fix $i \in \{0, 1, \ldots, n-1\}$. Write $r$ as $u/v$ where $u$ and $v$ are coprime. Then $c_n (u/v)^n + c_{n-1} (u/v)^{n-1} + \cdots + c_0 = 0$, so $c_n u^n + c_{n-1} u^{n-1} v + \cdots + c_0 v^n = 0$, so $c_n u^n + c_{n-1} u^{n-1} v + \cdots + c_{i+1} u^{i+1} v^{n-i-1} = -c_i u^i v^{n-i} - c_{i-1} u^{i-1} v^{n-i+1} - \cdots - c_0 v^n$ is a multiple of $v^{n-i}$; so $c_n u^{n-i} + c_{n-1} u^{n-i-1} v + \cdots + c_{i+1} u v^{n-i-1}$ is a multiple of $v^{n-i}$ since $u^i$ and $v^{n-i}$ are coprime; so $c_n (u/v)^{n-i} + c_{n-1} (u/v)^{n-i-1} + \cdots + c_{i+1} (u/v)$ is an integer as claimed.

**Problem B2**

Let $m$ and $n$ be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m! \cdot n!}{m^m \cdot n^n}.$$

**Solution:** Define $g(x) = x \log(1+1/x)$ for $x > 0$. Then $g'(x) = \log(1+1/x) - (1/x^2)\log(1+1/x) - 1/(x+1)$, so $g''(x) = (-1/x^2)/(1+1/x) + 1/(x+1)^2 = (1-(x+1)/x)/(x+1)^2 < 0$. The limit of $g'(x)$ as $x \to \infty$ is 0, so $g'(x) > 0$ for all $x > 0$, so $g$ is strictly increasing.

Fix $m \geq 1$. Define $f(n) = (m+n)!m^n n^n/(m+n)^{m+n} m! n!$ for $n \geq 1$. Then $f(1) = m^n/(m+1)^m < 1$, and $f(n+1)/f(n) = (m+n)^{m+n} (n+1)^n/(m+n+1)^{m+n} n^n = g(n)/g(m+n) < 1$, so $f(n) < 1$ for all $n$.

Alternate proof: Use Stirling’s bounds $1 < n!(\exp n)/n^n \sqrt{2\pi n} < \exp(1/12n)$ for $n \geq 1$ to see that $f(n)$ is in $[\exp(-1/12(m-1/12n), \exp(1/12(m+n))] \sqrt{(m+n)/2\pi mn}$ and hence is below $\exp(1/24)\sqrt{1/\pi}$, which is smaller than 1 since $\exp(1/12) < \exp 1 < \pi$.

Alternate proof: The binomial expansion of $(m+n)^{m+n}$ includes at least two terms since the exponent is positive; all the terms are positive, and one of them is $\binom{m+n}{m} m^m n^n$.

**Problem B3**

Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0, a]$ with the property that the region

$$R = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq f(x)\}$$
has perimeter $k$ units and area $k$ square units for some real number $k$.

**Solution:** The answer is $\{a : a > 2\}$.

For $a > 2$: Define $f(x) = 2a/(a - 2)$ and $k = 2a^2/(a - 2)$. Then $f$ is continuous; the region $R$ is a rectangle of width $a$ and height $2a/(a - 2)$; the area of $R$ is $2a^2/(a - 2) = k$; and the perimeter of $R$ is $2a + 4a/(a - 2) = (2a(a - 2) + 4a)/(a - 2) = k$.

For $a \leq 2$: Suppose that there exists such a function $f$. Because $f$ is continuous, it has a maximum value on the compact interval $[0, a]$; say $f(m)$ is the maximum. The area under $f$ is at most $af(m)$. The perimeter of the region is at least $f(m)$, to get from $(0, 0)$ to $(m, f(m))$; plus at least another $f(m)$, to get from $(m, f(m))$ to $(a, 0)$; plus $a$, to get from $(a, 0)$ to $(0, 0)$; for a total of $a + 2f(m) \geq a + af(m) > af(m)$. Contradiction.

**Problem B4**

Let $n$ be a positive integer, $n \geq 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k; 0)$ in the $xy$-plane, for $k = 1, 2, \ldots, n$. Let $R_k$ be the map that rotates the plane counterclockwise by the angle $\theta$ about the point $P_k$. Let $R$ denote the map obtained by applying, in order, $R_1$, then $R_2$, . . ., then $R_n$. For an arbitrary point $(x, y)$, find, and simplify, the coordinates of $R(x, y)$.

**Solution:** $R(x, y) = (x + n, y)$.

Put $(x, y)$ into the complex plane as $z = x + iy$, and put $P_1, P_2, \ldots, P_n$ into the complex plane as $1, 2, \ldots, n$. Define $z_0 = z$, and define $z_k$ for $k \geq 1$ as the result of rotating $z_{k-1}$ by $\theta$ around $k$. Then $z_k = \zeta(z_{k-1} - k) + k$ where $\zeta = \cos \theta + i \sin \theta$, so, by induction, $z_k = \zeta^k z - \zeta^k - \zeta^{k-1} - \ldots - \zeta + k$. In particular, $z_n = \zeta^n z - (\zeta^n + \zeta^{n-1} + \ldots + \zeta) + n = z + n$.

**Problem B5**

Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(1 + \frac{x^{n+1}}{1 + x^n}\right)^{x^n}.$$

**Solution:** The answer is $2/\exp 1$.

The logarithm of $((1 + x^{n+1})/(1 + x^n))^{x^n} = (1 + x^n(x - 1))/(1 + x^n)$ is approximately $x^{2n}(x - 1)/(1 + x^n)$. There are approximately $\delta/u(1 - x)$ nonnegative integers $n$ for which $u \leq x^n < u + \delta$. The sum of $x^{2n}(x - 1)/(1 + x^n)$ over those $n$’s is approximately $-(\delta/u)u^2/(1 + u) = -(u/(1 + u))\delta$. Hence the sum of $x^{2n}(x - 1)/(1 + x^n)$ over all $n$’s is approximately $-\int_0^1 (u/(1 + u)) \, du = \log(1 + u) - u|_0^1 = \log 2 - 1$.

To turn this into a complete proof, write down explicit error bounds (using explicit log bounds as in B2) instead of just saying “approximately”; then observe that the error converges to 0 as $x$ approaches 1.
Problem B6

Let $\mathcal{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathcal{A}$ not exceeding $x$. Let $\mathcal{B}$ denote the set of positive integers $b$ that can be written in the form $b = a - a' + a''$ with $a \in \mathcal{A}$ and $a' \in \mathcal{B}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathcal{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then $\lim_{x \to \infty} N(x)/x = 0$.

Solution: Find the smallest positive integer $g_1$ such that $g_1 \notin \mathcal{B}$.

Find the smallest positive integer $h_2$ such that $h_2, h_2 + 1, h_2 + 2, \ldots, h_2 + 6g_1 \notin \mathcal{B}$. Define $g_2 = 2g_1(1 + [h_2/2g_1])$. Then $g_2 - 2g_1, g_2 - 2g_1 + 1, g_2 - 2g_1 + 2, \ldots, g_2 + 2g_1 \notin \mathcal{B}$, and $g_2$ is a multiple of $2g_1$.

Find the smallest positive integer $h_3$ such that $h_3, h_3 + 1, h_3 + 2, \ldots, h_3 + 6g_2 \notin \mathcal{B}$. Define $g_3 = 2g_2(1 + [h_3/2g_2])$. Then $g_3 - 2g_2, g_3 - 2g_2 + 1, g_3 - 2g_2 + 2, \ldots, g_3 + 2g_2 \notin \mathcal{B}$, and $g_3$ is a multiple of $2g_2$.

Similarly define $g_4, g_5, \ldots$

If $k$ and $m$ are positive integers then, by Lemma 1 below, $\mathcal{A} \cap \{1, 2, \ldots, 2mg_k\}$ has at most $2mg_k/2^k$ elements. Hence $\mathcal{A} \cap [1, x]$ has at most $2g_k + 2[x/2g_k]g_k/2^k$ elements; i.e., $N(x) \leq 2g_k + x/2^k$. Thus $\limsup_{x \to \infty} N(x)/x \leq 1/2^k$. This is true for every $k$, so $\lim_{x \to \infty} N(x)/x = 0$.

Lemma 1: For all integers $k \geq 1$ and $n \geq 0$, the set $\mathcal{A} \cap \{n + 1, n + 2, \ldots, n + 2g_k\}$ has at most $2g_k/2^k$ elements.

Proof for $k = 1$: If $\mathcal{A}$ has both $n + 1$ and $n + g_1 + 1$ then $g_1 \in \mathcal{B}$, contradiction; thus $\mathcal{A}$ has at most one of $n + 1, n + g_1 + 1$. Similar comments apply to $n + 2, n + g_1 + 2; n + 3, n + g_1 + 3; \ldots; n + g_1, n + 2g_1$. Hence $\mathcal{A} \cap \{n + 1, n + 2, \ldots, n + 2g_1\}$ has at most $g_1 = 2g_k/2^k$ elements as claimed.

Proof for $k \geq 2$: Assume inductively that, for all $n$, the set $\mathcal{A} \cap \{n + 1, \ldots, n + 2g_{k-1}\}$ has at most $2g_{k-1}/2^{k-1}$ elements.

Consider the sets $S = \{n + 1, \ldots, n + 2g_{k-1}\}$ and $S' = \{n + g_k + 1, \ldots, n + g_k + 2g_{k-1}\}$. If $a \in S$ and $a' \in S'$ then $a' - a \in \{g_k - 2g_{k-1} + 1, \ldots, g_k + 2g_{k-1} - 1\}$, so $a' - a \notin \mathcal{B}$ by construction of $g_k$. Hence $\mathcal{A}$ cannot have elements in common with both $S$ and $S'$. Furthermore, $\mathcal{A} \cap S$ has at most $2g_{k-1}/2^{k-1}$ elements, and $\mathcal{A} \cap S'$ has at most $2g_{k-1}/2^{k-1}$ elements, so $\mathcal{A} \cap (S \cup S')$ has at most $2g_{k-1}/2^{k-1}$ elements.

Similar comments apply with $n$ shifted by $2g_{k-1}, 4g_{k-1}, \ldots, g_k - 2g_{k-1}$; recall here that $g_k$ is a multiple of $2g_{k-1}$. Hence $\mathcal{A} \cap \{n + 1, n + 2, \ldots, n + 2g_k\}$ has at most $(g_k/2g_{k-1})(2g_{k-1}/2^{k-1}) = 2g_k/2^k$ elements as claimed.