1998 Putnam problems and unofficial solutions

As usual, first come the problems, then the problems with solutions. Comments and criticism at the end.

Send any followup remarks to the USENET newsgroup sci.math.

Problems

Problem A1
A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

Problem A2
Let $s$ be any arc of the unit circle lying entirely in the first quadrant. Let $A$ be the area of the region lying below $s$ and above the $x$-axis and let $B$ be the area of the region lying to the right of the $y$-axis and to the left of $s$. Prove that $A + B$ depends only on the arc length, and not on the position, of $s$.

Problem A3
Let $f$ be a real function on the real line with continuous third derivative. Prove that there exists a point $a$ such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

Problem A4
Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number $A_n$ is defined by concatenating the decimal expansions of $A_{n-1}$ and $A_{n-2}$ from left to right. For example $A_3 = A_2 A_1 = 10$, $A_4 = A_3 A_2 = 101$, $A_5 = A_4 A_3 = 10110$, and so forth. Determine all $n$ such that 11 divides $A_n$.

Problem A5
Let $\mathcal{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E \subseteq \mathbb{R}^2$. Show that there is a pairwise disjoint subcollection $D_1, \ldots, D_n$ in $\mathcal{F}$ such that

$$\bigcup_{j=1}^n 3D_j \supseteq E.$$
Here, if $D$ is the disc of radius $r$ and center $P$, then $3D$ is the disc of radius $3r$ and center $P$.

**Problem A6**

Let $A, B, C$ denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$\left( |AB| + |BC| \right)^2 < 8 \cdot [ABC] + 1$$

then $A, B, C$ are three vertices of a square. Here $|XY|$ is the length of segment $XY$ and $[ABC]$ is the area of triangle $ABC$.

**Problem B1**

Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

**Problem B2**

Given a point $(a, b)$ with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at $(a, b)$, one on the $x$-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

**Problem B3**

Let $H$ be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, $C$ the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and $P$ a regular pentagon inscribed in $C$. Determine the surface area of that portion of $H$ lying over the planar region inside $P$, and write your answer in the form $A \sin \alpha + B \cos \beta$, were $A, B, \alpha$, and $\beta$ are real numbers.

**Problem B4**

Find necessary and sufficient conditions on positive integers $m$ and $n$ so that

$$\sum_{i=0}^{mn-1} (-1)^{|i/m| + |i/n|} = 0.$$

**Problem B5**

Let $N$ be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = \underbrace{1111 \cdots 11}_{1998 \text{ digits}}.$$
Find the thousandth digit after the decimal point of \( \sqrt{N} \).

**Problem B6**

Prove that, for any integers \( a, b, c \), there exists a positive integer \( n \) such that

\[
\sqrt{n^3 + an^2 + bn + c}
\]

is not an integer.

**Unofficial solutions**

**Problem A1**

A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

**Solution:** Let \( x \) be the side-length of the cube. The top of the cube is at depth \( 3 - x \) from the top of the cone, and it has a diagonal of length \( x\sqrt{2} \) touching opposite sides of the cone, so \( 3 - x = (3/2)x\sqrt{2} \), so \( x = 6/(2 + 3\sqrt{2}) \).

**Problem A2**

Let \( s \) be any arc of the unit circle lying entirely in the first quadrant. Let \( A \) be the area of the region lying below \( s \) and above the \( x \)-axis and let \( B \) be the area of the region lying to the right of the \( y \)-axis and to the left of \( s \). Prove that \( A + B \) depends only on the arc length, and not on the position, of \( s \).

**Solution:** Say the arc is from angle \( \alpha \) to angle \( \beta \), with \( \alpha < \beta \). Then

\[
A + B = \int_{\cos \alpha}^{\cos \beta} y \, dx + \int_{\sin \alpha}^{\sin \beta} x \, dy = \int_{\alpha}^{\beta} \sin^2 \theta \, d\theta + \int_{\alpha}^{\beta} \cos^2 \theta \, d\theta = \int_{\alpha}^{\beta} 1 \, d\theta = \beta - \alpha.
\]

Alternative solution: \( A \) is the area of the circular sector from \( \alpha \) to \( \beta \), plus the area of the triangle \( (0, 0), (\cos \alpha, 0), (\cos \alpha, \sin \alpha) \), minus the area of the triangle \( (0, 0), (\cos \beta, 0), (\cos \beta, \sin \beta) \); \( B \) is the area of the same circular sector, plus the area of the triangle \( (0, 0), (0, \sin \beta), (\cos \beta, \sin \beta) \), minus the area of the triangle \( (0, 0), (0, \sin \alpha), (\cos \alpha, \sin \alpha) \). Thus \( A + B \) is twice the area of the circular sector; the triangle areas cancel.

**Problem A3**

Let \( f \) be a real function on the real line with continuous third derivative. Prove that there exists a point \( a \) such that

\[
f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.
\]
Solution: If \( f(0)f''(0) < 0 \) or \( f'(0)f''(0) < 0 \) then, by Lemma 1, there exists \( a \) with \( f(a)f'(a)f''(a) = 0 \). Otherwise \( f(0)f'(0)f''(0) \geq 0 \), so take \( a = 0 \).

Lemma 1: If \( g'' \) is continuous and \( g(0)g''(0) < 0 \) then there exists \( a \) with \( g(a) = 0 \) or \( g'(a) = 0 \) or \( g''(a) = 0 \). Proof: If \( g'(0) = 0 \) then take \( a = 0 \). Otherwise define \( x = -g(0)/g'(0) \). By the mean value theorem there is some \( c \) between \( 0 \) and \( x \) for which \( g'(c)x = g(x) - g(0) \), and there is some \( b \) between \( 0 \) and \( c \) for which \( g''(b)c = g'(c) - g'(0) \). Thus \( g''(b)cx = g'(c)x - g'(0)x = g(x) - g(0) + g(0) = g(x) \). If \( g''(0)g''(b) \leq 0 \) then by the intermediate value theorem \( g' \) has a root; otherwise \( g(0)g''(b) < 0 \) so \( g(0)g'(x) = g(0)g''(b)cx < 0 \) so by the intermediate value theorem \( g \) has a root.

Problem A4

Let \( A_1 = 0 \) and \( A_2 = 1 \). For \( n > 2 \), the number \( A_n \) is defined by concatenating the decimal expansions of \( A_{n-1} \) and \( A_{n-2} \) from left to right. For example \( A_3 = A_2A_1 = 10, A_4 = A_3A_2 = 101, A_5 = A_4A_3 = 10110 \), and so forth. Determine all \( n \) such that 11 divides \( A_n \).

Solution: Define \( d_n \) as the number of digits in \( A_n \). Then \( d_1 = 1, d_2 = 1, \) and \( d_n = d_{n-1} + d_{n-2} \) for \( n \geq 3 \). By induction \( d_n \) mod 2 is \( 0, 1, 1 \) when \( n \) mod 3 is \( 0, 1, 2 \).

Now \( A_n = 10^{d_{n-2}}A_{n-1} + A_{n-2} \) for \( n \geq 3 \), so \( A_n \equiv (-1)^{d_{n-2}}A_{n-1} + A_{n-2} \) (mod 11). By induction \( A_n \) mod 11 is \( 1, 0, 1, 10, 2, 1 \) when \( n \) mod 6 is \( 0, 1, 2, 3, 4, 5 \).

Thus \( A_n \) is divisible by 11 if and only if \( n \) mod 6 = 1.

Problem A5

Let \( \mathcal{F} \) be a finite collection of open discs in \( \mathbb{R}^2 \) whose union contains a set \( E \subseteq \mathbb{R}^2 \). Show that there is a pairwise disjoint subcollection \( D_1, \ldots, D_n \) in \( \mathcal{F} \) such that

\[
\bigcup_{j=1}^n 3D_j \supseteq E.
\]

Here, if \( D \) is the disc of radius \( r \) and center \( P \), then \( 3D \) is the disc of radius \( 3r \) and center \( P \).

Solution: Select \( D_1 \in \mathcal{F} \) of maximal radius. Select \( D_2 \in \mathcal{F} \), not intersecting \( D_1 \), of maximal radius. Select \( D_3 \in \mathcal{F} \), not intersecting \( D_1 \) or \( D_2 \), of maximal radius. Continue until every disc in \( \mathcal{F} \) intersects one of \( D_1, D_2, \ldots, D_n \).

For any point \( x \in E \) there is a disc \( C \in \mathcal{F} \) with \( x \in C \). Find the smallest integer \( k \geq 1 \) such that \( C \) intersects \( D_k \). Then \( C \) does not intersect \( D_1, D_2, \ldots, D_{k-1} \); by construction of \( D_k \), the radius of \( C \) is at most the radius of \( D_k \). The distance from \( x \) to the center of \( D_k \) is less than twice the radius of \( C \) plus the radius of \( D_k \), hence less than three times the radius of \( D_k \); so \( x \in 3D_k \subseteq 3D_1 \cup 3D_2 \cup \cdots \cup 3D_n \) as desired.
**Problem A6**

Let $A, B, C$ denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$\left( |AB| + |BC| \right)^2 < 8 \cdot |ABC| + 1$$

then $A, B, C$ are three vertices of a square. Here $|XY|$ is the length of segment $XY$ and $|ABC|$ is the area of triangle $ABC$.

**Solution:** Define $x = |AB|$ and $y = |BC|$, and let $T$ be the area of the triangle. Then $2T \leq xy$ so $x^2 + y^2 + 4T \leq (x + y)^2 < 8T + 1$ so $x^2 + y^2 < 4T + 1$. Both $x^2 + y^2$ and $4T$ are integers, so $x^2 + y^2 \leq 4T$, so $0 \leq (x - y)^2 \leq 4T - 2xy \leq 0$. Thus $2T = xy$ and $x = y$.

**Problem B1**

Find the minimum value of

$$\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}$$

for $x > 0$.

**Solution:** The ratio is $3(x + 1/x)$, so its minimum value is $6$.

**Problem B2**

Given a point $(a, b)$ with $0 < b < a$, determine the minimum perimeter of a triangle with one vertex at $(a, b)$, one on the $x$-axis, and one on the line $y = x$. You may assume that a triangle of minimum perimeter exists.

**Solution:** (Thanks to Martin Tangora for pointing out this solution to me.) The perimeter of the triangle $(a, b)$, $(d, d)$, $(c, 0)$ equals the sum of distances from $(b, a)$ to $(d, d)$ to $(c, 0)$ to $(a, -b)$, which is at least the distance from $(b, a)$ to $(a, -b)$, namely $\sqrt{2a^2 + 2b^2}$. This bound can be achieved: the line segment from $(b, a)$ to $(a, -b)$ intersects the lines $y = x$ and $y = 0$ by the hypotheses on $a$ and $b$.

**Problem B3**

Let $H$ be the unit hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}$, $C$ the unit circle $\{(x, y, 0) : x^2 + y^2 = 1\}$, and $P$ a regular pentagon inscribed in $C$. Determine the surface area of that portion of $H$ lying over the planar region inside $P$, and write your answer in the form $A \sin \alpha + B \cos \beta$, were $A$, $B$, $\alpha$, and $\beta$ are real numbers.

**Solution:** The surface area of the hemisphere is $2\pi$. The portion not over the pentagon consists of five copies of the surface obtained by rotating $\{(\cos \theta, \sin \theta) : 0 \leq \theta \leq \pi/5\}$ halfway around the $x$-axis. That surface has area $\pi \int_0^{\pi/5} (\sin \theta) \sqrt{\sin^2 \theta + \cos^2 \theta} \, d\theta = \pi (1 - \cos(\pi/5))$, so the answer is $5\pi \cos(\pi/5) - 3\pi$. One solution in the stated form is $(A, \alpha, B, \beta) = (-3\pi, \pi/2, 5\pi, \pi/5)$. 
Problem B4

Find necessary and sufficient conditions on positive integers $m$ and $n$ so that

$$\sum_{i=0}^{m-1} (-1)^{[i/m]+[i/n]} = 0.$$  

**Solution:** Define $S(m, n) = \sum_{0 \leq i < mn} (-1)^{[i/m]+[i/n]}$. Write $\text{ord}_2 t$ for the number of powers of 2 dividing $t$.

If $\text{ord}_2 m = \text{ord}_2 n$ then $S(m, n) \neq 0$. Indeed, write $m = 2^e u$ and $n = 2^e v$, with $u$ and $v$ odd. Then $S(u, v)$ has an odd number of odd terms, so it is nonzero; and $S(m, n) = 2^{2e} S(u, v)$ by Lemma 1.

If $\text{ord}_2 m \neq \text{ord}_2 n$ then $S(m, n) = 0$. Indeed, write $m = 2^d x$ and $n = 2^e v$, with $x$ and $v$ odd; without loss of generality assume $d > e$. Write $u = 2^{d - e} x$, and substitute $i = uv - 1 - j$ in $S(u, v)$:

$$S(u, v) = \sum_{0 \leq j < uv} (-1)^{[(uv - 1 - j)/u] + [(uv - 1 - j)/v]}$$

$$= \sum_{0 \leq j < uv} (-1)^{u - 1 + v - 1 - j/u - j/v}$$

$$= \sum_{0 \leq j < uv} (-1)^{1 + [j/u] + [j/v]} = -S(u, v).$$

Thus $S(u, v) = 0$; and $S(m, n)$ is a multiple of $S(u, v)$ by Lemma 1.

Lemma 1. $S(bu, bv)$ is a multiple of $S(u, v)$; if $u + v$ is even then $S(bu, bv) = b^2 S(u, v)$.

**Proof:** Write $i = buvq + br + s$, with $0 \leq r < uv$ and $0 \leq s < b$. Then

$$S(bu, bv) = \sum_{0 \leq q < b} \sum_{0 \leq r < uv} \sum_{0 \leq s < b} (-1)^{[(buvq + br + s)/u] + [(buvq + br + s)/v]}$$

$$= \sum_{0 \leq q < b} \sum_{0 \leq r < uv} \sum_{0 \leq s < b} (-1)^{(u + v)q + [r/u] + [r/v]}$$

$$= bS(u, v) \sum_{0 \leq q < b} (-1)^{(u + v)q}.$$  

If $u + v$ is even then $\sum_{0 \leq q < b} (-1)^{(u + v)q} = b$.

**Problem B5**

Let $N$ be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = \underbrace{1111 \cdots 11}_{1998 \text{ digits}}.$$
Find the thousandth digit after the decimal point of $\sqrt{N}$.

**Solution:** The answer is 1.

More generally, for any positive integer $h$, write $N = (10^{2h} - 1)/9$. By Lemma 1,

$$10^{h+1} \sqrt{N} = 10^{2h+1} \sqrt{1 - 10^{-2h}/3}$$

$$= 10^{2h+1} (1 - 10^{-2h}/2 - 10^{-4h} x/4)/3$$

$$= 10^{2h+1}/3 - 10/6 - 10^{1-2h} x/12$$

$$= (10^{2h+1} - 10)/3 + 1 + 2/3 - 10^{1-2h} x/12$$

for some $x$ between 0 and 1. Now $0 < 2/3 - 10^{1-2h} x/12 < 1$, and $(10^{2h+1} - 10)/3$ is a multiple of 10, so $\lfloor 10^{h+1} \sqrt{N} \rfloor \mod 10 = 1$.

For example, $\sqrt{11} \approx 3.3166$, and $\sqrt{1111} \approx 33.3316666$.

**Lemma 1.** If $0 < d < 1/2$ then $1 - d/2 - d^2/4 < \sqrt{1 - d} < 1 - d/2$. Proof: $d + d^2/4 < 1$ so $(1 - d/2 - d^2/4)^2 = 1 - d - (1 - d - d^2/4)d^2/4 < 1 - d < 1 - d + d^2/4 = (1 - d/2)^2$.

**Problem B6**

Prove that, for any integers $a, b, c$, there exists a positive integer $n$ such that

$$\sqrt{n^3 + an^2 + bn + c}$$

is not an integer.

**Solution:** Write $f(n) = n^3 + an^2 + bn + c$. Find an integer $m$ large enough that $8m^3$ exceeds both $(a^2 - 4b)m^2 - am - c + 1/4$ and $(4b - a^2)m^2 - am + c - 1/4$. Then

$$(8m^3 + am - 1/2)^2 < f(4m^2) < (8m^3 + am + 1/2)^2,$$

so if $f(4m^2)$ is a square then $f(4m^2) = (8m^3 + am)^2$, i.e., $(a^2 - 4b)m^2 = c$. This cannot hold for more than one value of $m$ unless $4b = a^2$ and $c = 0$. But then $f(n) = n(n+a/2)^2$; this is not a square if $n$ is a non-square different from $-a/2$. 
Comments

In A1, is one required to prove that the \(6/(2 + 3\sqrt{2})\) cube fits inside the cone, and that larger cubes do not? In A2, can one assume that the arc length from \((\cos \alpha, \sin \alpha)\) to \((\cos \beta, \sin \beta)\) is \(\beta - \alpha\)? In A6, can one take “parallelogram with equal sides and with area equal to product of sides” as the definition of a square? In B3, can one use standard facts about areas of portions of a sphere? Contestants should not have to guess how their work will be graded.

The form of answer for B3—“\(A \sin \alpha + B \cos \beta\), where \(A, B, \alpha,\) and \(\beta\) are real numbers”—was rather strange. Did the question writers mean to say “\(A\pi + B\pi \cos(C\pi)\), where \(A, B, C\) are rational numbers”?

There are many solutions to B6; what makes the problem difficult is the weakness of its conclusion. Here is a very fast proof by Brian R. Hunt: \(f(4k + b + 1) - f(4k + b - 1) \equiv 2 \pmod{4}\) so it is not possible for both \(f(4k + b + 1) \mod{4}\) and \(f(4k + b - 1) \mod{4}\) to be in \(\{0,1\}\).

—Daniel J. Bernstein, djb@cr.yp.to, 7 December 1998