Solutions of the Diophantine Equation $x^3 + y^3 = z^3 - d$

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1. Introduction. In 1955, at the suggestion of Professor L. J. Mordell, Miller and Woollett [1] investigated the integer solutions of the equation

$$x^3 + y^3 + z^3 = d \tag{1.1}$$

for all integers $0 < d \leqq 100$. These authors carried out a numerical search in the range

$$|x| \leqq |y| \leqq |z| \leqq 3200 \tag{1.2}$$

with the help of the EDSAC computer at Cambridge University; their results are tabulated in [1].

Mordell's original interest in this equation centered on the case $d = 3$; in particular, he wanted to know whether there existed solutions in addition to the known triples $x = y = z = 1$ and $x = y = 4$, $z = -5$. For the range they considered, Miller and Woollett showed that in fact no further solutions existed. As a result of their happy decision to extend the search to other values of d, they discovered several other interesting properties of equation (1.1). Perhaps the two most striking facts were the following:

(a) For $d = 2$, all solutions in the range (1.2) belong to the family:

$$-6t^2, \qquad -6t^3 + 1, \qquad 6t^3 + 1. \tag{1.3}$$

(b) Over the range considered, equation (1.1) has no solutions for the values $d = 30, 33, 39, 42, 52, 74, 75, 84, 87$.

With regard to (b), it should be remarked that, while it has long been known [2] that equation (1.1) has no solutions if d is an integer of the form $9m \pm 4$, there is no known reason for excluding any other integer (except, of course, $d = 0$). One might be tempted to conjecture that all integers (except zero) not of the form $9m \pm 4$ can be expressed as the sum of three cubes, minus signs allowed. (If this conjecture were true, it would solve the so-called "Easier Waring's Problem" for cubes [2], since it would then follow that all integers can be expressed as the sum of at most four cubes.) Miller and Woollett's results seemed to cast a certain doubt on the soundness of such a conjecture; as will be seen below, further numerical experimentation has served to make it unlikely that the conjecture is true.

2. The Present Calculation. In the fall of 1961, Professor S. Chowla suggested to one of us (P.R.S.) that it would be of interest to investigate the case $d = 3$ for a much larger range of (x, y, z) values. Early in 1963 this suggestion was taken up and a program was written for the Laboratory's I.B.M. STRETCH Computer to

Received November 13, 1963. Revised December 11, 1963. This paper was performed under the auspices of the United States Atomic Energy Commission at the Los Alamos Scientific Laboratory of The University of California, Los Alamos, New Mexico.

search for solutions of equation (1.1) in the equivalent form:

$$x^3 + y^3 = z^3 - d. \tag{2.1}$$

The range chosen was:

$$0 \leqq x \leqq y \leqq 2^{16} = 65{,}536,$$
$$0 < N \leqq 2^{16}, \qquad N \equiv z - x, \tag{2.2}$$
$$0 < |d| \leqq 999.$$

It will be observed that this range excludes negative values of x, y, z, although d may have either sign. The small number of solutions (217 such) thus omitted were calculated separately on the MANIAC II Computer.

3. Results. The actual solutions found are collected in a large table, a copy of which has been deposited in the UMT file. A limited number of copies have been retained by the authors for distribution to interested mathematicians. The table is divided into three parts; Table I and Table II cover, respectively, the ranges $-999 \leqq d \leqq -2$ and $2 \leqq d \leqq 999$. Only "primitive" solutions are tabulated; these are solutions in which x, y, z have no common factor. All "derived" solutions (the terms go back to Miller and Woollett) can be recovered by multiplication: e.g., if d, x, y, z is a primitive solution, the associated derived solutions are $d' = m^3 d$, $x' = mx$, $y' = my$, $z' = mz$, $m = -1, \pm 2, \pm 3, \cdots$. Values of $|d|$ which are themselves cubes have been omitted from the present calculation. There is a large number of solutions for each such case, and it was felt that their inclusion would make the tables too long. The additional solutions mentioned in Section 2, those for which one or two members of the triple (x, y, z) are negative, have been arbitrarily assigned to Table II (positive d). With these conventions, the total number of primitive solutions found is 1873 for negative d and 2148 for positive d. Except for the cubes $|d| = 1, 8, 27, 64$ (which we do not list), our results are in exact agreement with those of Miller and Woollett over the range they considered.

Finally, we have included a third table (Table III), which serves as a summary of our results. With the cubes omitted, the column labelled "d" lists all integers $2 \leqq d \leqq 999$ which are not of the form $9m \pm 4$. For each such entry, the column labelled N_+ indicates the number of primitive solutions of (2.1) with d positive, while N_- gives the corresponding number of solutions for negative d. This table is reproduced in the present paper (Table A).

4. Discussion. (a) For the range considered, there are 70 values of $|d|$ for which there exist no solutions of equation (2.1). In Table A these are indicated by an asterisk. There are in addition 12 values of $|d|$ which have only derived solutions, viz:

$$|d| = 24, 80, 192, 250, 375, 384, 480, 624, 744, 768, 808, 960.$$

52 of the 70 "excluded" integers are of the form $9m \pm 3$, 13 are of the form $9m \pm 2$, 4 are of the form $9m \pm 1$ and one ($|d| = 180$) is divisible by 9.

One of the "excluded" values noted by Miller and Woollett has gone away,

TABLE A

d	N_+	N_-	d	N_+	N_-	d	N_+	N_-	d	N_+	N_-
2	23	0	90	9	10	174	2	3	260	6	4
3	1	1	91	4	3	177	1	1	261	3	2
6	4	1	92	9	2	178	1	1	262	3	0
7	1	2	93	2	0	179	1	4	263	1	1
9	2	1	96	0	1	180*	0	0	264	0	3
10	3	1	97	5	4	181	11	8	267	2	1
11	4	1	98	2	2	182	2	2	268	0	1
12	0	1	99	7	11	183	3	6	269	4	3
15	2	1	100	3	0	186	1	1	270	2	1
16	1	0	101	4	2	187	1	2	271	5	2
17	4	3	102	0	1	188	6	3	272	4	5
18	3	4	105	1	0	189	4	2	273	1	3
19	2	2	106	3	2	190	16	11	276	0	1
20	4	4	107	1	2	191	2	2	277	5	0
21	3	2	108	3	1	192	0	0	278	3	5
24	0	0	109	7	5	195*	0	0	279	7	9
25	2	2	110*	0	0	196	2	4	280	3	5
26	1	4	111	2	2	197	6	6	281	7	4
28	3	3	114*	0	0	198	4	2	282	0	1
29	4	3	115	2	1	199	2	3	285	1	2
30*	0	0	116	2	0	200	1	1	286	2	2
33*	0	0	117	3	0	201	0	1	287	6	7
34	5	5	118	6	10	204	1	1	288	3	8
35	5	2	119	4	0	205	2	1	289	2	1
36	3	3	120	0	2	206	1	2	290*	0	0
37	2	1	123	2	0	207	4	5	291	1	0
38	1	2	124	1	1	208	1	3	294	1	3
39*	0	0	126	8	2	209	8	10	295	5	6
42*	0	0	127	5	4	210	1	1	296	2	1
43	4	4	128	13	3	213	1	1	297	1	2
44	1	0	129	2	1	214	2	2	298	6	2
45	1	3	132	1	0	215	4	1	299	1	2
46	2	2	133	3	4	217	6	4	300	0	4
47	2	2	134	4	5	218	4	2	303	1	1
48	2	0	135	2	1	219	1	1	304	2	4
51	0	1	136	0	1	222	0	1	305	3	3
52*	0	0	137	5	0	223	5	3	306	4	4
53	3	2	138	1	0	224	2	3	307	11	11
54	4	1	141	2	1	225	10	5	308	1	1
55	8	6	142	1	0	226	2	5	309	3	2
56	1	2	143*	0	0	227	0	1	312*	0	0
57	2	8	144	4	2	228	1	0	313	2	0
60	2	1	145	1	1	231*	0	0	314	12	3
61	1	1	146	2	3	232	4	3	315	7	4
62	5	5	147	1	1	233	2	2	316	9	4
63	5	3	150	0	1	234	0	1	317	0	2
65	4	4	151	2	2	235	2	4	318*	0	0
66	1	0	152	2	1	236	2	1	321*	0	0
69	1	3	153	9	5	237	0	2	322	5	2
70	2	4	154	3	6	240	1	0	323	7	15
71	9	4	155	5	13	241	0	1	324	2	3
72	1	1	156*	0	0	242	5	3	325	3	2
73	5	2	159	1	4	243	1	1	326	1	1
74*	0	0	160	4	2	244	6	3	327	1	0
75*	0	0	161	8	8	245	2	3	330	5	1
78	1	1	162	3	0	246	2	4	331	4	0
79	3	0	163	4	2	249	1	1	332	1	4
80	0	0	164	4	0	250	0	0	333	4	6
81	2	1	165*	0	0	251	9	7	334	1	2
82	1	2	168	1	2	252	8	5	335	2	5
83	12	4	169	4	0	253	7	4	336	0	2
84*	0	0	170	3	5	254	1	0	339	2	3
87	1	0	171	7	3	255	3	1	340	1	0
88	3	2	172	0	1	258	3	0	341	6	3
89	0	3	173	1	0	259	3	5	342	6	15

TABLE A—*Continued*

d	N_+	N_-	d	N_+	N_-	d	N_+	N_-	d	N_+	N_-
344	3	2	429	0	2	513	2	1	596	4	1
345	1	0	430	2	2	514	1	6	597	0	1
348	0	2	431	1	2	515	2	2	600*	0	0
349	4	6	432	1	2	516*	0	0	601	8	4
350	4	5	433	5	12	519	6	4	602	4	6
351	4	2	434	9	3	520	2	0	603	13	6
352	2	1	435*	0	0	521	7	8	604	6	4
353	2	0	438	1	1	522	1	1	605	1	2
354	1	0	439*	0	0	523	2	3	606*	0	0
357	2	1	440	1	1	524	8	2	609*	0	0
358	1	6	441	10	3	525	2	2	610	6	5
359	5	3	442	6	3	528	0	1	611	0	3
360	3	1	443	1	0	529	1	0	612	2	4
361	3	2	444*	0	0	530*	0	0	613	3	2
362	1	1	447	4	2	531	8	10	614	0	1
363	1	1	448	1	3	532	10	10	615	4	1
366*	0	0	449	11	9	533	1	5	618*	0	0
367*	0	0	450	2	3	534*	0	0	619	1	2
368	1	2	451	1	2	537	0	2	620	2	1
369	5	3	452*	0	0	538	6	1	621	2	0
370	3	5	453	0	1	539	6	4	622	5	2
371	2	0	456	2	2	540	5	7	623	6	4
372	2	3	457	2	2	541	8	5	624	0	0
375	0	0	458	2	1	542*	0	0	627*	0	0
376	1	0	459	2	2	543	1	1	628	0	2
377	4	2	460	3	1	546	1	1	629	12	9
378	7	3	461	5	6	547	3	7	630	7	1
379	11	7	462*	0	0	548	0	3	631	6	6
380	3	0	465	0	3	549	3	1	632	1	3
381	1	3	466	1	1	550	2	1	633*	0	0
384	0	0	467	6	12	551	3	2	636	1	0
385	5	1	468	6	3	552	0	1	637	2	2
386	5	1	469	6	8	555	0	1	638	11	13
387	5	4	470	4	2	556*	0	0	639	5	5
388	4	3	471	1	0	557	3	3	640	1	3
389	1	1	474	1	1	558	8	5	641	1	0
390*	0	0	475	2	9	559	6	6	642	1	0
393	1	2	476	5	7	560	6	4	645	4	3
394	0	1	477	5	3	561	2	2	646	2	0
395	1	2	478*	0	0	564*	0	0	647	6	6
396	2	4	479	1	1	565	0	1	648	1	2
397	5	1	480	0	0	566	4	3	649	5	6
398	5	8	483	1	0	567	4	4	650	2	0
399	2	0	484	3	5	568	1	2	651	5	2
402	0	1	485	1	1	569	2	2	654	1	1
403	1	1	486	3	6	570	0	4	655	1	2
404	2	1	487	6	2	573	1	1	656	4	3
405	15	7	488	0	1	574	0	3	657	16	5
406	6	2	489	2	5	575	7	8	658	8	5
407	6	2	492	1	0	576	1	2	659	4	2
408	1	0	493	2	3	577	8	1	660*	0	0
411	1	2	494	1	3	578	1	2	663*	0	0
412	1	2	495	7	4	579*	0	0	664	5	0
413	8	2	496	5	6	582	2	1	665	7	10
414	6	4	497	1	3	583	1	0	666	5	4
415	4	1	498	2	1	584	0	3	667	3	4
416	1	0	501*	0	0	585	1	1	668	2	2
417	0	3	502	1	0	586	3	2	669	0	2
420*	0	0	503	7	3	587	3	4	672	1	1
421	6	8	504	1	1	588*	0	0	673	3	5
422	1	0	505	5	8	591	2	0	674	4	3
423	1	1	506	2	2	592	0	1	675	0	1
424	3	2	507	1	2	593	4	1	676	1	0
425	1	1	510	3	1	594	6	3	677	3	1
426	2	0	511	7	4	595	2	2	678	2	5

TABLE A—*Continued*

d	N_+	N_-	d	N_+	N_-	d	N_+	N_-	d	N_+	N_-
681	1	0	765	0	1	848	7	1	933*	0	0
682	2	1	766	5	4	849	2	0	934	0	1
683	6	2	767*	0	0	852	0	1	935	7	2
684	6	2	768	0	0	853	6	5	936	2	1
685	8	5	771	4	2	854	6	2	937	10	7
686	1	0	772	2	0	855	12	15	938	4	4
687	2	4	773	0	4	856	2	5	939	0	2
690	2	0	774	0	2	857	5	2	942	0	1
691	1	1	775	1	5	858	1	0	943	3	6
692	5	4	776	3	0	861*	0	0	944	1	5
693	3	6	777*	0	0	862	3	7	945	2	2
694	10	12	780	1	0	863	4	3	946	8	3
695	1	1	781	2	2	864	3	1	947	2	1
696	0	1	782	0	2	865	1	2	948*	0	0
699	1	1	783	7	3	866	2	1	951	2	0
700	2	2	784	0	2	867	4	3	952	2	0
701	13	5	785	8	5	870*	0	0	953	5	5
702	0	2	786*	0	0	871	0	1	954	2	3
703	3	7	789*	0	0	872	6	2	955	2	0
704	1	0	790	7	1	873	5	3	956	3	0
705	0	1	791	8	1	874	12	11	957	4	1
708	1	0	792	7	7	875	2	3	960	0	0
709	2	0	793	2	1	876	2	3	961	4	1
710	1	5	794	3	0	879	3	4	962	2	2
711	1	2	795*	0	0	880	0	2	963	5	2
712	3	0	798	1	5	881	10	12	964*	0	0
713	4	3	799	2	3	882	3	7	965	5	4
714	3	1	800	0	1	883	6	10	966	1	0
717	1	0	801	5	4	884	0	1	969*	0	0
718	0	1	802	3	1	885	2	3	970	2	1
719	2	2	803	2	1	888	1	0	971	0	1
720	7	6	804	3	2	889	1	1	972	4	3
721	5	7	807	1	0	890	2	3	973	5	4
722	2	9	808	0	0	891	3	0	974	1	7
723	1	1	809	6	6	892	5	3	975*	0	0
726	1	0	810	4	3	893	1	2	978	0	1
727	4	4	811	11	12	894*	0	0	979	4	4
728	5	1	812	1	2	897	0	1	980	5	5
730	3	0	813	2	2	898	2	0	981	4	9
731	1	2	816	1	0	899	6	6	982	1	0
732*	0	0	817	1	0	900	1	2	983	3	2
735*	0	0	818	8	14	901	4	2	984	3	0
736	5	6	819	7	2	902	5	4	987	1	2
737	1	2	820	13	13	903*	0	0	988	5	3
738	2	0	821	1	2	906*	0	0	989	7	3
739	3	2	822	1	0	907	2	2	990	3	4
740	3	2	825	4	3	908	0	1	991	4	4
741	2	0	826	1	0	909	5	7	992	0	1
744	0	0	827	10	7	910	1	0	993	2	5
745	1	1	828	4	2	911	13	8	996	0	1
746	2	5	829	3	3	912*	0	0	997	3	1
747	1	1	830*	0	0	915	0	1	998	4	1
748	5	8	831	0	2	916	1	2	999	2	0
749	2	4	834*	0	0	917	4	3			
750	1	0	835	2	1	918	10	5			
753	1	0	836	0	1	919	6	2			
754*	0	0	837	4	5	920	0	1			
755	17	11	838	3	1	921*	0	0			
756	0	2	839	8	1	924	3	1			
757	8	2	840	1	2	925	4	4			
758*	0	0	843	0	1	926	3	2			
759	0	1	844	1	2	927	1	3			
762	2	0	845	0	2	928	2	1			
763	0	5	846	4	7	929	0	1			
764	5	4	847	6	2	930	0	3			

namely $|d| = 87$. The solution lies slightly beyond the range they considered:

$$(1972)^3 + (4126)^3 = (4271)^3 - 87.$$

For $|d| = 96$, Miller and Woollett found only one derived solution. There is actually a primitive solution for this case, but it lies well beyond their range:

$$(10853)^3 + (13139)^3 = (15250)^3 + 96.$$

In general, it is rather risky to draw conclusions from the experimental evidence, even with a search as extensive as the present one. For example, $|d| = 227$ and $|d| = 971$ each have only a single solution, lying relatively close to the boundary of the search region:

$$(24579)^3 + (51748)^3 = (53534)^3 + 227$$
$$(7423)^3 + (55643)^3 = (55687)^3 + 971.$$

Many other such examples can be found in our large table of solutions. Nevertheless, it is in our opinion rather unlikely that all the missing $|d|$'s will turn out to be expressible as sums of three cubes. It would be of interest to attempt a proof that, say, 30 cannot be so expressed.

(b) All solutions for $|d| = 2$ were found to belong to the parametric family (1.3). So far we have only succeeded in identifying one other family which is, in fact, a simple extension of (1.3). For $|d| = 128$, all solutions are given by the formula:

$$x = 6t^2, \qquad y = -4 + 3t^3, \qquad z = 4 + 3t^3, \qquad d = 128. \tag{4.1}$$

If t is even, the solutions are the derived ones associated with $d = 2$, $m = 4$, but for odd t we get a new primitive family. (The existence of this parametric family was noted in [1].)

(c) The case $|d| = 3$ was found to have no new solutions. $|d| = 12$ may also be of theoretical interest; it is the smallest integer that appears to have only a single solution:

$$7^3 + 10^3 = 11^3 + 12.$$

The next interesting case is $|d| = 24$, which, in fact, has only the derived solutions:

$$(-2)^3 + (-2)^3 = (2)^3 - 24,$$
$$8^3 + 8^3 = 10^3 + 24.$$

Then, of course, comes $|d| = 30$, the smallest integer for which no solution whatsoever has been found.

(d) As a final remark, we point out that our table affords an explicit decomposition into 4 or fewer cubes for every integer from 1 to 999. In particular, every number of the form $9m \pm 4$ in our range turns out to differ by a cube from a number for which one or more decompositions into 3 cubes has been found.

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1. J. C. P. Miller & M. F. C. Woollett, "Solutions of the Diophantine equation $x^3 + y^3 + z^3 = k$," *J. London Math. Soc.*, v. 30, 1955, p. 101–110.

2. G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, Third Edition, Clarendon Press, Oxford, 1954, p. 327.