A short proof of the unpredictability of cipher block chaining

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Abstract. Let u be a uniform random function from b-bit strings to b-bit strings. Fix $m \geq 1$. Define

$$u_m^+(g_1, g_2, \ldots, g_m) = u(u(\cdots u(u(g_1) + g_2) + \cdots) + g_m).$$

This paper presents a short proof that u_m^+ is unpredictable: specifically, if A is an algorithm that performs at most q oracle queries, and v is a uniform random function from mb-bit strings to b-bit strings, then the A-distance from u_m^+ to v is at most $mq(mq-1)/2^{b+1}$. It was already known that u_m^+ was unpredictable, but previous proofs were much more complicated.

Keywords: mode of operation, CBC, provable security

1 Introduction

Let u be a uniform random function from b-bit strings to b-bit strings; in other words, let $u(0), u(1), u(2), \ldots, u(2^b - 1)$ be independent uniform random b-bit strings. Define

$$u^+(g_1, g_2, \ldots, g_m) = u_m^+(g_1, g_2, \ldots, g_m) = u(u(\cdots u(u(g_1) + g_2) + \cdots) + g_m)$$

for each integer $m \geq 0$ and each mb-bit string $(g_1, g_2, \ldots, g_m)$. For example, $u^+() = u_0^+() = 0$, and $u^+(g_1, g_2) = u_2^+(g_1, g_2) = u(u(g_1) + g_2)$.

This paper presents a short proof that u_m^+ is unpredictable for $m \geq 1$—i.e., u_m^+ is indistinguishable from a uniform random function from mb-bit strings to

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b-bit strings. More precisely, if A is an algorithm that performs at most q oracle queries, and v is a uniform random function from mb-bit strings to b-bit strings, then the A-distance from u_m^+ to v is at most $mq(mq-1)/2^{b+1}$. Here the **A-distance from u_m^+ to v** is $|\Pr[A(u_m^+)=1]-\Pr[A(v)=1]|$, where $A(f)$ means the output of A using an oracle for f.

The heart of the proof—see Section 2—is that u_m^+ has large interpolation probabilities: if $x_1, x_2, \ldots, x_k$ are distinct mb-bit strings, and $y_1, y_2, \ldots, y_k$ are b-bit strings, then $(u_m^+(x_1), u_m^+(x_2), \ldots, u_m^+(x_k)) = (y_1, y_2, \ldots, y_k)$ with probability at least $(1-\epsilon)/2^{bk}$ where $\epsilon = mk(mk-1)/2^{b+1}$. The rest of the proof—see Section 3—is a broad principle having nothing to do with the details of u_m^+: any random function with large interpolation probabilities is unpredictable. Section 4 discusses a few standard consequences of the unpredictability of u_m^+.

History

The construction of u_m^+ is called "cipher block chaining." The unpredictability of CBC is not a new result: Bellare, Kilian, and Rogaway proved in [2, Theorem 3.1] that the q-query distance from u_m^+ to v is at most $3m^2q^2/2^{b+1}$. Their proof is vastly more complicated than the proof here.

In reaction to a draft of [2], I wrote [3], explaining a much simpler way to prove unpredictability. [3, Theorem 3.1] is the same as Theorem 3.1 in this paper. I illustrated the theorem with a construction different from CBC, but commented at the end of [3, Section 5] that the theorem would also allow an easy proof of unpredictability for CBC. This paper presents that proof.

A subsequent Bellare-Rogaway preprint "The Game-Playing Technique," now at Draft 0.4 after the correction of some serious errors, presents (among other things) another proof of unpredictability for CBC. The authors describe their proof as "elementary"; I agree that it is an improvement over the proof in [2], but it is still much more complicated than necessary.

Bellare and Rogaway say that their approach "can lead to more easily verified, less error-prone proofs than those grounded in more conventional probabilistic language." I see no justification for that claim. I see many cryptographic proofs that are unnecessarily complicated because the authors simply don't *know* the standard language of probability theory,[1] let alone how to competently use it;[2] but the obvious solution is to educate people, not to reinvent the wheel.

[1] There's much more to the language than the simplified concepts of "event" (subset of a finite universe) and "probability" (subset size divided by universe size) that we teach to undergraduates. Most importantly, the concept of a "random variable" has had a standard mathematical definition for seventy years and is a tremendous time-saver in probabilistic definitions, theorems, and proofs. Warning to undergraduates: "random" does not imply "uniform" or "discrete" or "independent of everything else." For definitions see, e.g., [3].

[2] For example, many cryptographers appear to believe that figuring out the success probability of a protocol requires separately analyzing the success probability of the first step, the conditional success probability of the second step, etc. See, e.g., the CBC proofs in [2] and [7].

2 CBC has large interpolation probabilities

Theorem 2.1. *Let G be a finite commutative group. Let u be a uniform random function from G to G. Define $u^+(g_1,\dots,g_i) = u(u(\cdots u(g_1)+\cdots)+g_i)$ for all $(g_1,\dots,g_i) \in G^0 \cup G^1 \cup G^2 \cup \cdots$. Let m and k be integers with $m \ge 1$ and $k \ge 0$. Let $x_1, x_2, \dots, x_k$ be distinct elements of G^m. Let $y_1, y_2, \dots, y_k$ be elements of G. Then $(u^+(x_1), u^+(x_2), \dots, u^+(x_k)) = (y_1, y_2, \dots, y_k)$ with probability at least $(1-\epsilon)/\#G^k$ where $\epsilon = mk(mk-1)/2\#G$.*

In other words, every k-interpolation probability of u^+_m is at least $(1-\epsilon)/\#G^k$.

Proof. Define $S = \{x_1, x_2, \dots, x_k\}$. Define $P \subseteq G^1 \cup \cdots \cup G^m$ as the set of nonempty prefixes of $x_1, x_2, \dots, x_k$. Note that $\#P \le mk$.

Each element of P can be written uniquely as (q, g) with $g \in G$ and $q \in G^0 \cup P$. Define $\operatorname{chop}(q,g) = q$ and $\operatorname{last}(q,g) = g$.

Define a function $f : G^0 \cup P \to G$ as **admissible** if $f() = 0$, $f(x_i) = y_i$ for all i, and the function $p \mapsto f(\operatorname{chop} p) + \operatorname{last} p$ from P to G is injective. Define f as being **compatible with** u if $u(f(\operatorname{chop} p) + \operatorname{last} p) = f(p)$ for every $p \in P$.

Observe that each admissible function f has probability $1/\#G^{\#P}$ of being compatible with u. (Proof: $p \mapsto u(f(\operatorname{chop} p)+\operatorname{last} p)$ is a uniform random function from P to G, so it has probability $1/\#G^{\#P}$ of matching f.) Furthermore, if an admissible function f is compatible with u, then $(u^+(x_1), u^+(x_2), \dots, u^+(x_k)) = (y_1, y_2, \dots, y_k)$; in fact, $u^+(p) = f(p)$ for every $p \in G^0 \cup P$. (Proof: $u^+() = 0 = f()$. For $p \in P$, assume inductively that $u^+(\operatorname{chop} p) = f(\operatorname{chop} p)$. Then $u^+(p) = u(u^+(\operatorname{chop} p) + \operatorname{last} p) = u(f(\operatorname{chop} p) + \operatorname{last} p) = f(p)$.)

If two different admissible functions f, f' are compatible with u then $f(p) = u^+(p) = f'(p)$ for every $p \in G^0 \cup P$, contradiction. I will show in a moment that there are at least $(1-\epsilon)\#G^{\#P-k}$ admissible functions f. Therefore, with probability at least $(1-\epsilon)\#G^{-k}$, some admissible function f is compatible with u, and in particular $(u^+(x_1), u^+(x_2), \dots, u^+(x_k)) = (y_1, y_2, \dots, y_k)$ as claimed.

To count admissible functions, consider a uniform random function $f : G^0 \cup P \to G$. Each of the conditions $f() = 0$, $f(x_1) = y_1$, $\dots$, $f(x_k) = y_k$ is satisfied with probability $1/\#G$. These conditions are independent, since $x_1, \dots, x_k$ are distinct and $m \ge 1$; thus f satisfies all the conditions with probability $\#G^{-1-k}$.

If p, p' are distinct elements of P then $f(\operatorname{chop} p) + \operatorname{last} p = f(\operatorname{chop} p') + \operatorname{last} p'$ with conditional probability at most $1/\#G$. (If $\operatorname{chop} p = \operatorname{chop} p'$ and $\operatorname{last} p = \operatorname{last} p'$ then $p = p'$, contradiction. If $\operatorname{chop} p = \operatorname{chop} p'$ and $\operatorname{last} p \ne \operatorname{last} p'$ then $f(\operatorname{chop} p) + \operatorname{last} p$ cannot equal $f(\operatorname{chop} p') + \operatorname{last} p'$. If $\operatorname{chop} p \ne \operatorname{chop} p'$ then at least one of $\operatorname{chop} p, \operatorname{chop} p'$, let's say $\operatorname{chop} p$, is distinct from $()$; thus $f(\operatorname{chop} p)$ is conditionally uniform, so it equals $f(\operatorname{chop} p') + \operatorname{last} p' - \operatorname{last} p$ with probability $1/\#G$. Note that requiring G to be a commutative group is overkill here.)

Hence the conditional probability of any collisions in $p \mapsto f(\operatorname{chop} p) + \operatorname{last} p$ is at most $\#P(\#P-1)/2\#G \le \epsilon$; i.e., f is admissible with probability at least $(1-\epsilon)\#G^{-1-k}$; i.e., there are at least $(1-\epsilon)\#G^{-1-k}\#G^{\#P+1} = (1-\epsilon)\#G^{\#P-k}$ admissible functions f. $\square$

Example

Say $G = \mathbf{Z}/10^6$, $m = 3$, $k = 3$, $x_1 = (1,2,3)$, $x_2 = (1,2,4)$, and $x_3 = (3,1,4)$. Then $S = \{(1,2,3),(1,2,4),(3,1,4)\}$ and

$$P = \{(1),(3),(1,2),(3,1),(1,2,3),(1,2,4),(3,1,4)\}.$$

There are at most $mk = 9$ elements of P: in fact, only 7, since $(1,2,3)$ and $(1,2,4)$ share some prefixes.

A function $f : G^0 \cup P \to G$ is admissible if and only if $f() = 0$, $f(1,2,3) = y_1$, $f(1,2,4) = y_2$, $f(3,1,4) = y_3$, and the seven quantities

$$f()+1, f()+3, f(1)+2, f(3)+1, f(1,2)+3, f(1,2)+4, f(3,1)+4$$

are distinct. There are $\#G^4$ functions satisfying the equations (i.e., $\#G^4$ choices of $f(1), f(3), f(1,2), f(3,1)$), and there are $7(7-1)/2 = 21$ inequalities each eliminating at most $\#G^3$ functions, so there are at least $\#G^4 - 21\#G^3$ admissible functions.

An admissible function f is compatible with u if and only if $u(f()+1) = f(1)$, $u(f()+3) = f(3)$, $u(f(1)+2) = f(1,2)$, $u(f(3)+1) = f(3,1)$, $u(f(1,2)+3) = f(1,2,3)$, $u(f(1,2)+4) = f(1,2,4)$, and $u(f(3,1)+4) = f(3,1,4)$, This occurs with probability exactly $1/\#G^7$ for each f, and if it does occur then $u^+(1,2,3) = y_1$, $u^+(1,2,4) = y_2$, $u^+(3,1,4) = y_3$. It cannot occur for two f's simultaneously, so it occurs with probability at least $(\#G^4 - 21\#G^3)/\#G^7 = (1 - 21/\#G)/\#G^3$.

3 Large interpolation probabilities imply unpredictability

Theorem 3.1. *Let φ be a random function from a set S to a finite set T. Let q be an integer with $q \ge 0$. Let A be an algorithm that performs at most q distinct oracle queries. Assume, for all $k \in \{0,1,2,\ldots,q\}$, all $y_1, y_2, \ldots, y_k \in T$, and all distinct $x_1, x_2, \ldots, x_k \in S$, that $(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_k)) = (y_1, y_2, \ldots, y_k)$ with probability at least $(1-\epsilon)/\#T^k$. Then the A-distance between φ and uniform is at most ϵ.*

In other words, if every k-interpolation probability of φ is at least $(1-\epsilon)/\#T^k$ for all $k \in \{0,1,2,\ldots,q\}$, then φ cannot be predicted with probability larger than ϵ by an algorithm that performs at most q oracle queries. Note that this is an information-theoretic statement: the run time of the algorithm is irrelevant.

Theorem 3.1 appears in my paper [3]. I have included a (slightly shorter) proof here for completeness.

Proof. For each $k \in \{0,1,2,\ldots,q\}$, each $y = (y_1, y_2, \ldots, y_k) \in T^k$, and each $x = (x_1, x_2, \ldots, x_k) \in S^k$ with $x_1, x_2, \ldots, x_k$ distinct, first define $\alpha(x,y)$ as the conditional probability that A's distinct oracle queries are exactly $x_1, x_2, \ldots, x_k$ and A's output is 1, given that the oracle responses are $y_1, y_2, \ldots, y_k$.

In other words, $\alpha(x,y)$ is the chance that A decides to issue oracle query x_1, then—given response y_1—to issue oracle query x_2, and so on.

Next define $\beta_f(x,y)$ as the probability that $(f(x_1),\ldots,f(x_k)) = (y_1,\ldots,y_k)$. Then $\alpha(x,y)\beta_f(x,y)$ is the probability that, when A uses f as an oracle, its distinct oracle queries are $x_1,x_2,\ldots,x_k$, the oracle responses are $y_1,y_2,\ldots,y_k$, and A's output is 1. Sum over all x,y to obtain the overall probability that A prints 1: namely, $\Pr[A(f)=1] = \sum_{x,y}\alpha(x,y)\beta_f(x,y)$.

By hypothesis $\beta_\varphi(x,y) \ge (1-\epsilon)/\#T^k = (1-\epsilon)\beta_v(x,y)$ where v is a uniform random function from S to T. Hence $\Pr[A(\varphi)=1] = \sum_{x,y}\alpha(x,y)\beta_\varphi(x,y) \ge (1-\epsilon)\sum_{x,y}\alpha(x,y)\beta_v(x,y) = (1-\epsilon)\Pr[A(v)=1] \ge \Pr[A(v)=1]-\epsilon$. Similarly $\Pr[A(\varphi)\ne 1] \ge \Pr[A(v)\ne 1]-\epsilon$. Thus the A-distance between φ and v is at most ϵ. $\square$

Theorem 3.2. *Let m and q be integers with $m \ge 1$ and $q \ge 0$. Let G be a finite commutative group. Let u be a uniform random function from G to G. Define*

$$u_m^+(g_1,g_2,\ldots,g_m) = u(u(\cdots u(u(g_1)+g_2)+\cdots)+g_m)$$

for all $(g_1,g_2,\ldots,g_m) \in G^m$. Let A be an algorithm that performs at most q distinct oracle queries. Then the A-distance between u_m^+ and uniform is at most $mq(mq-1)/2\#G$.

Proof. If $k \in \{0,1,\ldots,q\}$ then $(u_m^+(x_1),u_m^+(x_2),\ldots,u_m^+(x_k)) = (y_1,y_2,\ldots,y_k)$ with probability at least $(1-mq(mq-1)/2\#G)/\#G^k$ by Theorem 2.1. Apply Theorem 3.1. $\square$

4 Standard consequences

From uniform to unpredictable

Say f is a uniform random permutation of the set of b-bit strings. It is difficult to distinguish f from u, so it is difficult to distinguish f_m^+ from u_m^+. More precisely, the q-query distance from f_m^+ to u_m^+ is at most the mq-query distance from f to u, which is at most $mq(mq-1)/2^{b+1}$. Hence f_m^+ is unpredictable: the q-query distance from f_m^+ to uniform is at most $mq(mq-1)/2^b$.

More generally, if f is a random function from b-bit strings to b-bit strings, and if f is unpredictable to all fast algorithms, then f_m^+ is unpredictable to all fast algorithms. For example, if k is a uniform random 128-bit string, then the random function AES_k from 128-bit strings to 128-bit strings is conjectured to be unpredictable, so the random function $(\mathrm{AES}_k)_{20}^+$ from 2560-bit strings to 128-bit strings is also conjectured to be unpredictable.

Message authentication

One way to securely authenticate a message t is to transmit it as $(t, v(t))$, where v is a secret uniform random function shared by the sender and receiver. This protocol remains secure when v is replaced with any unpredictable random function—in particular, u_m^+, or more generally f_m^+ when f is unpredictable.

Beware that it is not a good idea to use CBC to authenticate messages in practice:

- Old reason: f_m^+ takes inputs of a fixed positive length, namely mb bits, whereas most applications send variable-length messages. Switching from f_m^+ to f^+ is not safe: observe that $f^+() = 0$, for example, and $f^+(0) = f^+(0, -f^+(0))$. On the other hand, minor variants of f^+ are unpredictable.
- New reason: Other message-authentication codes are much faster and provide much stronger security guarantees. See, e.g., [4].

CBC nevertheless remains—thanks to its simplicity—an interesting test case for security-proof methodologies.

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