SafeCurves: choosing safe curves for elliptic-curve cryptography

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University of Illinois at Chicago &
Technische Universiteit Eindhoven

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Technische Universiteit Eindhoven

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http://safecurves.cr.yp.to

Cryptography

Public-key signatures:
e.g., RSA, DSA, ECDSA.
Some uses: signed OS updates, SSL certificates, e-passports.

Public-key encryption:
e.g., RSA, DH, ECDH.
Some uses: SSL key exchange, locked iPhone mail download.

Secret-key encryption:
e.g., AES, Salsa20.
Some uses: disk encryption, bulk SSL encryption.

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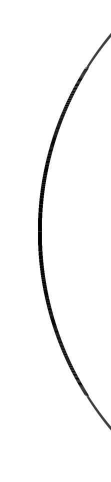

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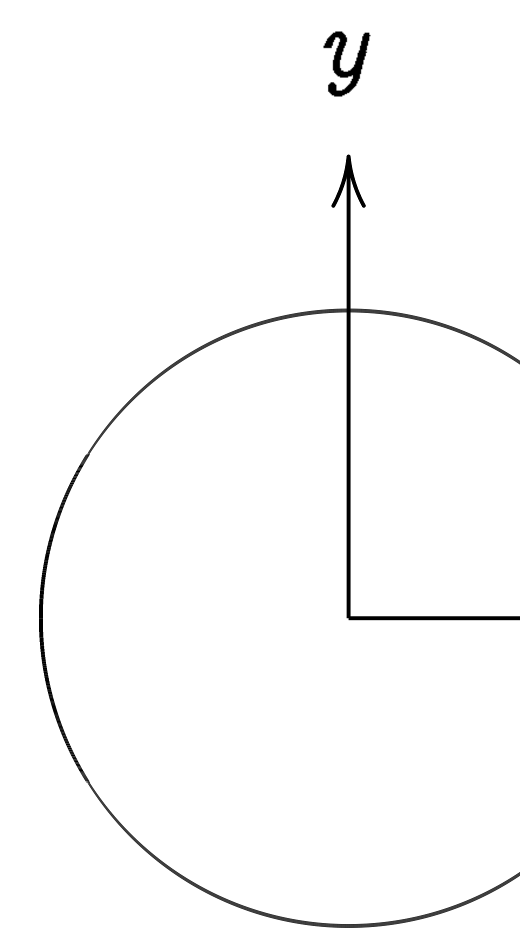

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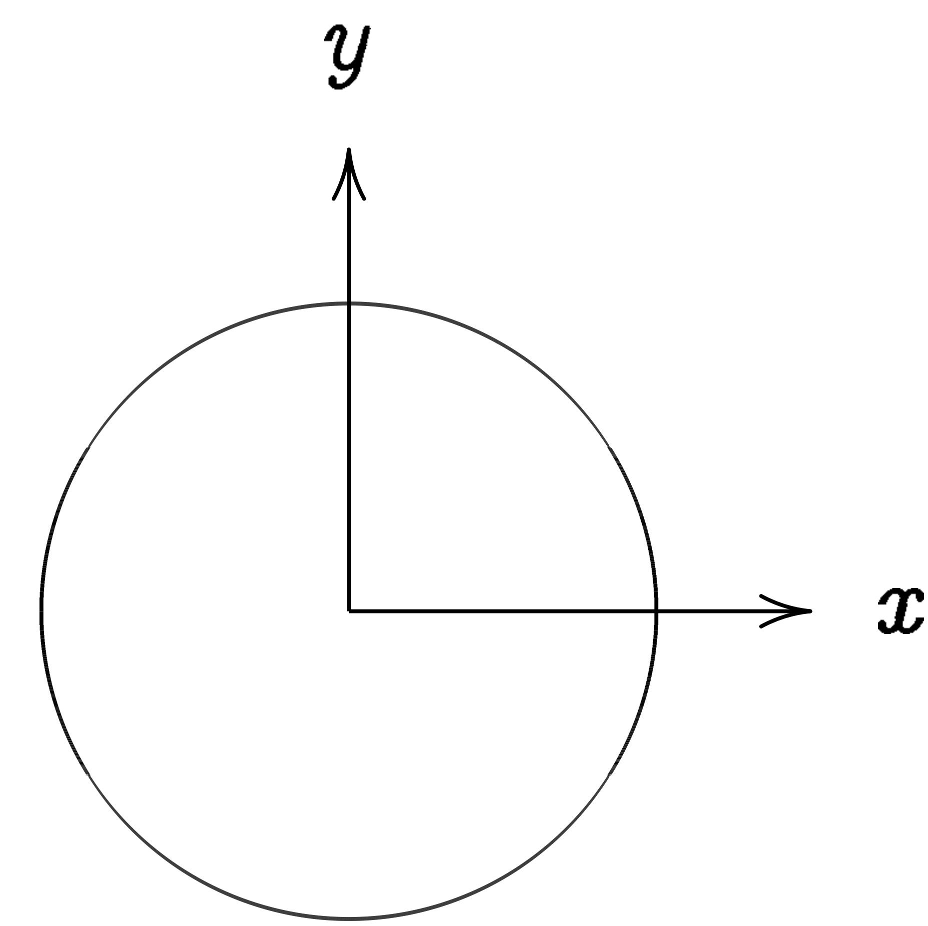

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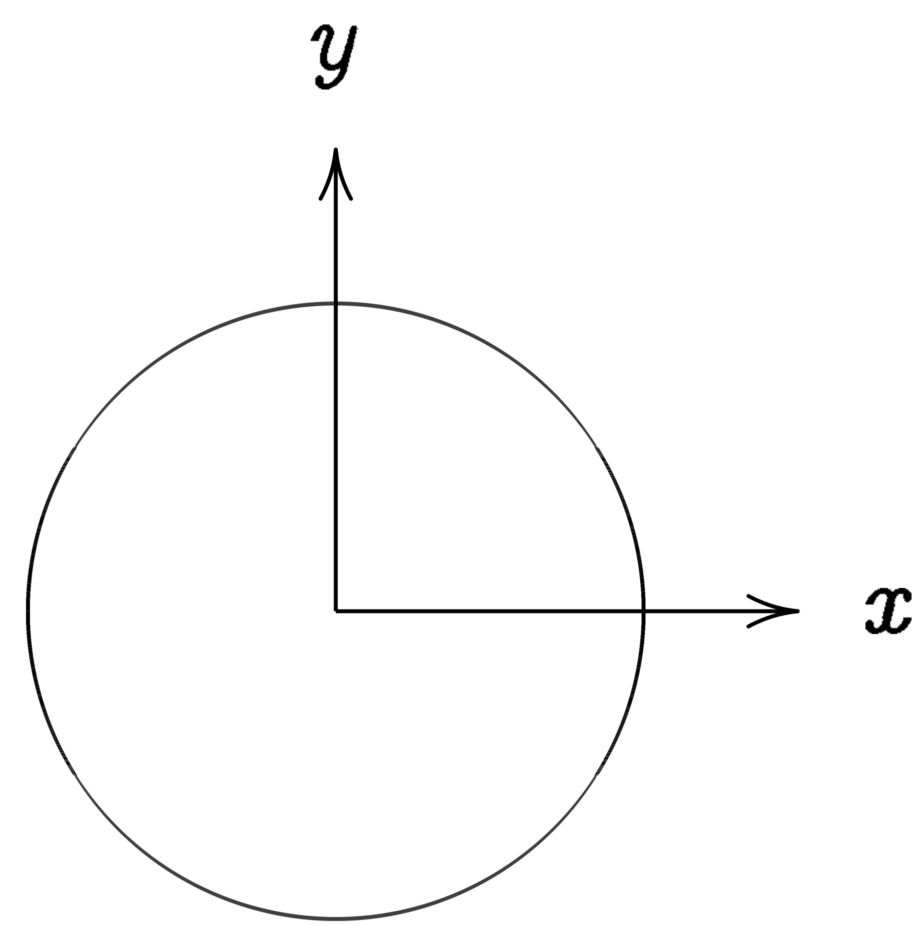

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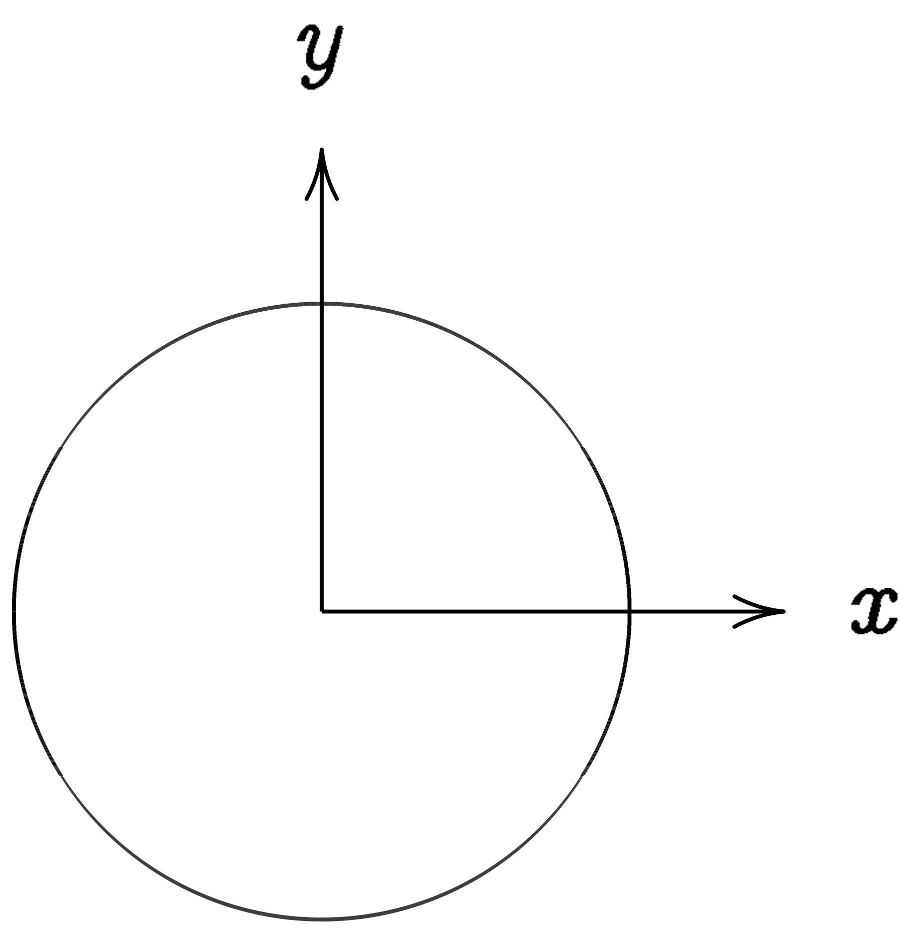

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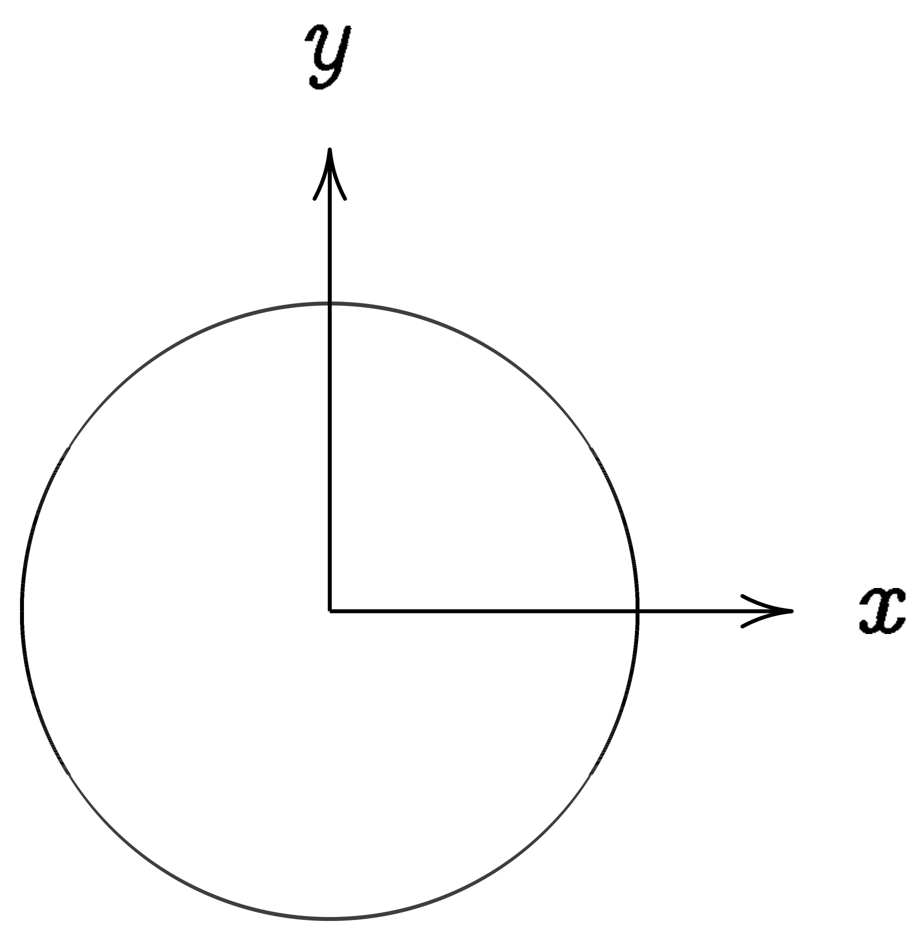

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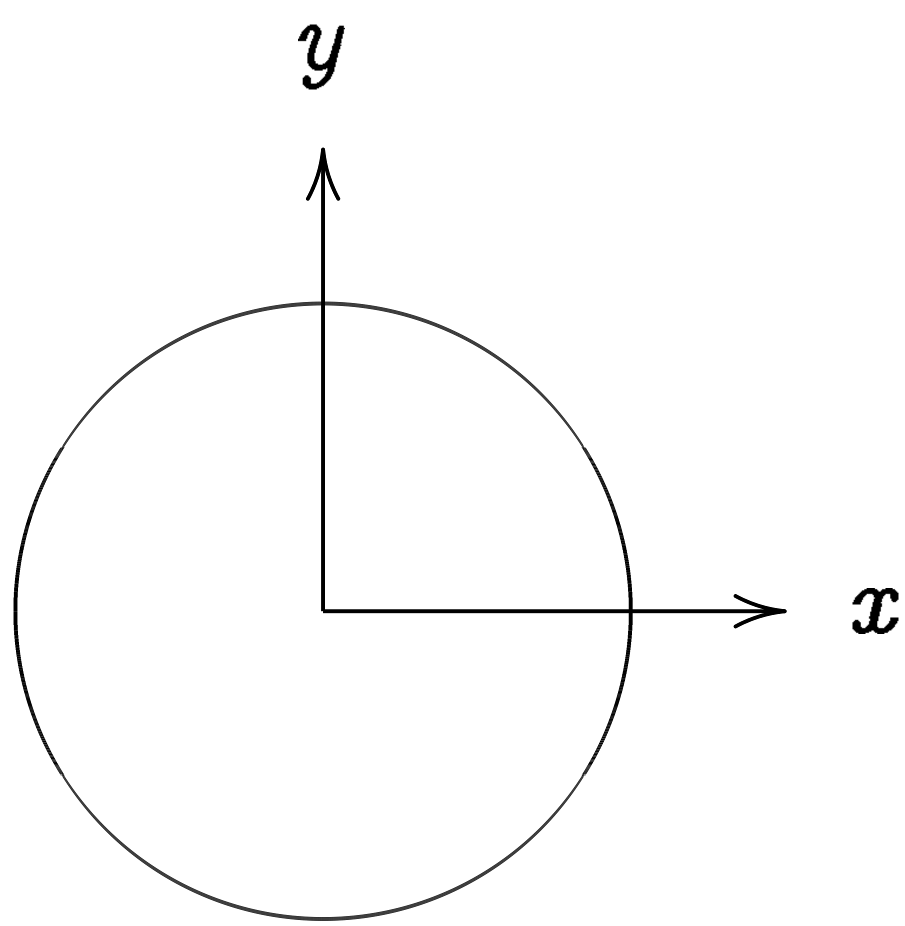

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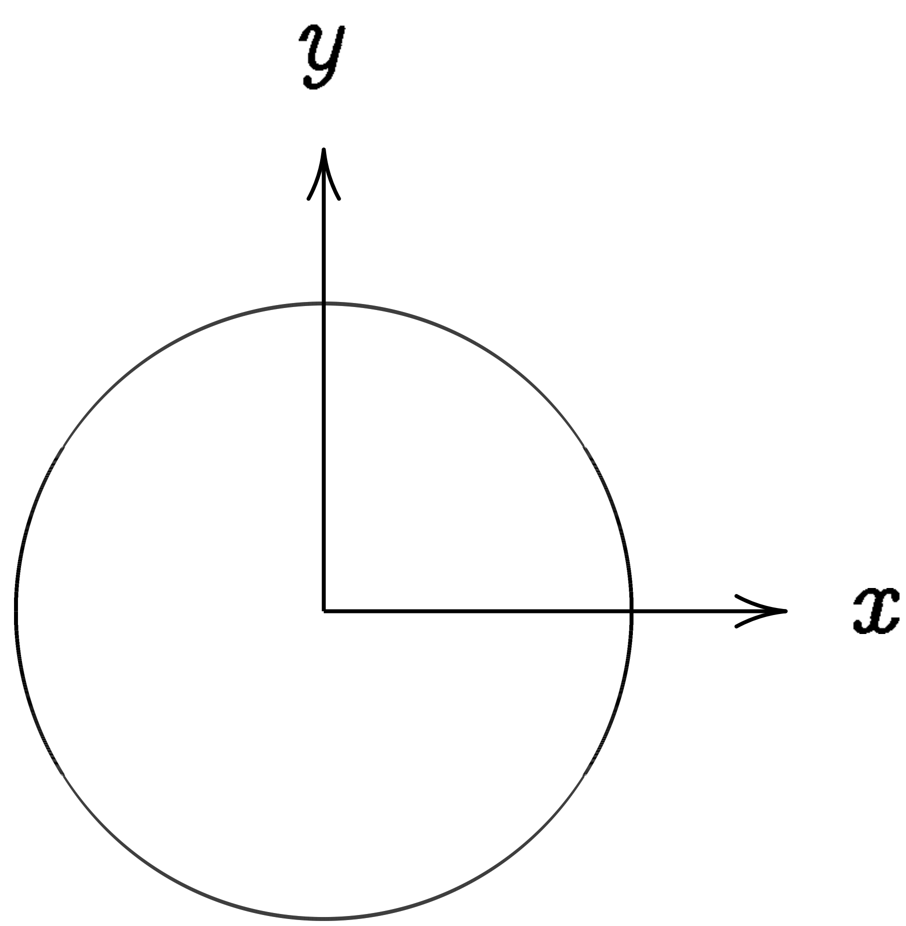

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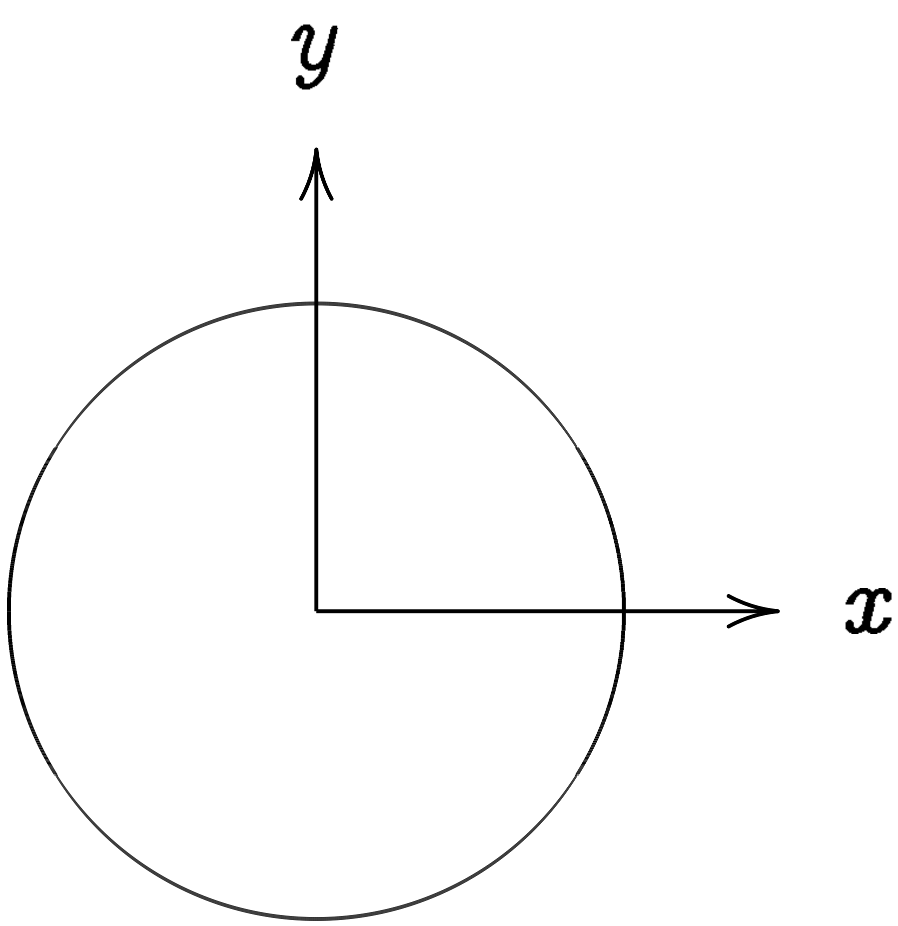

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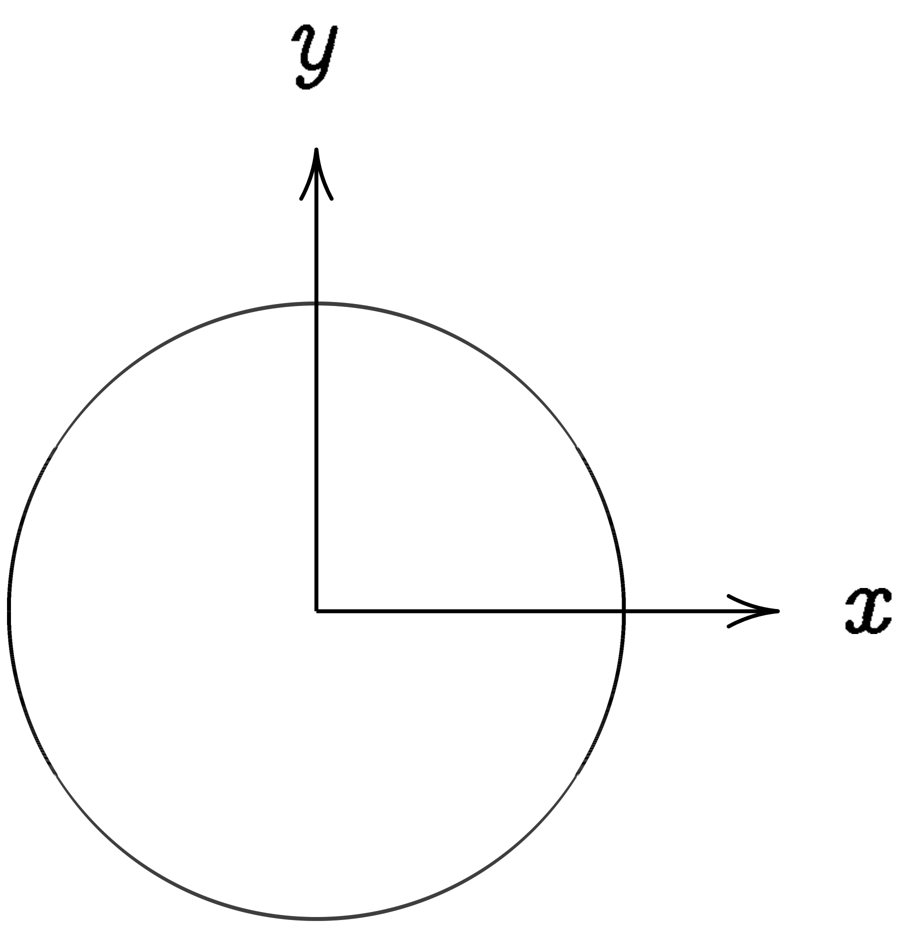

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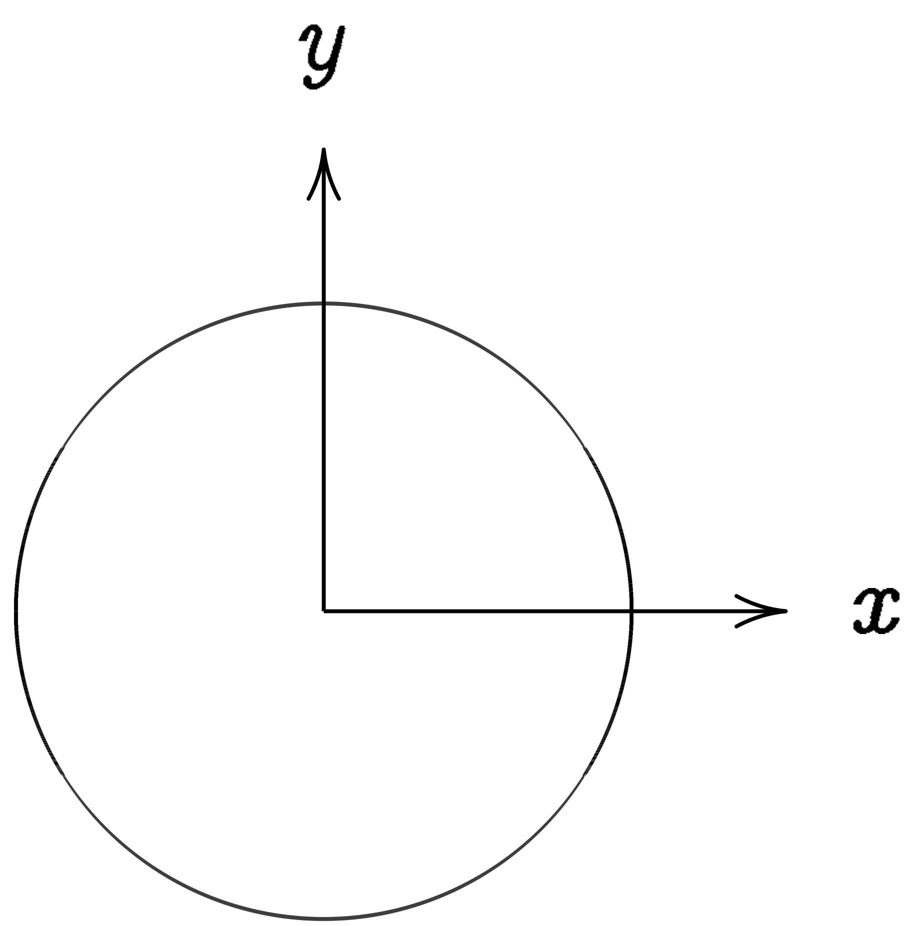

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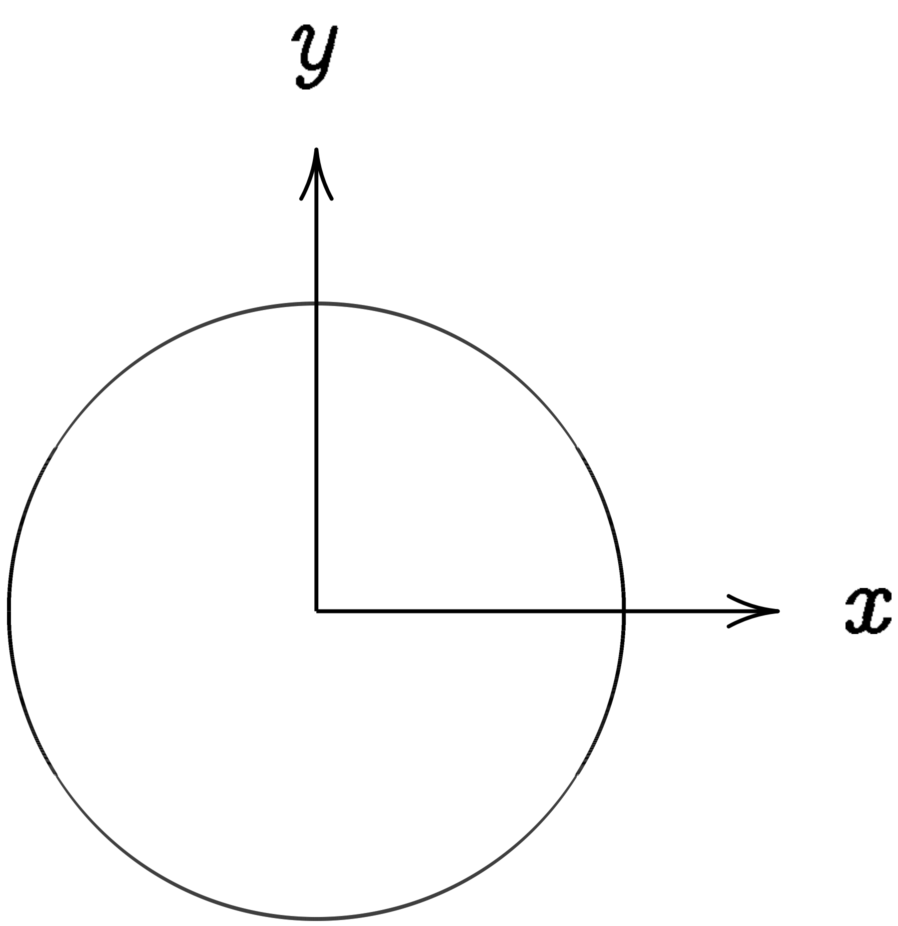

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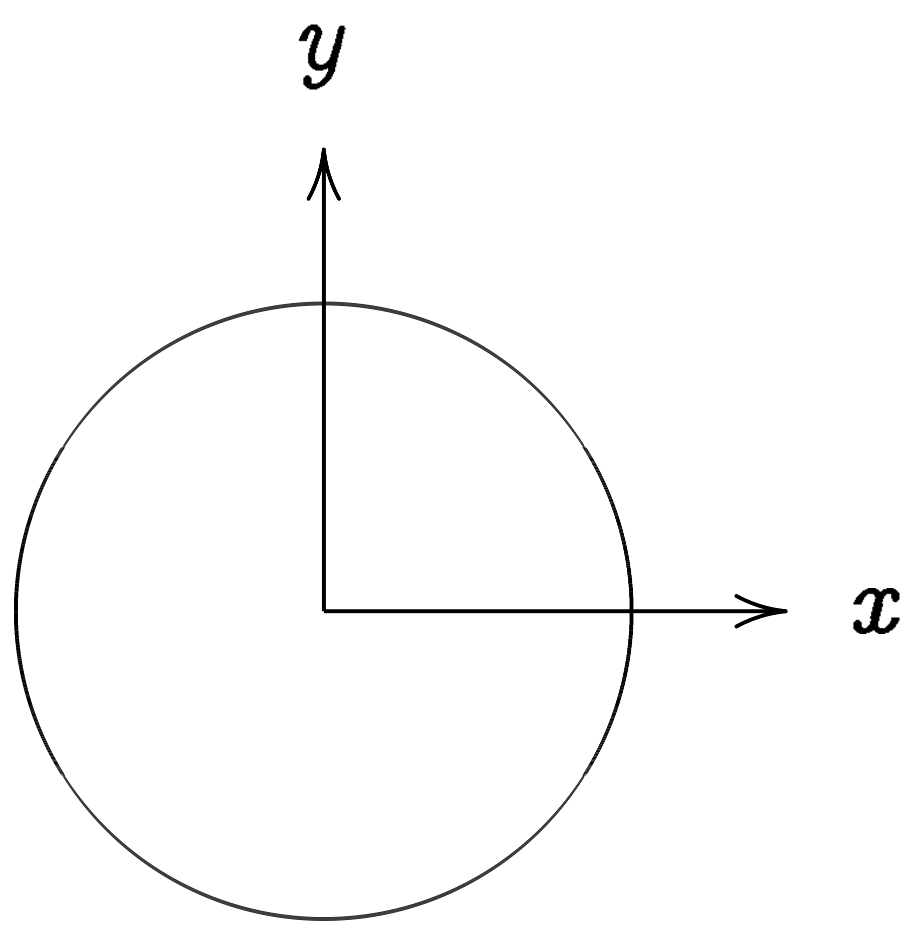

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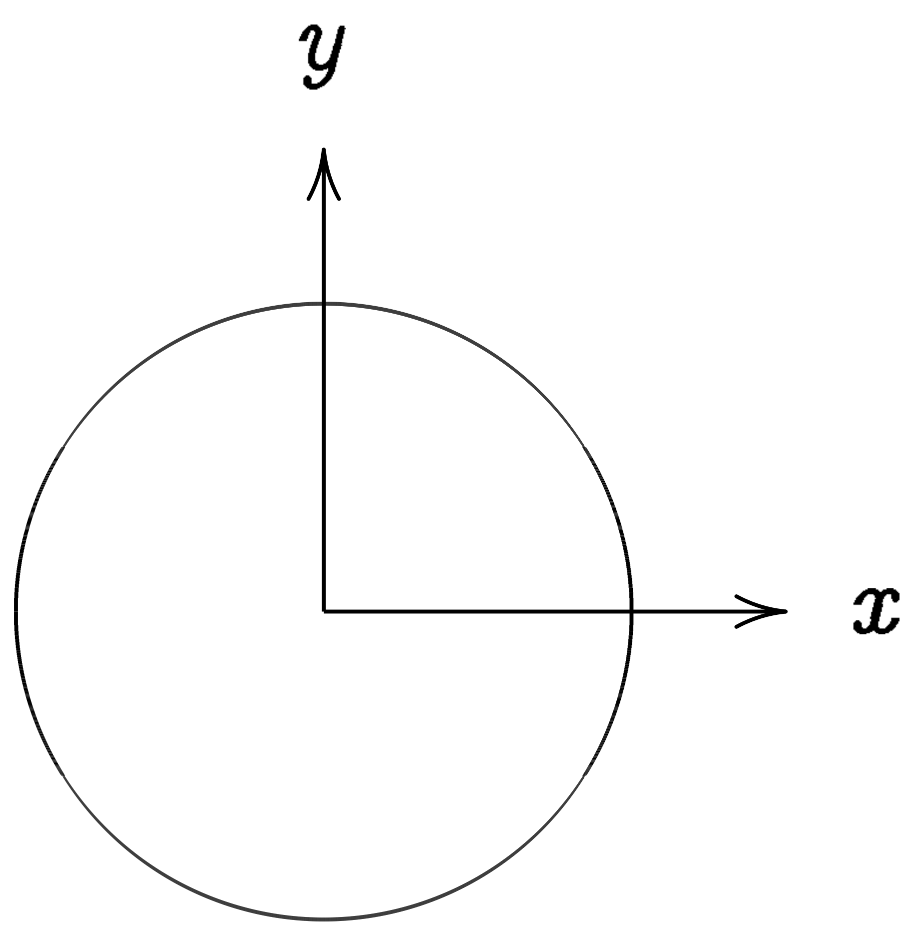

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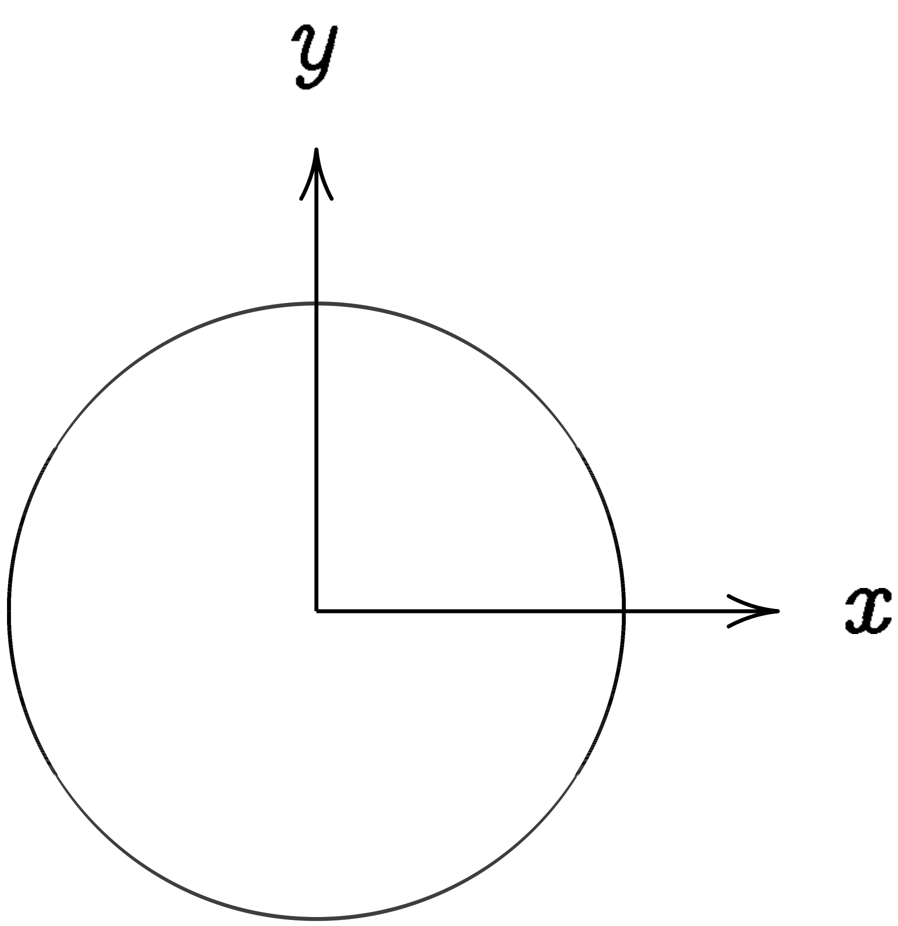

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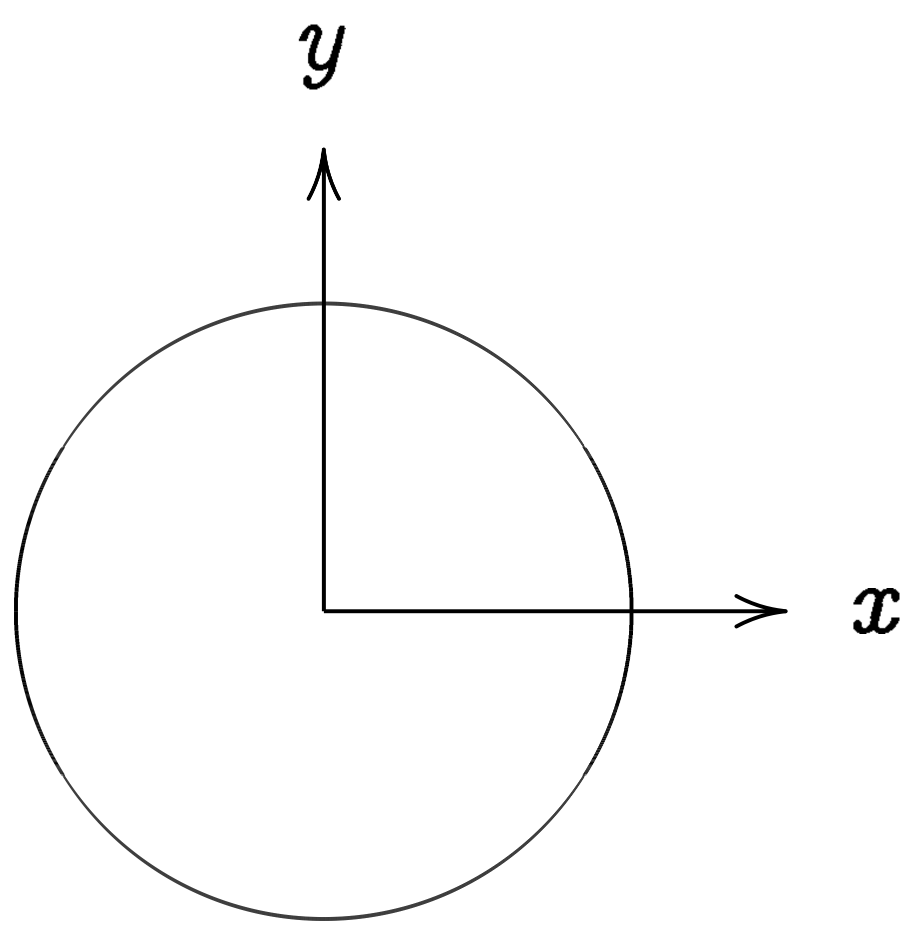

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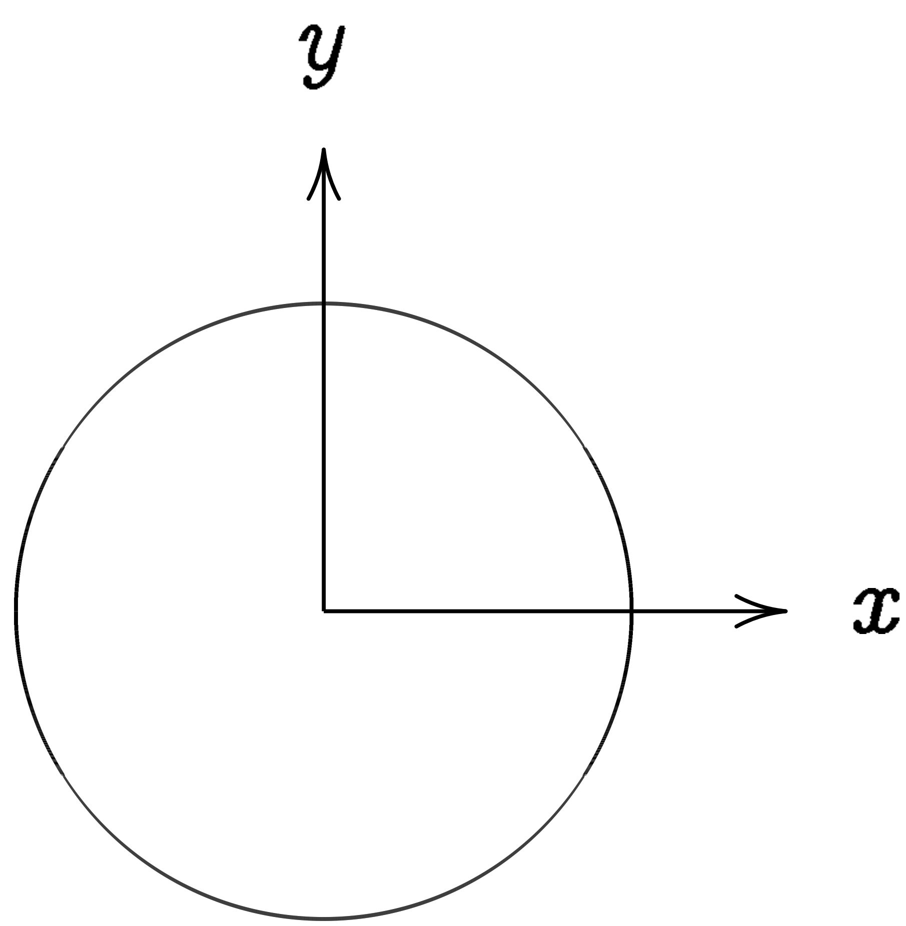

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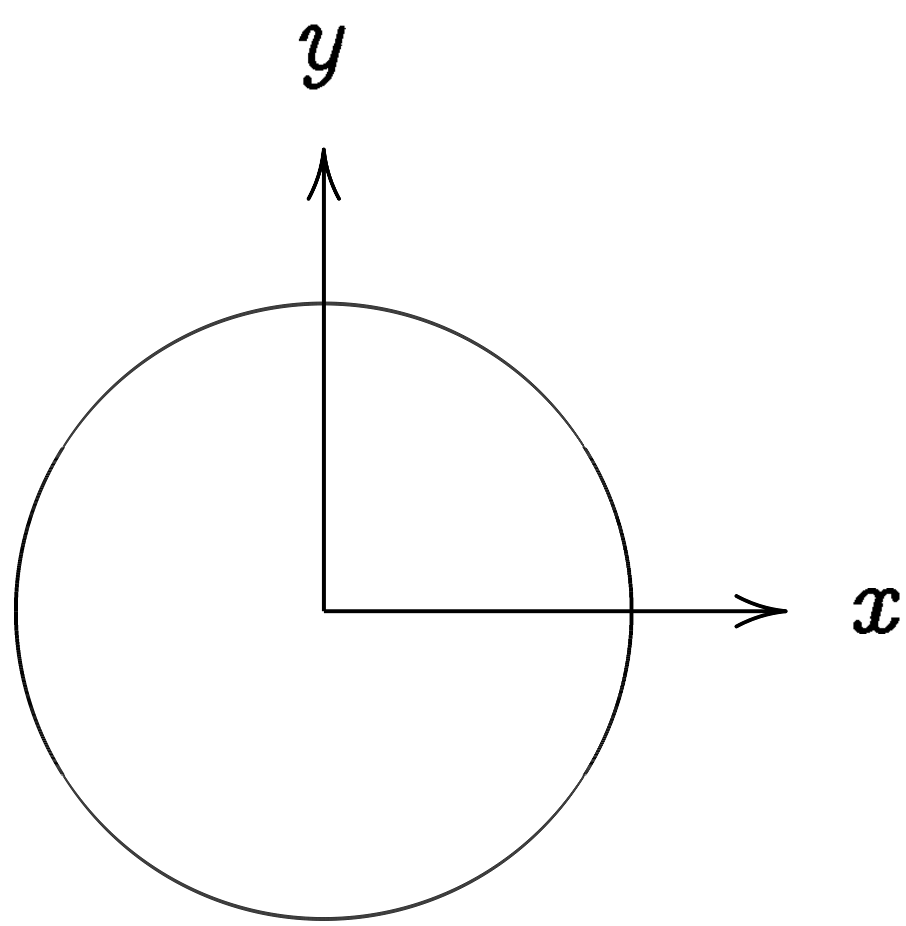

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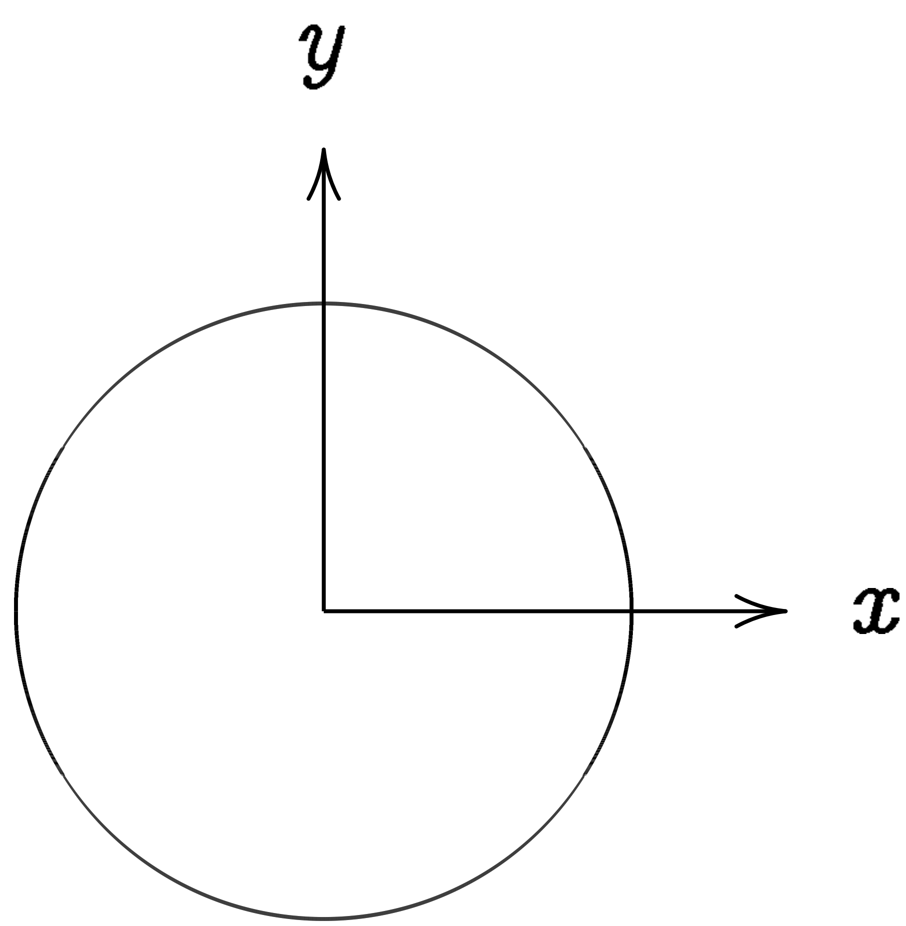

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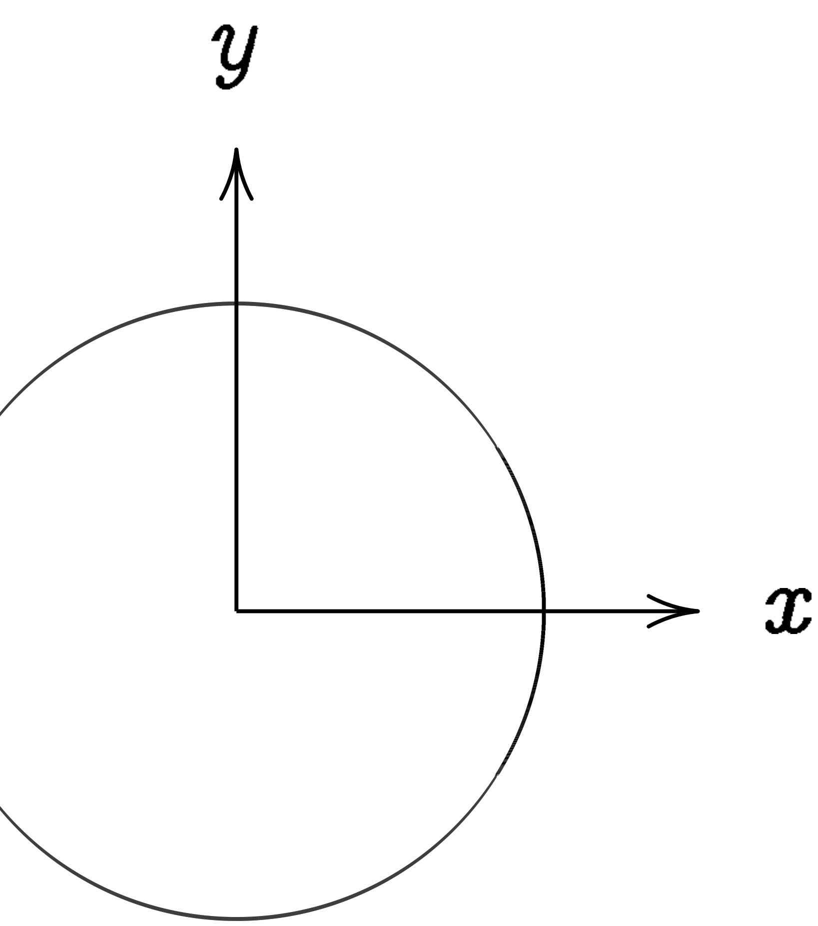

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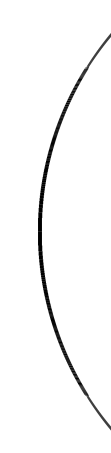
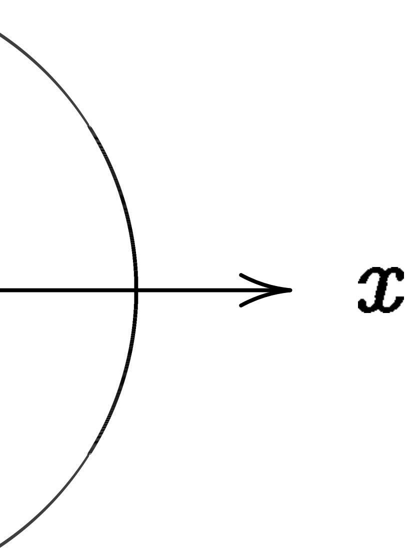
Addition



$x^2 + y^2$

$x = \sin$



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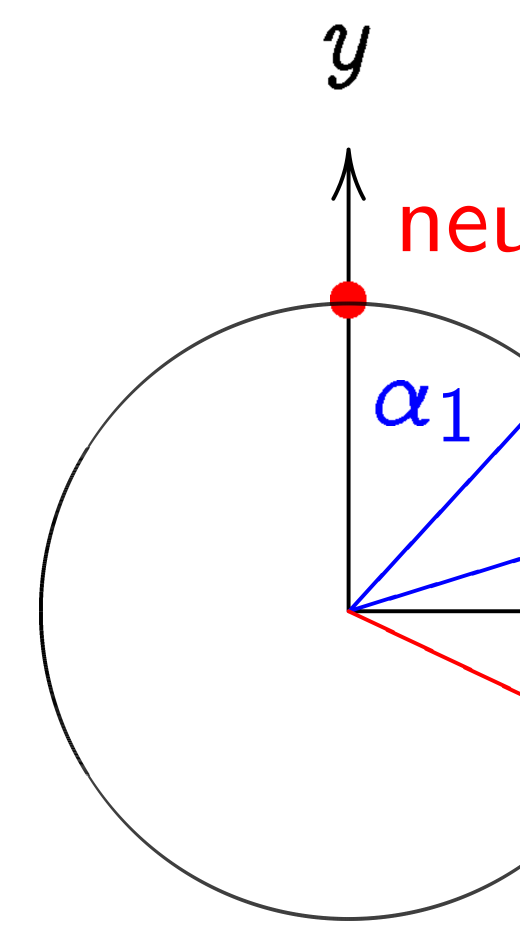

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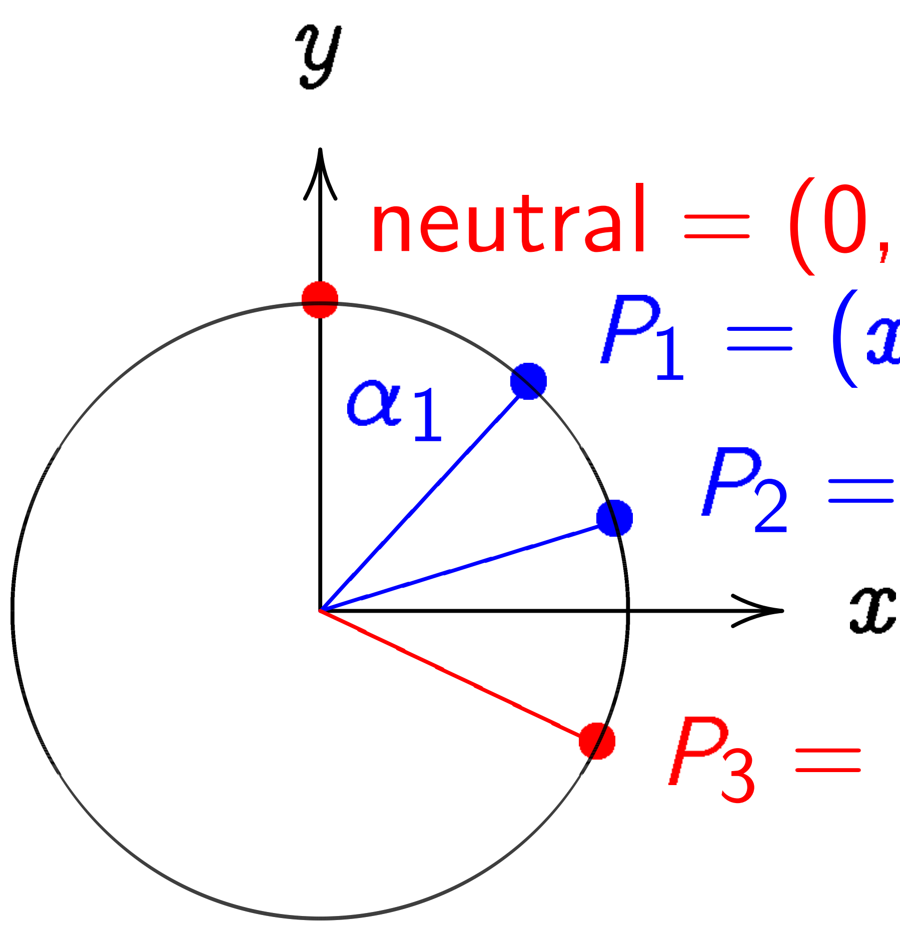

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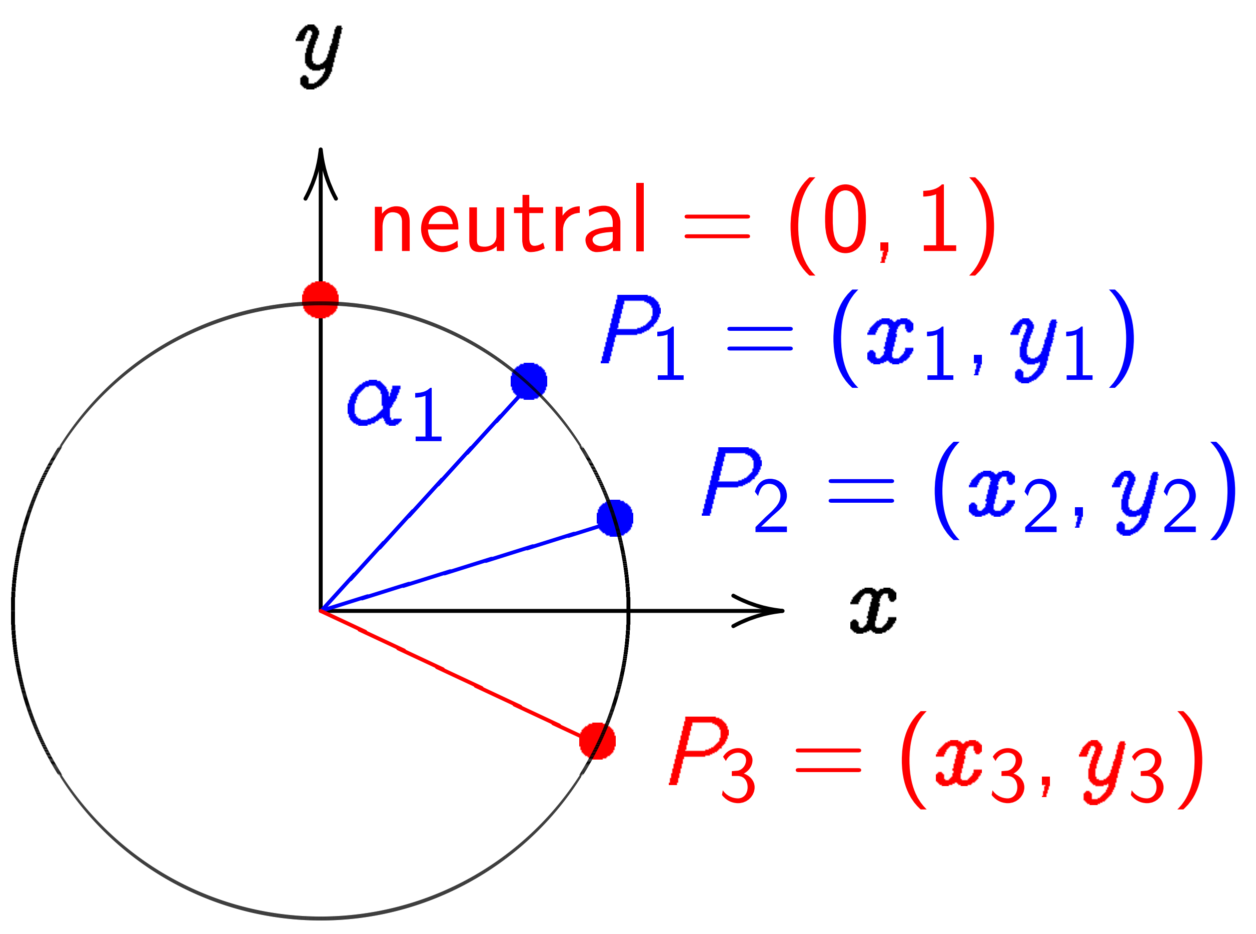

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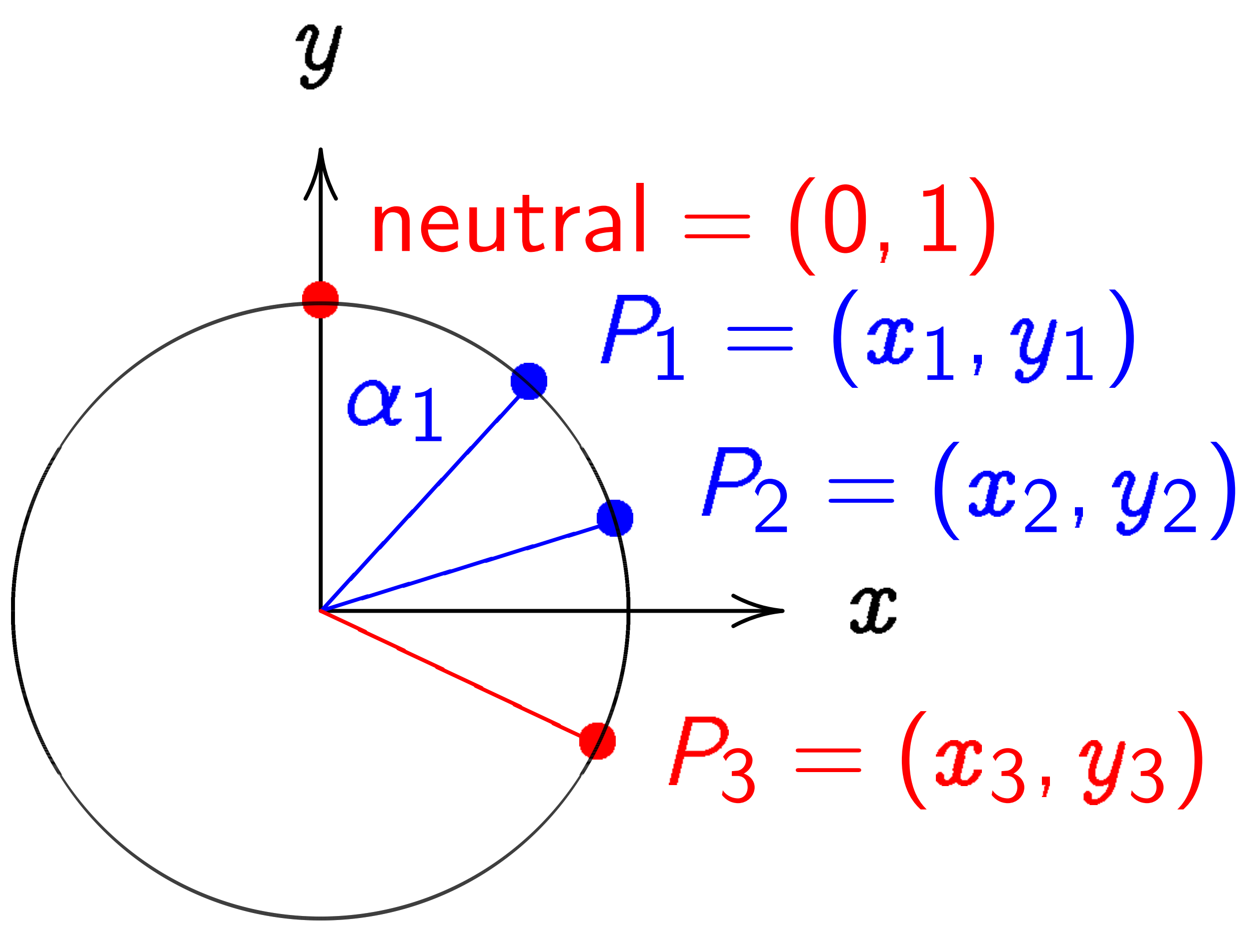

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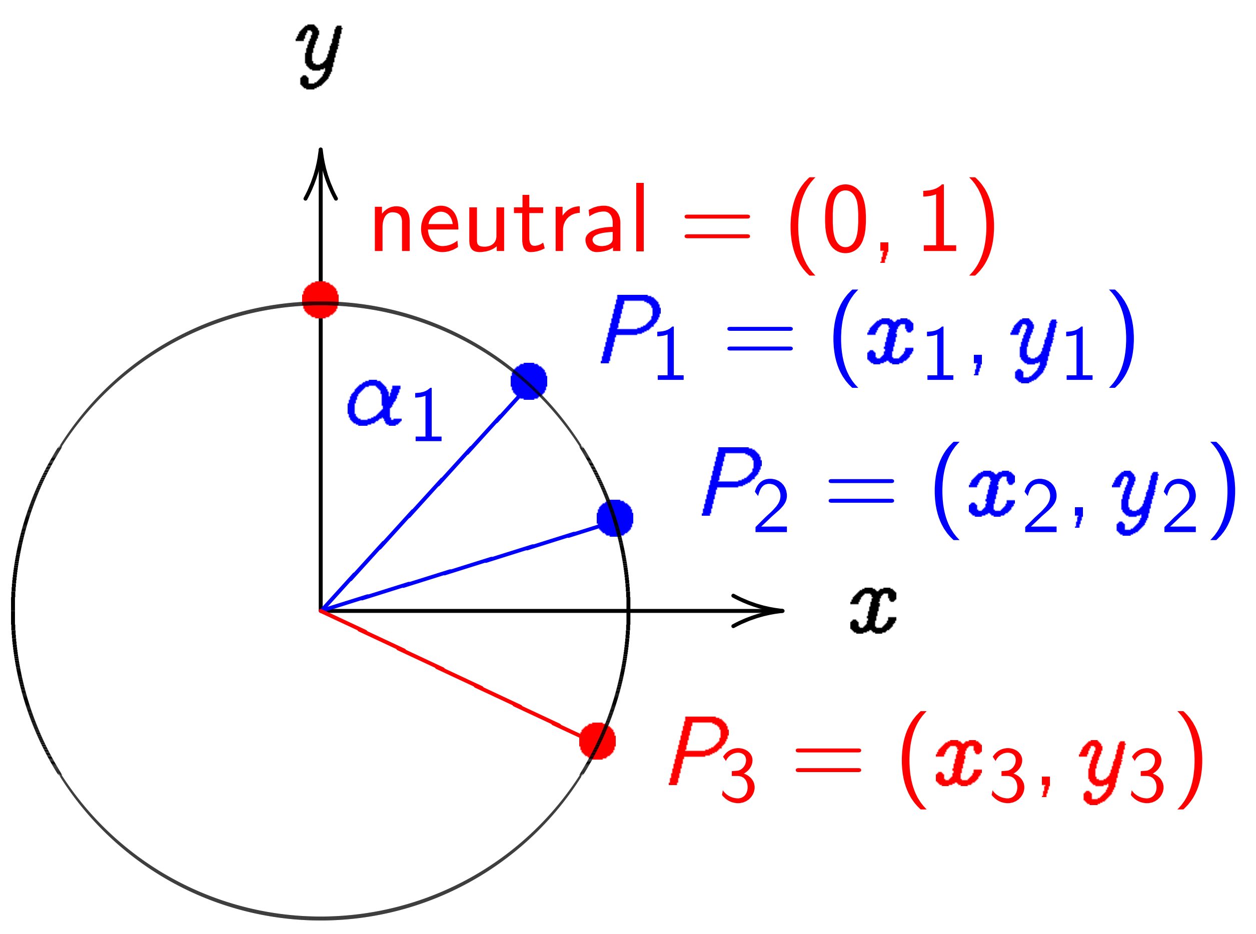

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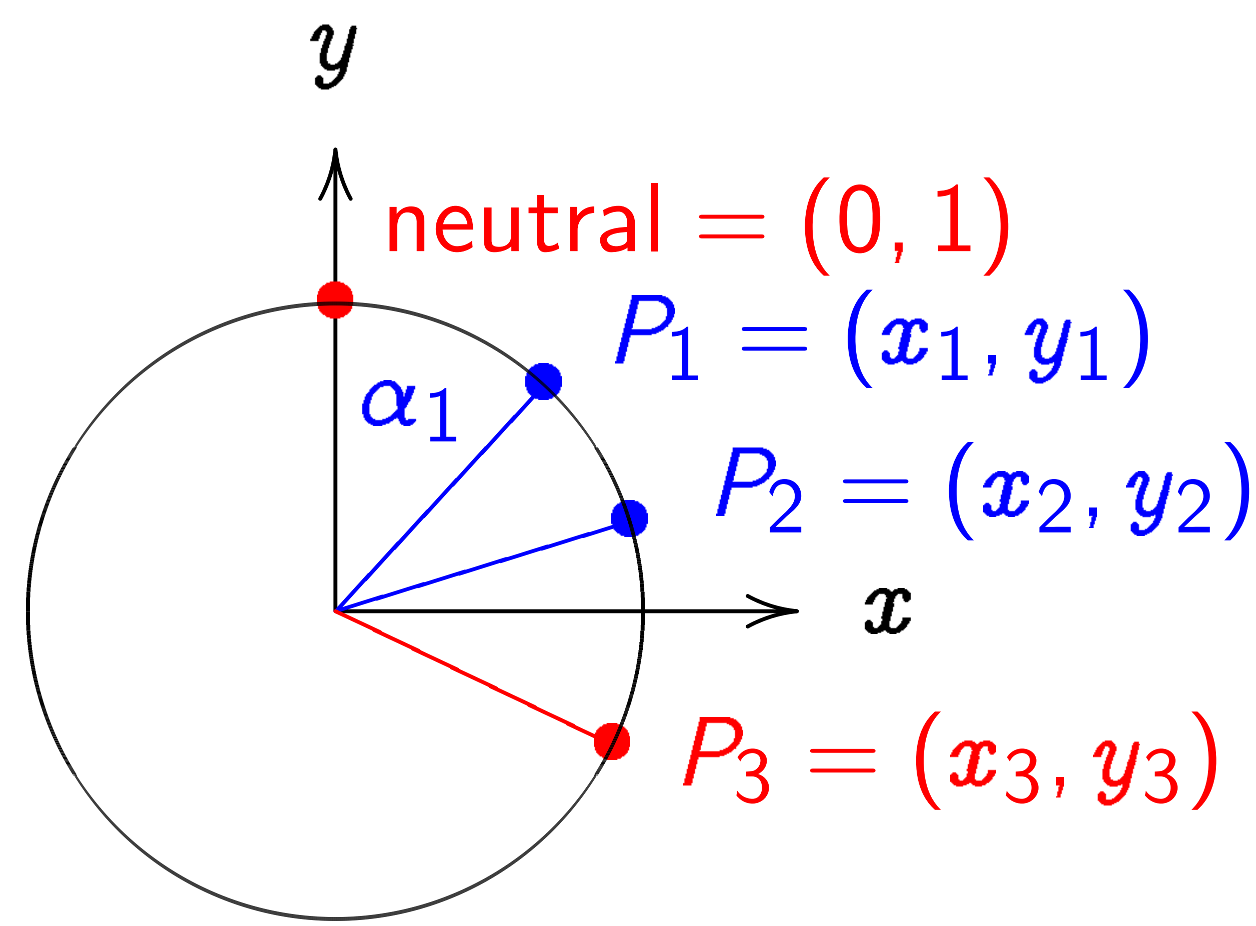

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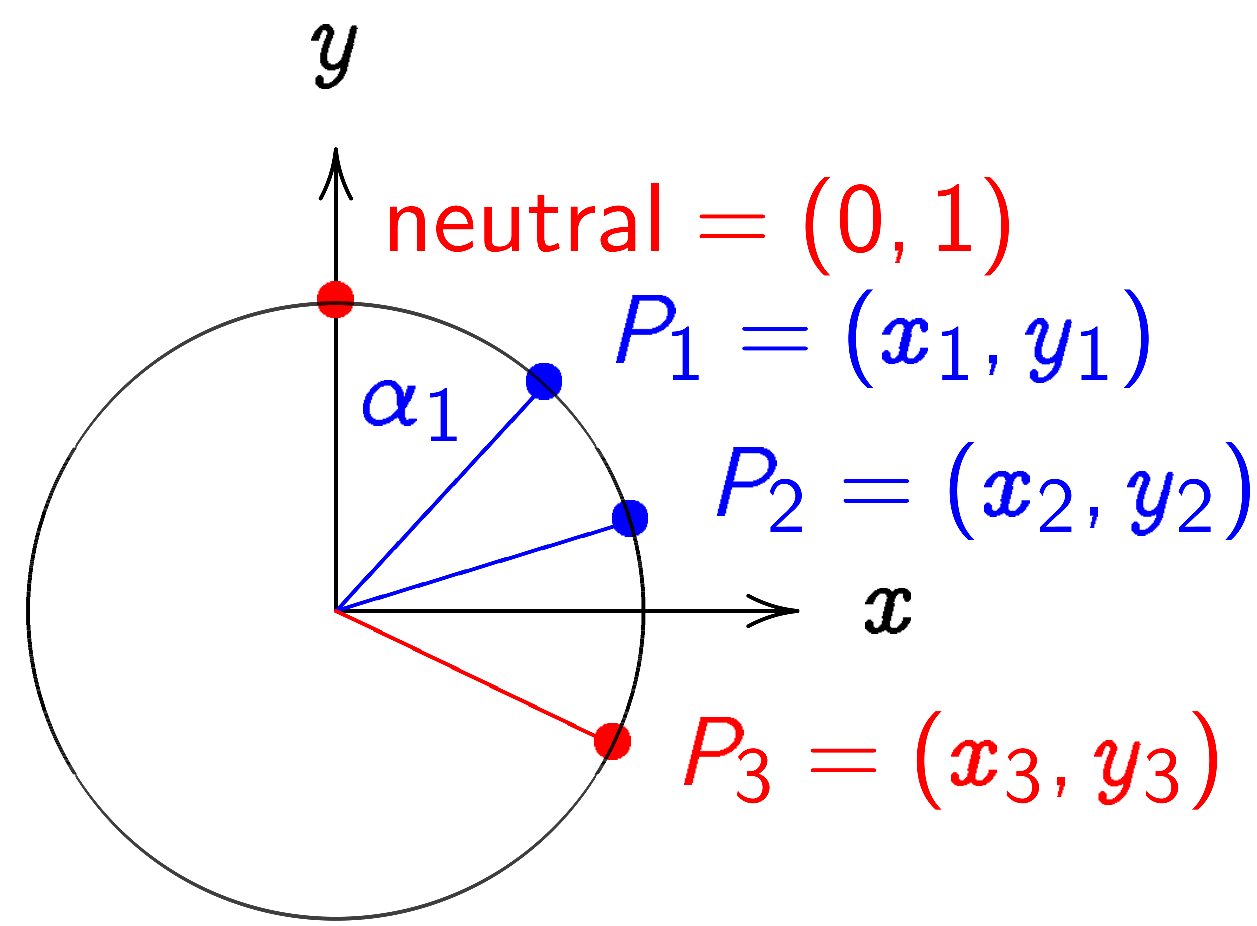

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Clock ac

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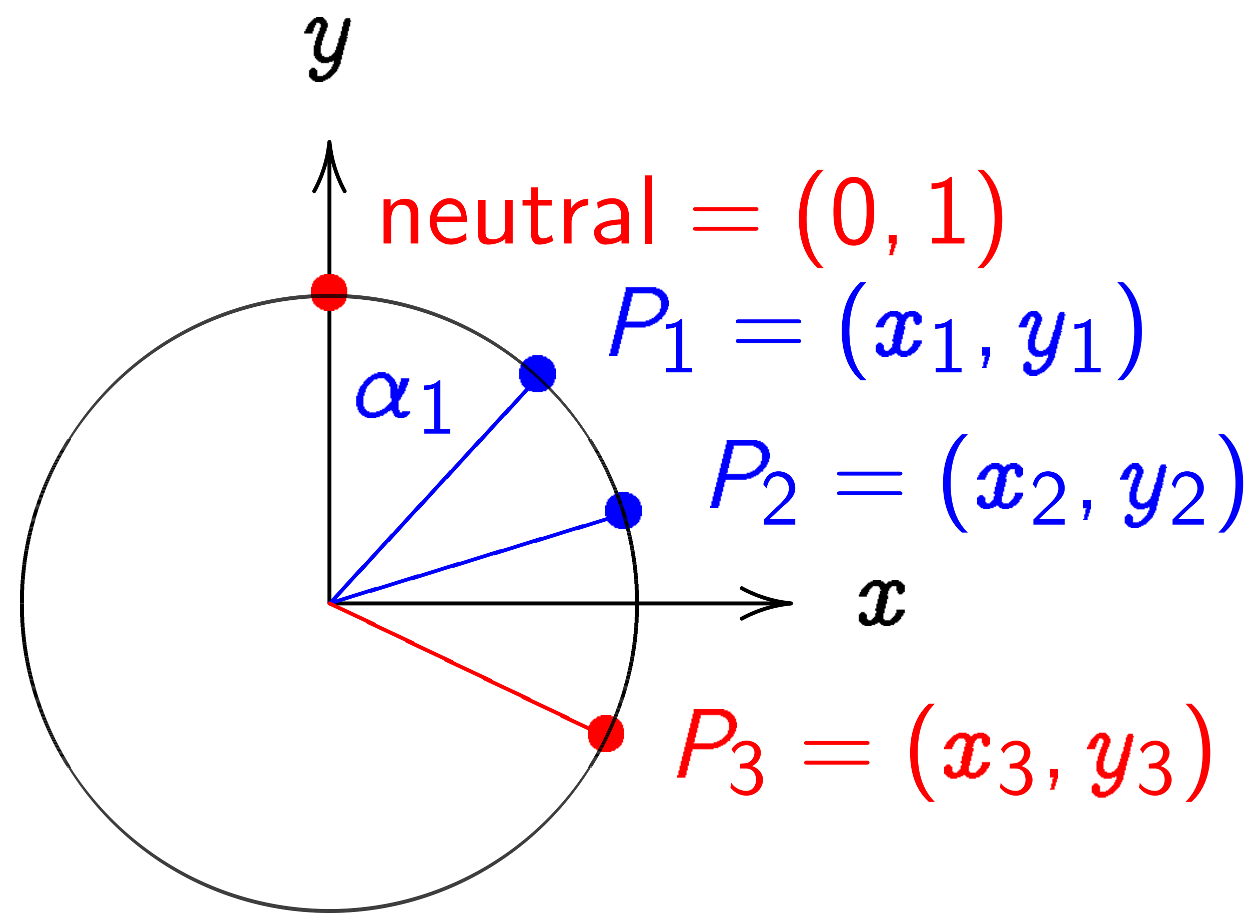

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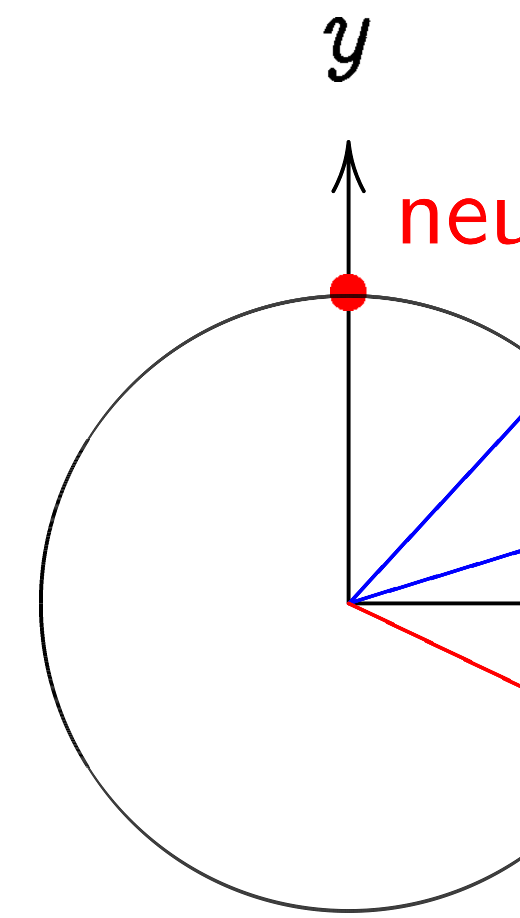

Use Cartesian coor

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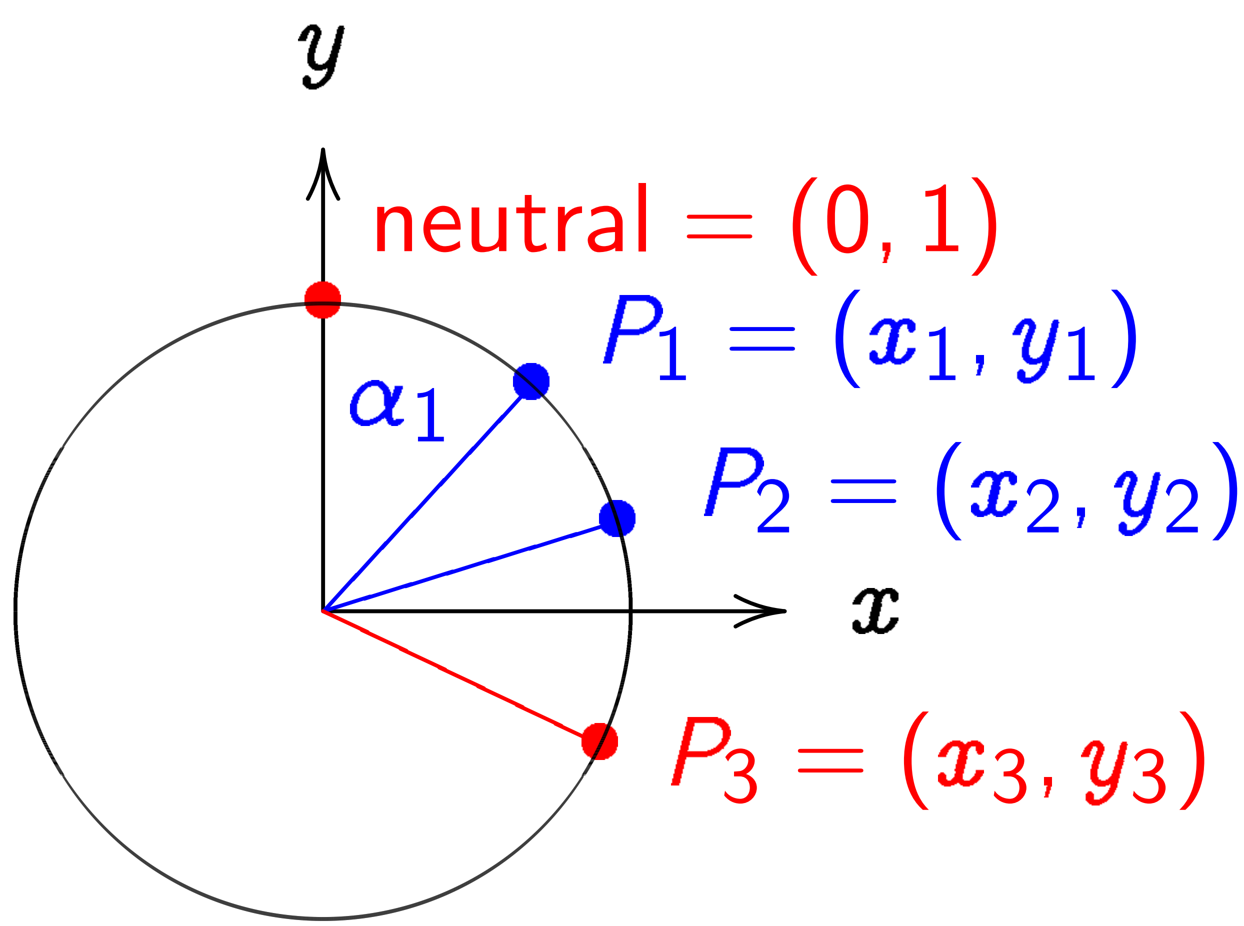
for the clock $x^2 +$

sum of (x_1, y_1) an

$(x_1 y_2 + y_1 x_2, y_1 y$

curve:

Addition on the clock:



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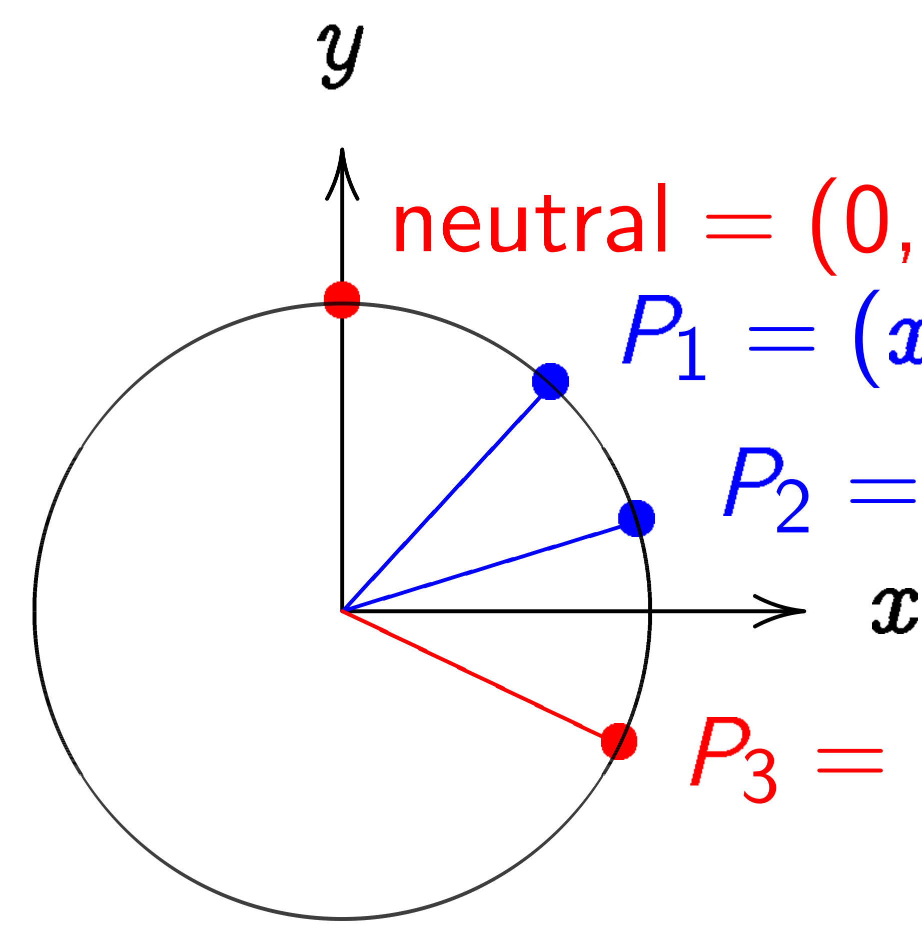
Clock addition without sin, c



Use Cartesian coordinates fo
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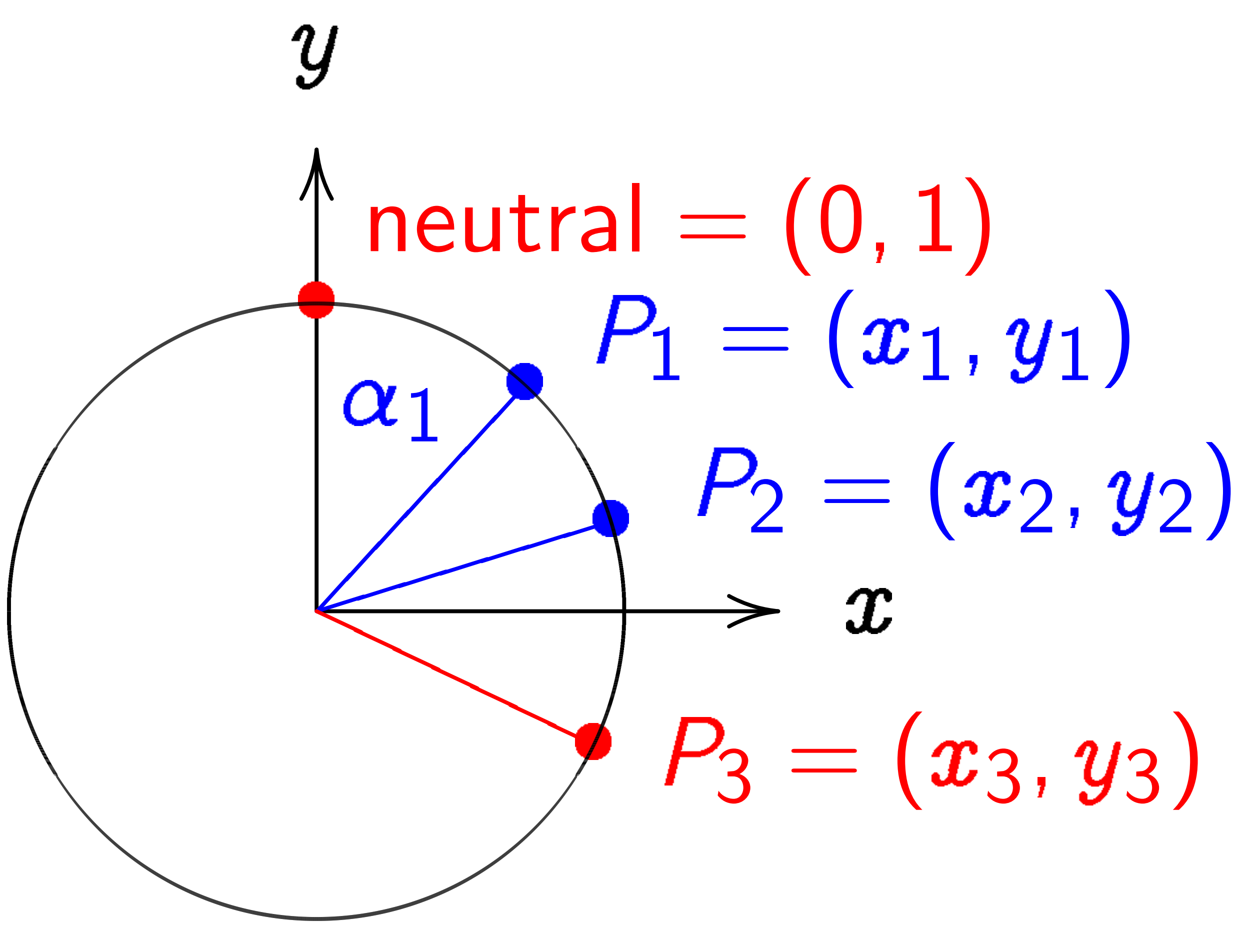

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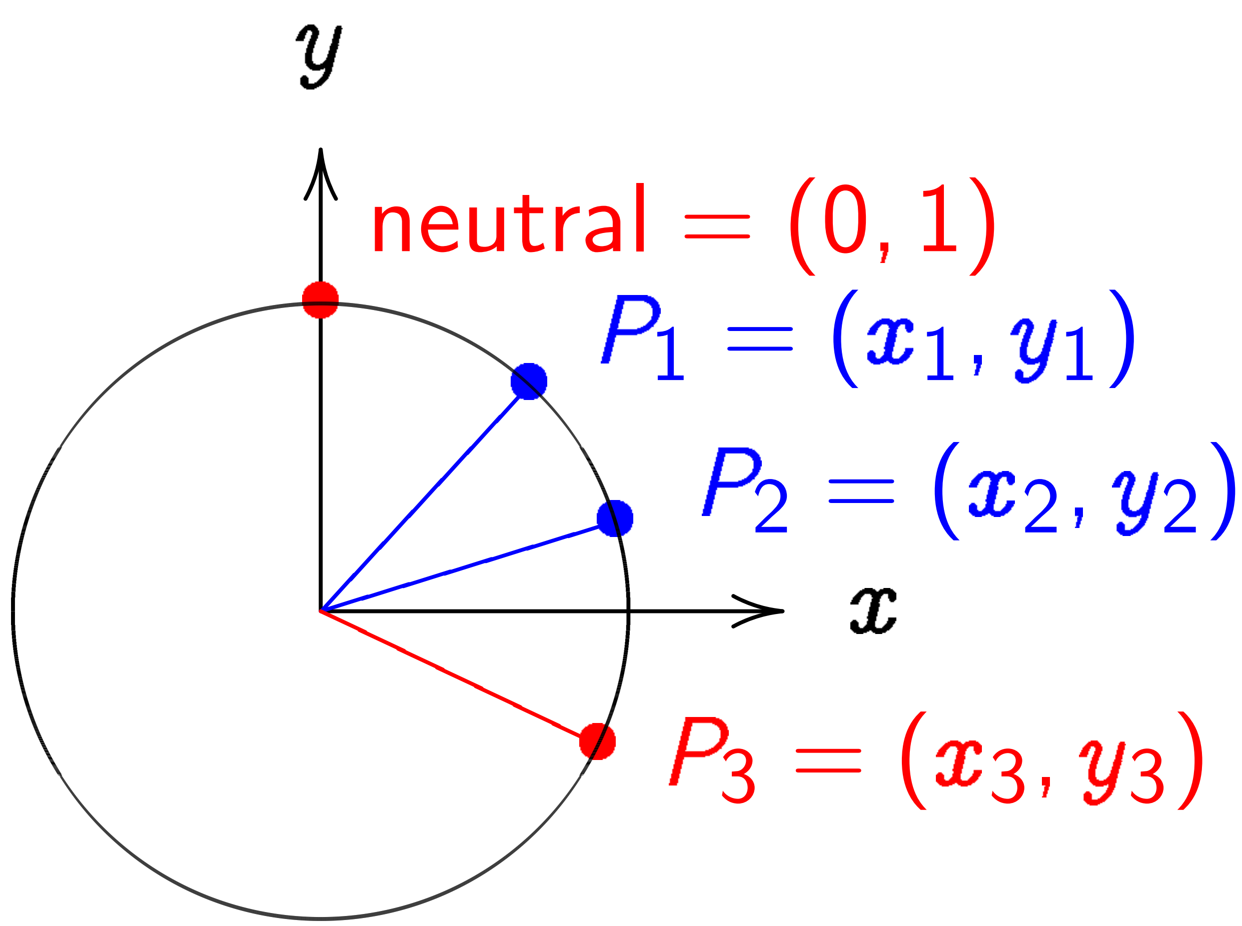

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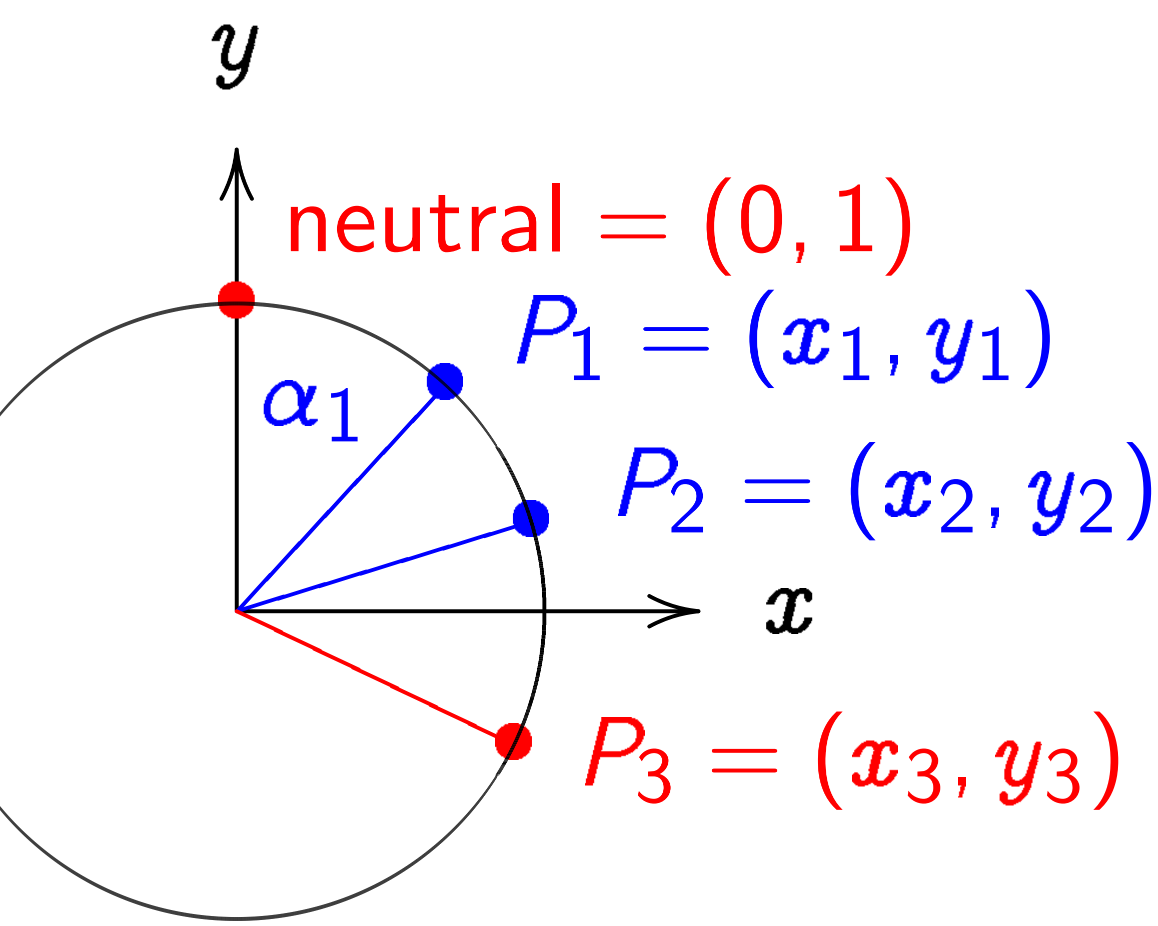

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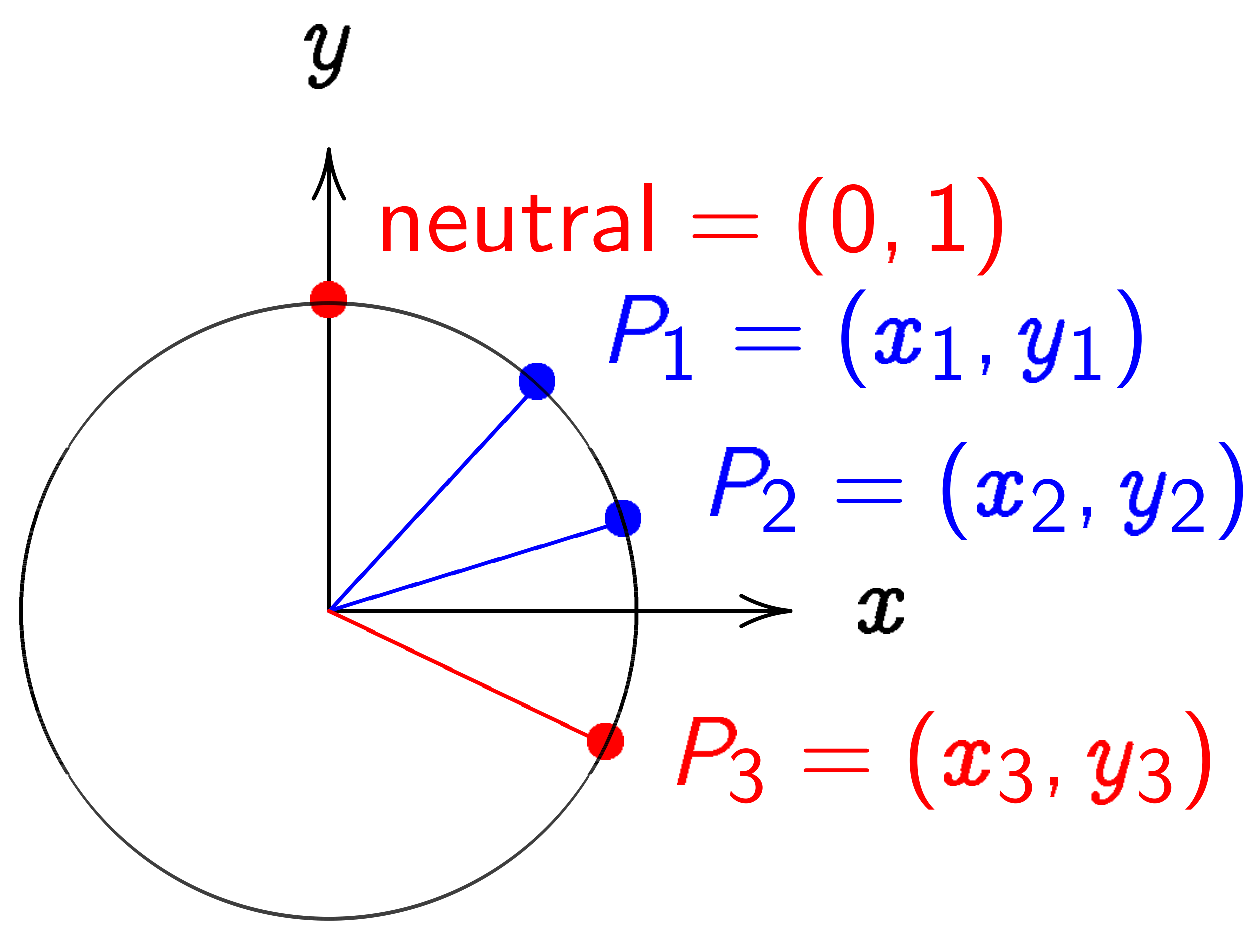

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Example

“2:00” +

$= (\sqrt{3/$

$= (-1/2$

“5:00” +

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$2\left(\frac{3}{5}, \frac{4}{5}\right.$

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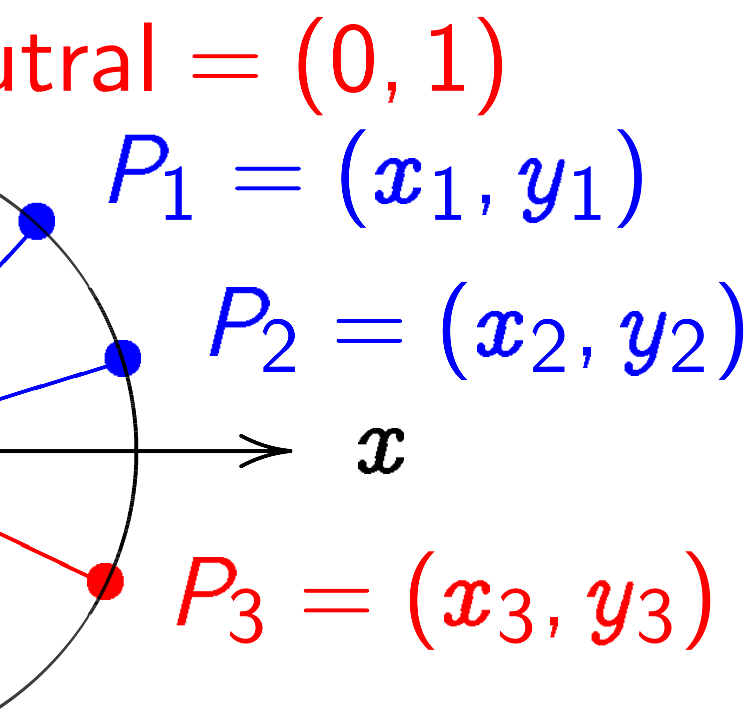

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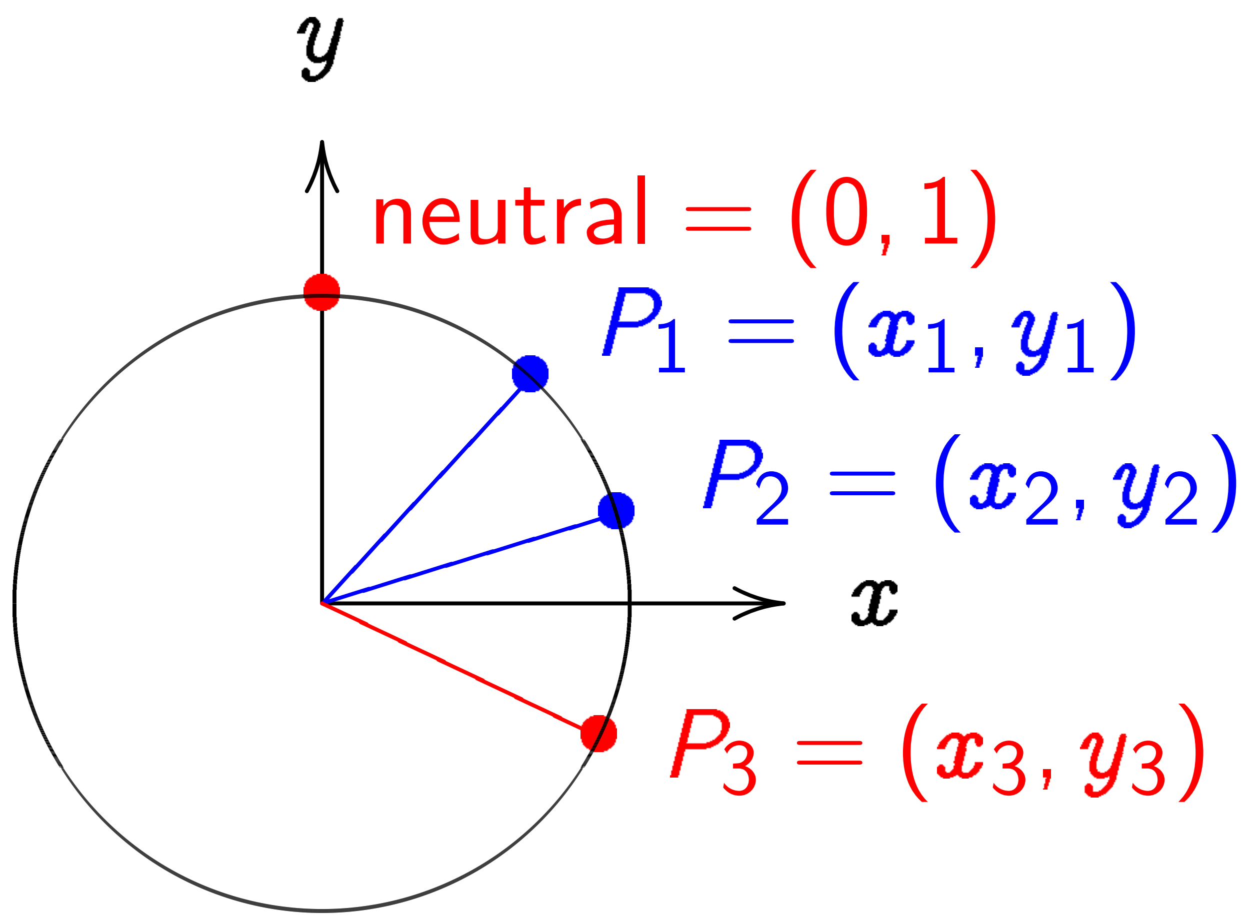

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Examples of clock

“2:00” + “5:00”
$= (\sqrt{3/4}, 1/2) +$
$= (-1/2, -\sqrt{3/4}$

“5:00” + “9:00”
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$= (\sqrt{3/4}, 1/2) =$

$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25},\right.$

1)

$x_1, y_1)$

(x_2, y_2)

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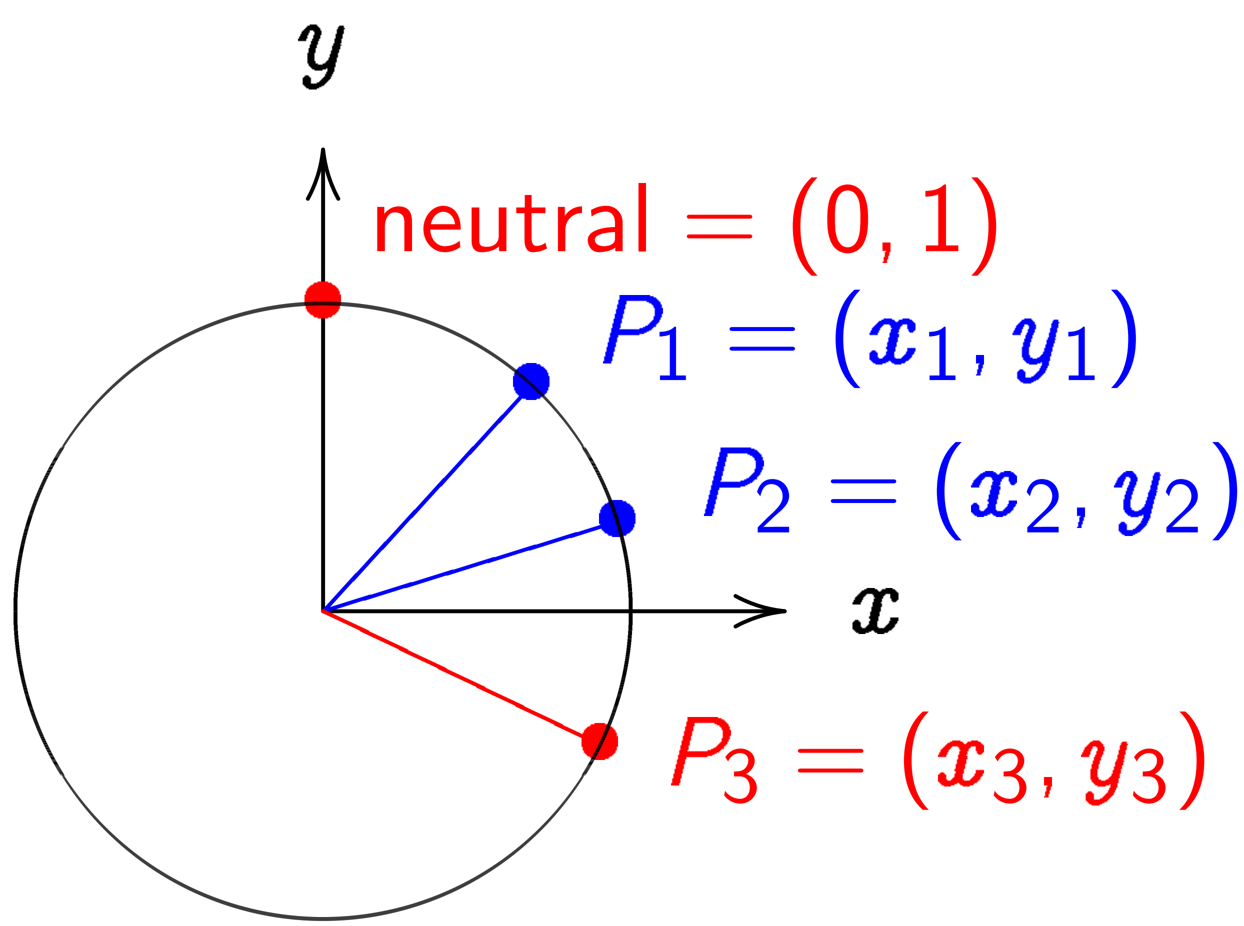
Clock addition without sin, cos:



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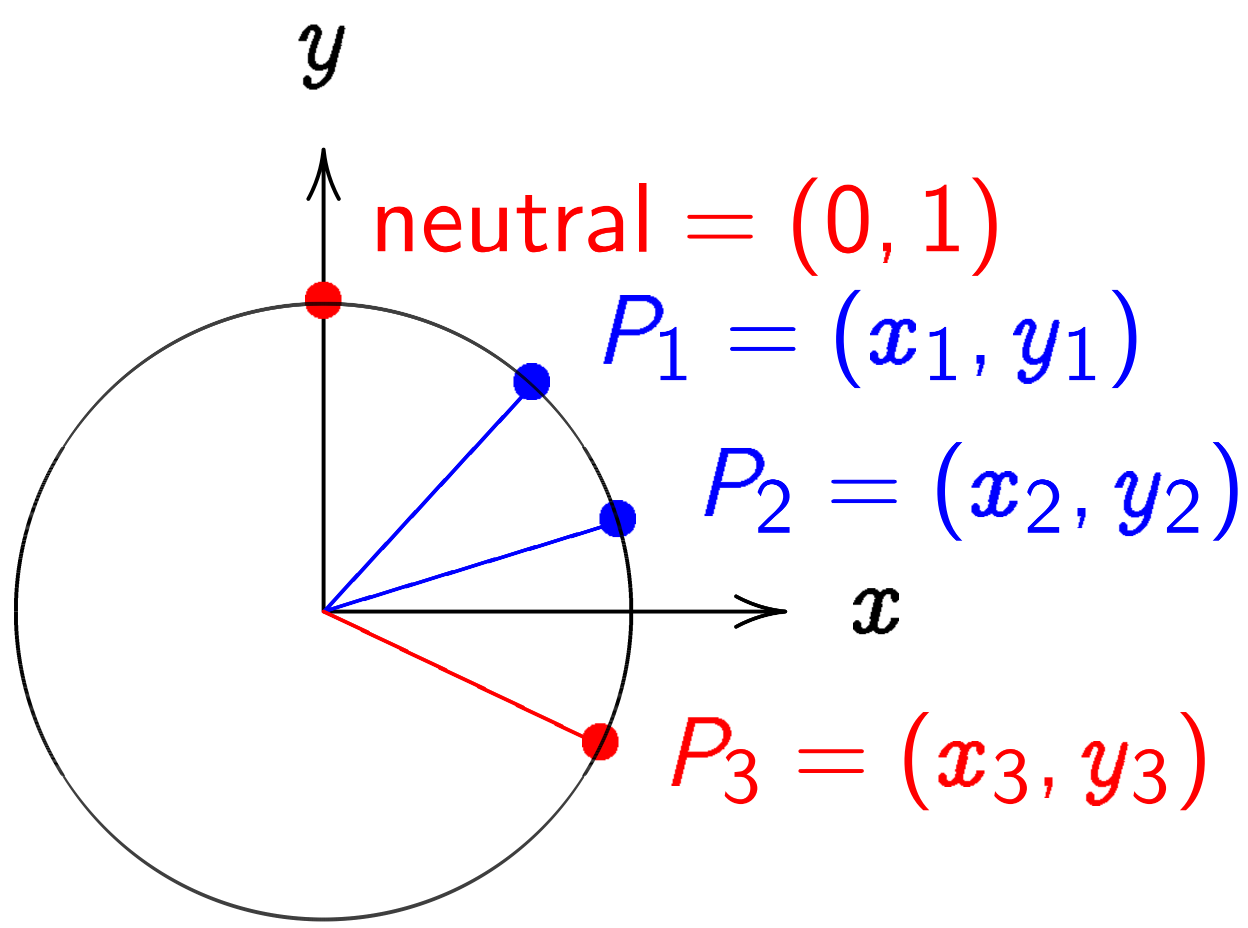

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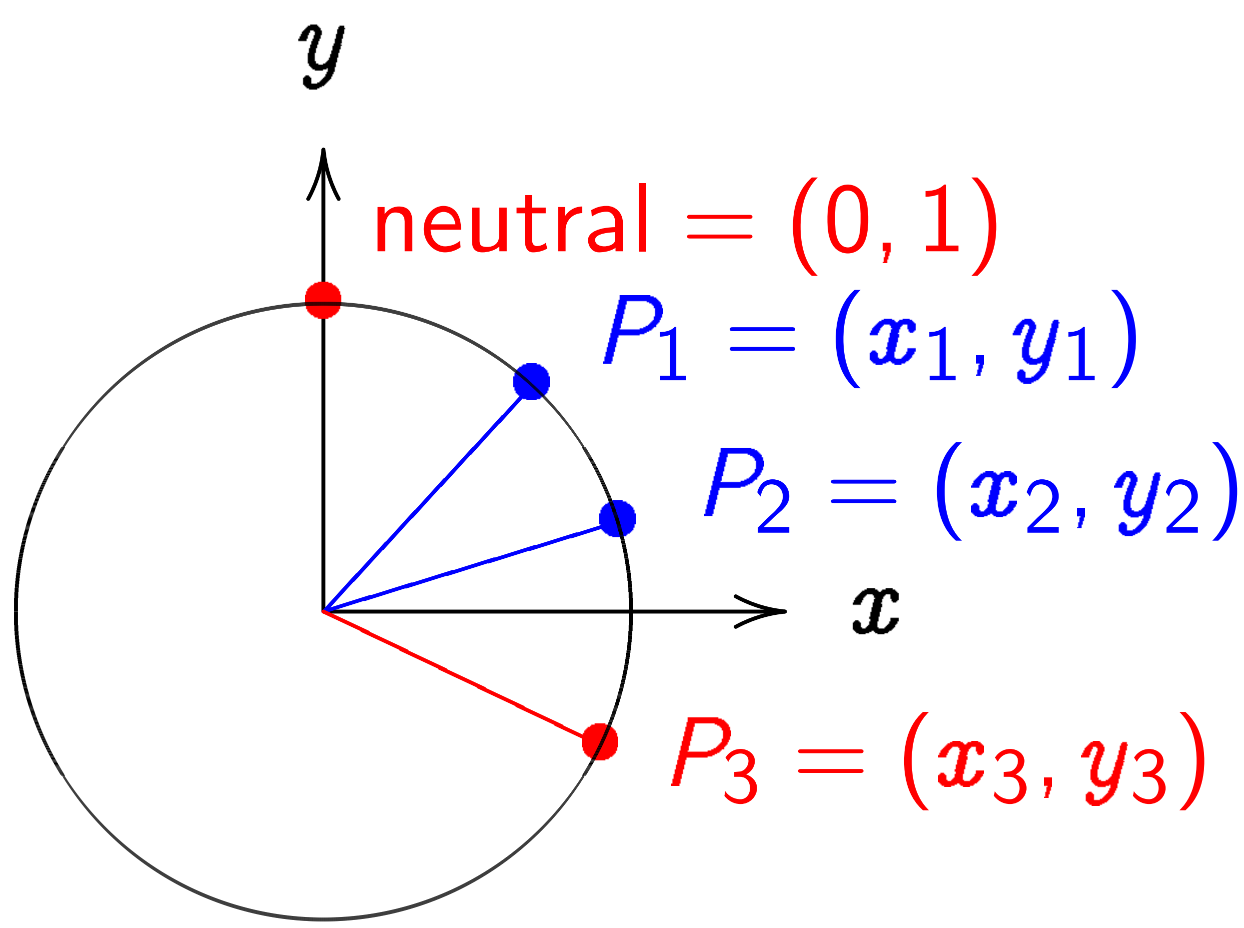

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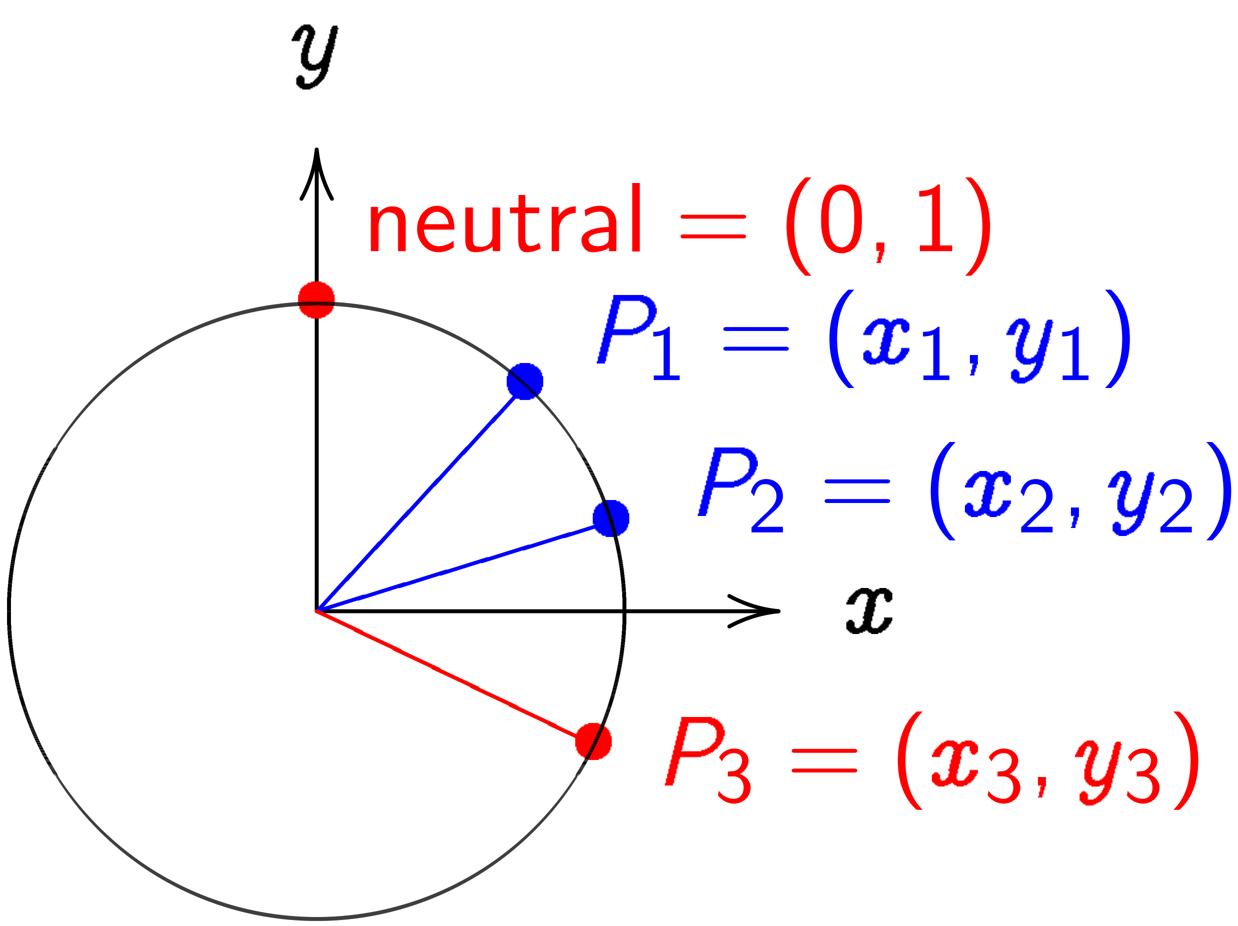

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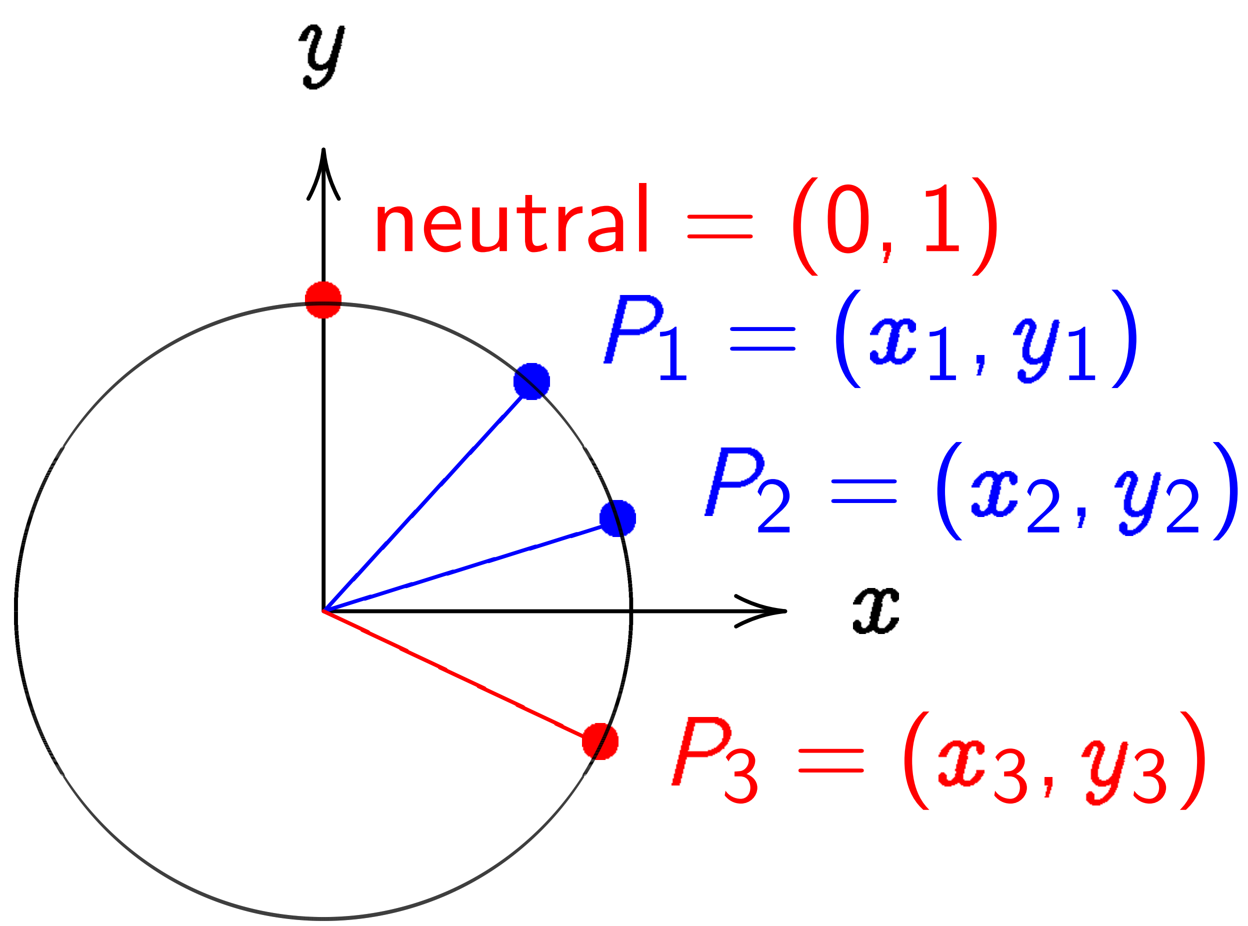

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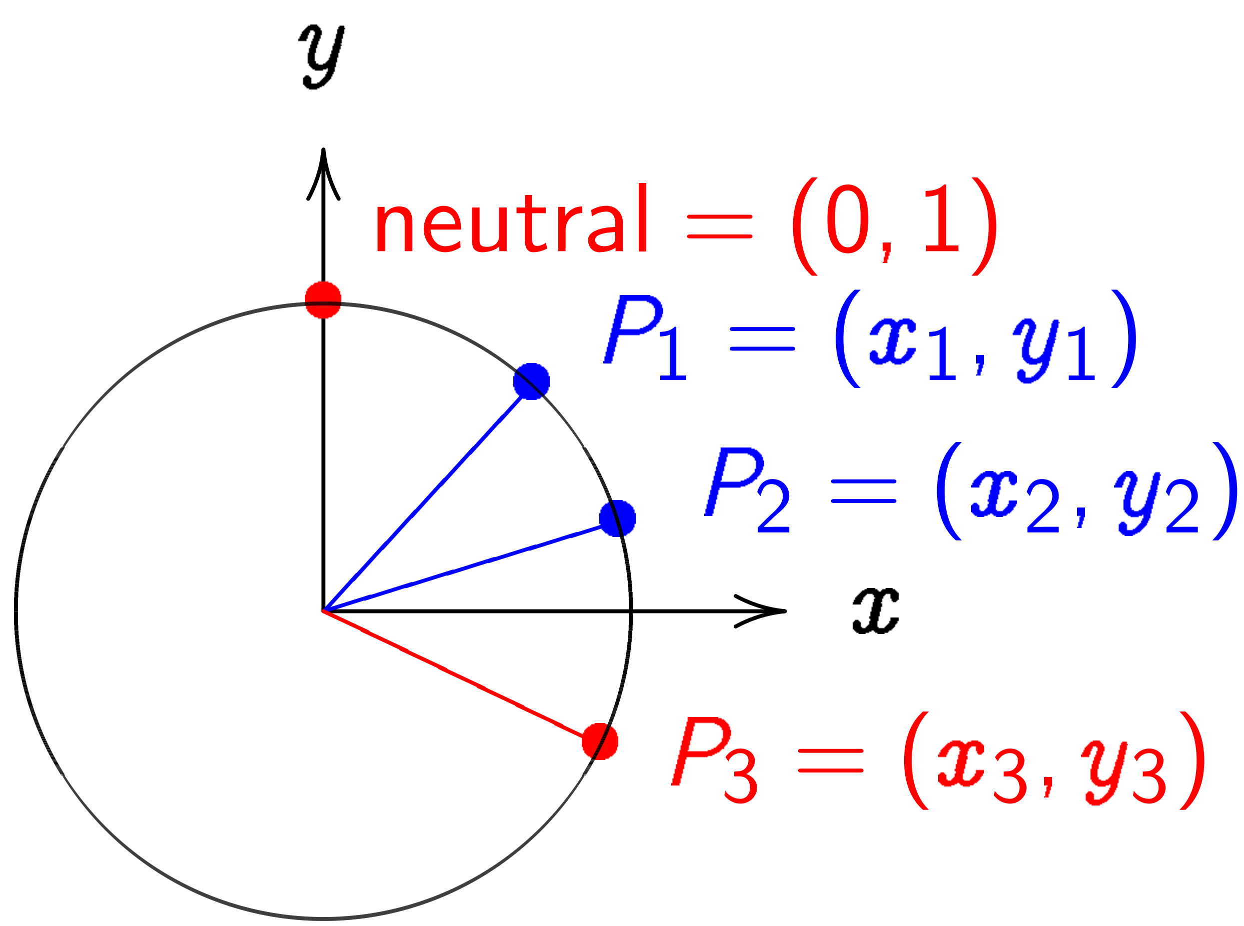

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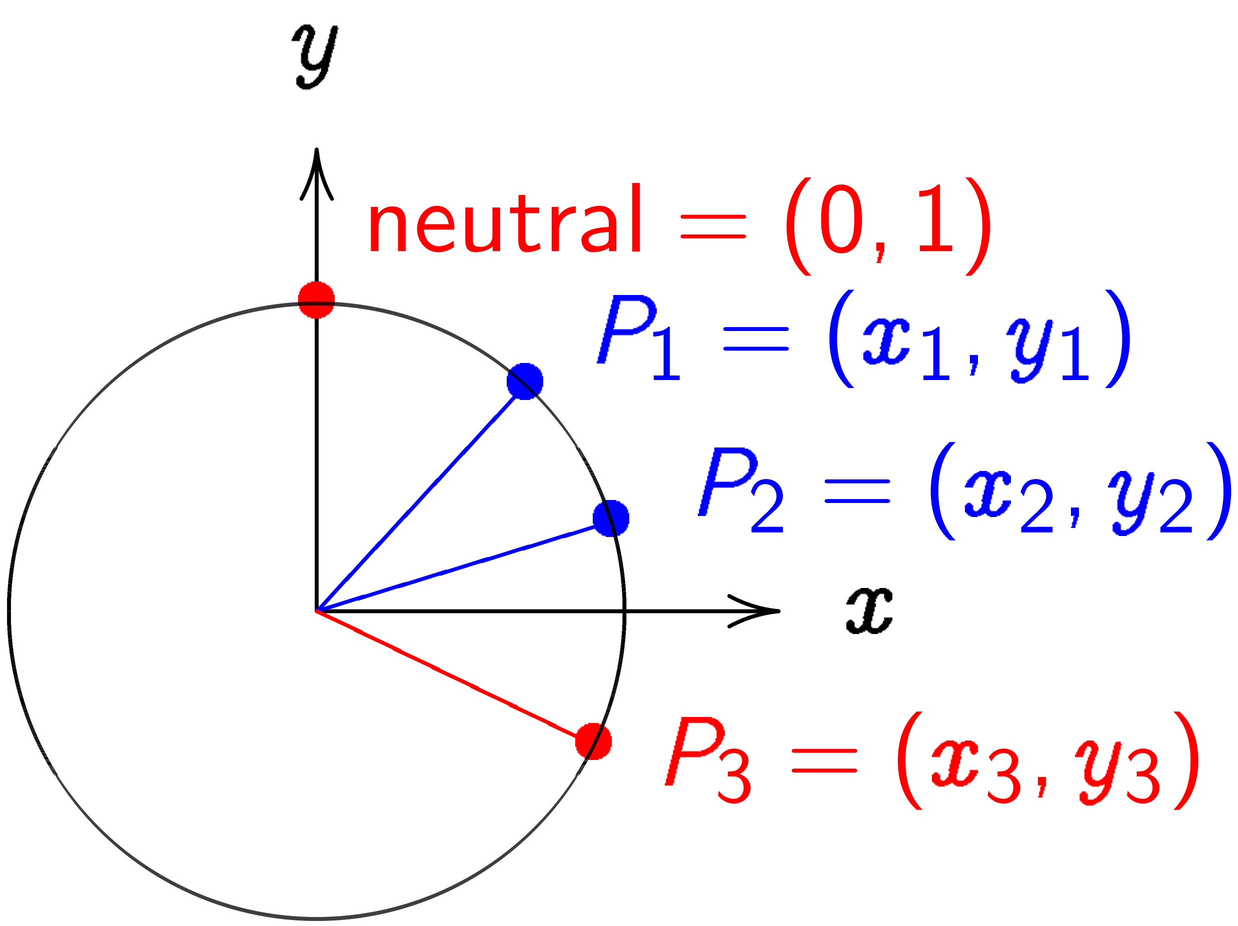

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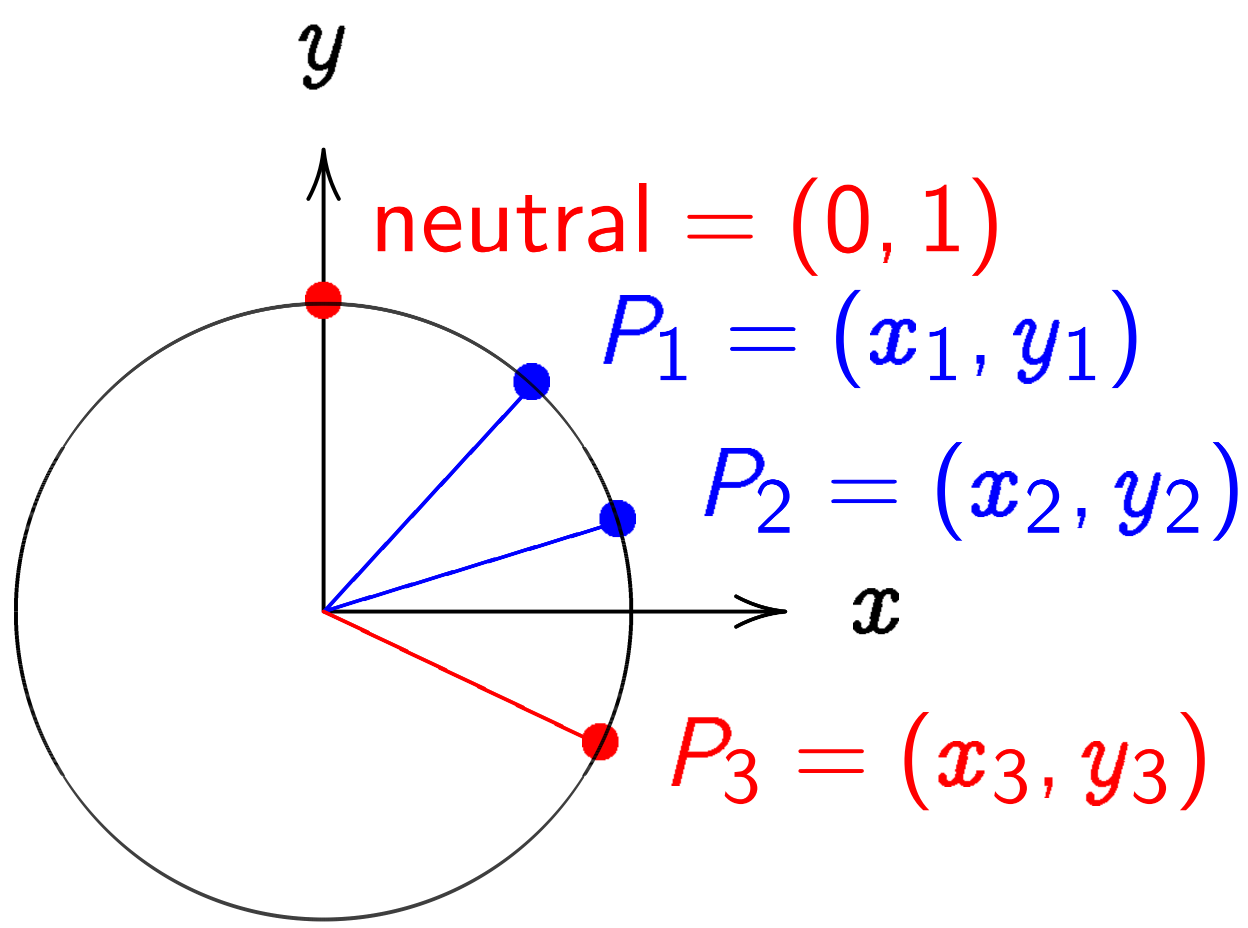
$(x_1, y_1) + (-x_1, y_1) =$

Clock addition without sin, cos:



Use Cartesian coordinates for addition. Addition formula for the clock $x^2 + y^2 = 1$: sum of (x_1, y_1) and (x_2, y_2) is $(x_1y_2 + y_1x_2, y_1y_2 - x_1x_2)$.

Examples of clock addition:

"2:00" + "5:00"
$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$
$= (-1/2, -\sqrt{3/4}) =$ "7:00".

"5:00" + "9:00"
$= (1/2, -\sqrt{3/4}) + (-1, 0)$
$= (\sqrt{3/4}, 1/2) =$ "2:00".

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

$$3\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{117}{125}, \frac{-44}{125}\right).$$

$$4\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{336}{625}, \frac{-527}{625}\right).$$

$(x_1, y_1) + (0, 1) = (x_1, y_1)$.

$(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

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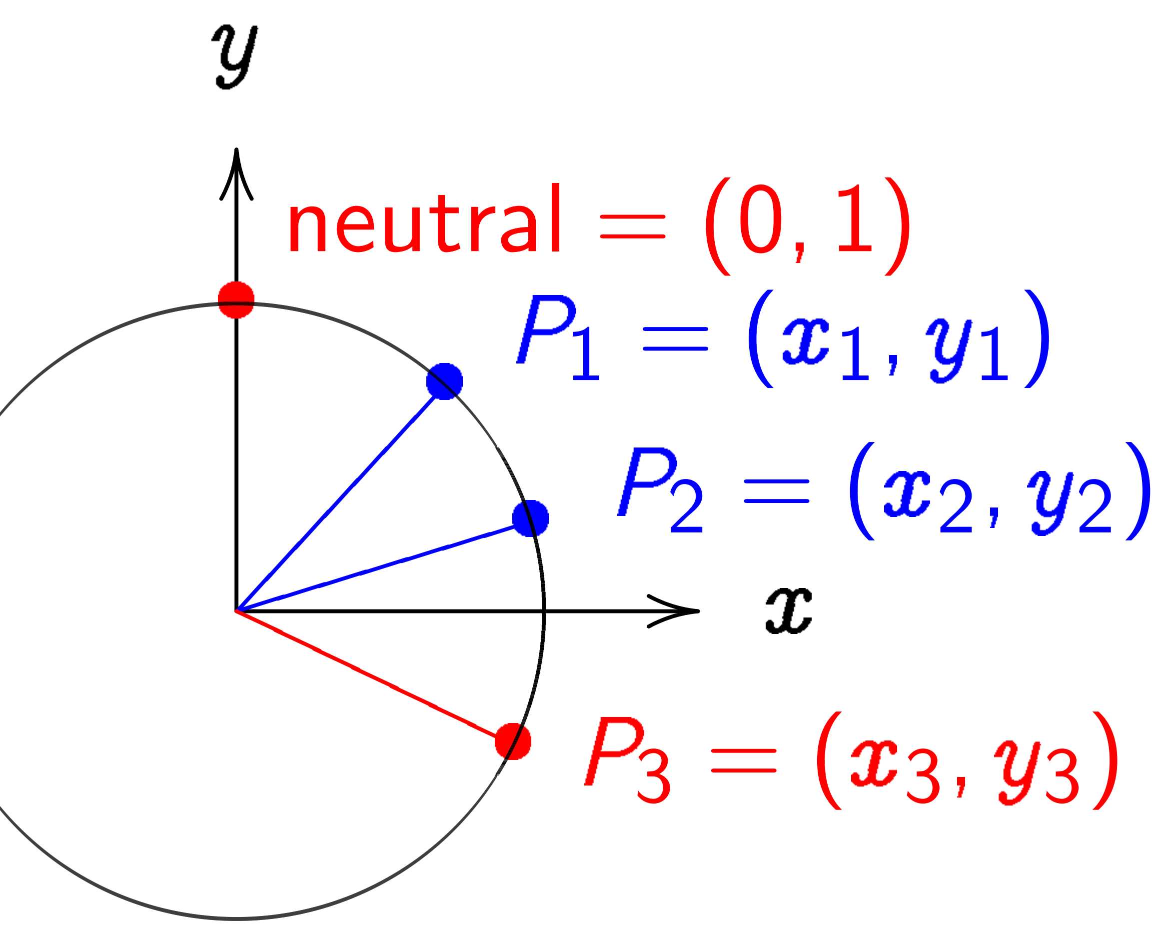

tesian coordinates for
. Addition formula
clock $x^2 + y^2 = 1$:
(x_1, y_1) and (x_2, y_2) is
$y_1 x_2, y_1 y_2 - x_1 x_2)$.

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Clocks o

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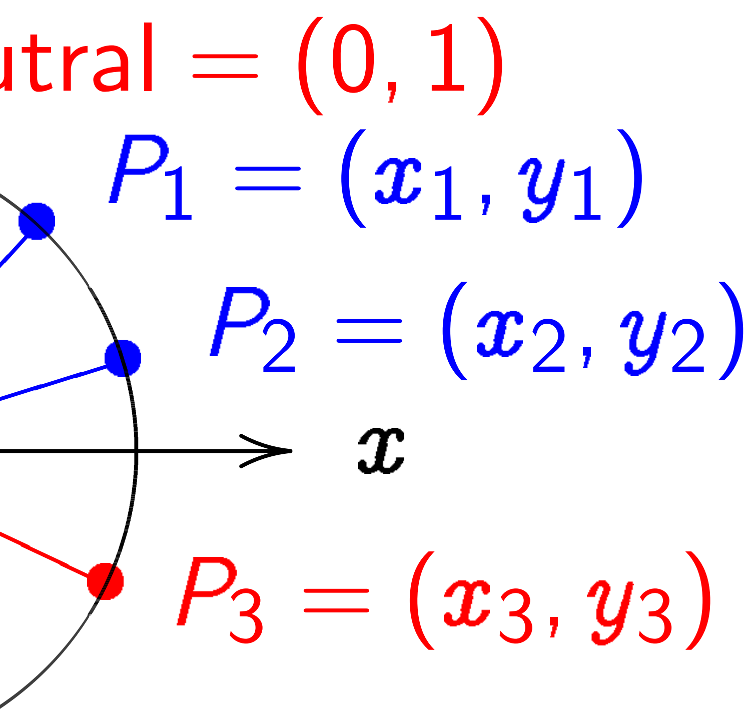
Clock($\mathbf{F}$
$\{(x, y) \in$
Here $\mathbf{F}_7$
$= \{0, 1,$
with arit
e.g. $2 \cdot 5$

thout sin, cos:



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$(x_1, y_1) + (0, 1) = (x_1, y_1)$.

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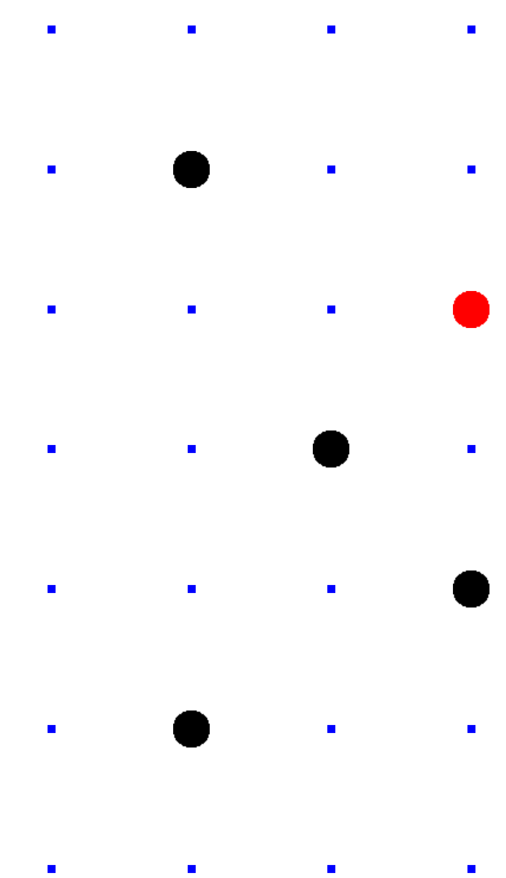
Clocks over finite



$\text{Clock}(\mathbf{F}_7) =$
$\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7$
Here $\mathbf{F}_7 = \{0, 1, 2$
$= \{0, 1, 2, 3, -3, -$
with arithmetic m
e.g. $2 \cdot 5 = 3$ and

cos:

$1)$

$x_1, y_1)$

(x_2, y_2)

(x_3, y_3)

or

is

.

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
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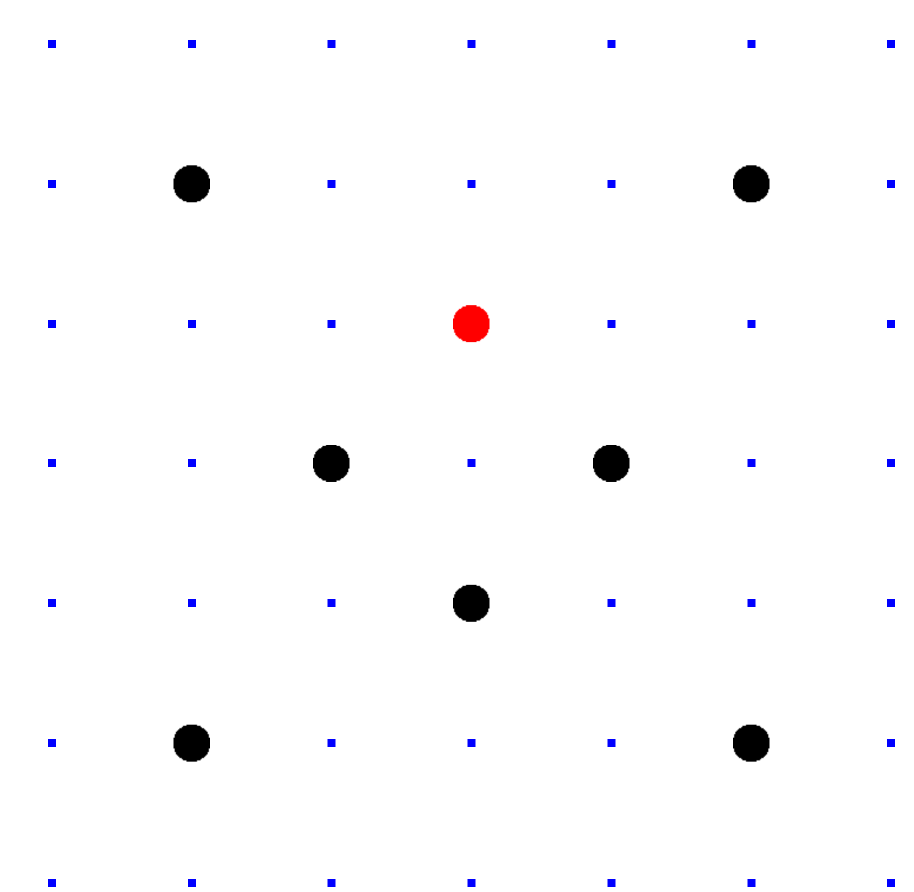
Clocks over finite fields



$\text{Clock}(\mathbf{F}_7) =$
$\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2$

Here $\mathbf{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$
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with arithmetic modulo 7.
e.g. $2 \cdot 5 = 3$ and $3/2 = 5$ i

Examples of clock addition:

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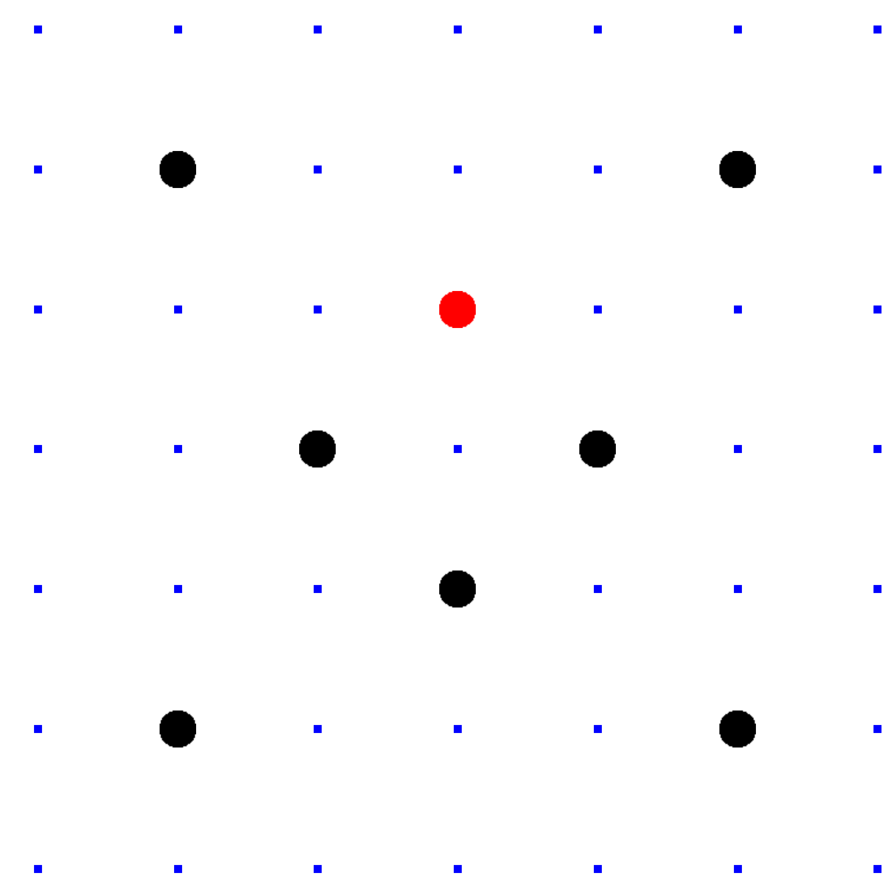
Clocks over finite fields



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– “5:00”

$4, 1/2) + (1/2, -\sqrt{3/4})$

$2, -\sqrt{3/4}) =$ “7:00”.

– “9:00”

$-\sqrt{3/4}) + (-1, 0)$

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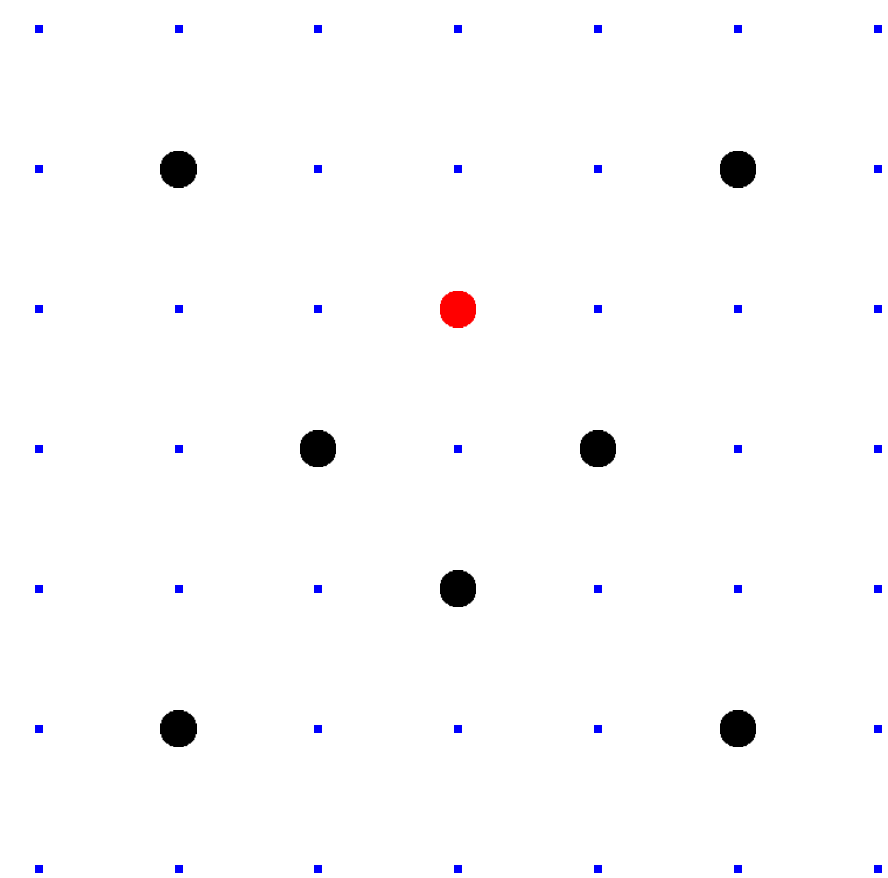
Clocks over finite fields



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Larger e

Example

on Clock

2(1000, 2

addition:

$(1/2, -\sqrt{3/4})$

$) =$ "7:00".

$+ (-1, 0)$

"2:00".

$\frac{7}{25}\Big)$.

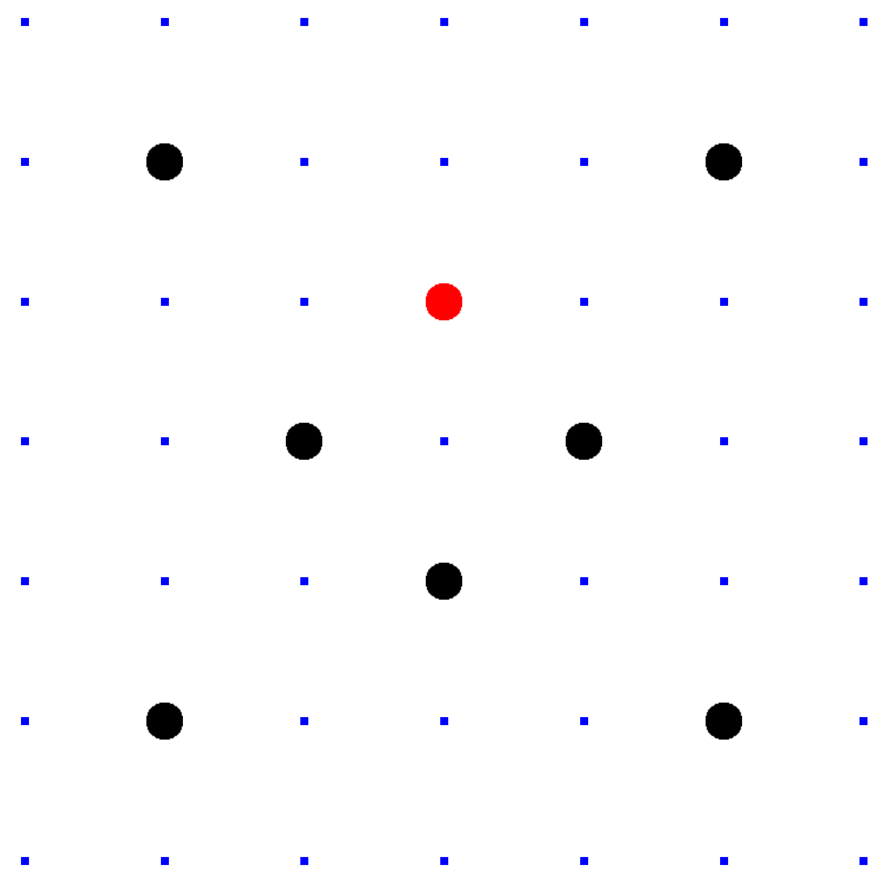
$, \frac{-44}{125}\Big)$.

$, \frac{-527}{625}\Big)$.

$= (x_1, y_1)$.

$_1) = (0, 1)$.

Clocks over finite fields



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Larger example: C

Examples of addit
on $\text{Clock}(\mathbf{F}_{1000003}$
$2(1000, 2) = (4000$

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3/4)
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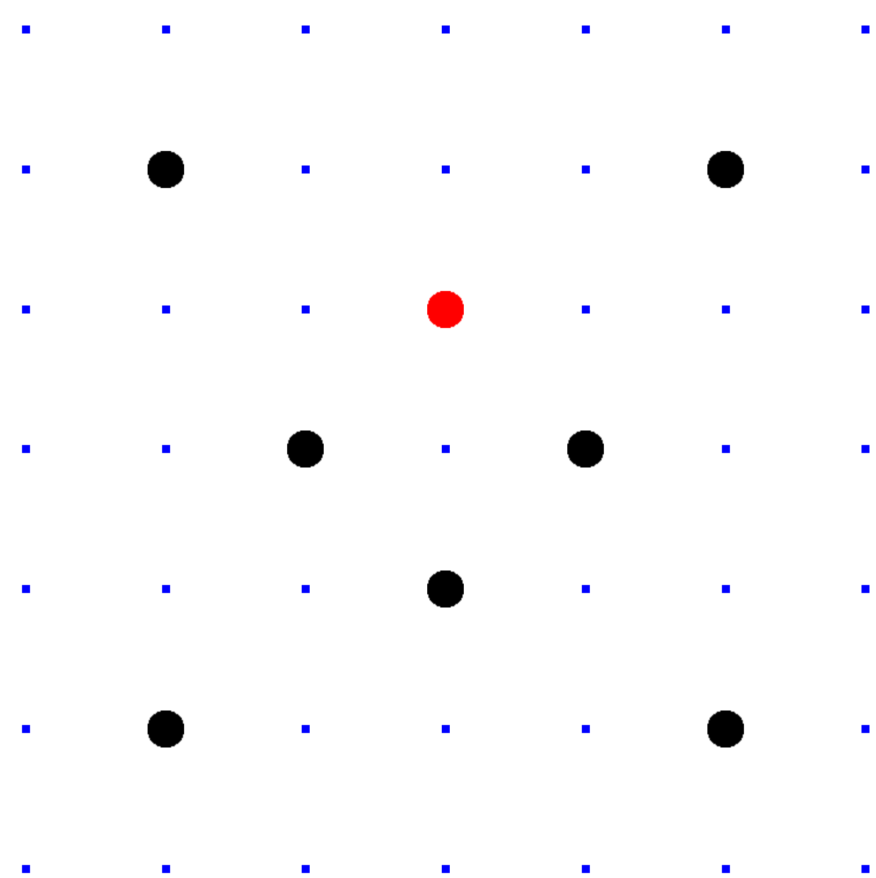
Clocks over finite fields



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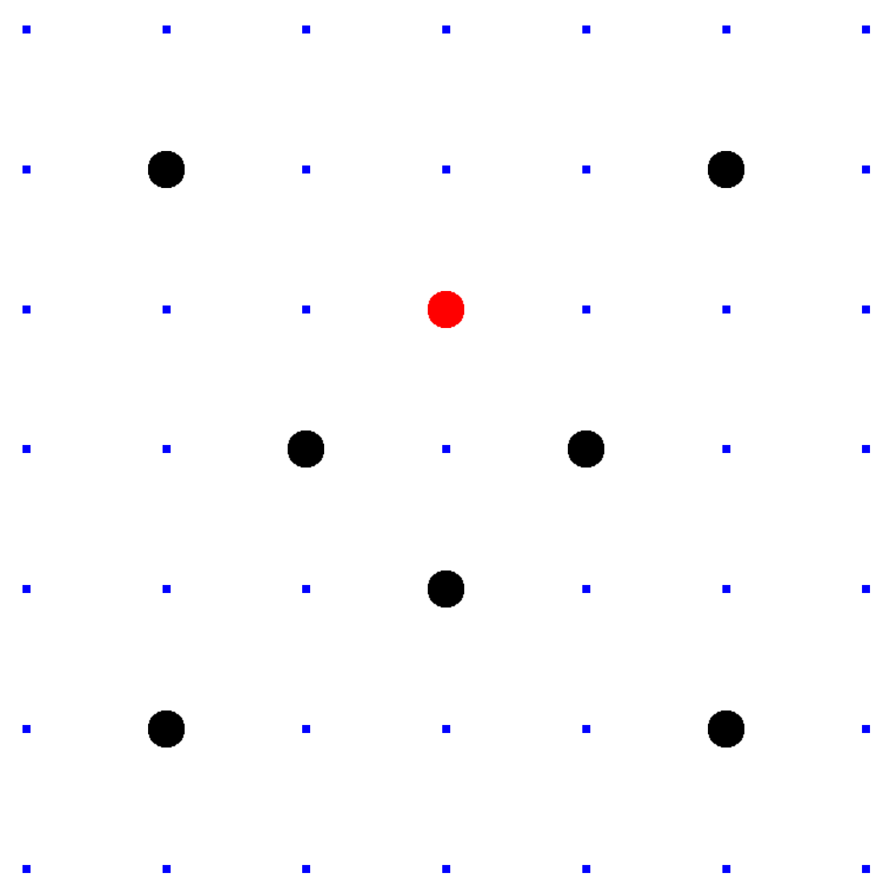
Larger example: Clock($\mathbf{F}_{100}$

Examples of addition
on $\text{Clock}(\mathbf{F}_{1000003})$:
$2(1000, 2) = (4000, 7)$.

Clocks over finite fields



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Larger example: Clock($\mathbf{F}_{1000003}$).

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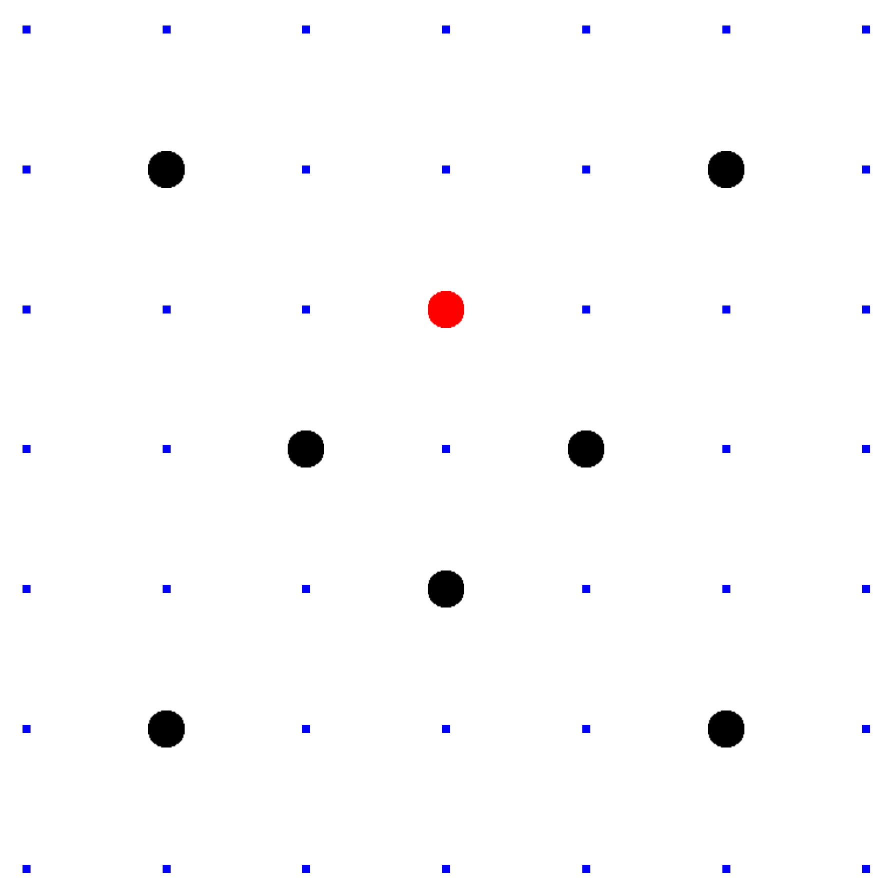
Clocks over finite fields



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Larger example: $\text{Clock}(\mathbf{F}_{1000003})$.

Examples of addition
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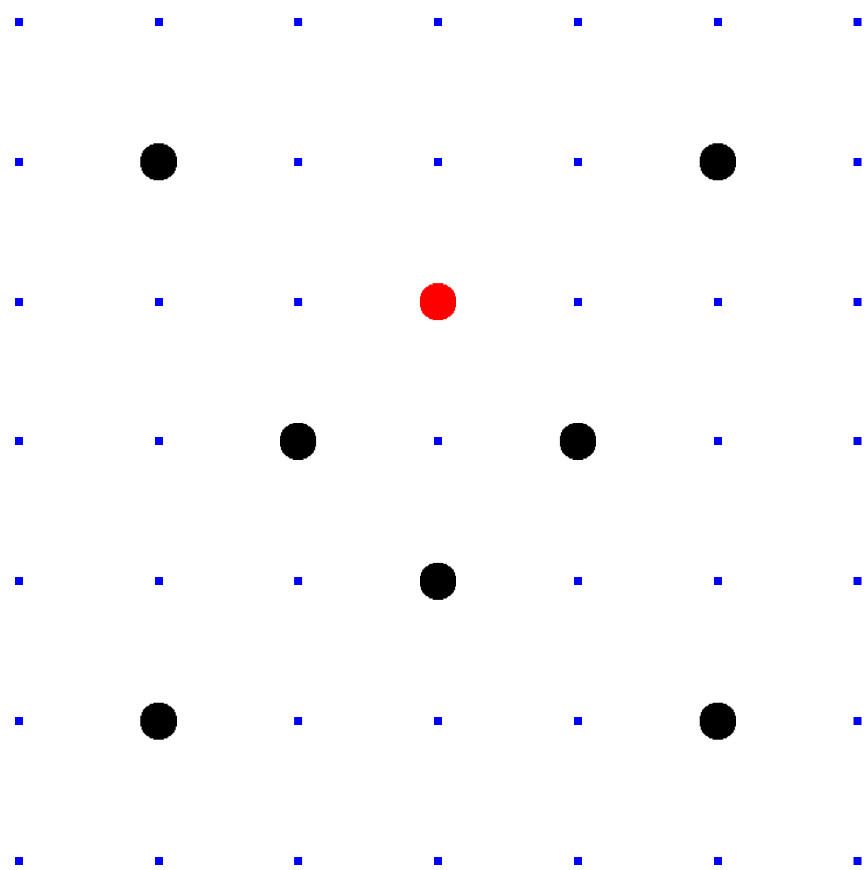
$2(1000, 2) = (4000, 7)$.
$4(1000, 2) = (56000, 97)$.

Clocks over finite fields



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Examples of addition
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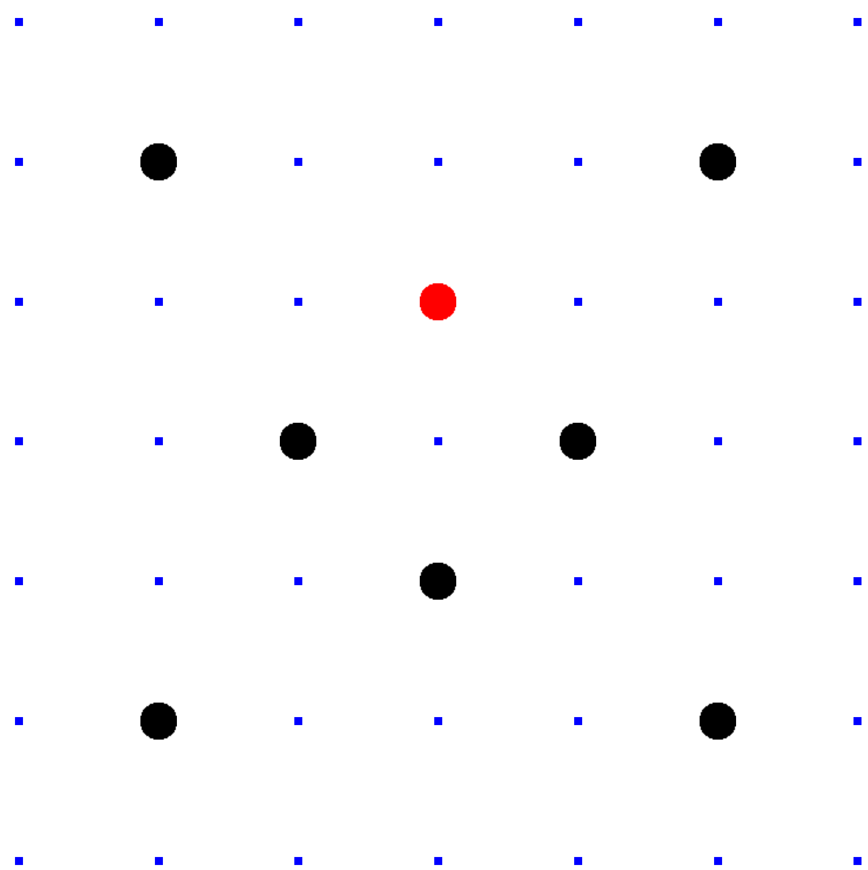

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$2(1000, 2) = (4000, 7)$.

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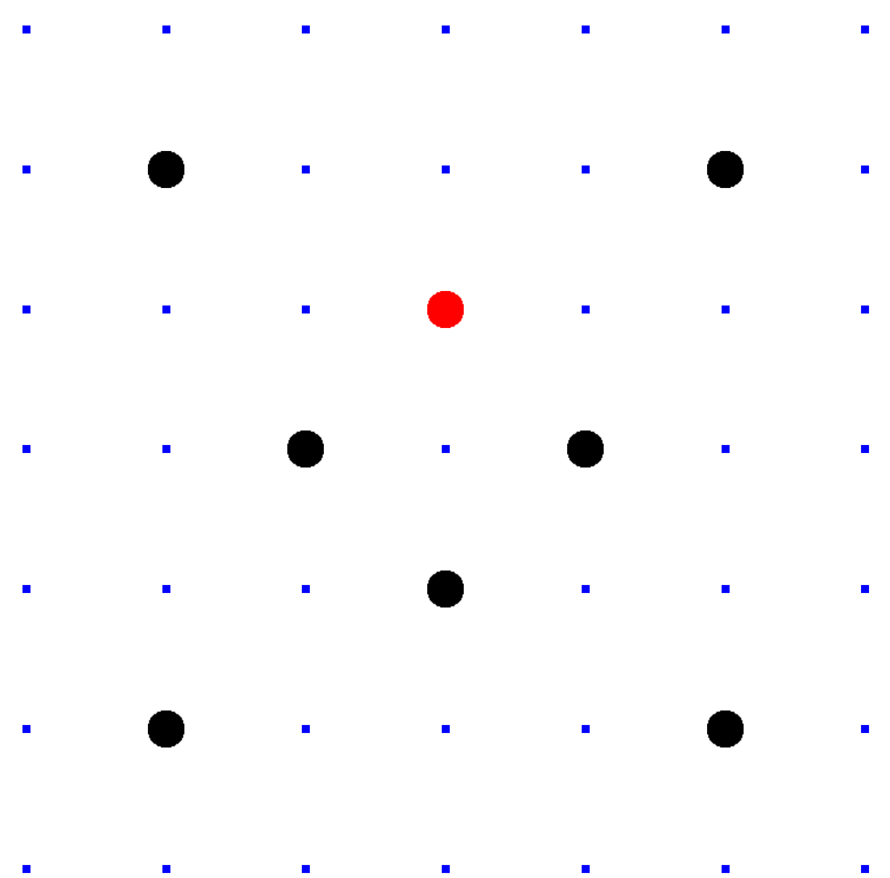
Clocks over finite fields



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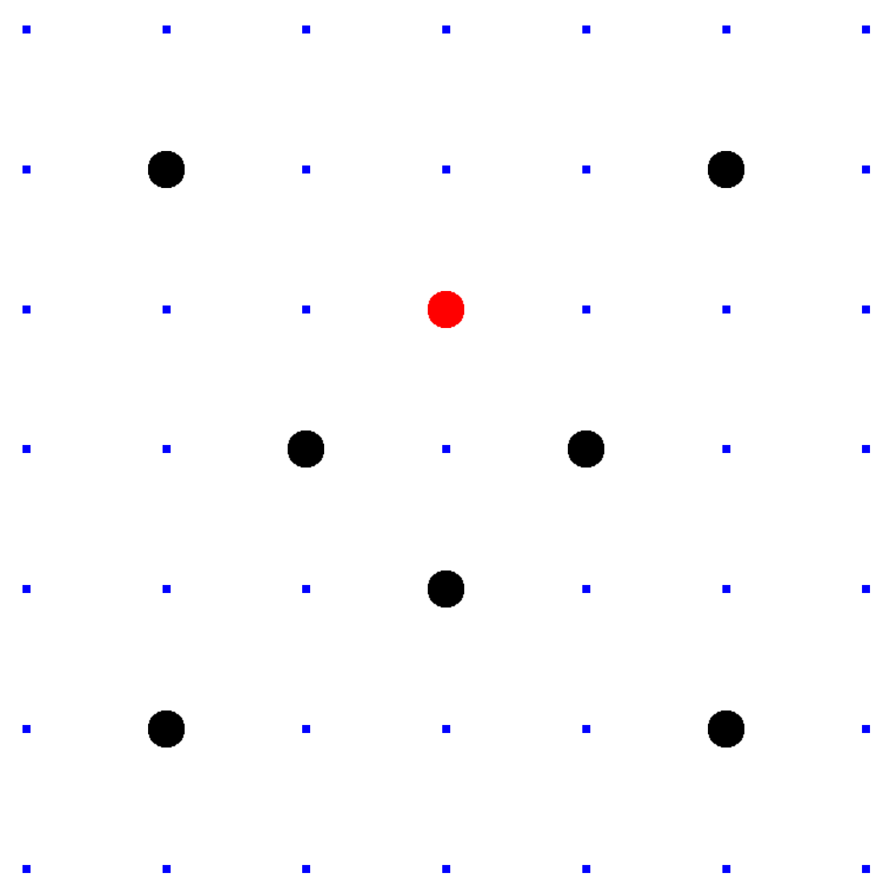

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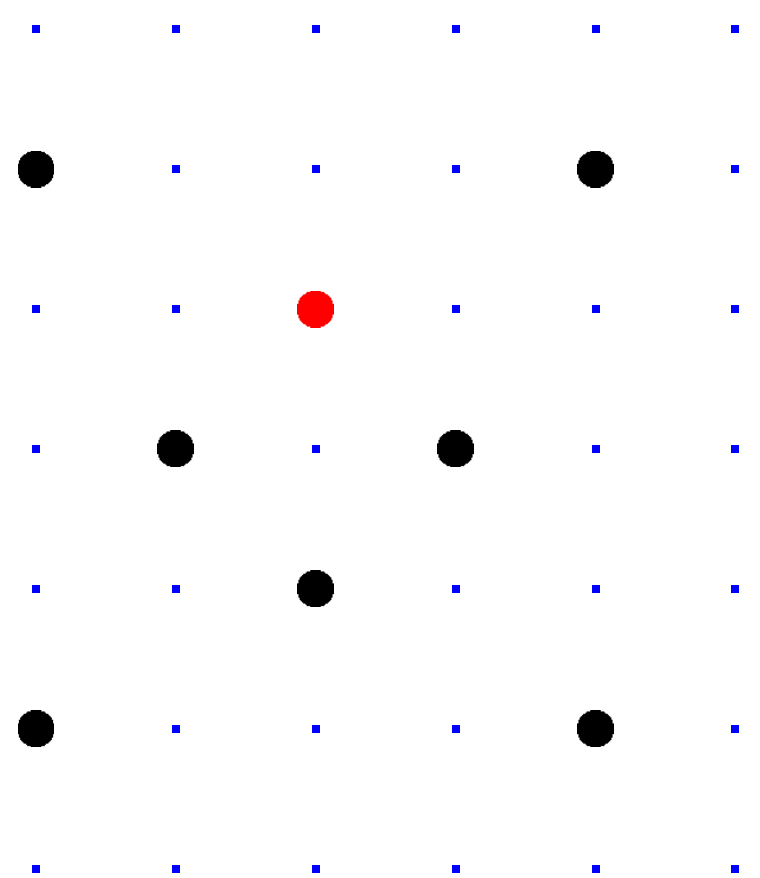
"Scalar multiplication"
on a clock:
Given integer $n \geq 0$
and clock point (x, y),
compute $n(x, y)$.

over finite fields



$_7) =$

$\in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2 = 1\}$.

$= \{0, 1, 2, 3, 4, 5, 6\}$

$2, 3, -3, -2, -1\}$

thmetic modulo 7.

$5 = 3$ and $3/2 = 5$ in $\mathbf{F}_7$.

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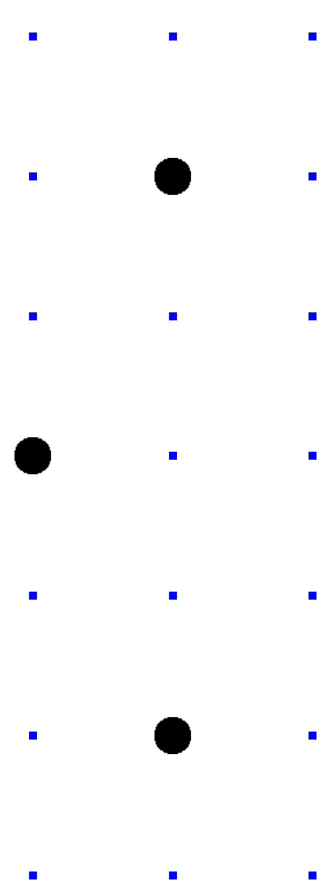

$: x^2 + y^2 = 1\}$.

$, 3, 4, 5, 6\}$

$-2, -1\}$

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If n is even, comp
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Otherwise comput
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"Binary method":
If n is even, compute $n(x, y$
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Otherwise compute $n(x, y)$
by adding (x, y) to $(n - 1)($
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If n is even, compute $n(x, y)$
by doubling $(n/2)(x, y)$.
Otherwise compute $n(x, y)$
by adding (x, y) to $(n - 1)(x, y)$.
This is very fast.

But figuring out n
given (x, y) and $n(x, y)$
is much more difficult.

With 30 clock additions
we computed
$n(1000, 2) = (947472, 736284)$
for some 6-digit n.
Can you figure out n?

xample: Clock($\mathbf{F}_{1000003}$).

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k($\mathbf{F}_{1000003}$):

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Can you figure out n?

Clock cryptograph

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If n is even, compute $n(x, y)$ by doubling $(n/2)(x, y)$.
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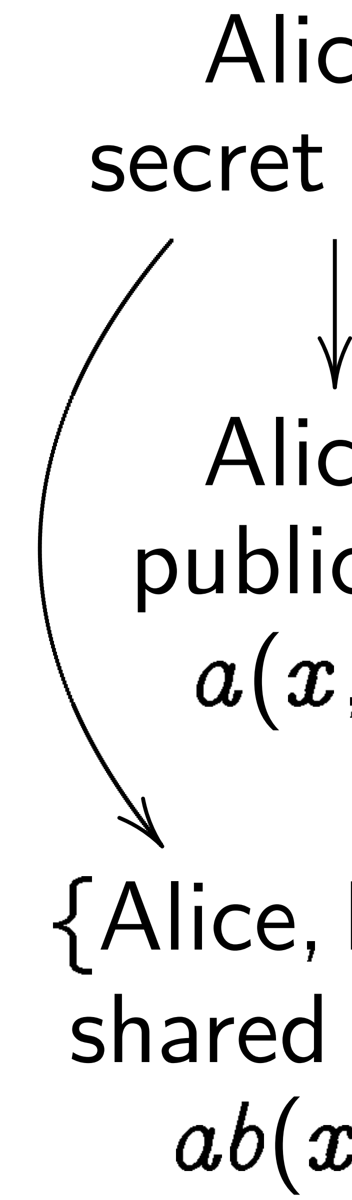

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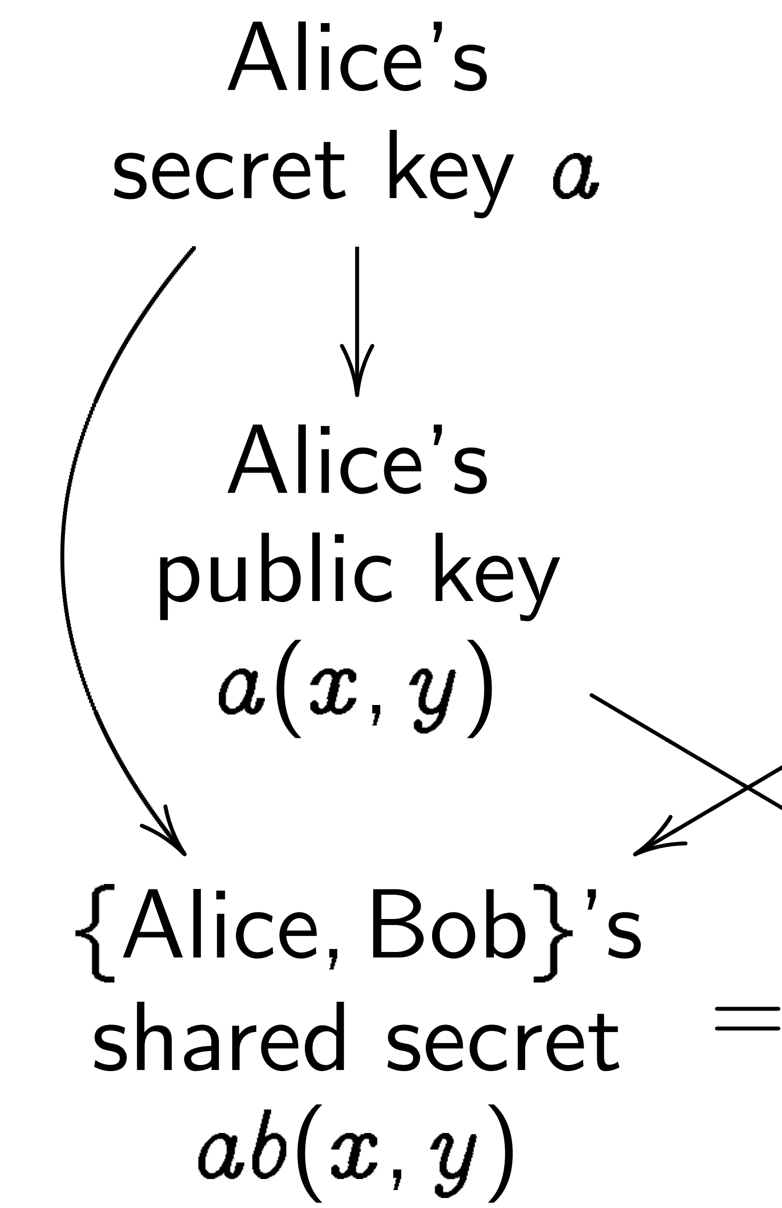

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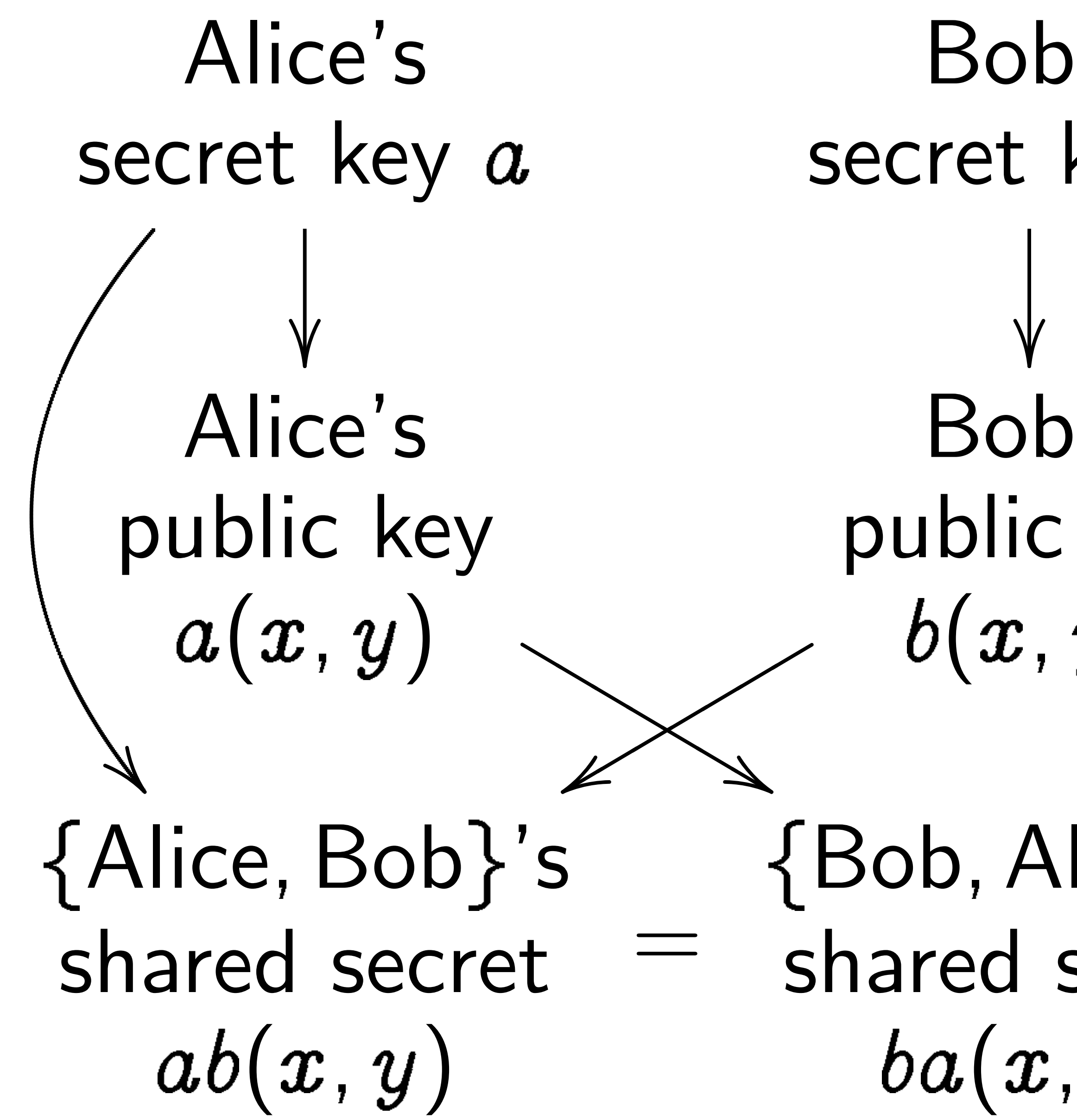

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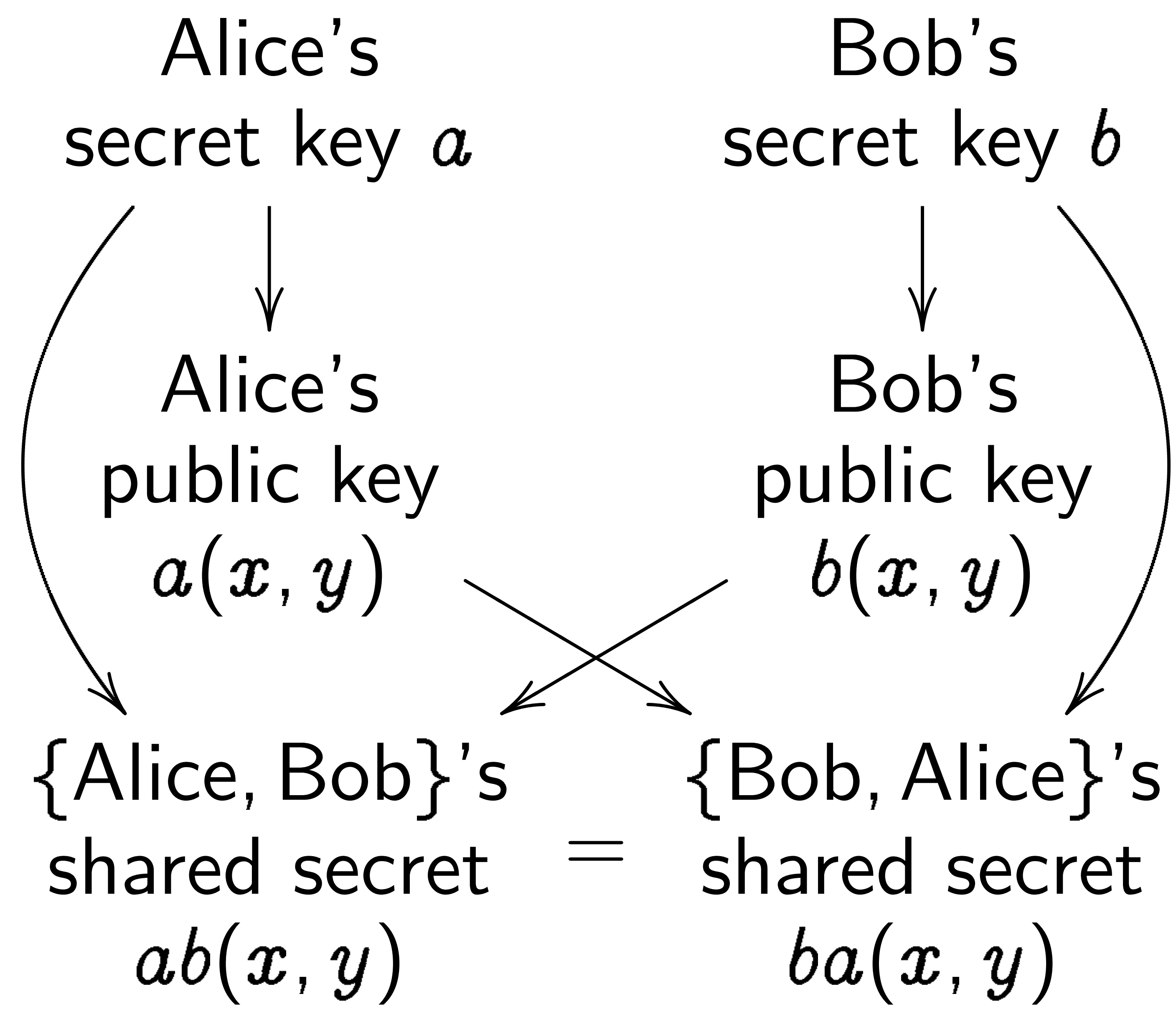

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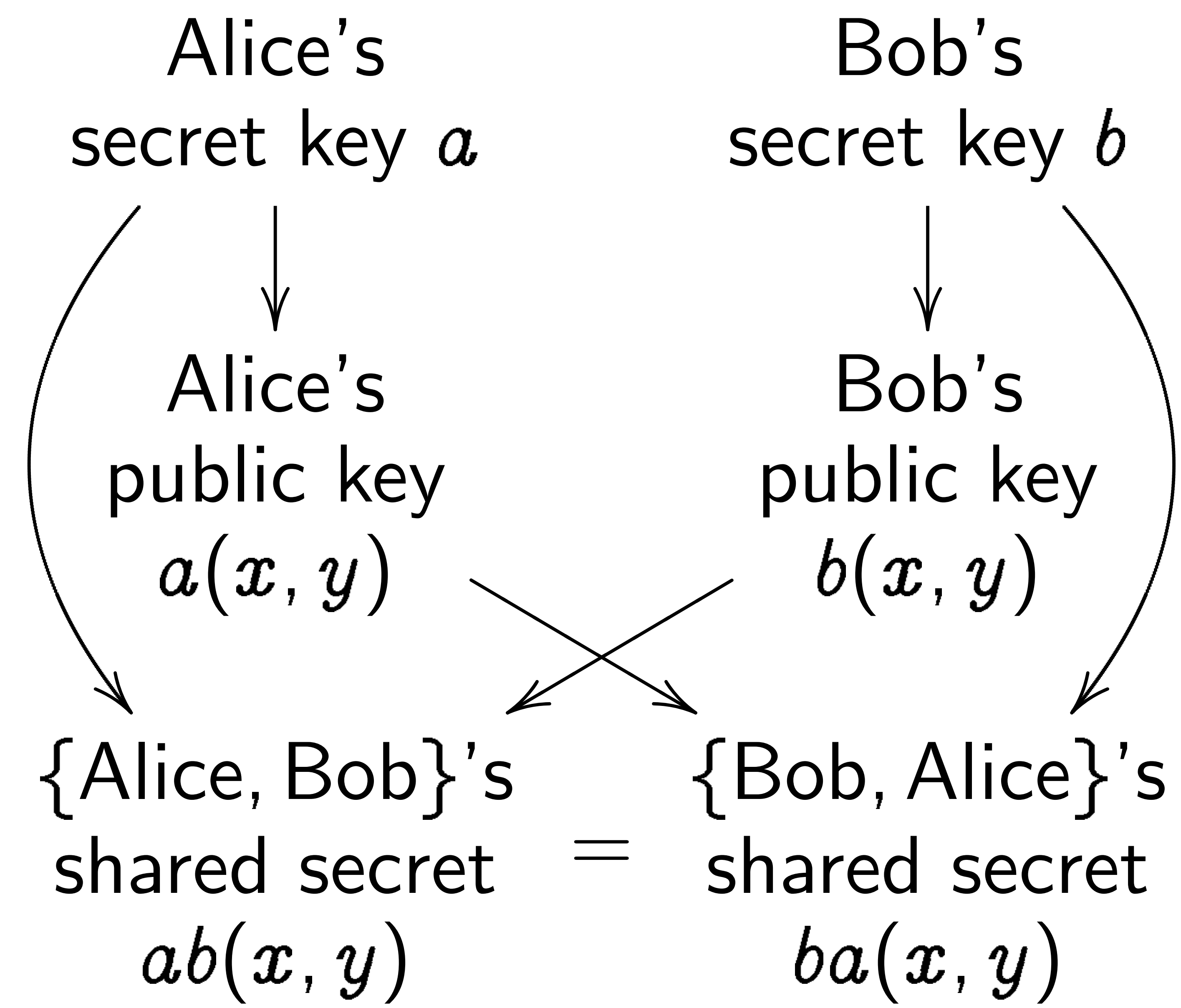

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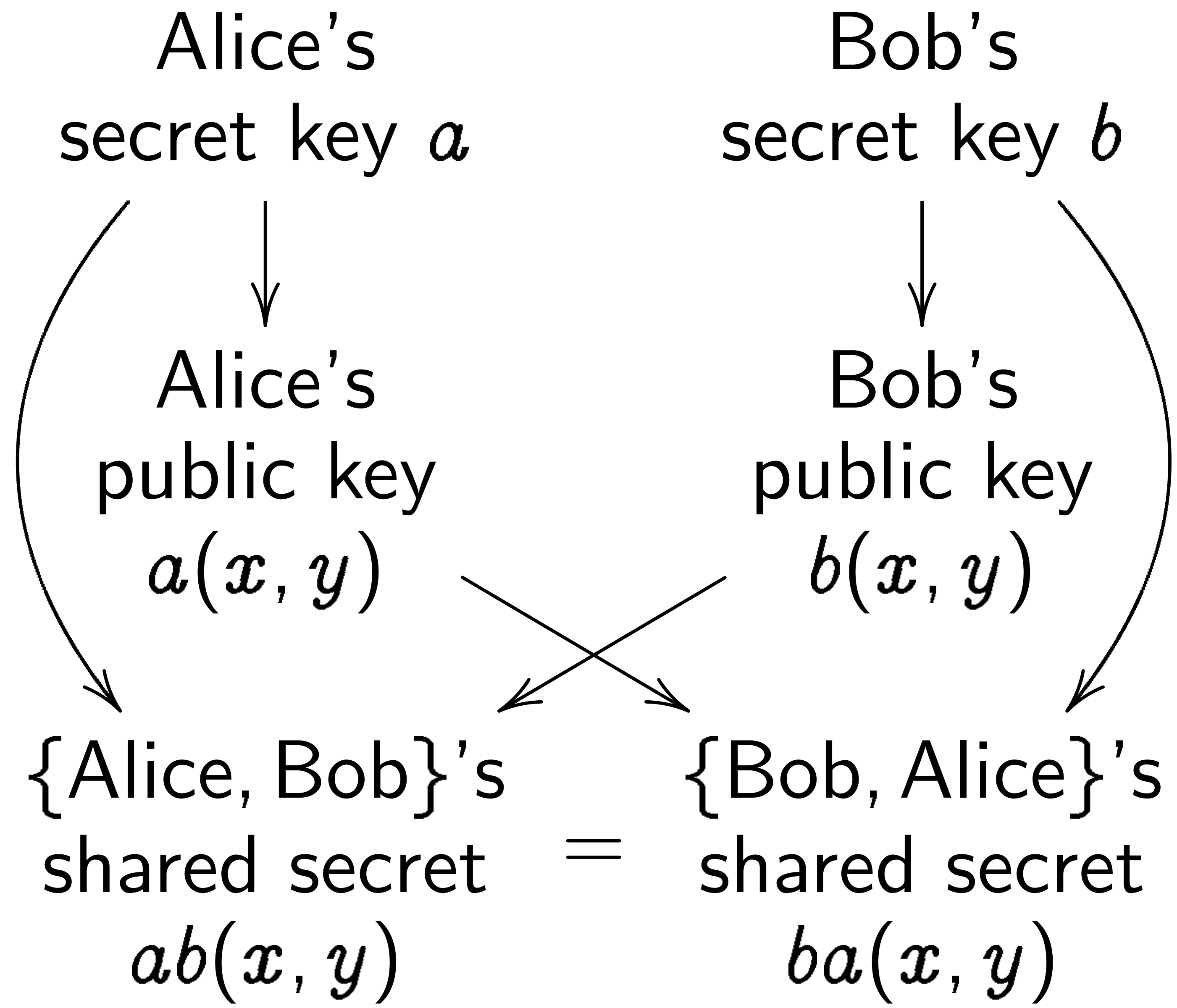

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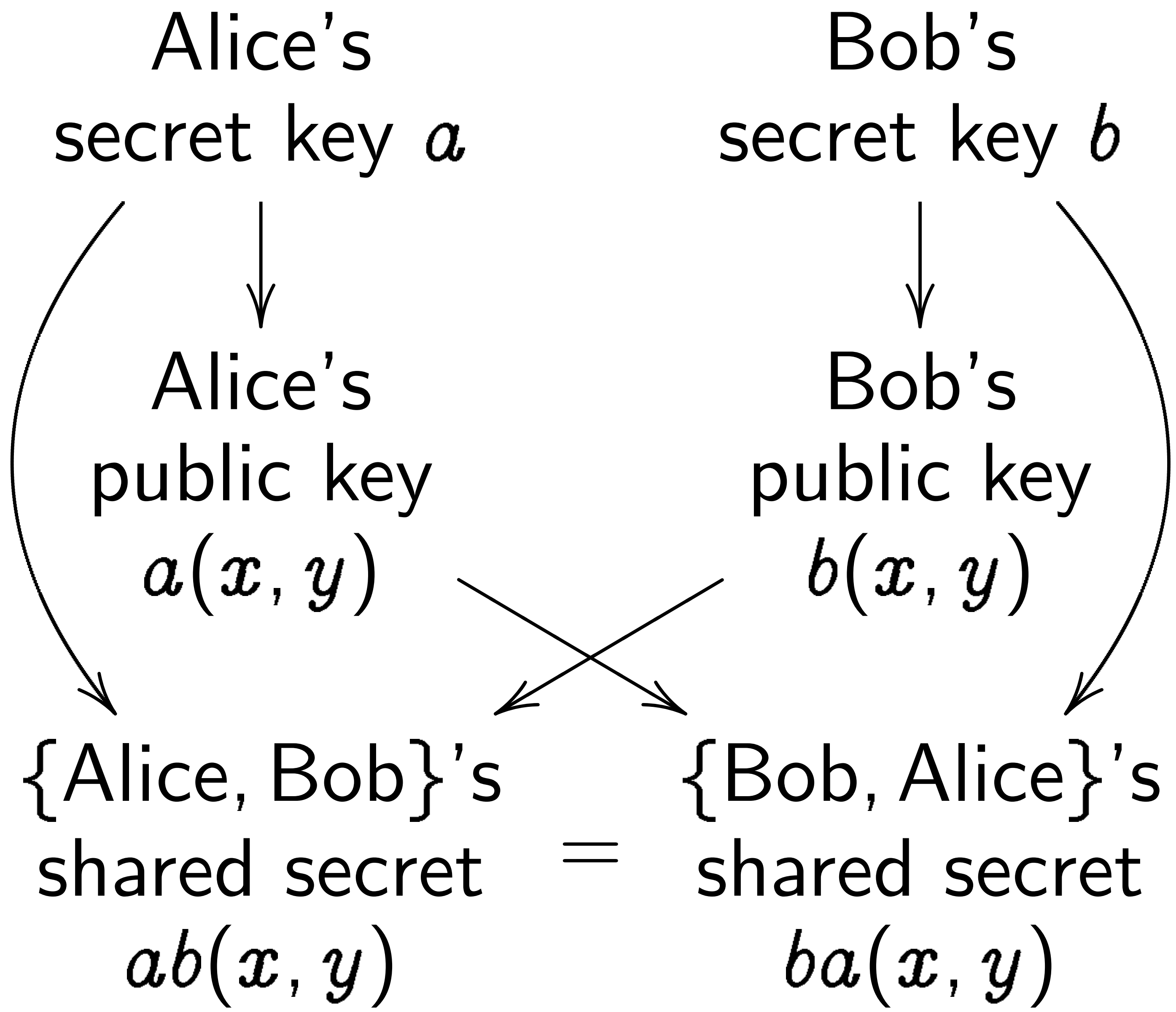

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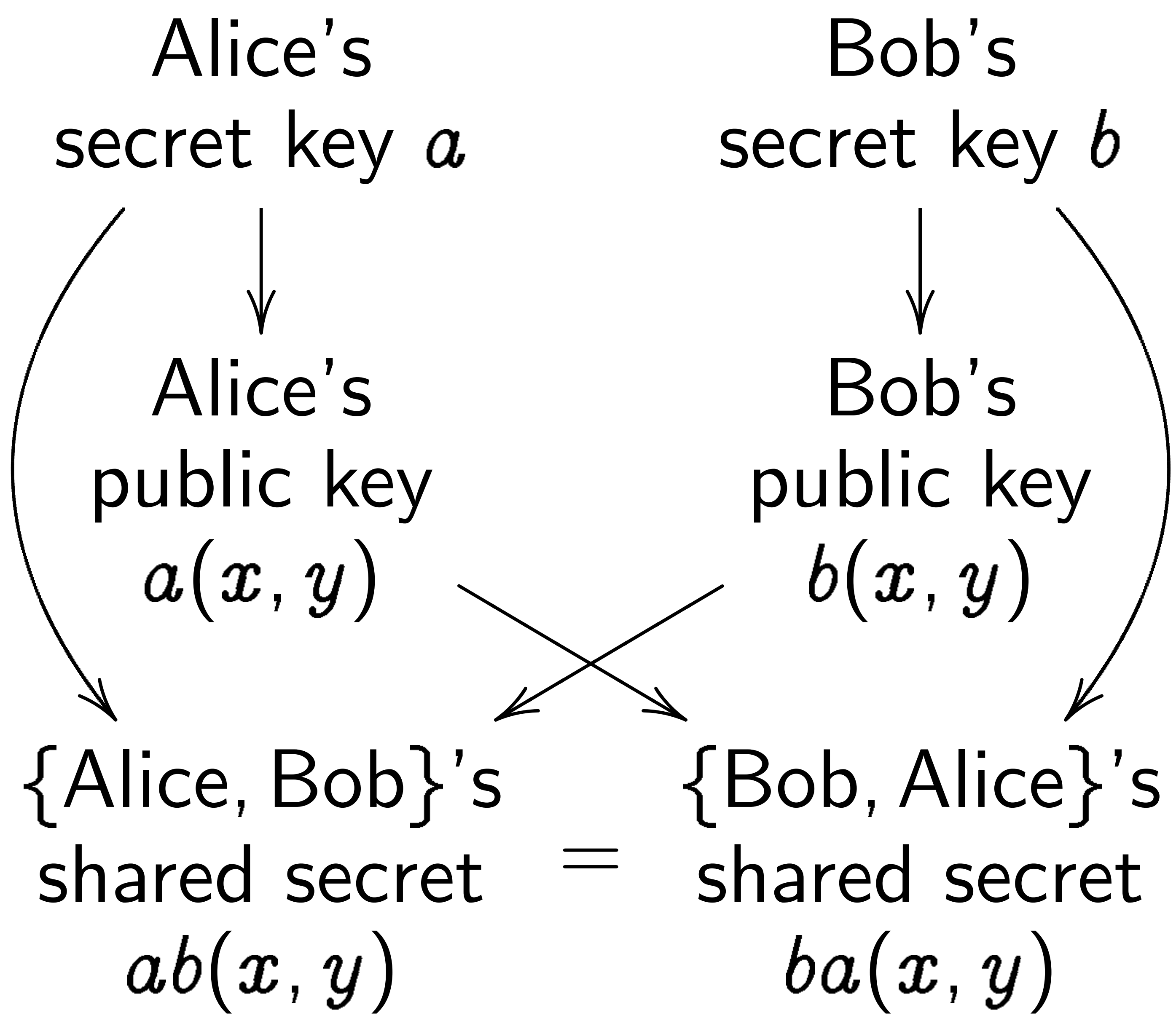

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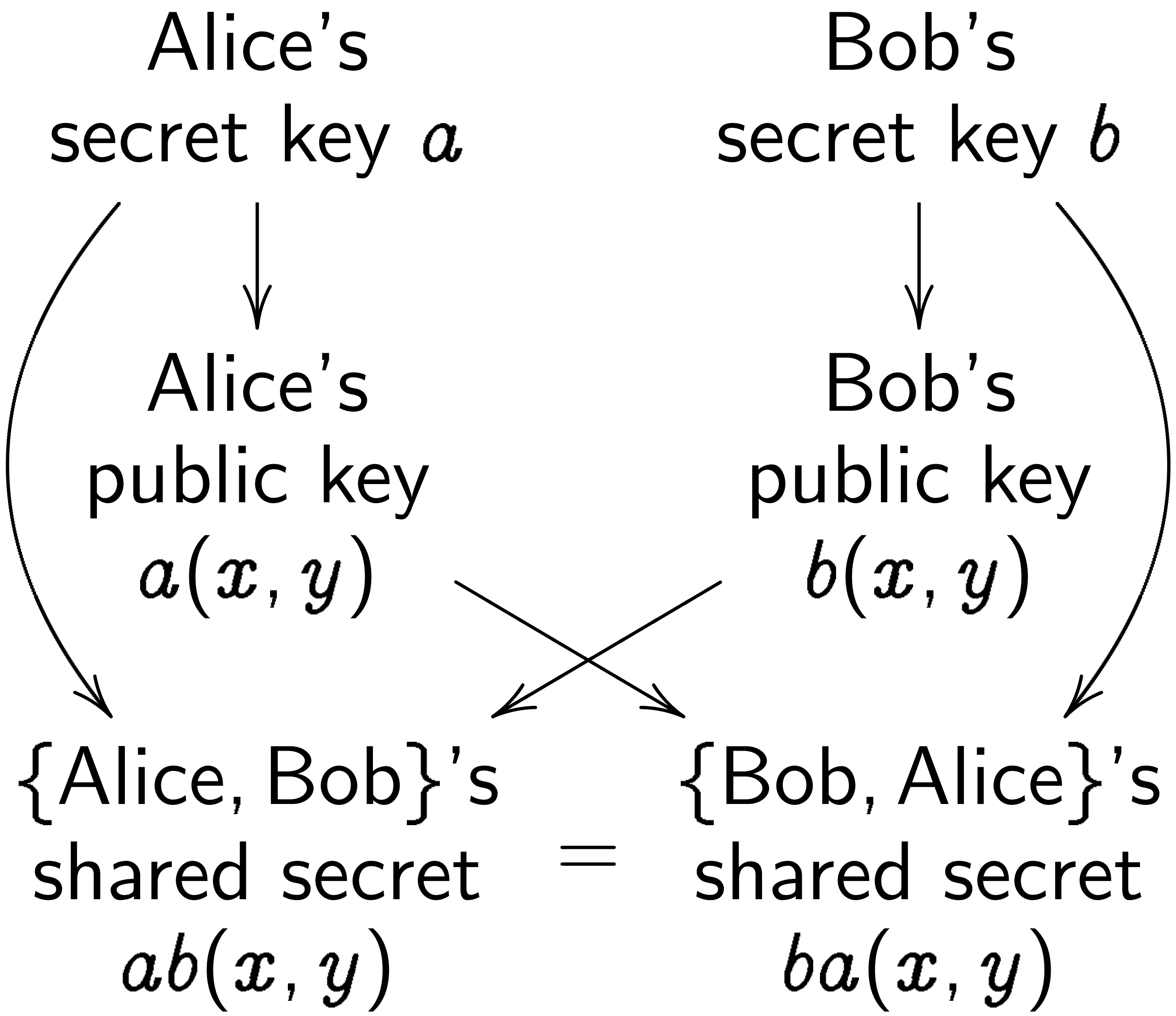

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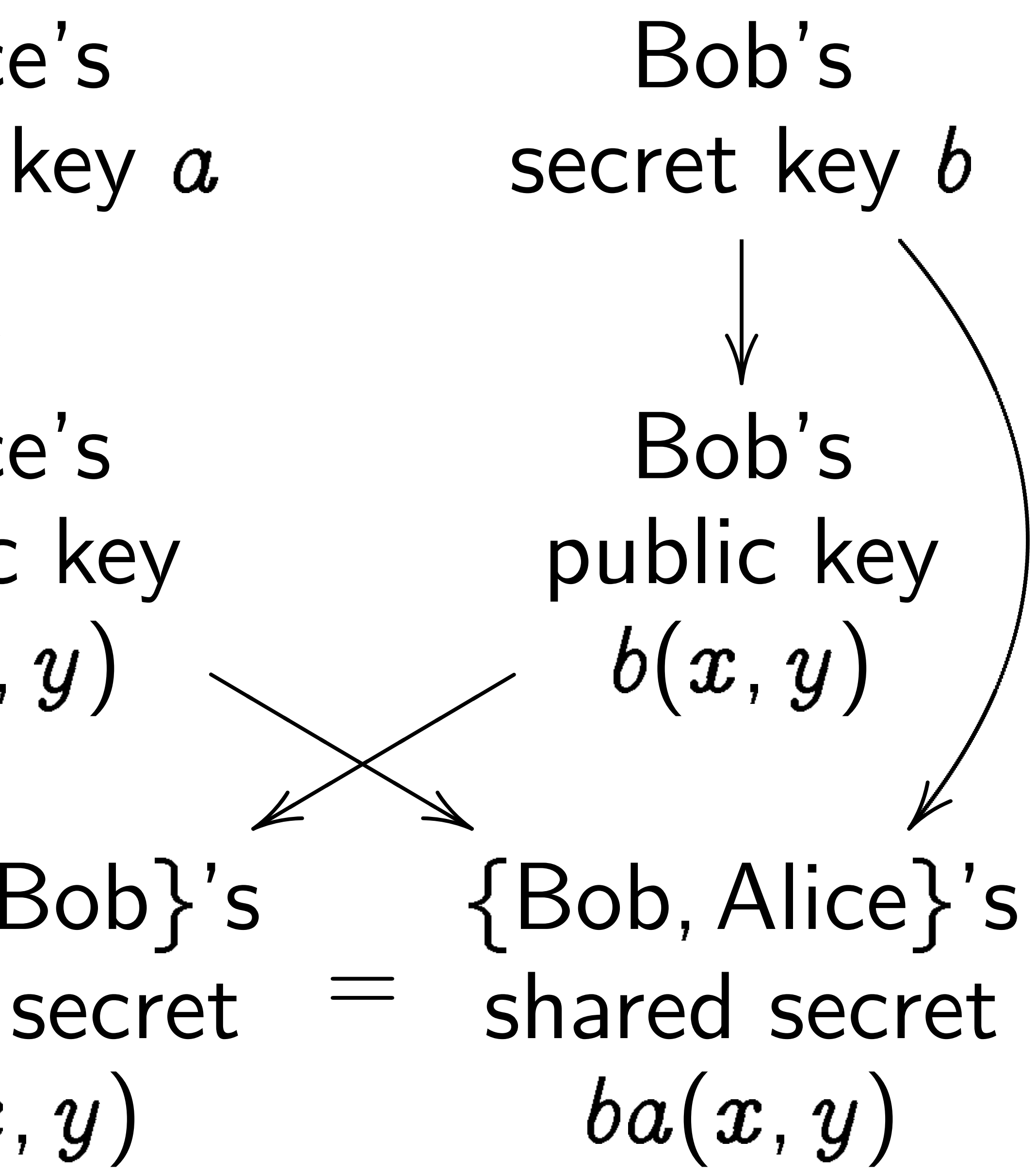

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Addition

$x^2 + y^2$
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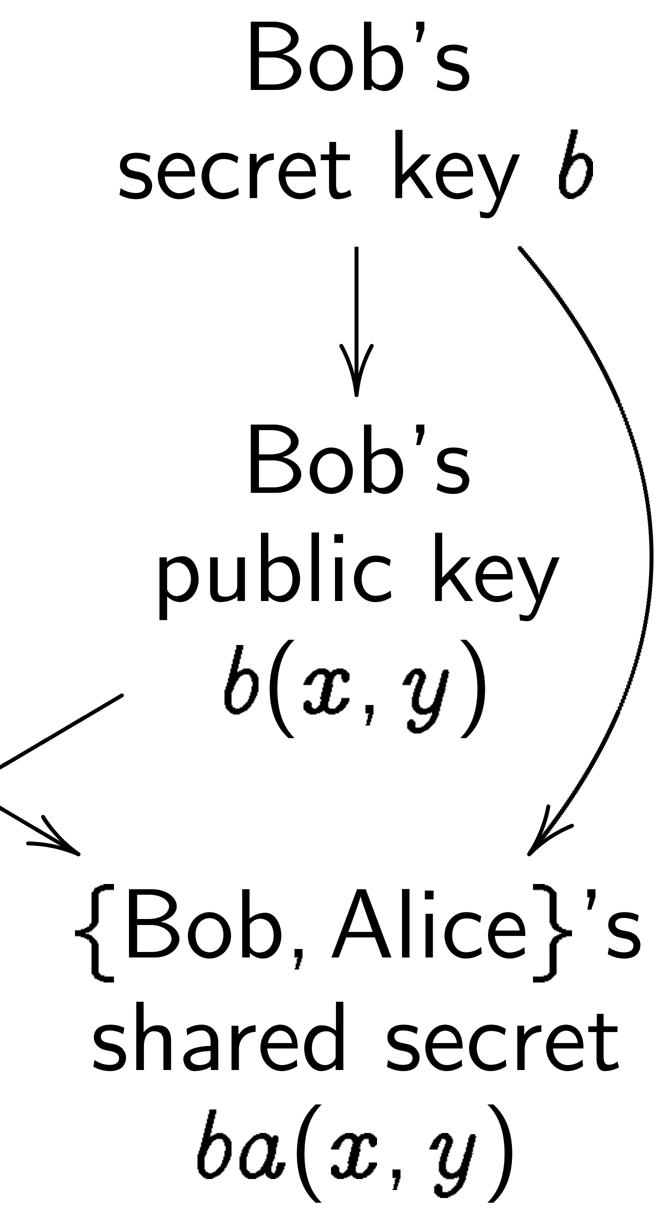

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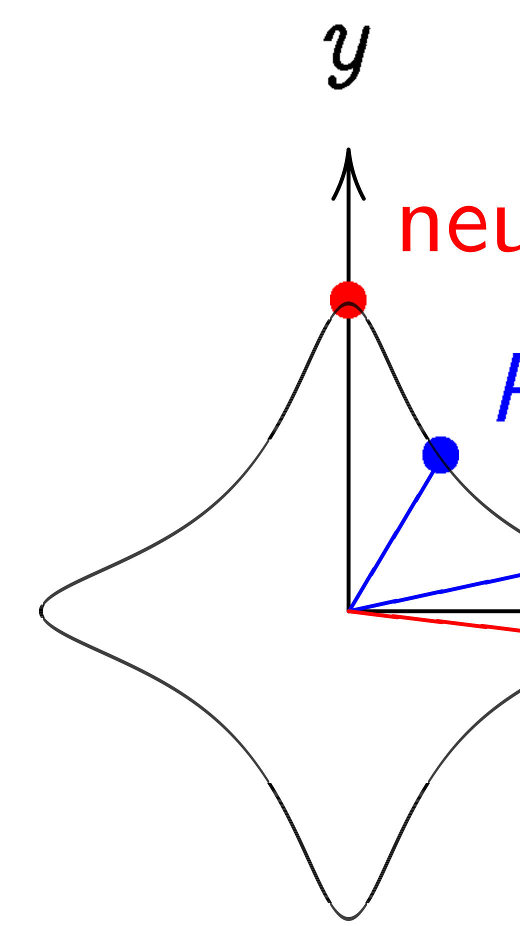

$x^2 + y^2 = 1 - 30x$

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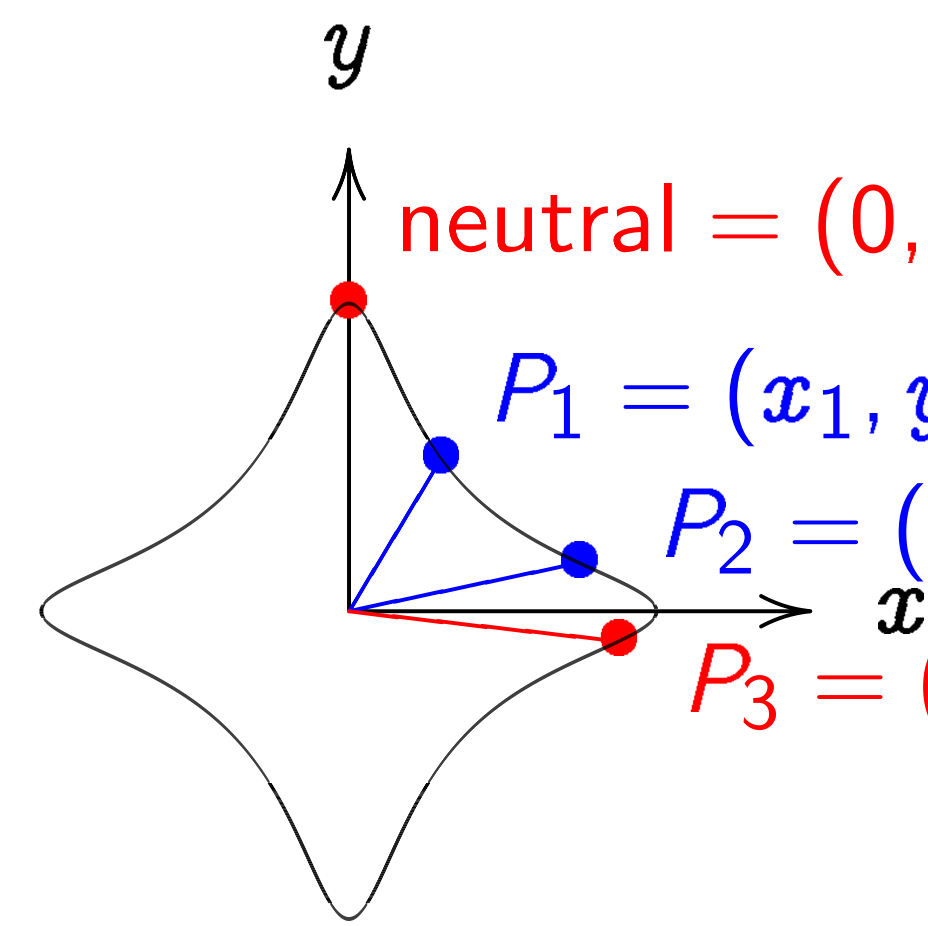
Addition on an elliptic curve



$x^2 + y^2 = 1 - 30x^2y^2$.

Sum of (x_1, y_1) and (x_2, y_2)

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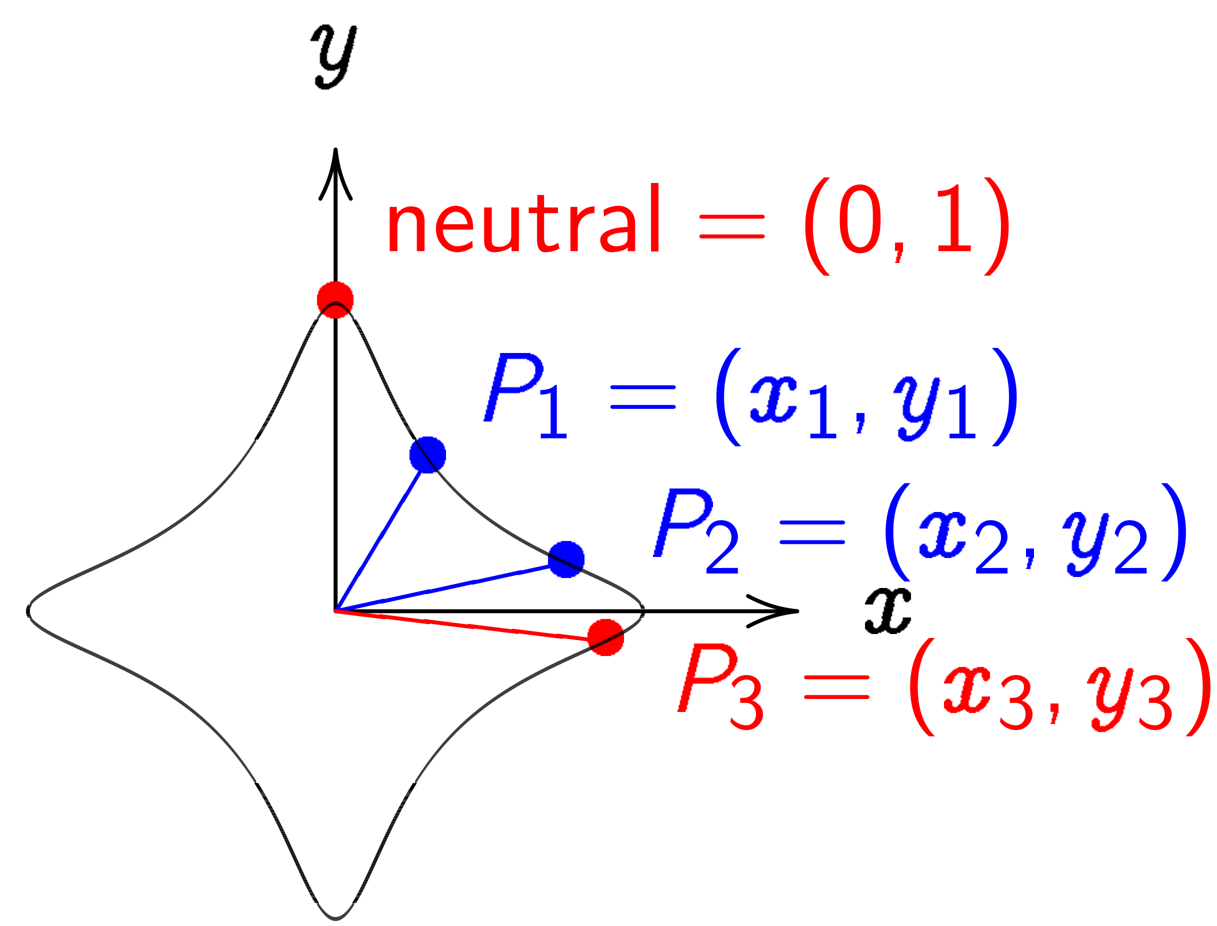

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Sum of (x_1, y_1) and (x_2, y_2) is $((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2)$, $(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

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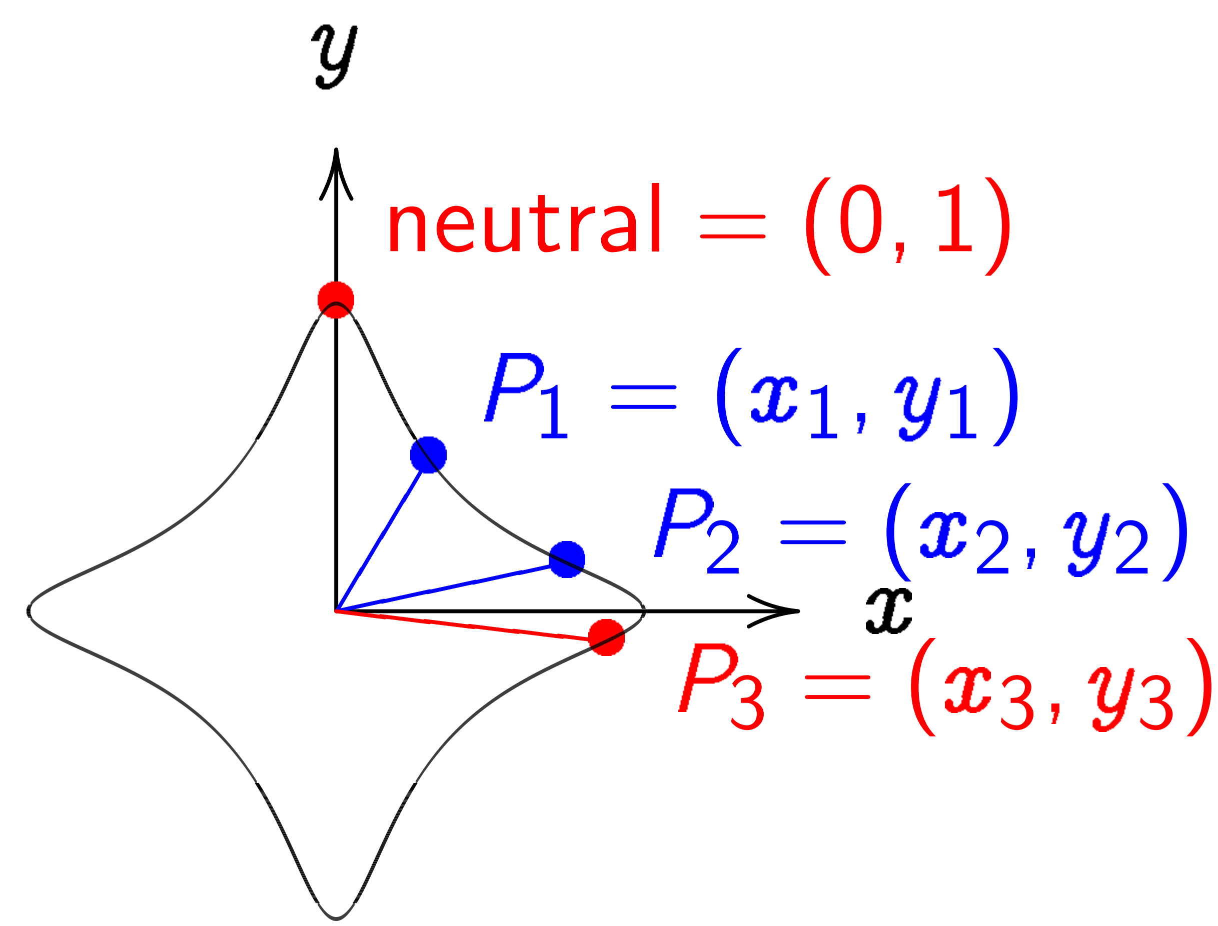

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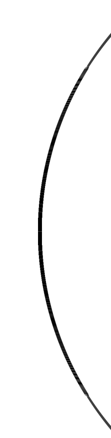

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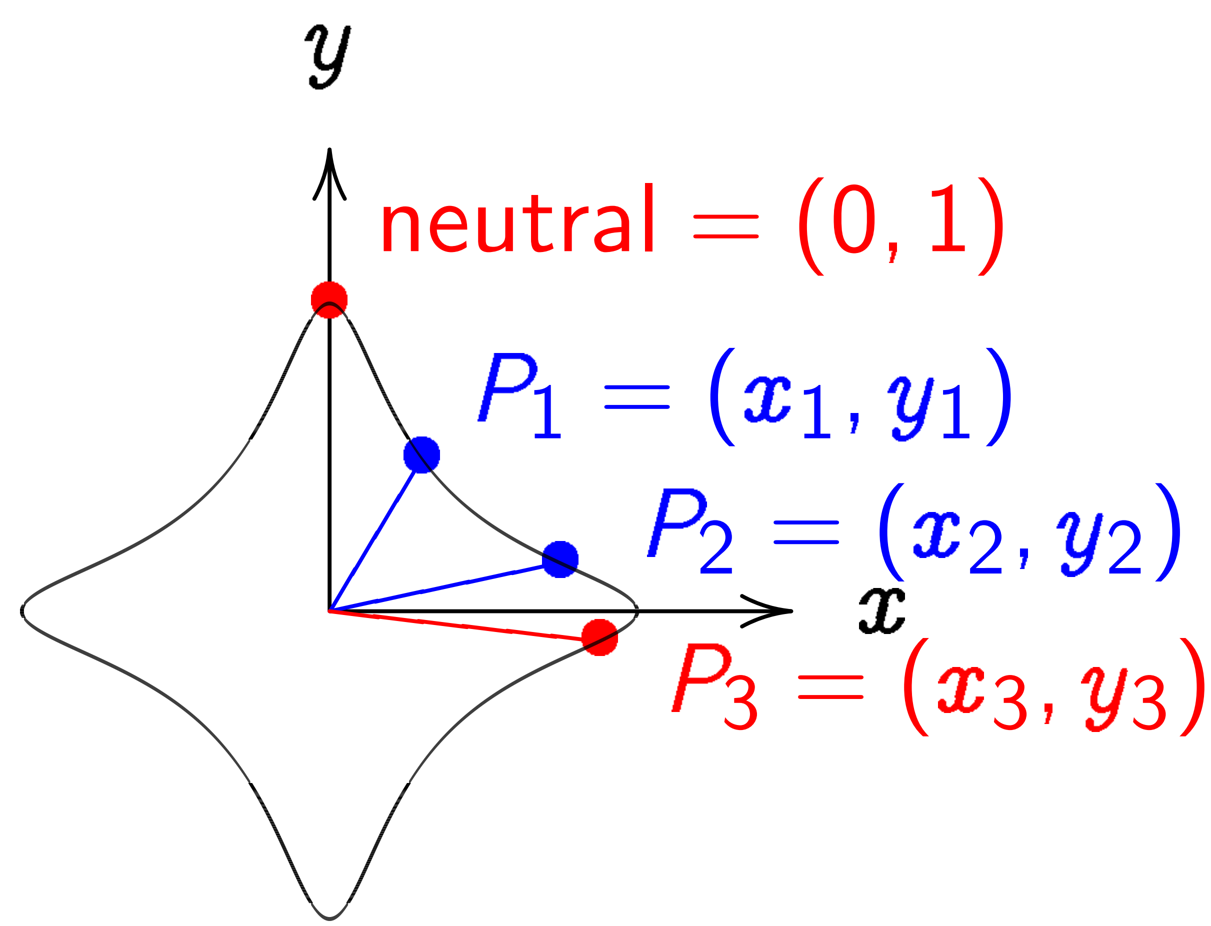
Addition on an elliptic curve



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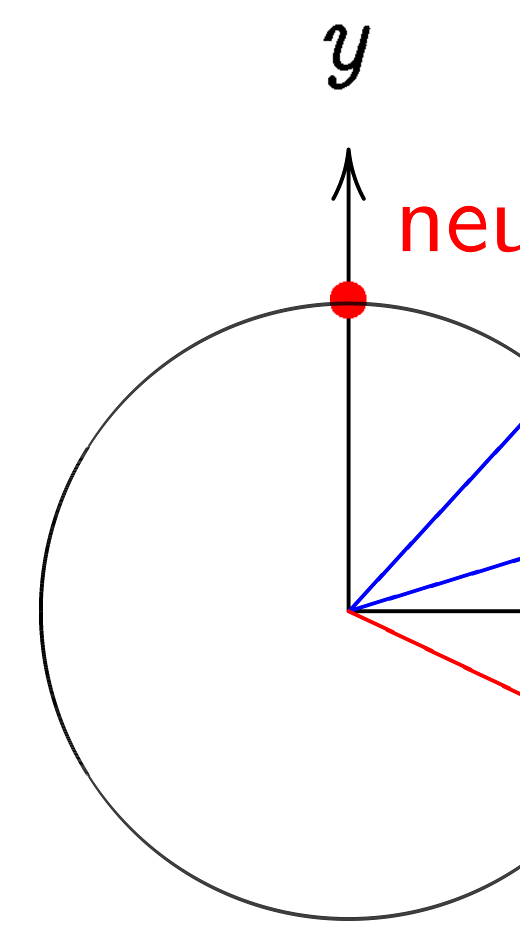

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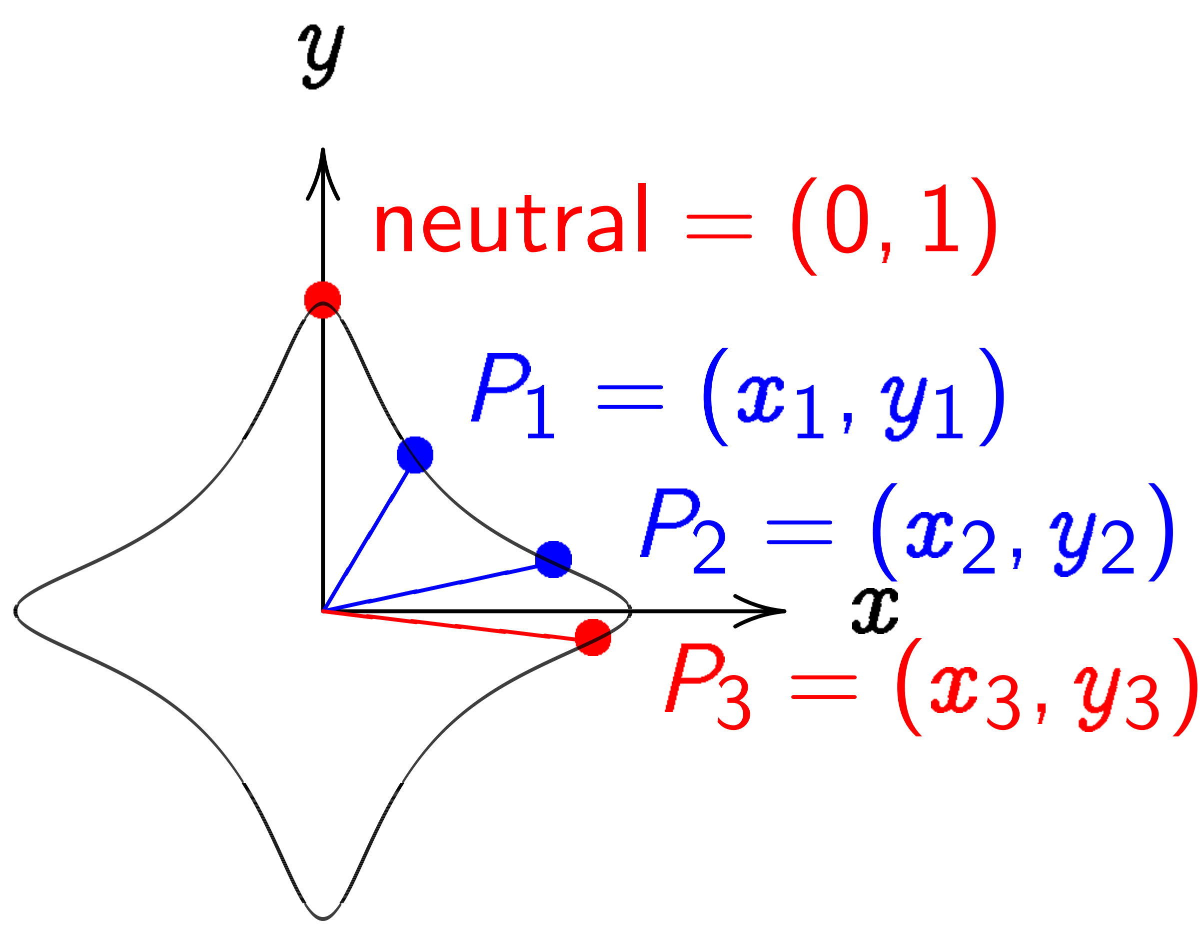

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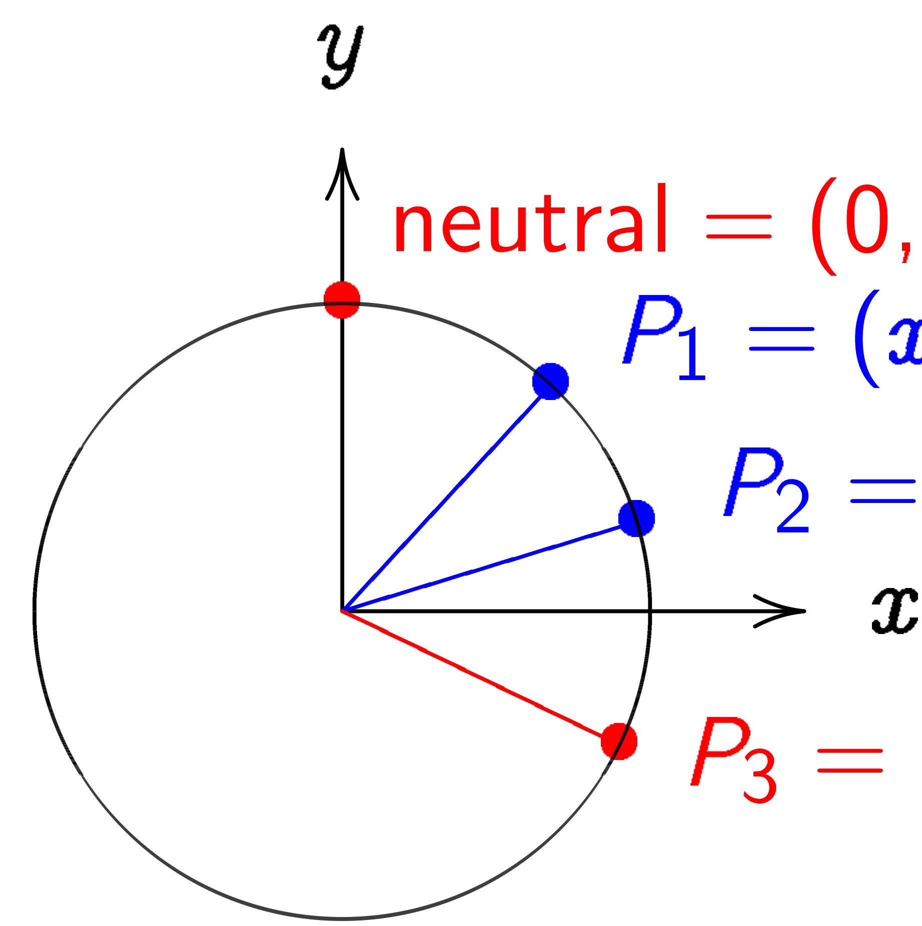

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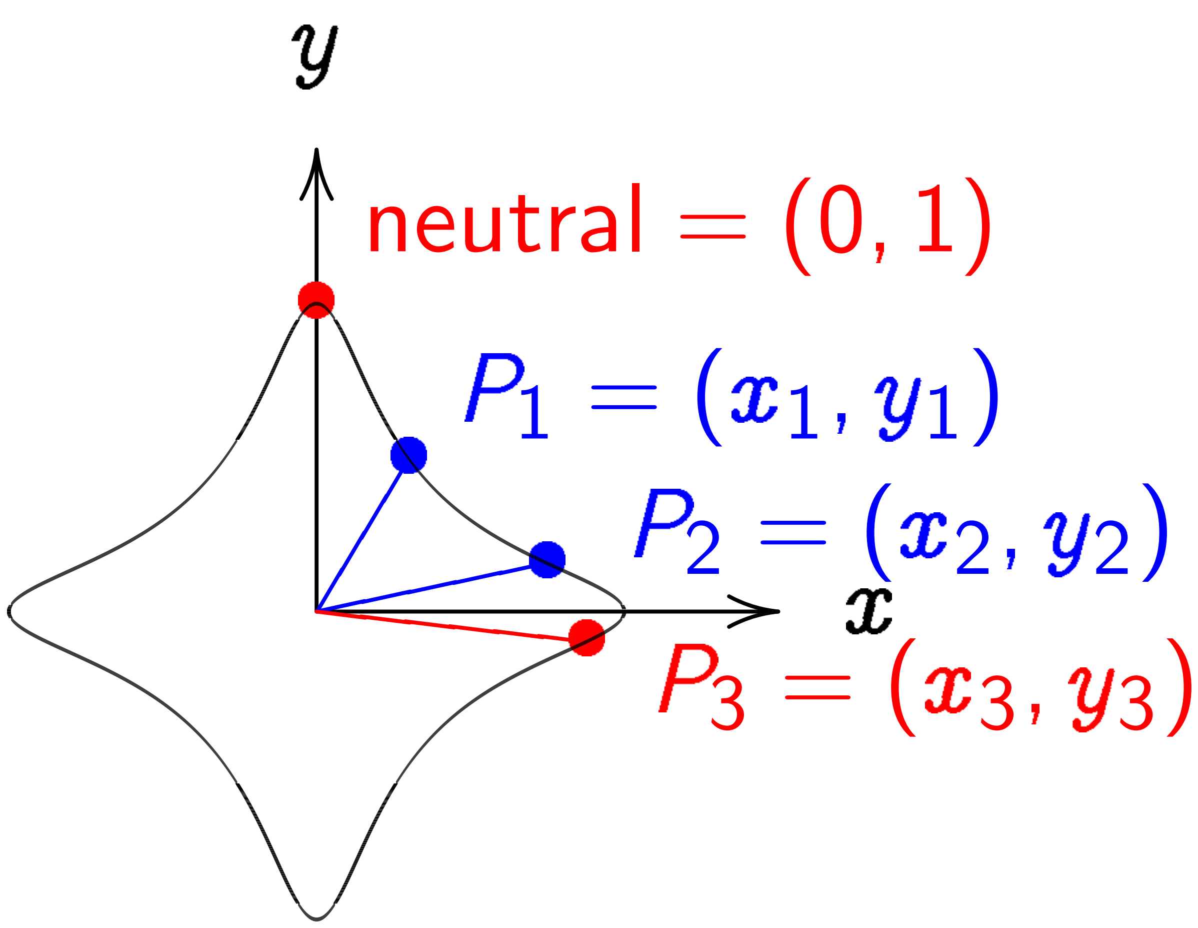

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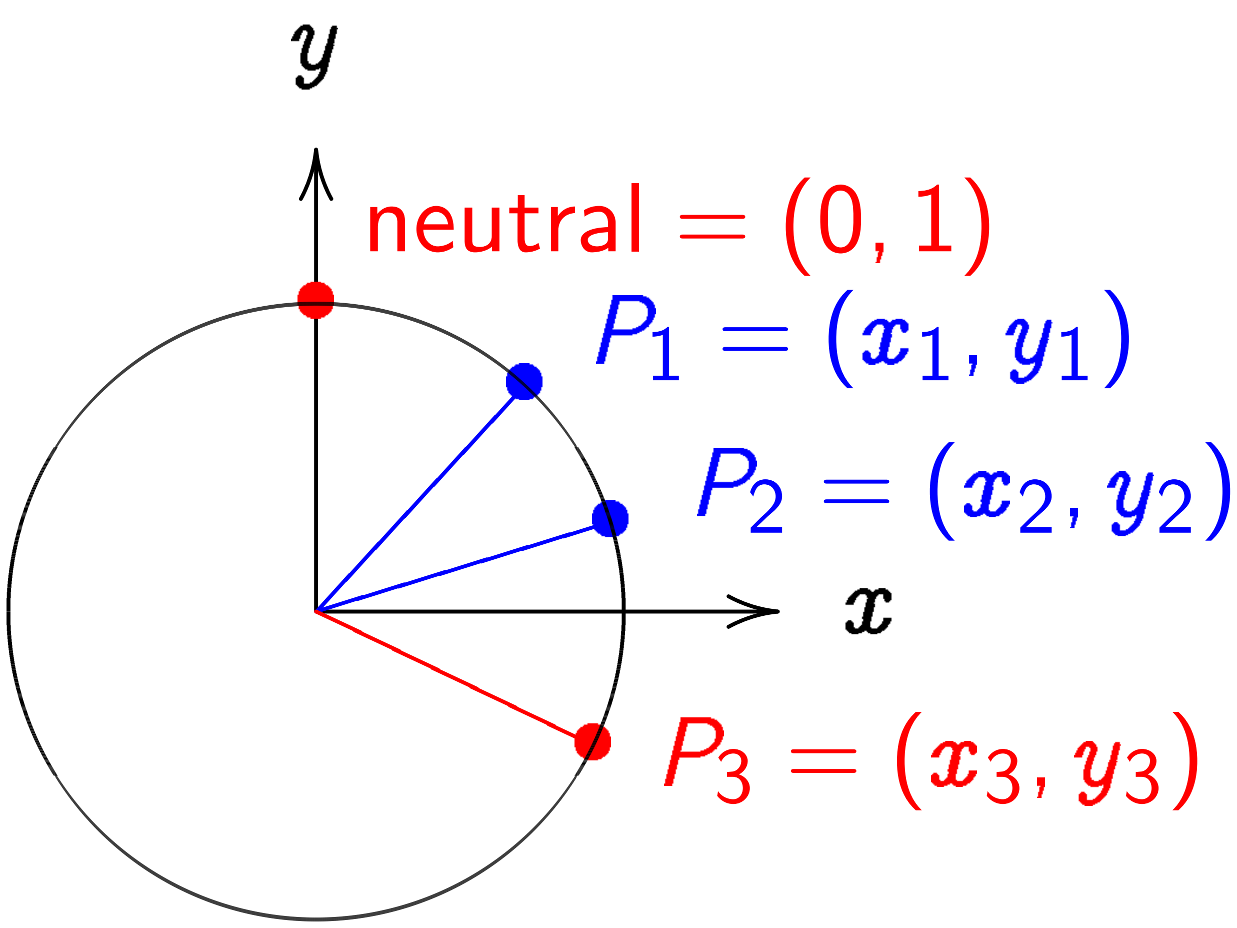

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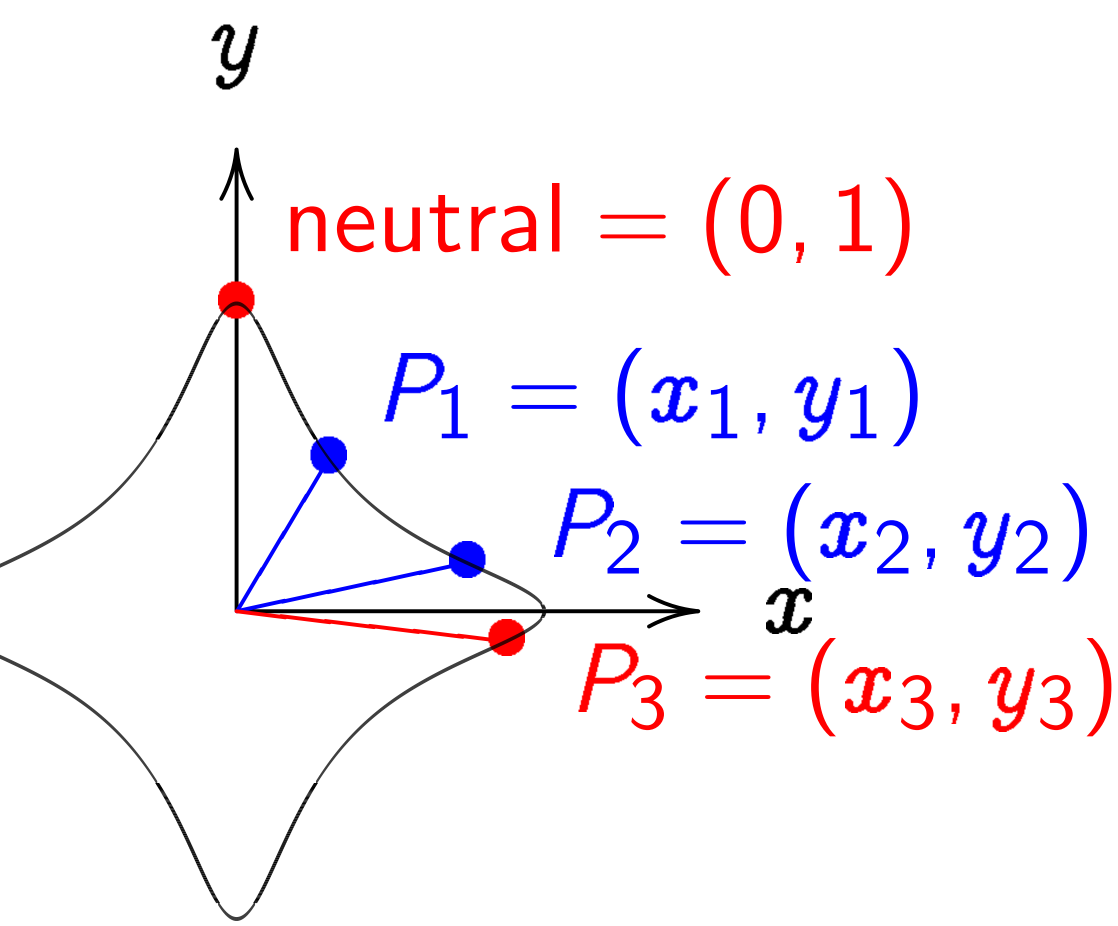

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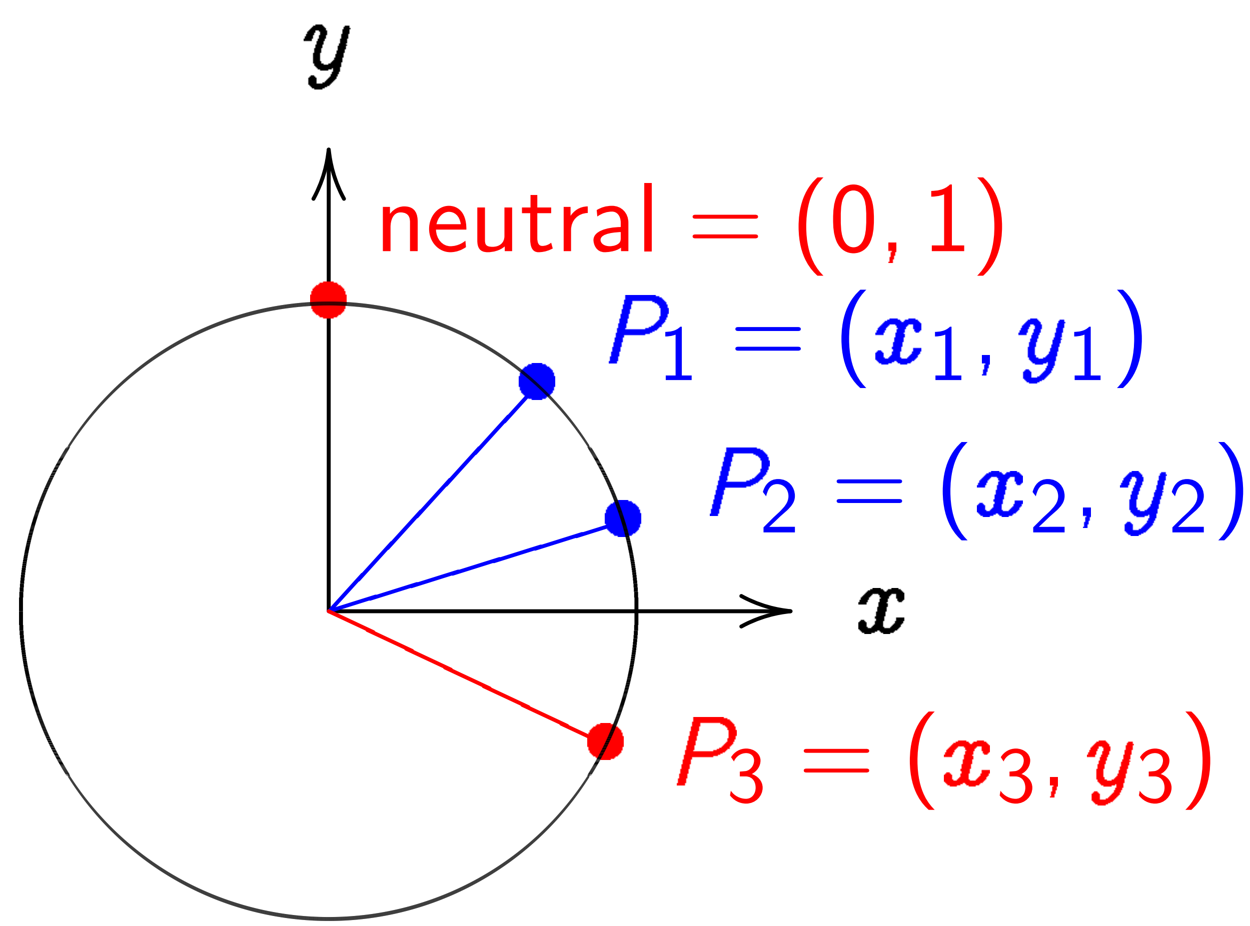

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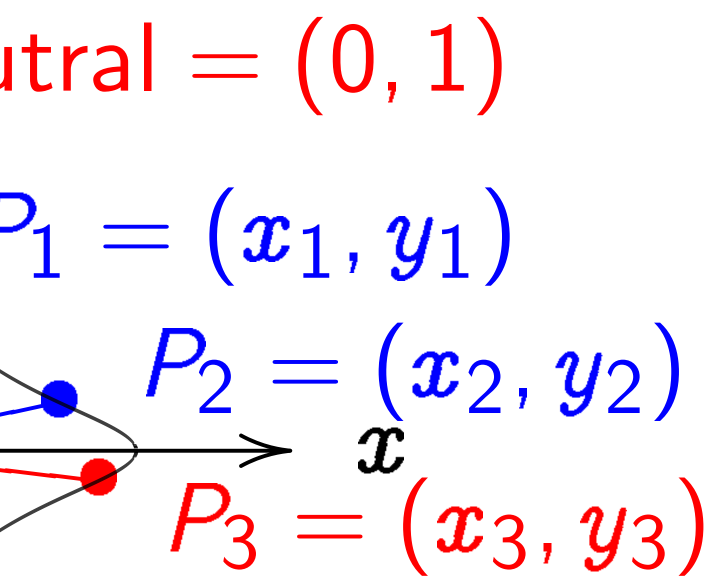

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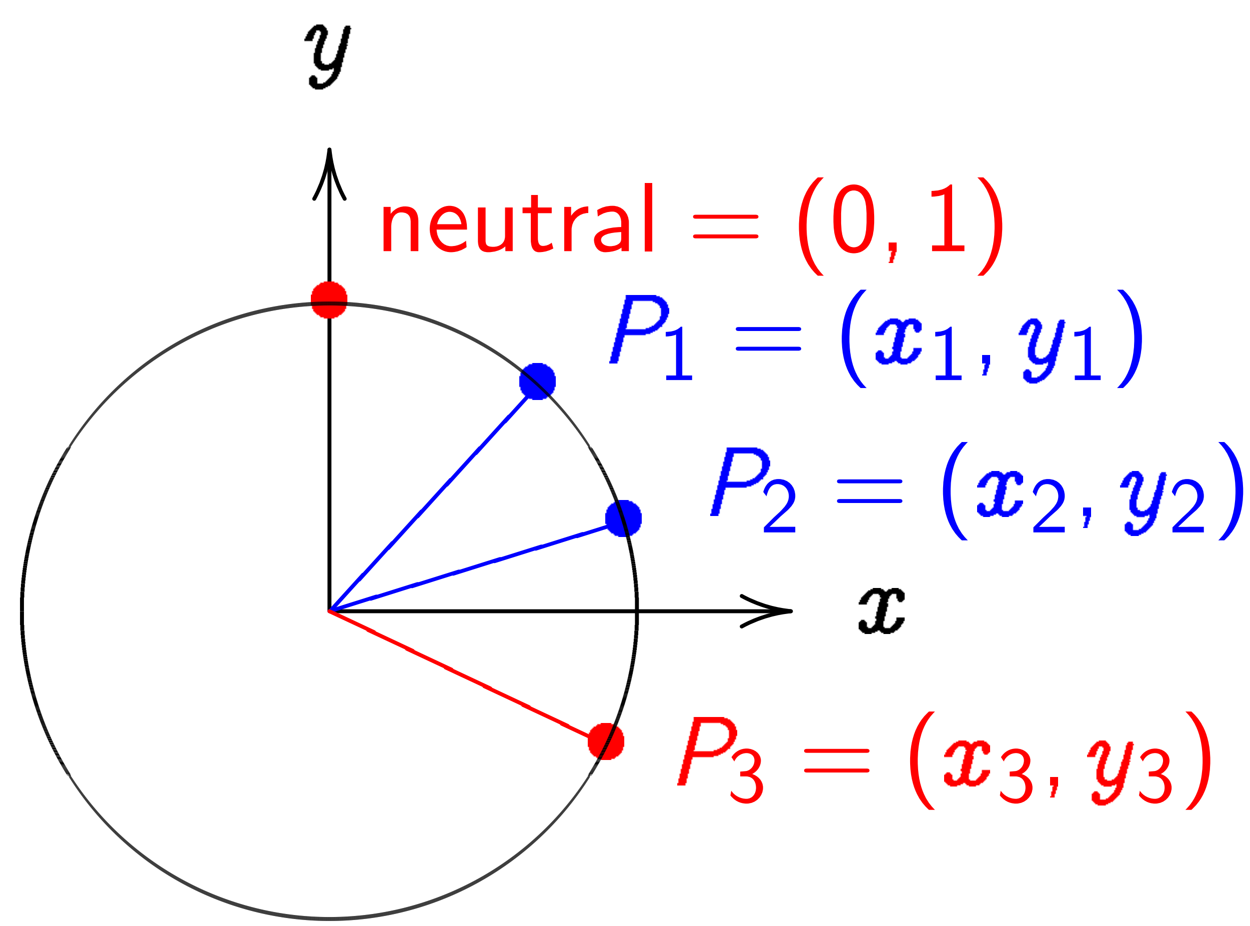

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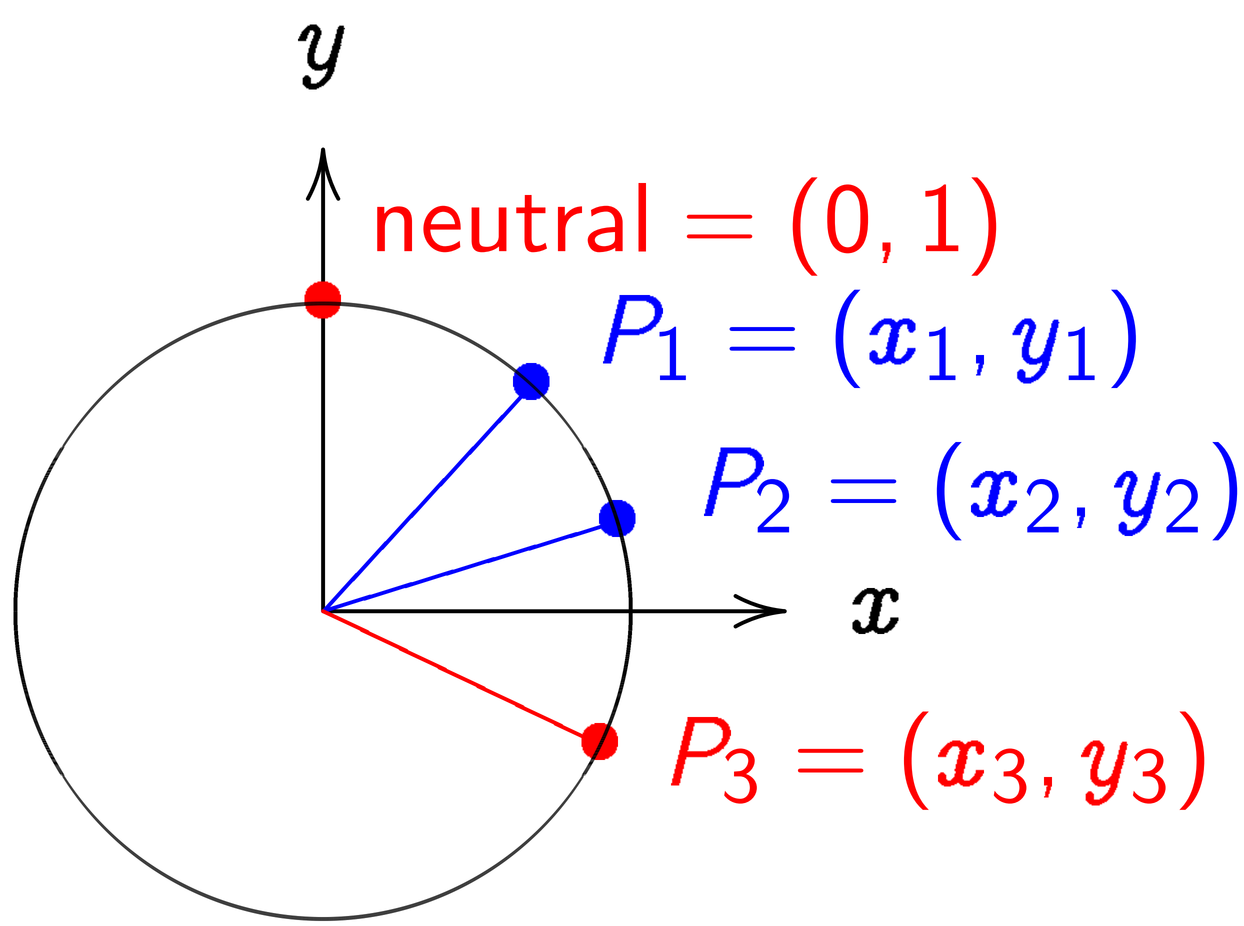

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Choose an odd prime p.

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$\quad x^2 + y^2 = 1 + dx^2y^2\}$

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"The Edwards addition law"

$(x_1, y_1) + (x_2, y_2) = (x_3, y_3$

where

$$x_3 = \frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2},$$

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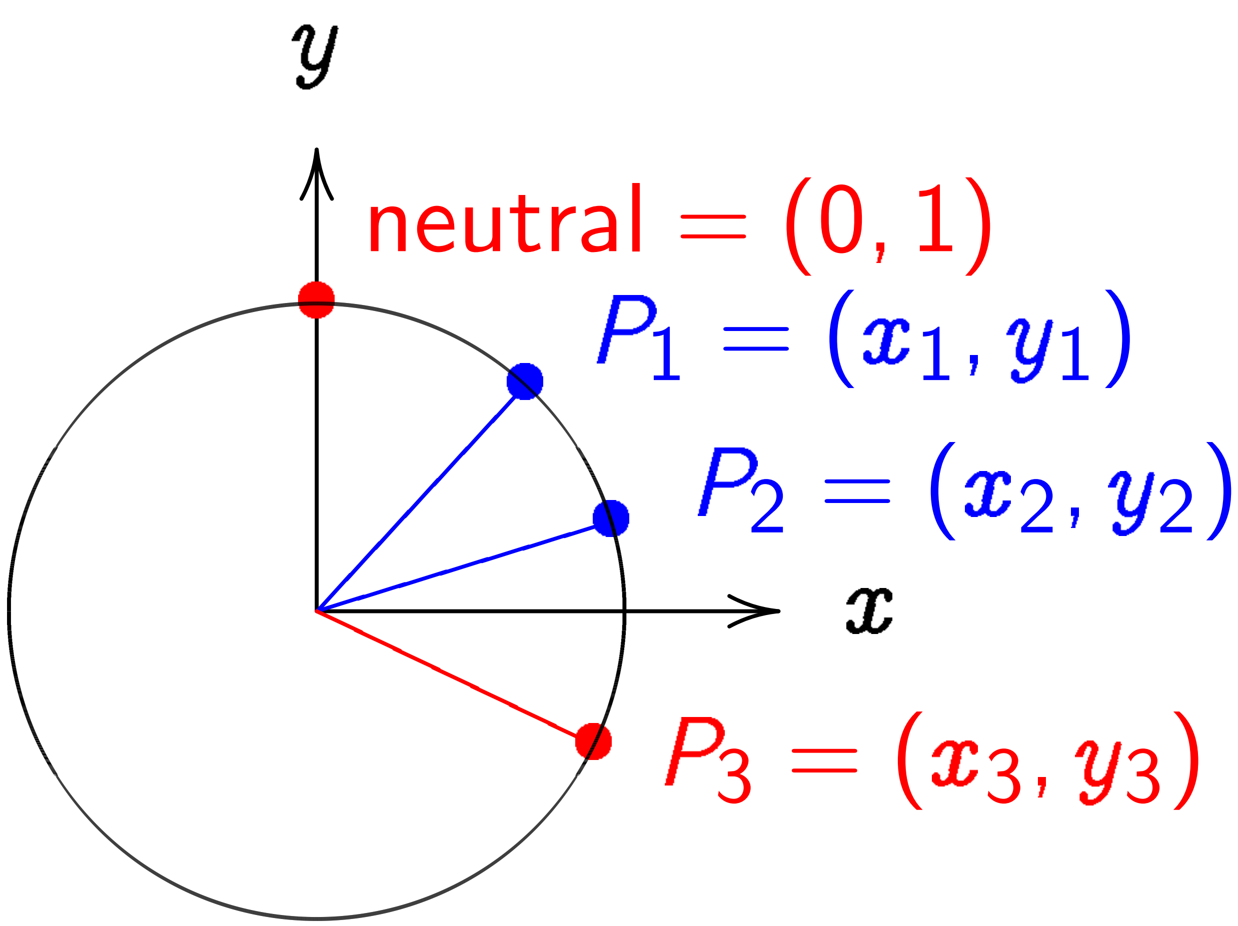

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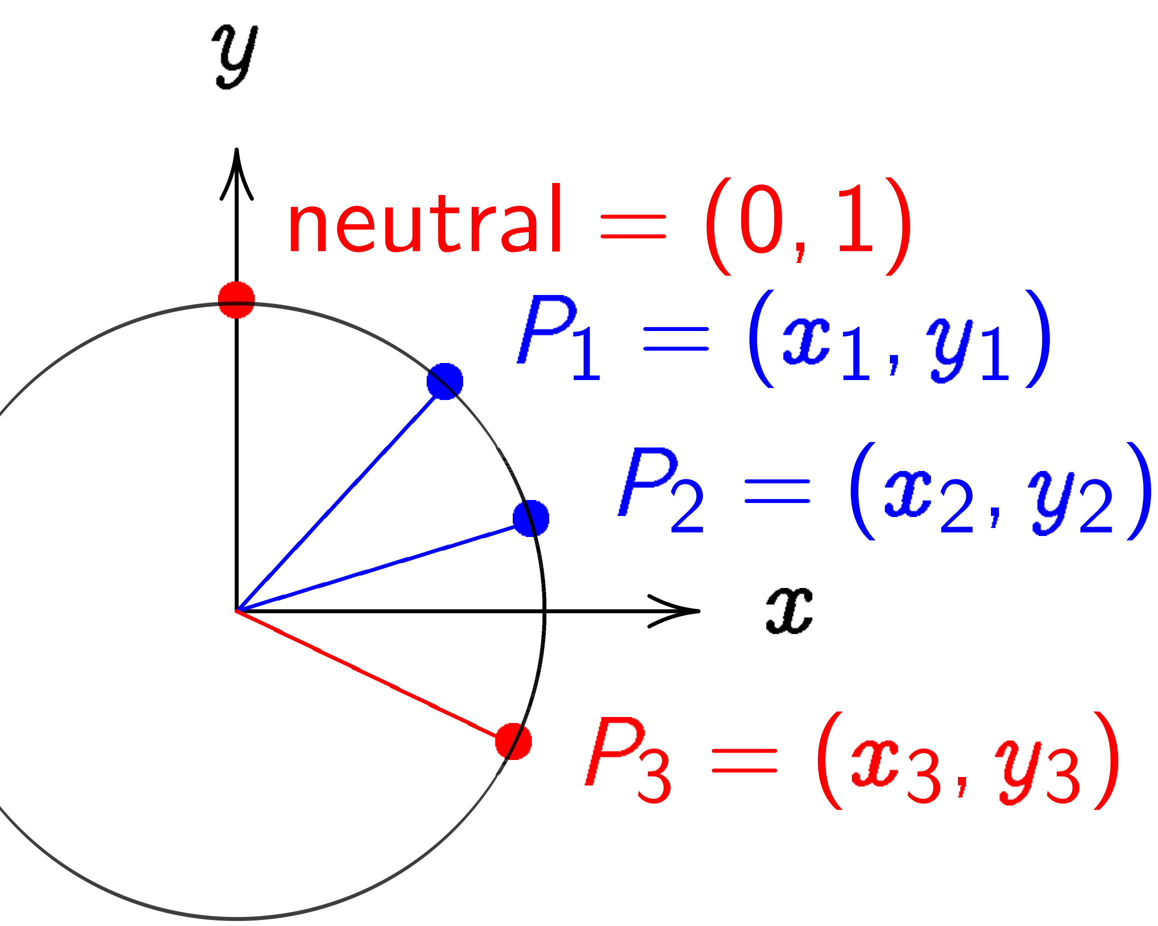

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<u>More elliptic curves</u>

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$$\{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p : \\ x^2 + y^2 = 1 + dx^2y^2\}$$

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"The Edwards addition law":

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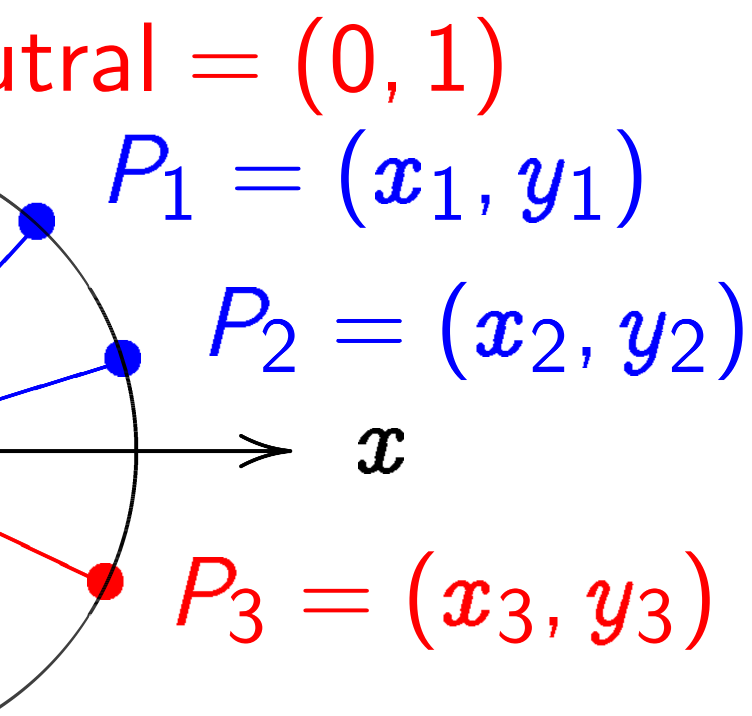

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Montgomery-curve ECDH u the "Montgomery ladder"— our recommended method fo Diffie–Hellman key exchange (e.g., for forward secrecy).

Montgomery ladder works only with u-coordinates of curve points P.

Montgomery ladder compute nP and $(n+1)P$ recursively $\lfloor n/2 \rfloor P$ and $(\lfloor n/2 \rfloor + 1)P$ using one bit of n with **no branches**.

Addition on Weierstrass curves $v^2 = u^3 + au + b$: for $u_1 \neq u_2$, $(u_1, v_1) + (u_2, v_2) = (u_3, v_3)$ with $u_3 = \lambda^2 - u_1 - u_2$, $v_3 = \lambda(u_1 - u_3) - v_1$, $\lambda = (v_2 - v_1)/(u_2 - u_1)$; for $v_1 \neq 0$, $(u_1, v_1) + (u_1, v_1) = (u_3, v_3)$ with $u_3 = \lambda^2 - u_1 - u_2$, $v_3 = \lambda(u_1 - u_3) - v_1$, $\lambda = (3u_1^2 + a)/2v_1$; $(u_1, v_1) + (u_1, -v_1) = \infty$; $(u_1, v_1) + \infty = (u_1, v_1)$; $\infty + (u_2, v_2) = (u_2, v_2)$; $\infty + \infty = \infty$.
Messy to implement and test.

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Example of new requirement: **twist security**.

If curve isn't twist-secure:
Twist attacks break ladder implementations that don't check whether input point is on curve.
Security-simplicity conflict.

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ecret data
gh timing;
ackers exploit this.

Better choices of curves
allow **simple** implementations
to be **secure** implementations.

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Curve
Anomalous
M-221
E-222
NIST P-224
Curve1174
Curve25519
BN(2,254)
brainpoolP2
ANSSI FRP25
NIST P-256
secp256k1
E-382
M-383
Curve38318
brainpoolP3
NIST P-384
Curve3617

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input point is on curve.
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Curve	Safe?	
Anomalous	False	Tr
M-221	True ✔	Tr
E-222	True ✔	Tr
NIST P-224	False	Tr
Curve1174	True ✔	Tr
Curve25519	True ✔	Tr
BN(2,254)	False	Tr
brainpoolP256t1	False	Tr
ANSSI FRP256v1	False	Tr
NIST P-256	False	Tr
secp256k1	False	Tr
E-382	True ✔	Tr
M-383	True ✔	Tr
Curve383187	True ✔	Tr
brainpoolP384t1	False	Tr
NIST P-384	False	Tr
Curve3617	True ✔	Tr

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If curve isn't twist-secure: Twist attacks break ladder implementations that don't check whether input point is on curve. Security-simplicity conflict.

		Parameter	
Curve	Safe?	field	equation
Anomalous	False	True ✔	True ✔
M-221	True ✔	True ✔	True ✔
E-222	True ✔	True ✔	True ✔
NIST P-224	False	True ✔	True ✔
Curve1174	True ✔	True ✔	True ✔
Curve25519	True ✔	True ✔	True ✔
BN(2,254)	False	True ✔	True ✔
brainpoolP256t1	False	True ✔	True ✔
ANSSI FRP256v1	False	True ✔	True ✔
NIST P-256	False	True ✔	True ✔
secp256k1	False	True ✔	True ✔
E-382	True ✔	True ✔	True ✔
M-383	True ✔	True ✔	True ✔
Curve383187	True ✔	True ✔	True ✔
brainpoolP384t1	False	True ✔	True ✔
NIST P-384	False	True ✔	True ✔
Curve3617	True ✔	True ✔	True ✔

Better choices of curves allow **simple** implementations to be **secure** implementations.

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Example of new requirement: **twist security**.

If curve isn't twist-secure: Twist attacks break ladder implementations that don't check whether input point is on curve. Security-simplicity conflict.

		Parameters:			
Curve	Safe?	field	equation	base	rho
Anomalous	False	True ✔	True ✔	True ✔	True ✔
M-221	True ✔	True ✔	True ✔	True ✔	True ✔
E-222	True ✔	True ✔	True ✔	True ✔	True ✔
NIST P-224	False	True ✔	True ✔	True ✔	True ✔
Curve1174	True ✔	True ✔	True ✔	True ✔	True ✔
Curve25519	True ✔	True ✔	True ✔	True ✔	True ✔
BN(2,254)	False	True ✔	True ✔	True ✔	True ✔
brainpoolP256t1	False	True ✔	True ✔	True ✔	True ✔
ANSSI FRP256v1	False	True ✔	True ✔	True ✔	True ✔
NIST P-256	False	True ✔	True ✔	True ✔	True ✔
secp256k1	False	True ✔	True ✔	True ✔	True ✔
E-382	True ✔	True ✔	True ✔	True ✔	True ✔
M-383	True ✔	True ✔	True ✔	True ✔	True ✔
Curve383187	True ✔	True ✔	True ✔	True ✔	True ✔
brainpoolP384t1	False	True ✔	True ✔	True ✔	True ✔
NIST P-384	False	True ✔	True ✔	True ✔	True ✔
Curve3617	True ✔	True ✔	True ✔	True ✔	True ✔

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		Parameters:			ECDLP securi		
Curve	**Safe?**	**field**	**equation**	**base**	**rho**	**transfer**	**dis**
Anomalous	**False**	True ✔	True ✔	True ✔	True ✔	**False**	**Fals**
M-221	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
E-222	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
NIST P-224	**False**	True ✔	True ✔	True ✔	True ✔	True ✔	True
Curve1174	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
Curve25519	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
BN(2,254)	**False**	True ✔	True ✔	True ✔	True ✔	**False**	**Fals**
brainpoolP256t1	**False**	True ✔	True ✔	True ✔	True ✔	True ✔	True
ANSSI FRP256v1	**False**	True ✔	True ✔	True ✔	True ✔	True ✔	True
NIST P-256	**False**	True ✔	True ✔	True ✔	True ✔	True ✔	True
secp256k1	**False**	True ✔	True ✔	True ✔	True ✔	True ✔	**Fals**
E-382	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
M-383	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
Curve383187	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
brainpoolP384t1	**False**	True ✔	True ✔	True ✔	True ✔	True ✔	True
NIST P-384	**False**	True ✔	True ✔	True ✔	True ✔	True ✔	True
Curve3617	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True

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		Parameters:			ECDLP security:				
Curve	**Safe?**	**field**	**equation**	**base**	**rho**	**transfer**	**disc**	**rigid**	**lad**
Anomalous	False	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	Fal
M-221	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
E-222	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
NIST P-224	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	Fal
Curve1174	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
Curve25519	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
BN(2,254)	False	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	Fal
brainpoolP256t1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Fal
ANSSI FRP256v1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	Fal
NIST P-256	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	Fal
secp256k1	False	True ✔	True ✔	True ✔	True ✔	True ✔	False	True ✔	Fal
E-382	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
M-383	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
Curve383187	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True
brainpoolP384t1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Fal
NIST P-384	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	Fal
Curve3617	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True

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		Parameters:			ECDLP security:				ECC secu		
Curve	**Safe?**	**field**	**equation**	**base**	**rho**	**transfer**	**disc**	**rigid**	**ladder**	**twist**	**com**
Anomalous	False	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	False	False	Fal
M-221	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Tru
E-222	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Tru
NIST P-224	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	False	Fal
Curve1174	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Tru
Curve25519	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Tru
BN(2,254)	False	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	False	False	Fal
brainpoolP256t1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	Fal
ANSSI FRP256v1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	False	Fal
NIST P-256	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	Fal
secp256k1	False	True ✔	True ✔	True ✔	True ✔	True ✔	False	True ✔	False	True ✔	Fal
E-382	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Tru
M-383	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Tru
Curve383187	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Tru
brainpoolP384t1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	True ✔	Fal
NIST P-384	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	Fal
Curve3617	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	Tru

		Parameters:			ECDLP security:				ECC security:			
Curve	**Safe?**	field	equation	base	rho	transfer	disc	rigid	ladder	twist	complete	ind
Anomalous	False	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	False	False	False	False
M-221	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔
E-222	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔
NIST P-224	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	False	False	False
Curve1174	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔
Curve25519	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔
BN(2,254)	False	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	False	False	False	False
brainpoolP256t1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	False	False
ANSSI FRP256v1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	False	False	False
NIST P-256	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	False	False
secp256k1	False	True ✔	True ✔	True ✔	True ✔	True ✔	False	True ✔	False	True ✔	False	False
E-382	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔
M-383	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔
Curve383187	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔
brainpoolP384t1	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	True ✔	False	False
NIST P-384	False	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	False	False	True ✔	False	False
Curve3617	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔	True ✔