High-speed cryptography, part 2: more elliptic-curve formulas; field arithmetic

Daniel J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Speed-oriented Jacobian standards

2000 IEEE "Std 1363" uses Weierstrass curves in Jacobian coordinates to "provide the fastest arithmetic on elliptic curves." Also specifies a method of choosing curves $y^2 = x^3 - 3x + b$.

2000 NIST "FIPS 186–2" standardizes five such curves.

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e.g. $2P$,
e.g. $3P$,
e.g. $6P$,

$2\mathbf{M} + 2\mathbf{S}$
$4\mathbf{M} + 2\mathbf{S}$
Save $1\mathbf{M}$

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e.g. $2P, P, P \mapsto 3$
e.g. $3P, 2P, P \mapsto$
e.g. $6P, 5P, P \mapsto$

$2\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ fo
$4\mathbf{M} + 2\mathbf{S}$ for DAD
Save $1\mathbf{M}$ if $Z_1 =$

Easily compute $n($
$\approx \lg n$ DBL, $\approx \lg$
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$Q, R, Q - R \mapsto Q + R$.

e.g. $2P, P, P \mapsto 3P$.
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e.g. $6P, 5P, P \mapsto 11P$.

$2\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for DBL.
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Save $1\mathbf{M}$ if $Z_1 = 1$.

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Relatively slow for $mP + nQ$ etc.

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Doubling

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Use y^2 =
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$2\mathbf{M} + 5\mathbf{S}$
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g $x = X/Z$.

$= B \cdot C$,

$a + 2)/4) \Rightarrow$

$Z_4)$.

$) = (X_1{:}Z_1)$,

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<u>Doubling-oriented</u>

2006 Doche–Icart–

Use $y^2 = x^3 + ax$
Choose small a.

Use $(X : Y : Z : Z$
to represent $(X/Z$

$3\mathbf{M} + 4\mathbf{S} + 2\mathbf{D}$ fo
How? Factor DBL
where φ is a 2-iso

2007 Bernstein–La
$2\mathbf{M} + 5\mathbf{S} + 2\mathbf{D}$ fo
on the same curve

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$\Rightarrow$

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<u>Doubling-oriented curves</u>

2006 Doche–Icart–Kohel:

Use $y^2 = x^3 + ax^2 + 16ax$.
Choose small a.

Use $(X : Y : Z : Z^2)$
to represent $(X/Z, Y/Z^2)$.

$3\mathbf{M} + 4\mathbf{S} + 2\mathbf{D}$ for DBL.
How? Factor DBL as $\hat{\varphi}(\varphi)$
where φ is a 2-isogeny.

2007 Bernstein–Lange:
$2\mathbf{M} + 5\mathbf{S} + 2\mathbf{D}$ for DBL
on the same curves.

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e.g. $6P, 5P, P \mapsto 11P$.

$2\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for DBL.
$4\mathbf{M} + 2\mathbf{S}$ for DADD.
Save $1\mathbf{M}$ if $Z_1 = 1$.

Easily compute $n(X_1 : Z_1)$ using
$\approx \lg n$ DBL, $\approx \lg n$ DADD.
Almost as fast as Edwards nP.
Relatively slow for $mP + nQ$ etc.

<u>Doubling-oriented curves</u>

2006 Doche–Icart–Kohel:

Use $y^2 = x^3 + ax^2 + 16ax$.
Choose small a.

Use $(X : Y : Z : Z^2)$
to represent $(X/Z, Y/Z^2)$.

$3\mathbf{M} + 4\mathbf{S} + 2\mathbf{D}$ for DBL.
How? Factor DBL as $\hat{\varphi}(\varphi)$
where φ is a 2-isogeny.

2007 Bernstein–Lange:
$2\mathbf{M} + 5\mathbf{S} + 2\mathbf{D}$ for DBL
on the same curves.

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"differential addition":

$- R \mapsto Q + R$.

$P, P \mapsto 3P$.

$2P, P \mapsto 5P$.

$5P, P \mapsto 11P$.

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$\mathbf{M}$ if $Z_1 = 1$.

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"Unified addition formulas,"
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But need to permute inputs
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Easily avoid permutation!

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$Z_2 \cdot X_1 Y_2 - Y_1 X_2 \cdot Z_1 X_2$,

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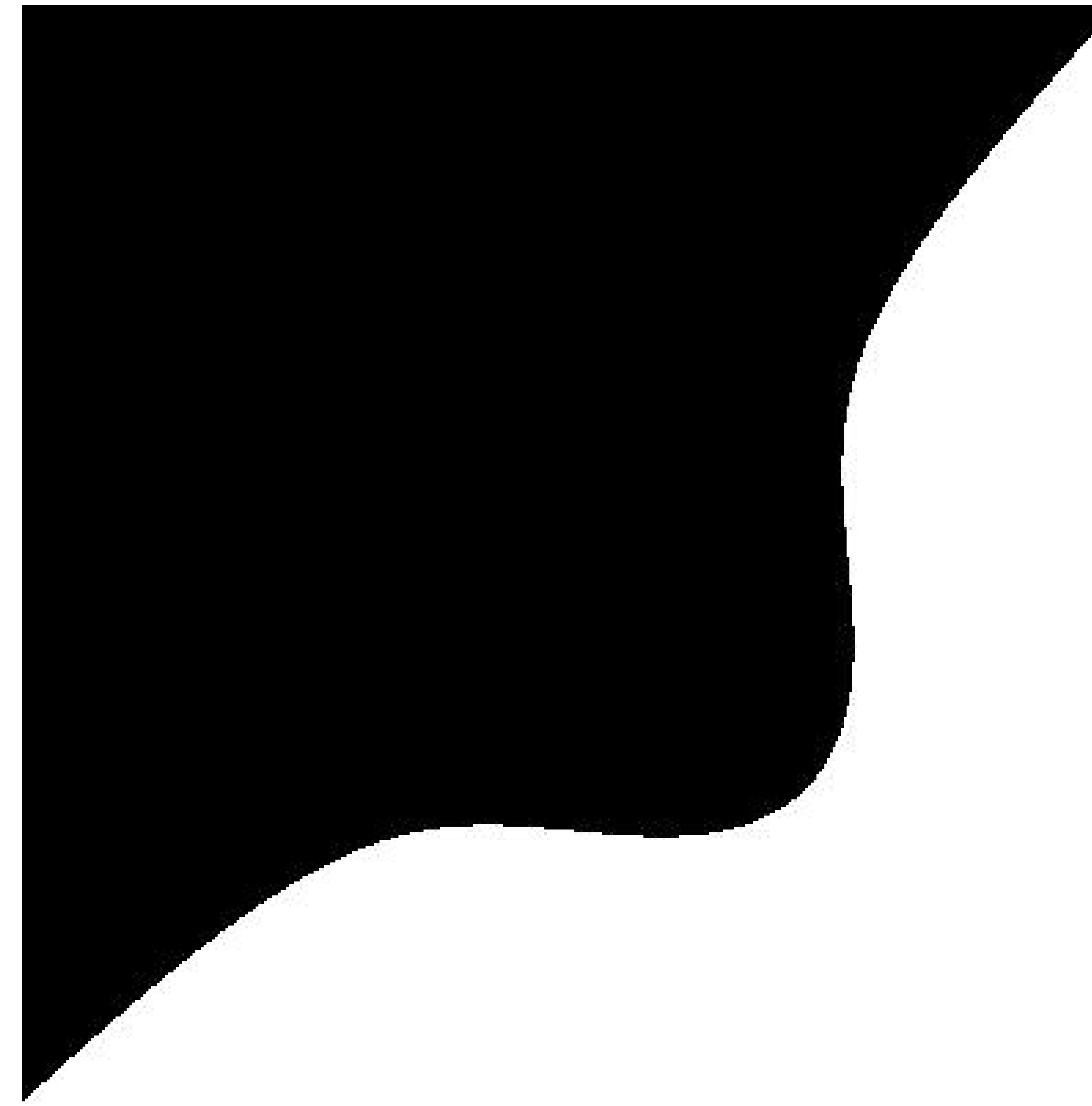

$x^3 - y^3 + 1 = 0.3xy$

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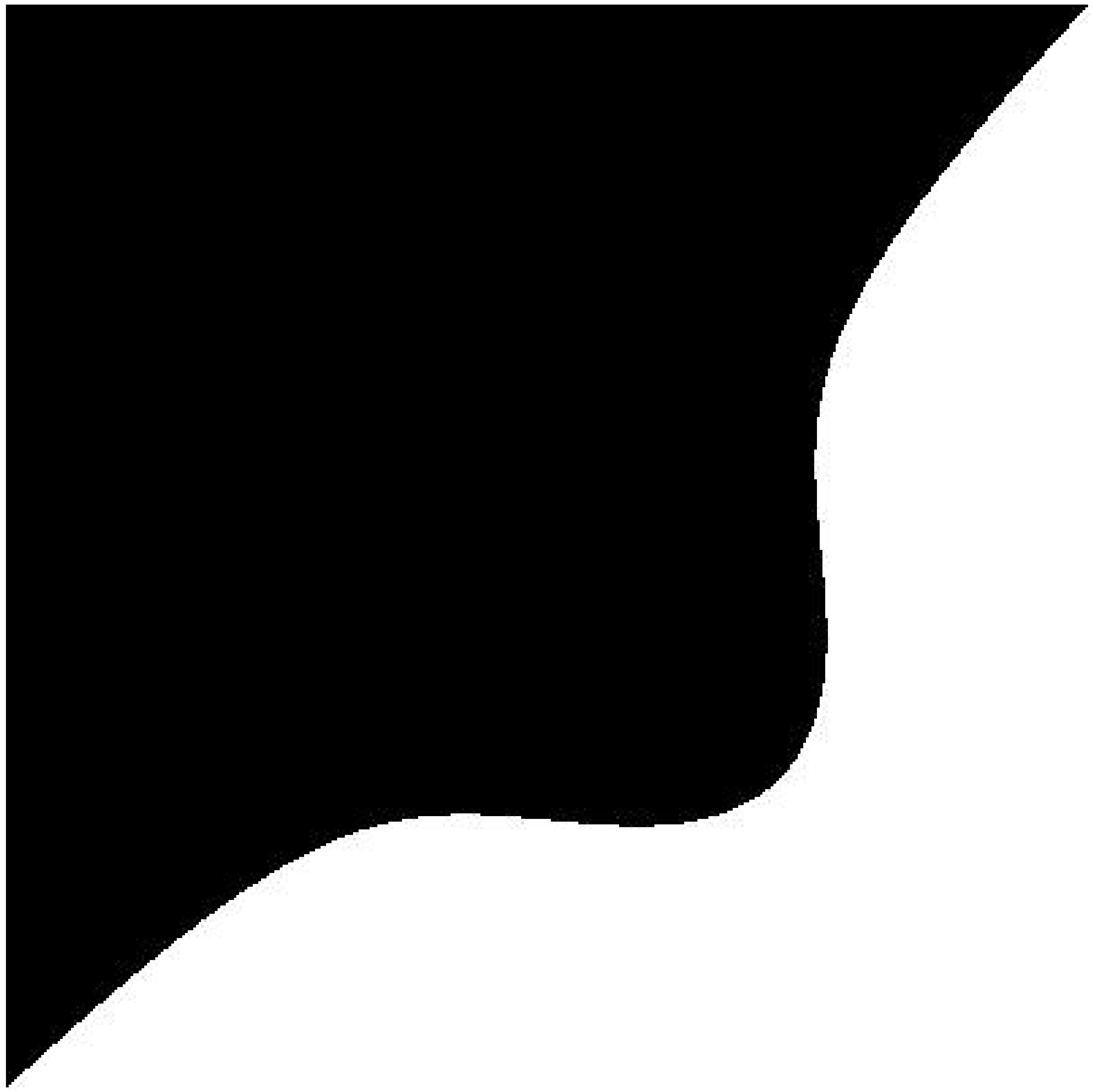

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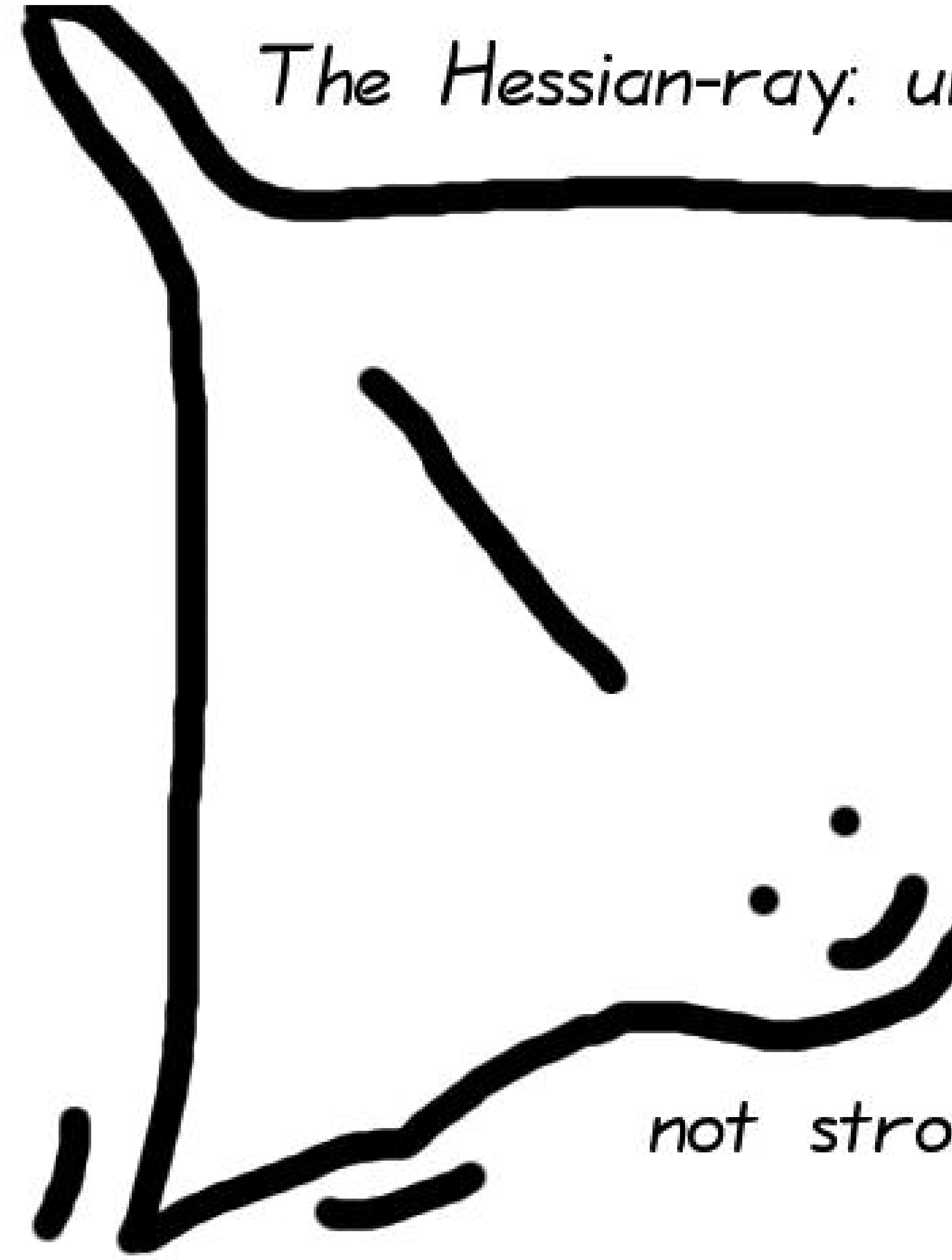

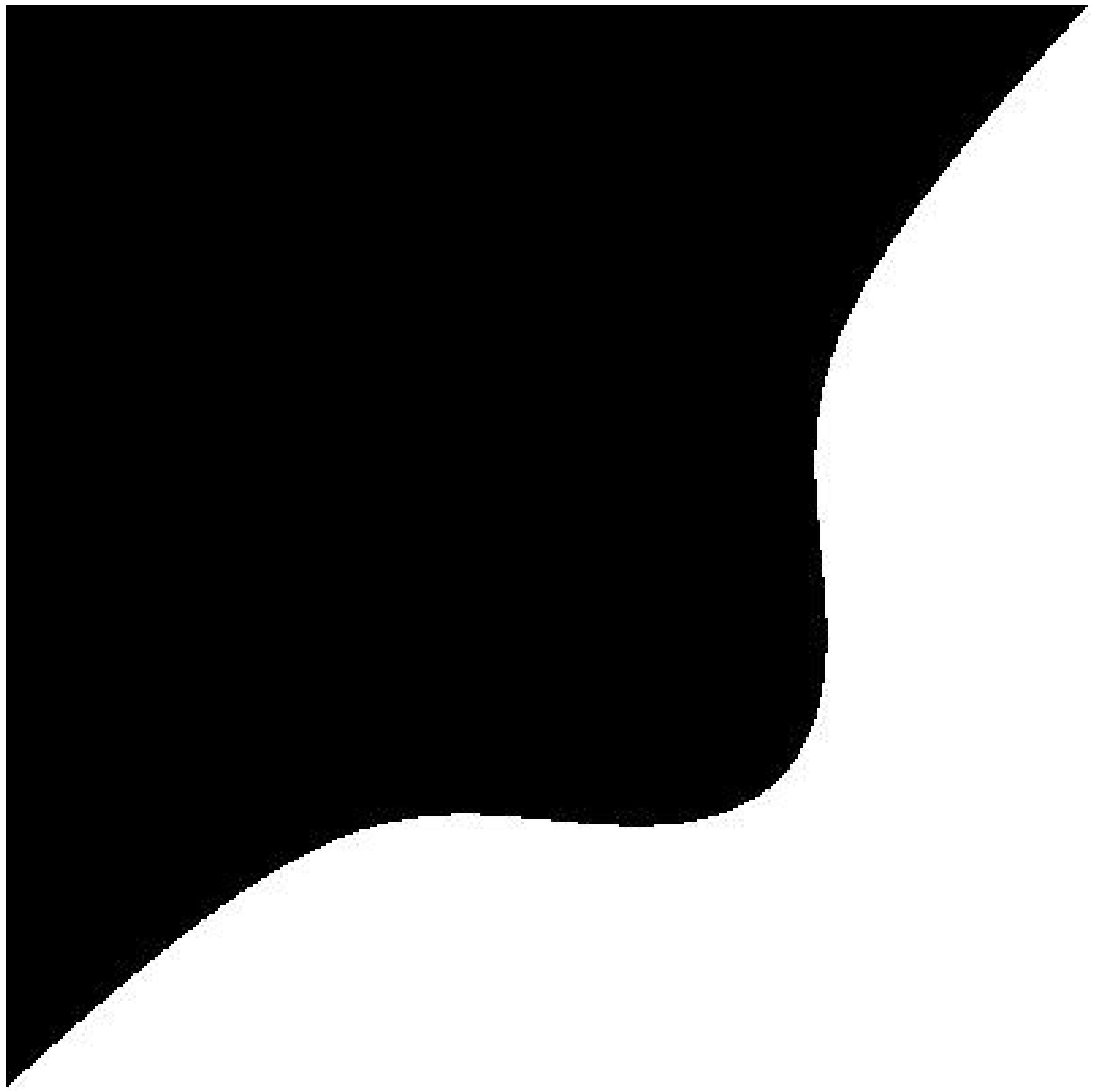

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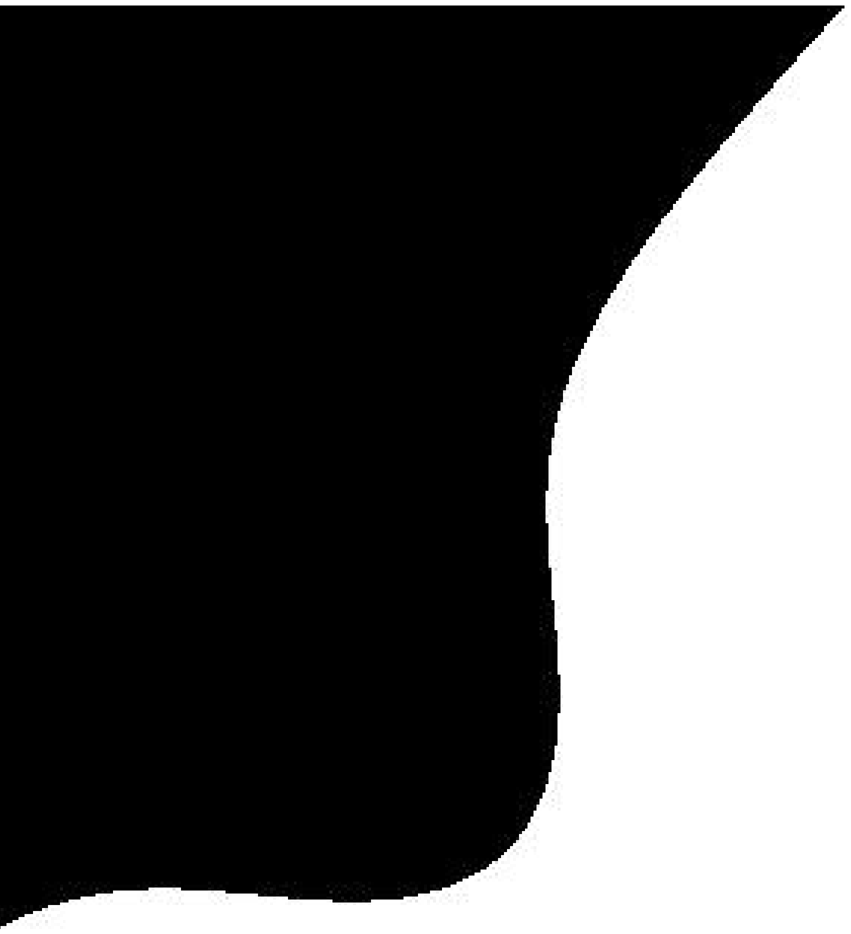

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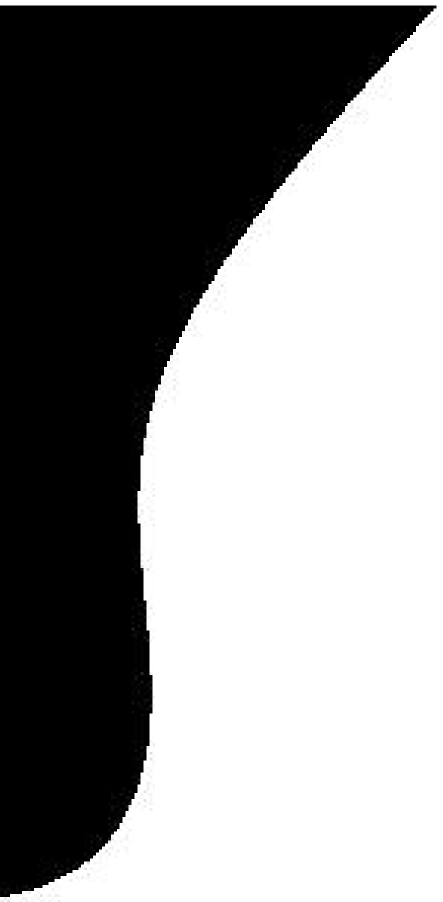

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$13\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
$4\mathbf{M} + 3\mathbf{S}$ for DBL.

2007 Bernstein–Lange:
$3\mathbf{M} + 4\mathbf{S}$ for DBL.

2008 Hisil–Wong–Carter–Da
$13\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.
Also $(S : C : D : Z : SC : D$
$11\mathbf{M} + 1\mathbf{S} + 2\mathbf{D}$ for ADD.
$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

Jacobi intersections

1986 Chudnovsky–Chudnovsky:

$(S : C : D : Z)$ represent
$(S/Z, C/Z, D/Z)$ on
$s^2 + c^2 = 1$, $as^2 + d^2 = 1$.

$14\mathbf{M} + 2\mathbf{S} + 1\mathbf{D}$ for ADD.
"Tremendous advantage"
of being strongly unified.

$5\mathbf{M} + 3\mathbf{S}$ for DBL.
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$2\mathbf{M} + 5\mathbf{S} + 1\mathbf{D}$ for DBL.

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$/Z, D/Z)$ on

$= 1$, $as^2 + d^2 = 1$.

$2\mathbf{S} + 1\mathbf{D}$ for ADD.

ndous advantage”

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Jacobi q

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Jacobi quartics

$(X{:}Y{:}Z)$ represent

on $y^2 = x^4 + 2ax$

1986 Chudnovsky–

$3\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ fo

Slow ADD.

2002 Billet–Joye:

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2007 Bernstein–La

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1986 Chudnovsky–Chudnovs
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Slow ADD.

2002 Billet–Joye:
New choice of neutral eleme
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strongly unified.

2007 Bernstein–Lange:
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Slow ADD.

2002 Billet–Joye:

New choice of neutral element.

$10\mathbf{M} + 3\mathbf{S} + 1\mathbf{D}$ for ADD, strongly unified.

2007 Bernstein–Lange:

$1\mathbf{M} + 9\mathbf{S} + 1\mathbf{D}$ for DBL.

ardet–Smart:

$2\mathbf{S} + 1\mathbf{D}$ for ADD.

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rnstein–Lange:

$\mathbf{S}$ for DBL.

sil–Wong–Carter–Dawson:

$1\mathbf{S} + 2\mathbf{D}$ for ADD.

$\mathbf{S} + 1\mathbf{D}$ for DBL.

$: C : D : Z : SC : DZ)$:

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2007 Hisil–Carter–
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2007 Feng–Wu:
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$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ fo
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2008 Hisil–Wong–C
use $(X : Y : Z : X$
or $(X : Y : Z : X^2$
Can combine with
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Jacobi quartics

$(X{:}Y{:}Z)$ represent $(X/Z, Y/Z^2)$
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2007 Bernstein–Lange:
1**M** + 9**S** + 1**D** for DBL.

2007 Hisil–Carter–Dawson:
2**M** + 6**S** + 2**D** for DBL.

2007 Feng–Wu:
2**M** + 6**S** + 1**D** for DBL.
1**M** + 7**S** + 3**D** for DBL
on curves chosen with $a^2 +$

More speedups: 2007 Duque
2007 Hisil–Carter–Dawson,
2008 Hisil–Wong–Carter–Da
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2X$
Can combine with Feng–Wu
Competitive with Edwards!

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$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.
$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL
on curves chosen with $a^2 + c^2 = 1$.

More speedups: 2007 Duquesne,
2007 Hisil–Carter–Dawson,
2008 Hisil–Wong–Carter–Dawson:
use $(X : Y : Z : X^2 : Z^2)$
or $(X : Y : Z : X^2 : Z^2 : 2XZ)$.
Can combine with Feng–Wu.
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uartics

) represent $(X/Z, Y/Z^2)$

$x^4 + 2ax^2 + 1$.

udnovsky–Chudnovsky:

S + 2**D** for DBL.

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3**S** + 1**D** for ADD,

unified.

rnstein–Lange:

S + 1**D** for DBL.

2007 Hisil–Carter–Dawson:

2**M** + 6**S** + 2**D** for DBL.

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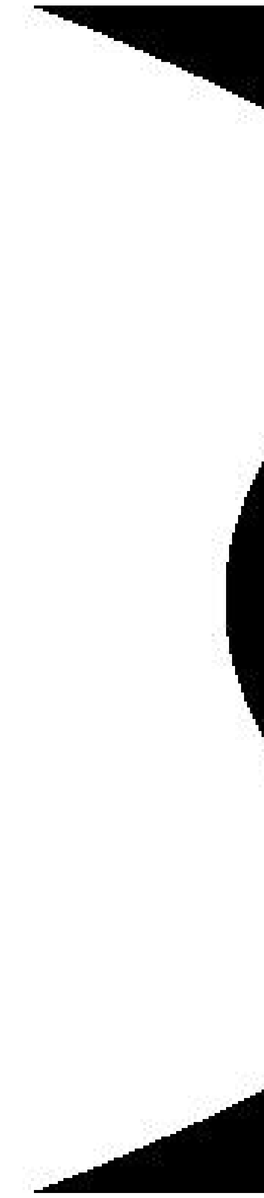

$x^2 = y^4$

t $(X/Z, Y/Z^2)$

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More speedups: 2007 Duquesne,
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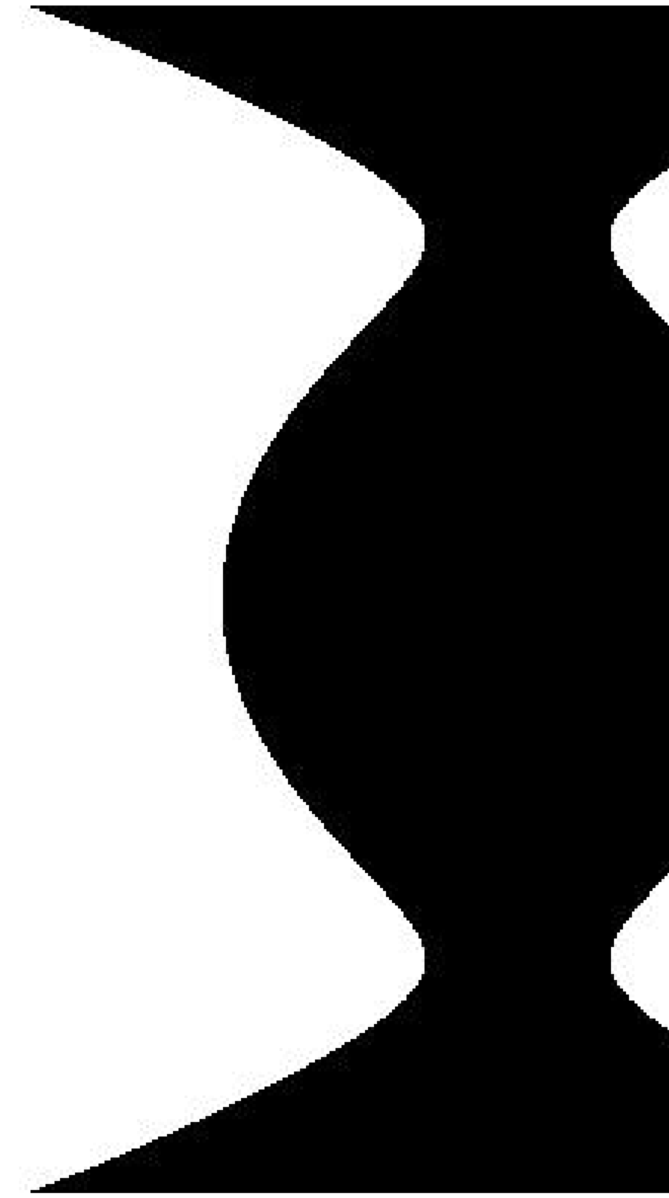

$x^2 = y^4 - 1.9y^2 +$

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2007 Hisil–Carter–Dawson:
$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

2007 Feng–Wu:
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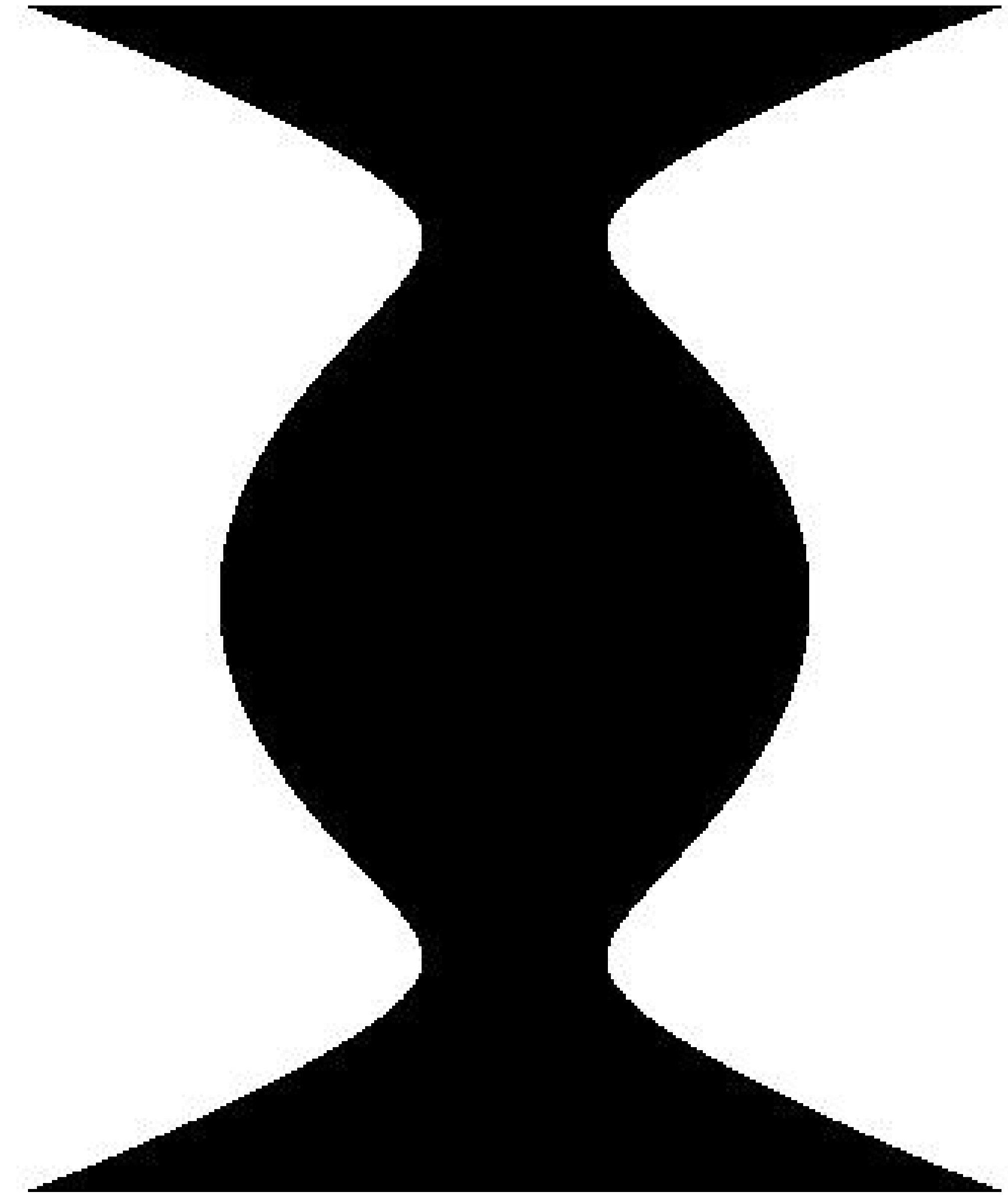


$x^2 = y^4 - 1.9y^2 + 1$

2007 Hisil–Carter–Dawson:

$2\mathbf{M} + 6\mathbf{S} + 2\mathbf{D}$ for DBL.

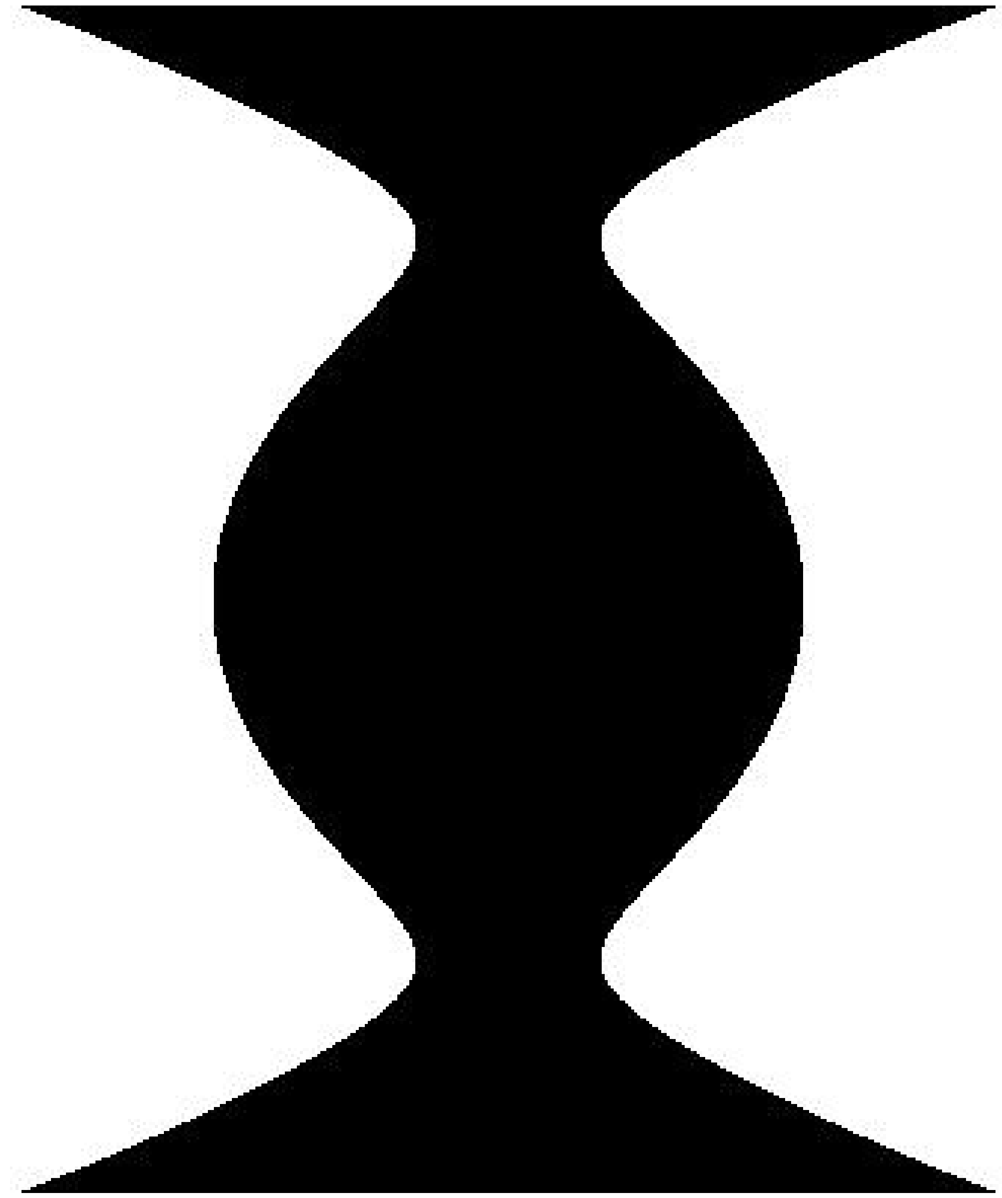
2007 Feng–Wu:

$2\mathbf{M} + 6\mathbf{S} + 1\mathbf{D}$ for DBL.

$1\mathbf{M} + 7\mathbf{S} + 3\mathbf{D}$ for DBL

on curves chosen with $a^2+c^2 = 1$.

More speedups: 2007 Duquesne, 2007 Hisil–Carter–Dawson, 2008 Hisil–Wong–Carter–Dawson: use $(X : Y : Z : X^2 : Z^2)$ or $(X : Y : Z : X^2 : Z^2 : 2XZ)$. Can combine with Feng–Wu. Competitive with Edwards!



$x^2 = y^4 - 1.9y^2 + 1$

sil–Carter–Dawson:

S $+ 2$**D** for DBL.

ng–Wu:

S $+ 1$**D** for DBL.

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es chosen with $a^2 + c^2 = 1$.

eedups: 2007 Duquesne,

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sil–Wong–Carter–Dawson:

$Y : Z : X^2 : Z^2)$

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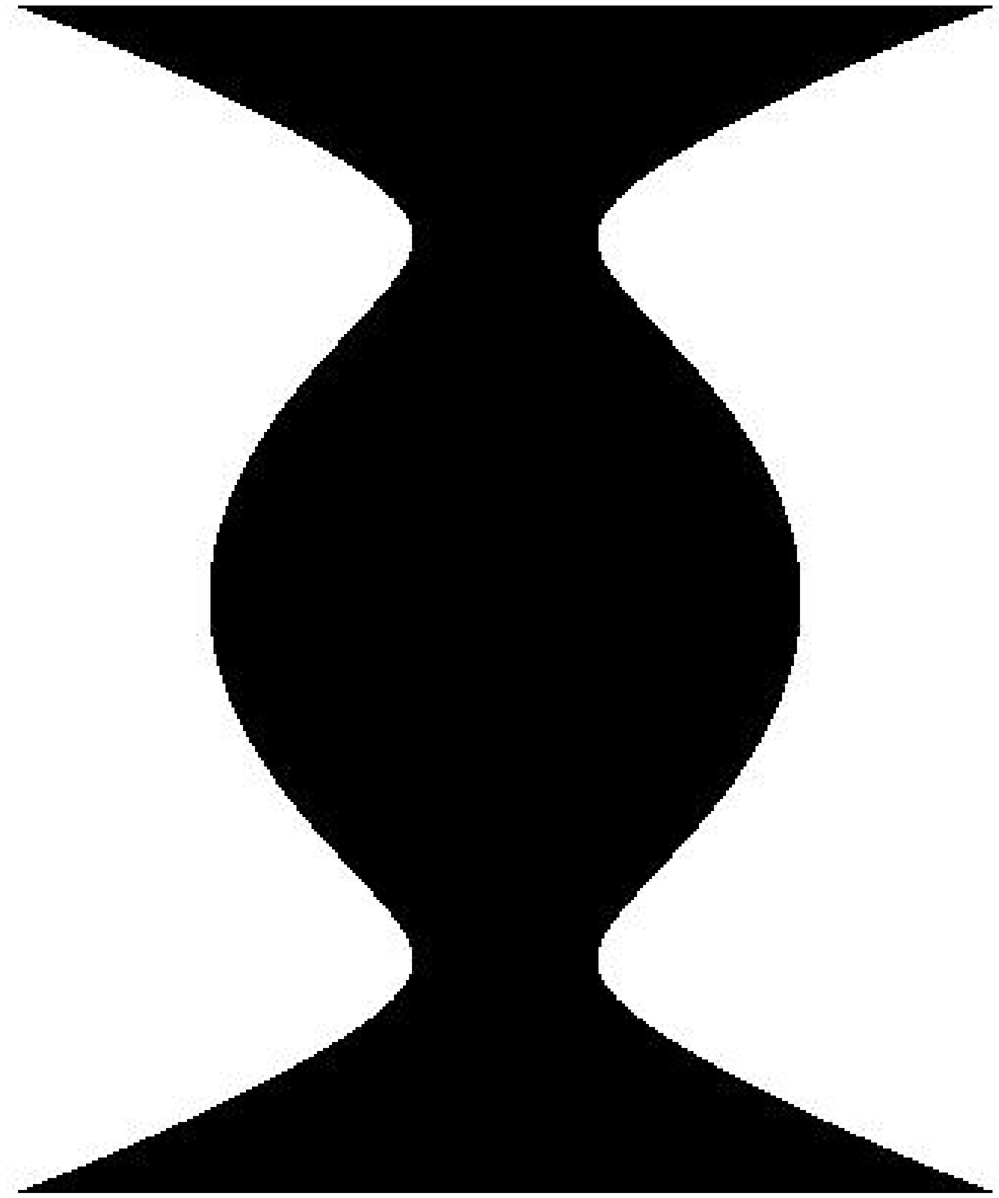

$x^2 = y^4 - 1.9y^2 + 1$

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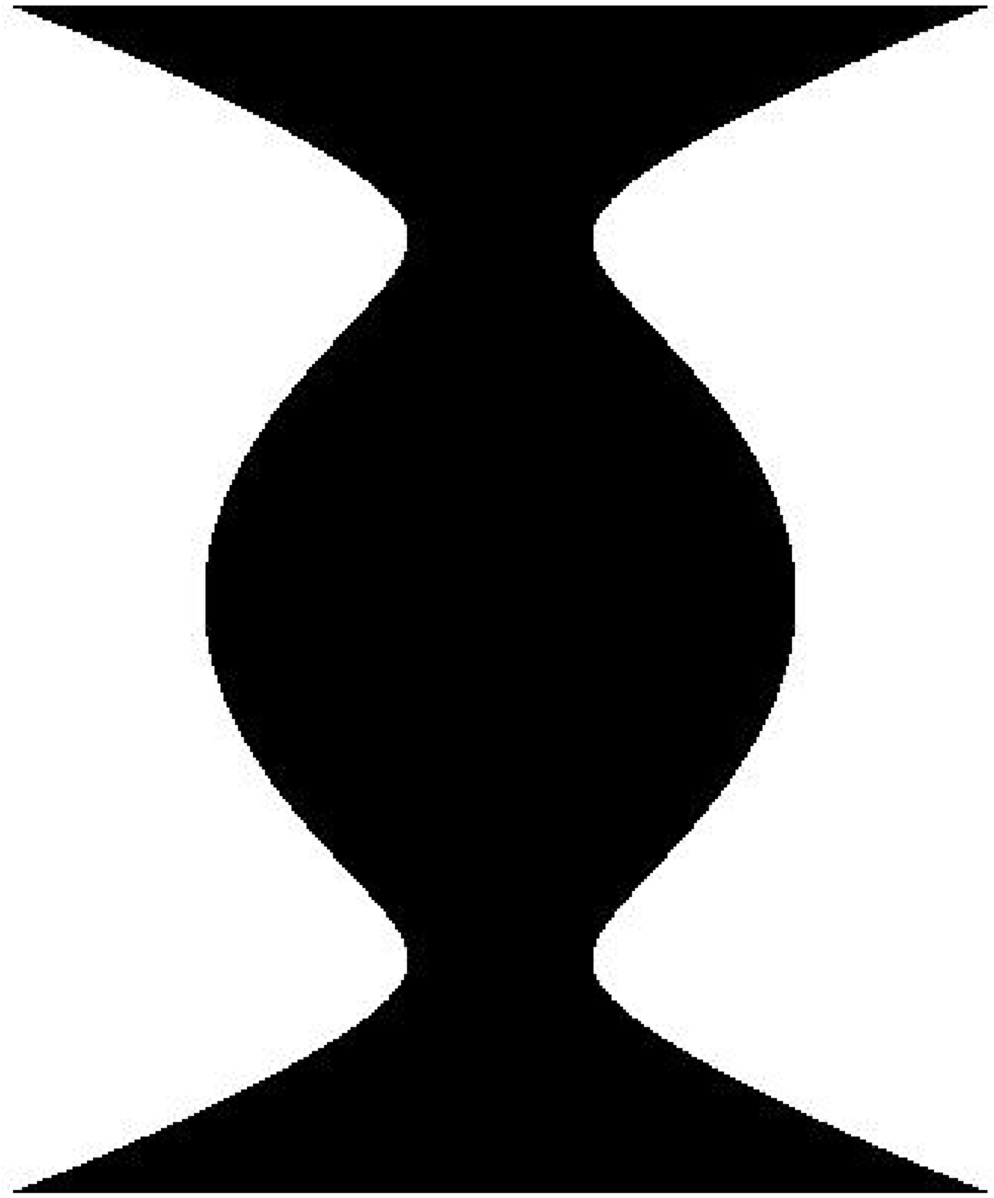
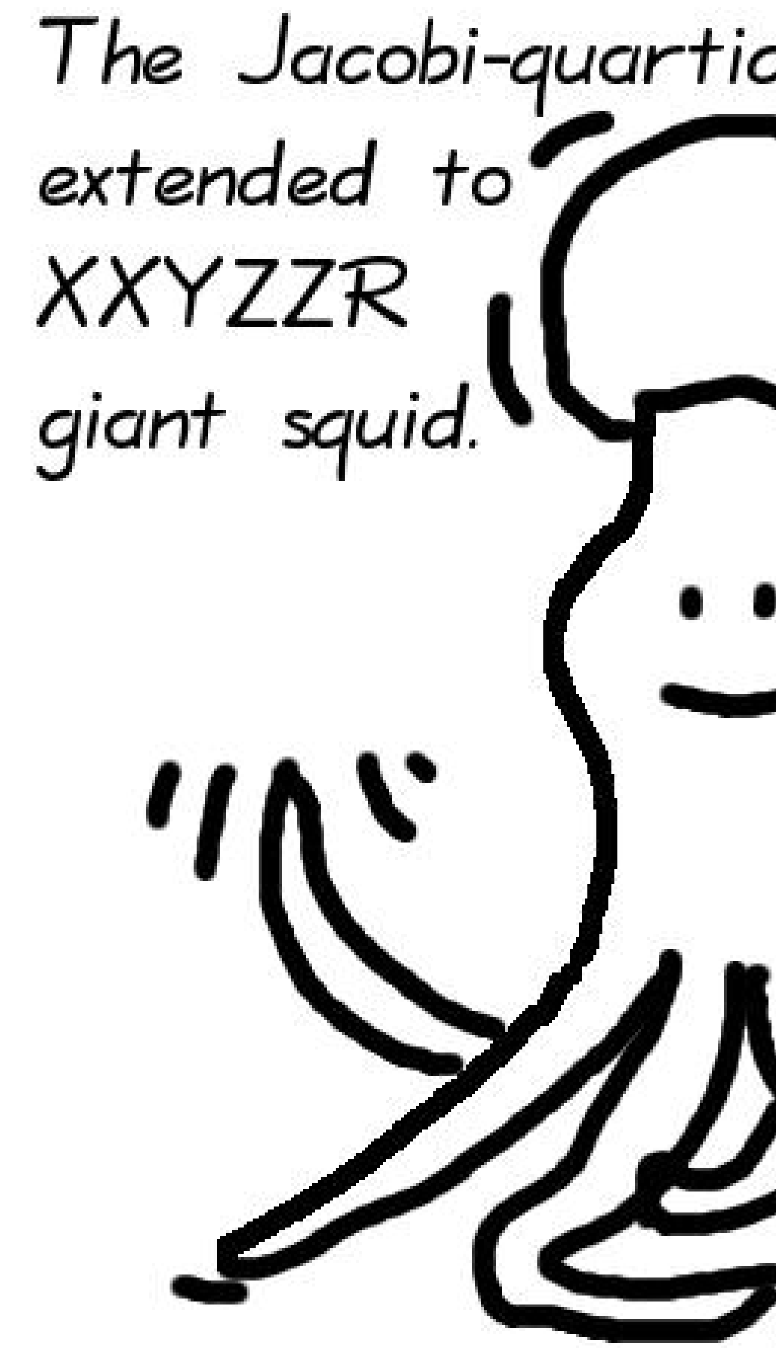


$x^2 = y^4 - 1.9y^2 + 1$



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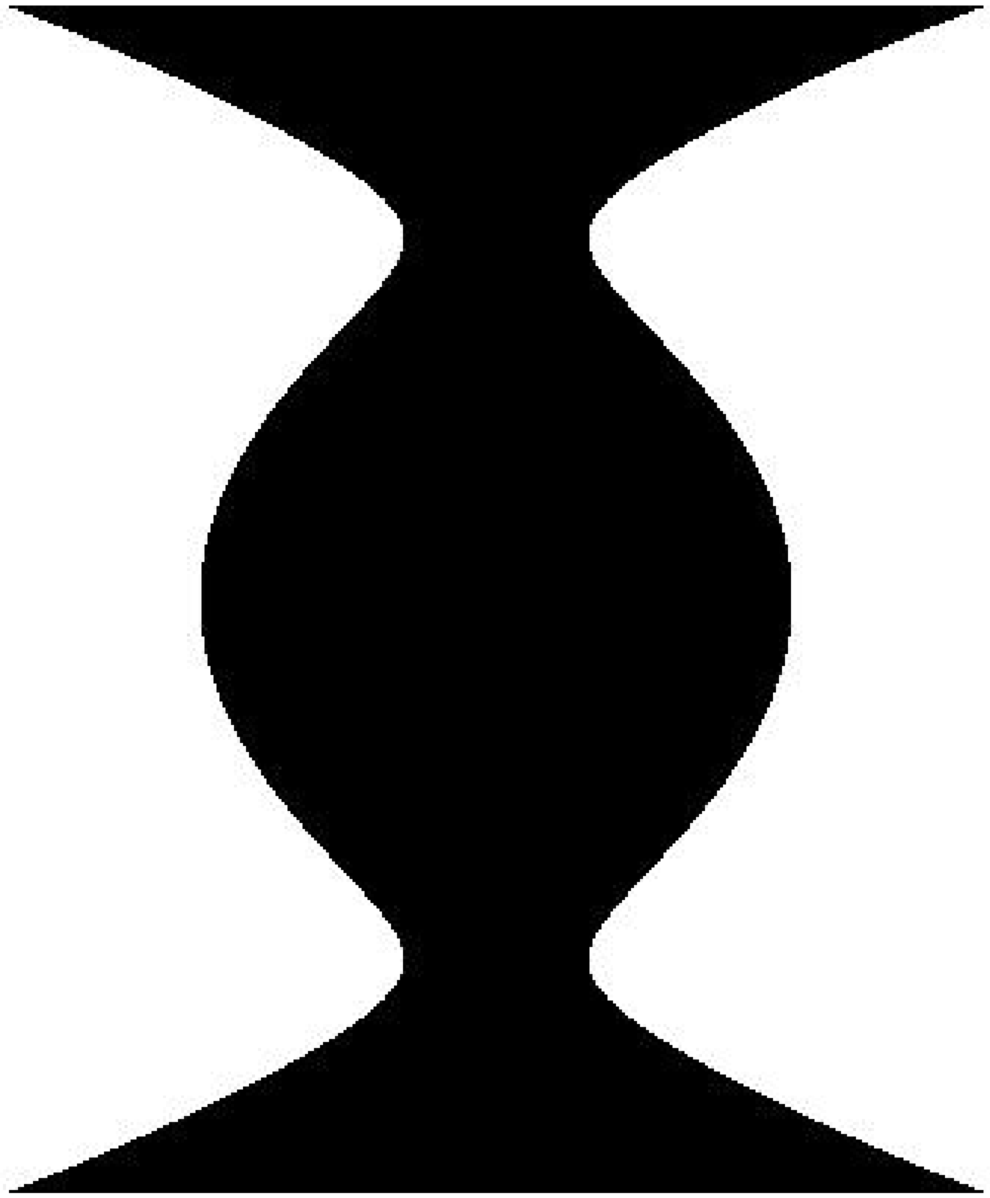
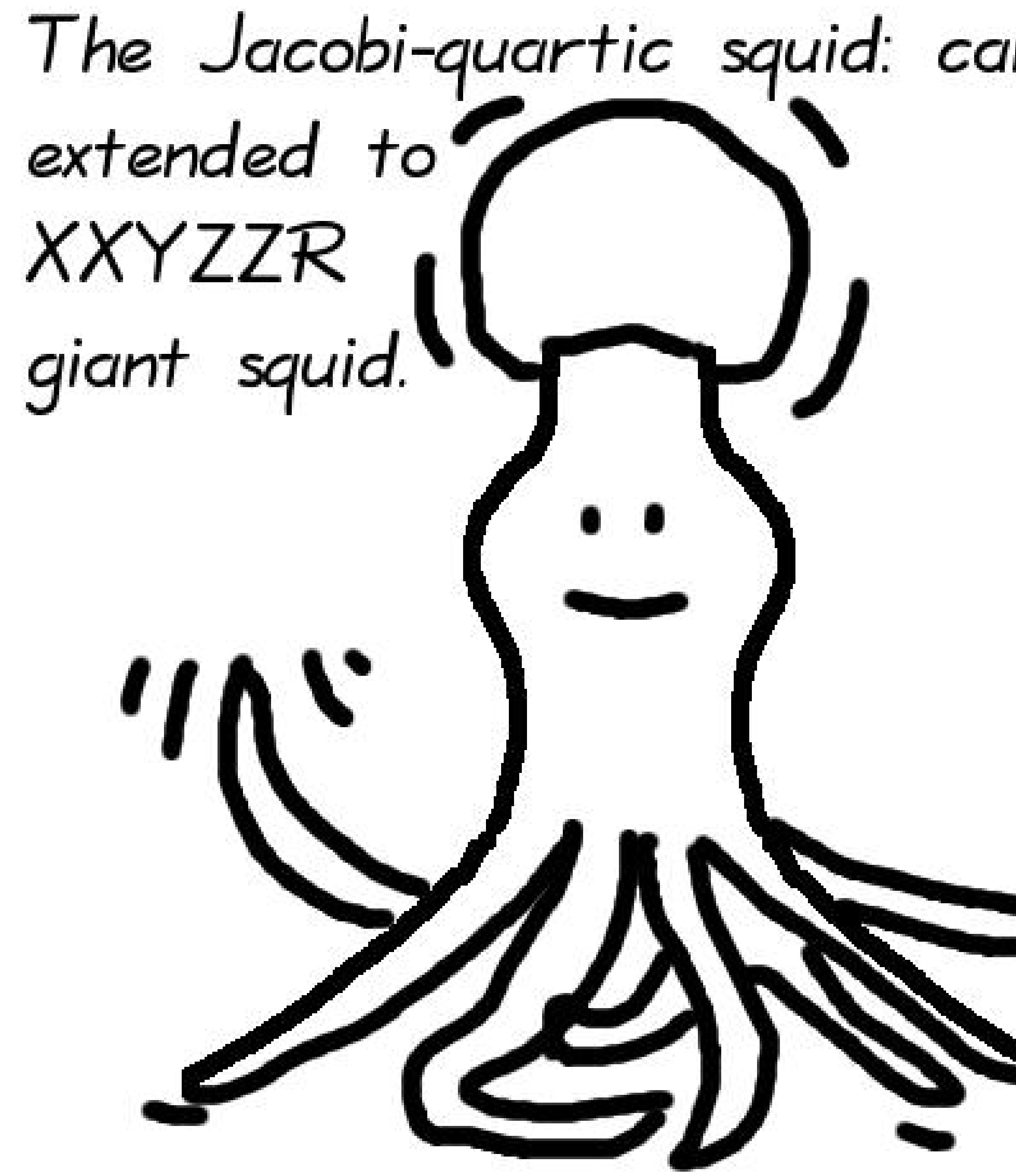
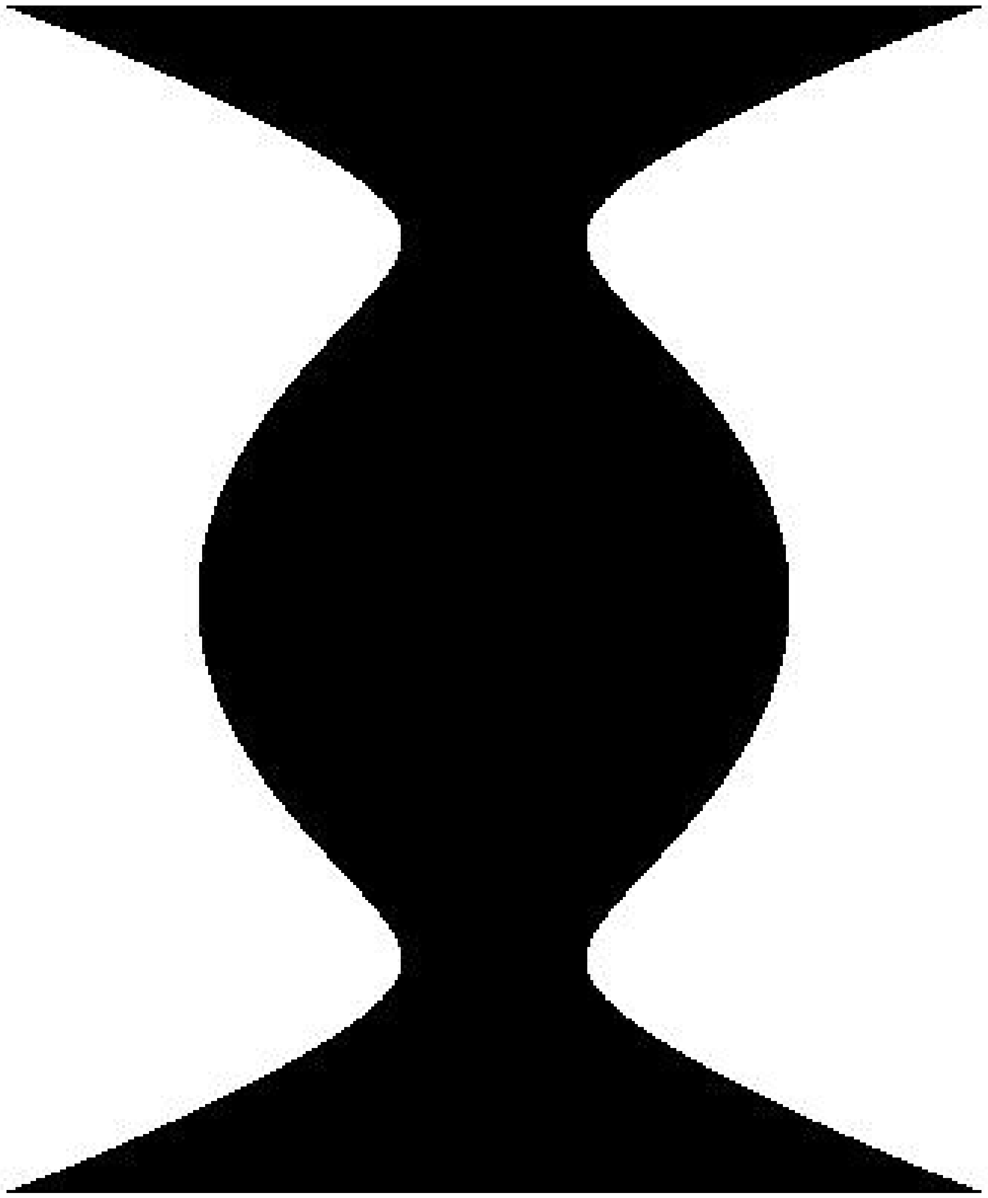


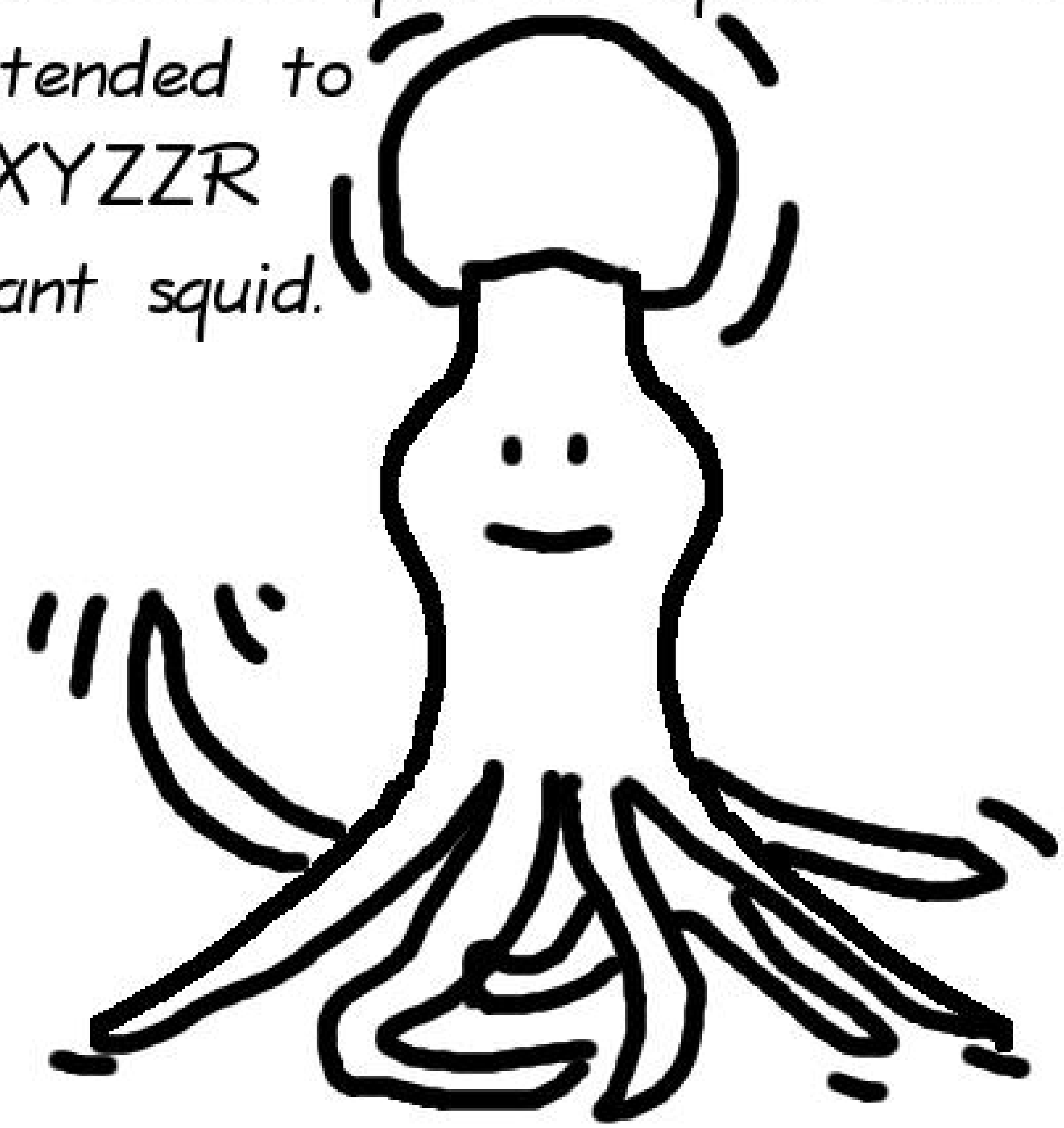
$x^2 = y^4 - 1.9y^2 + 1$

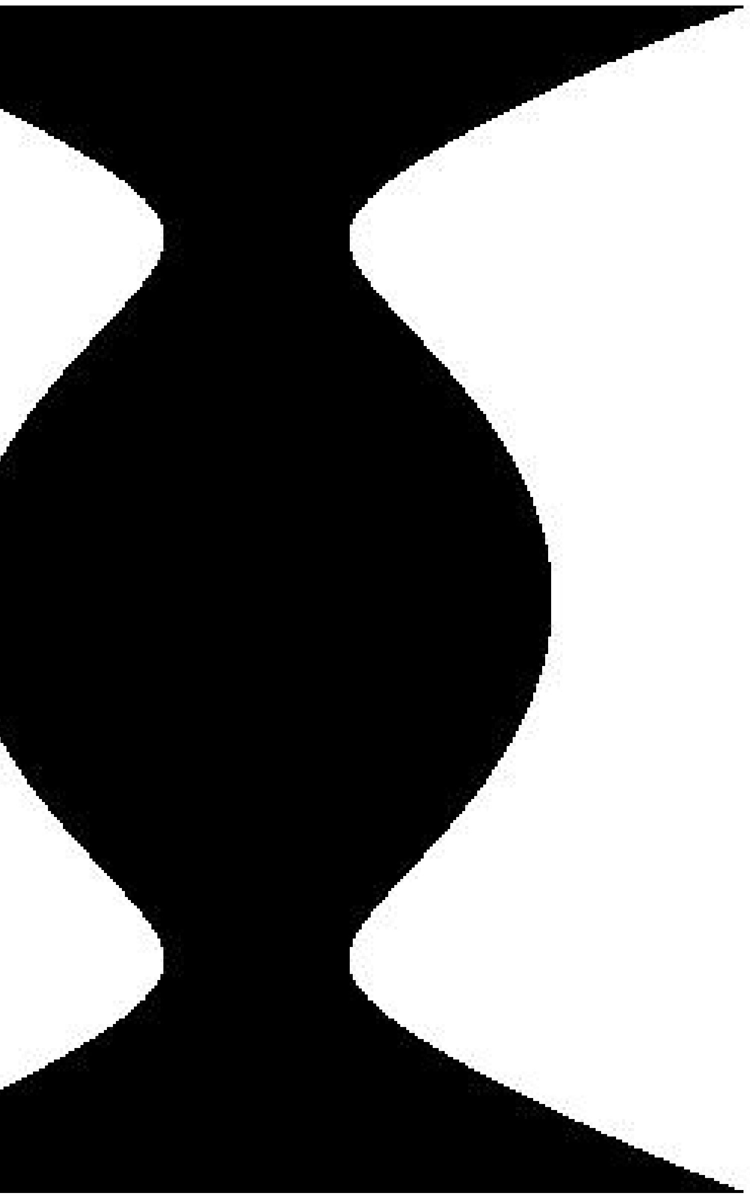




$$x^2 = y^4 - 1.9y^2 + 1$$

The Jacobi-quartic squid: can be extended to XXYZZR giant squid.





$- 1.9y^2 + 1$

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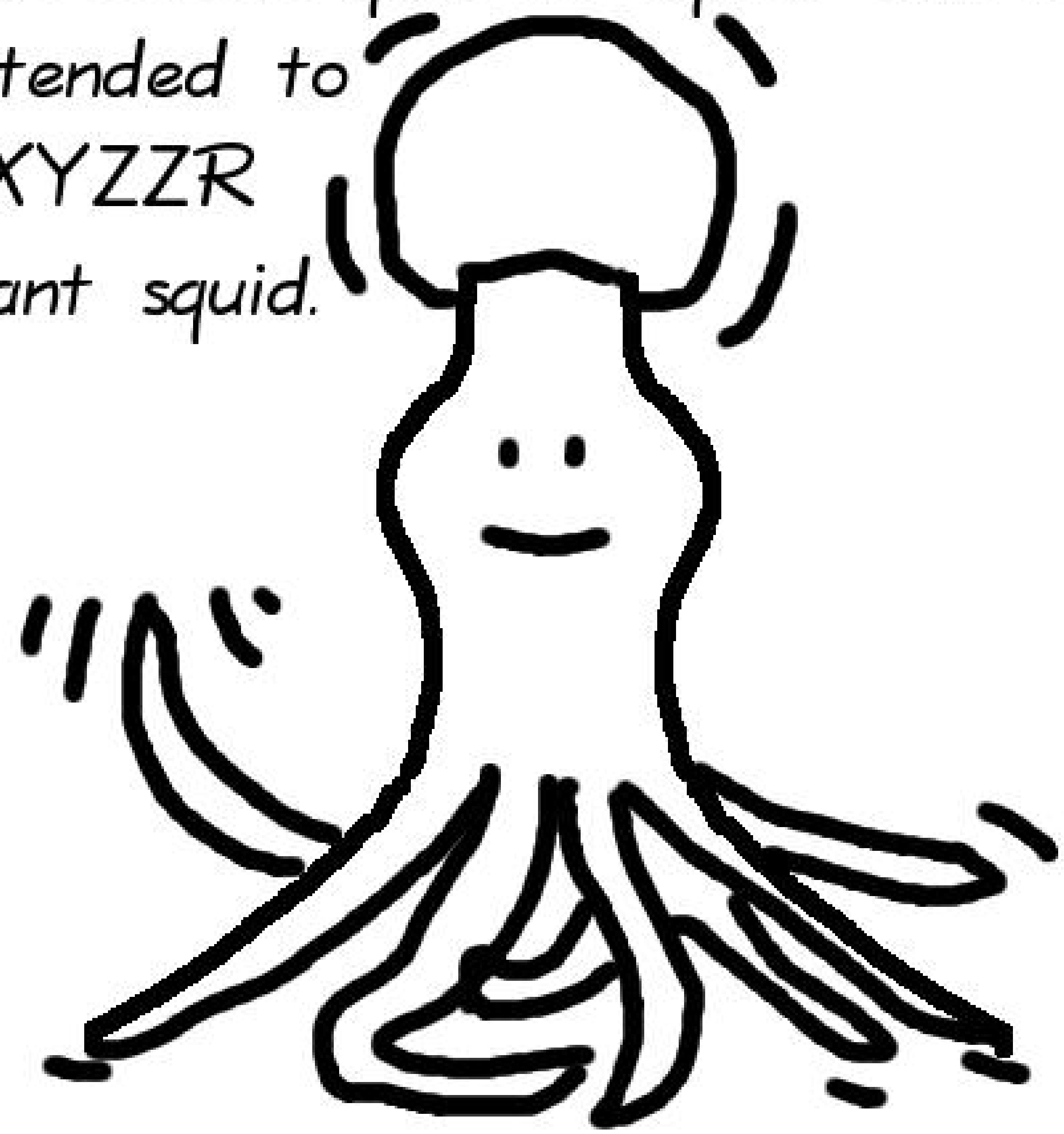

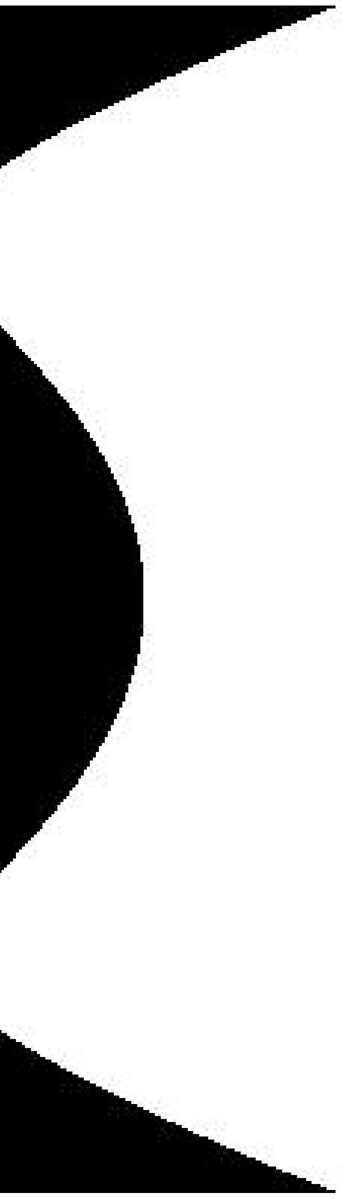

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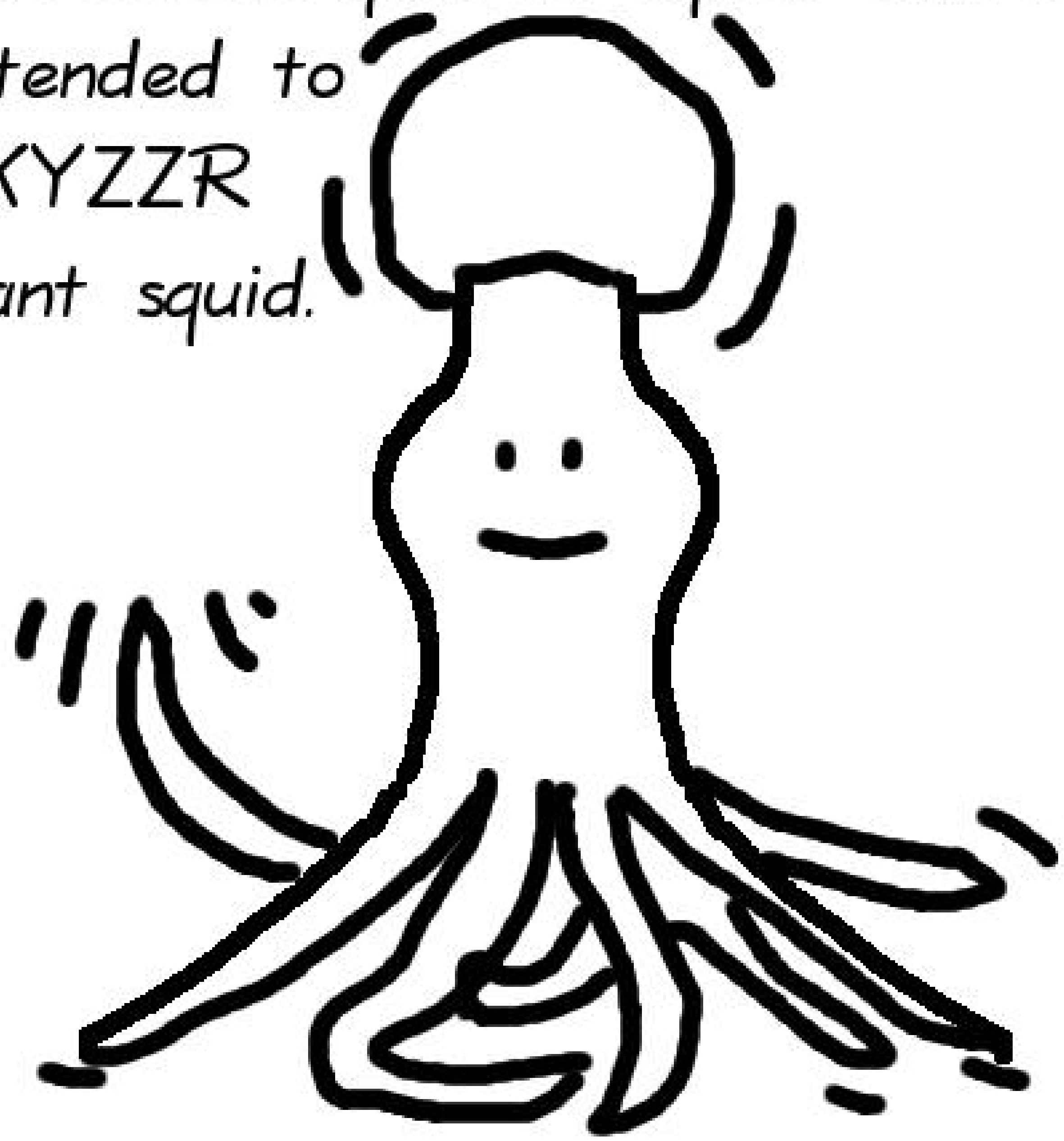

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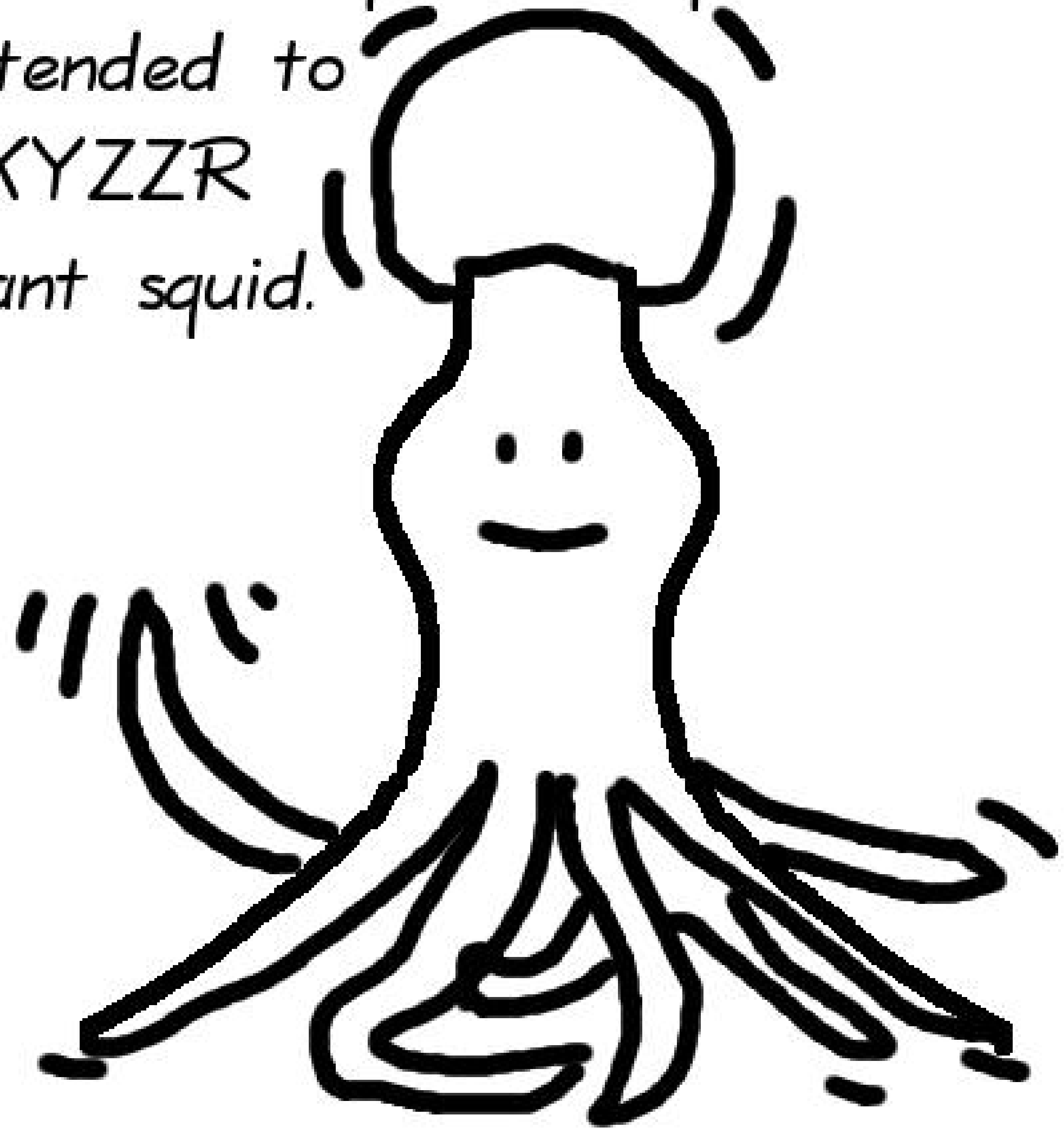

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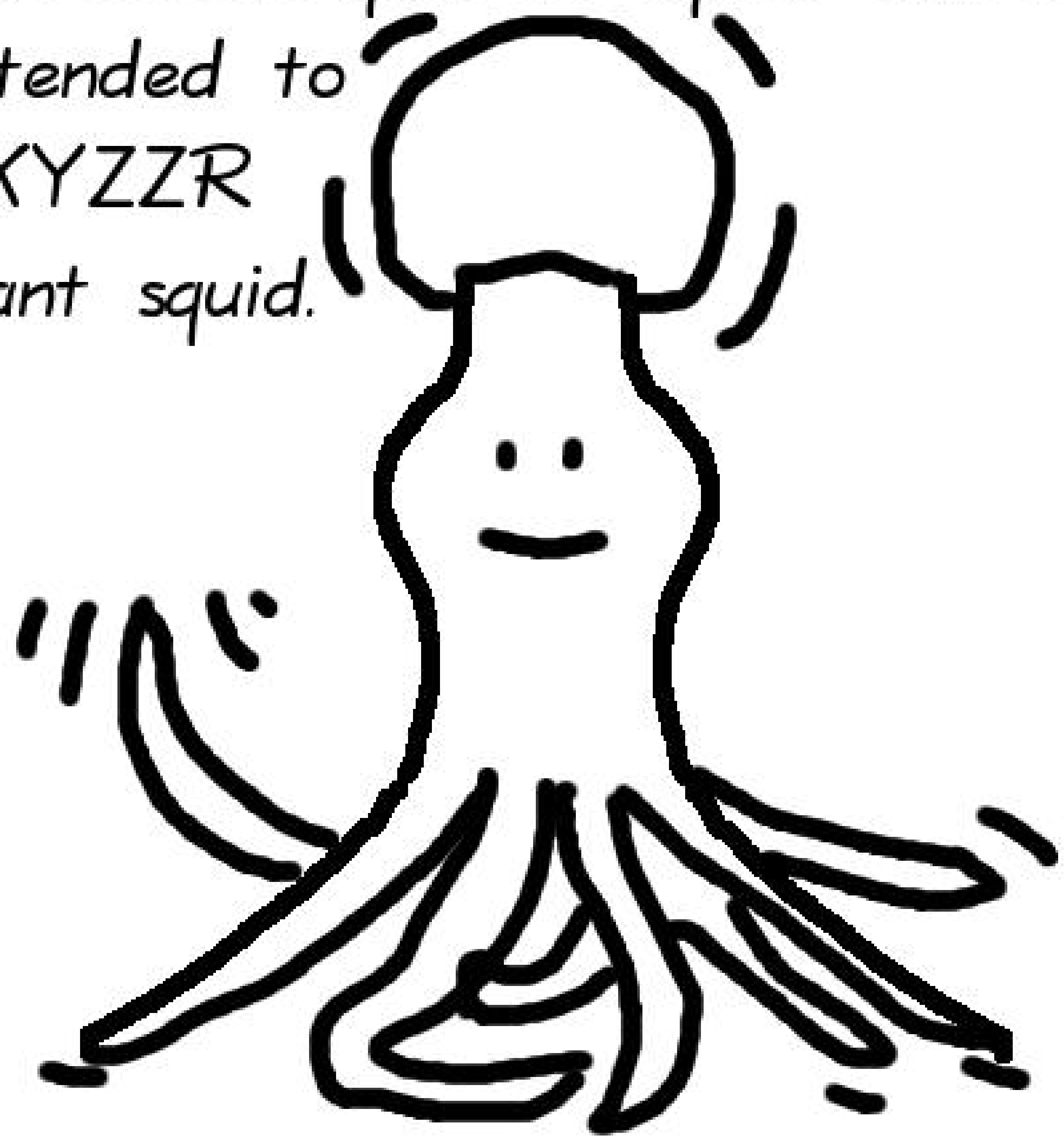

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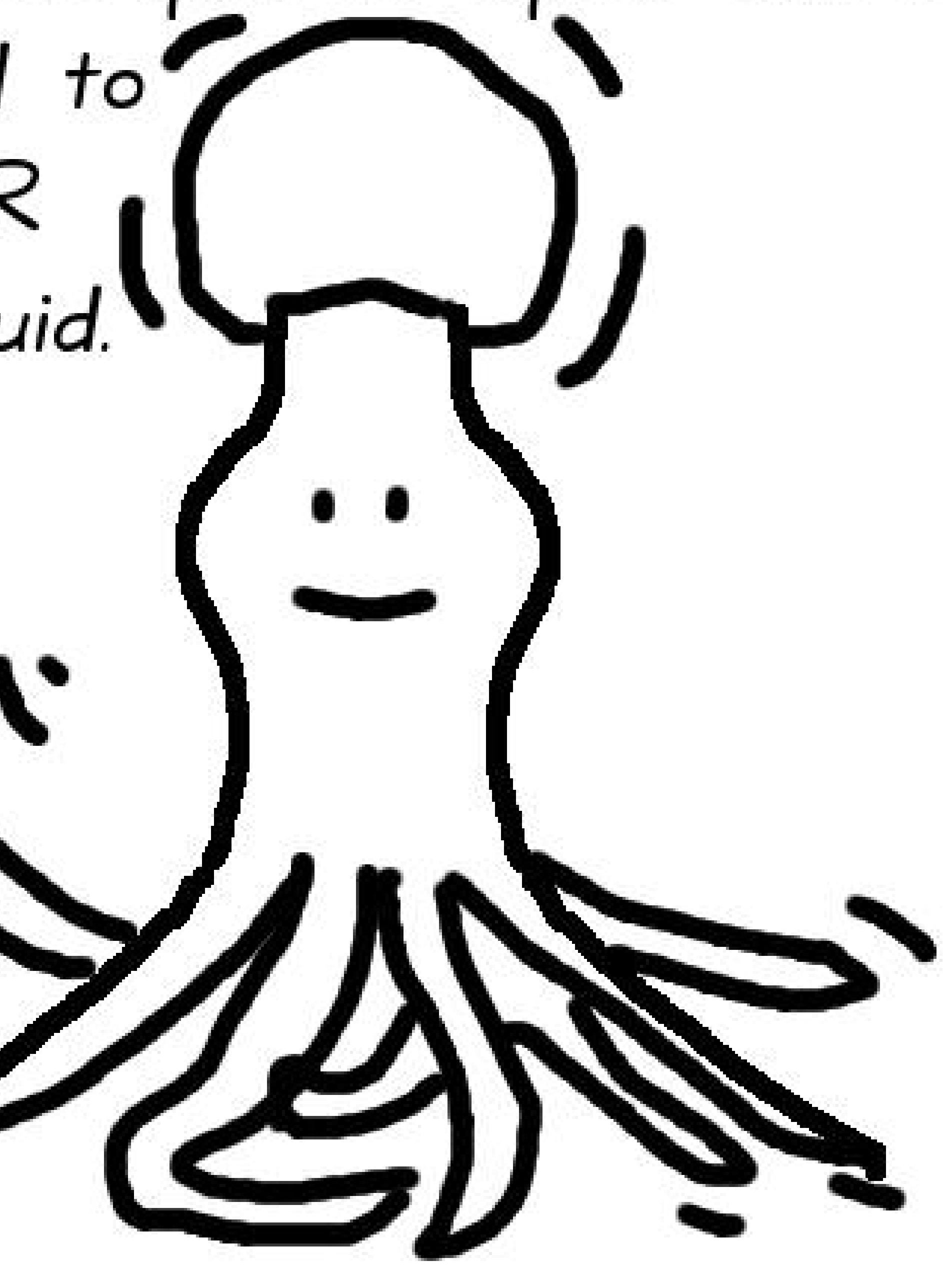

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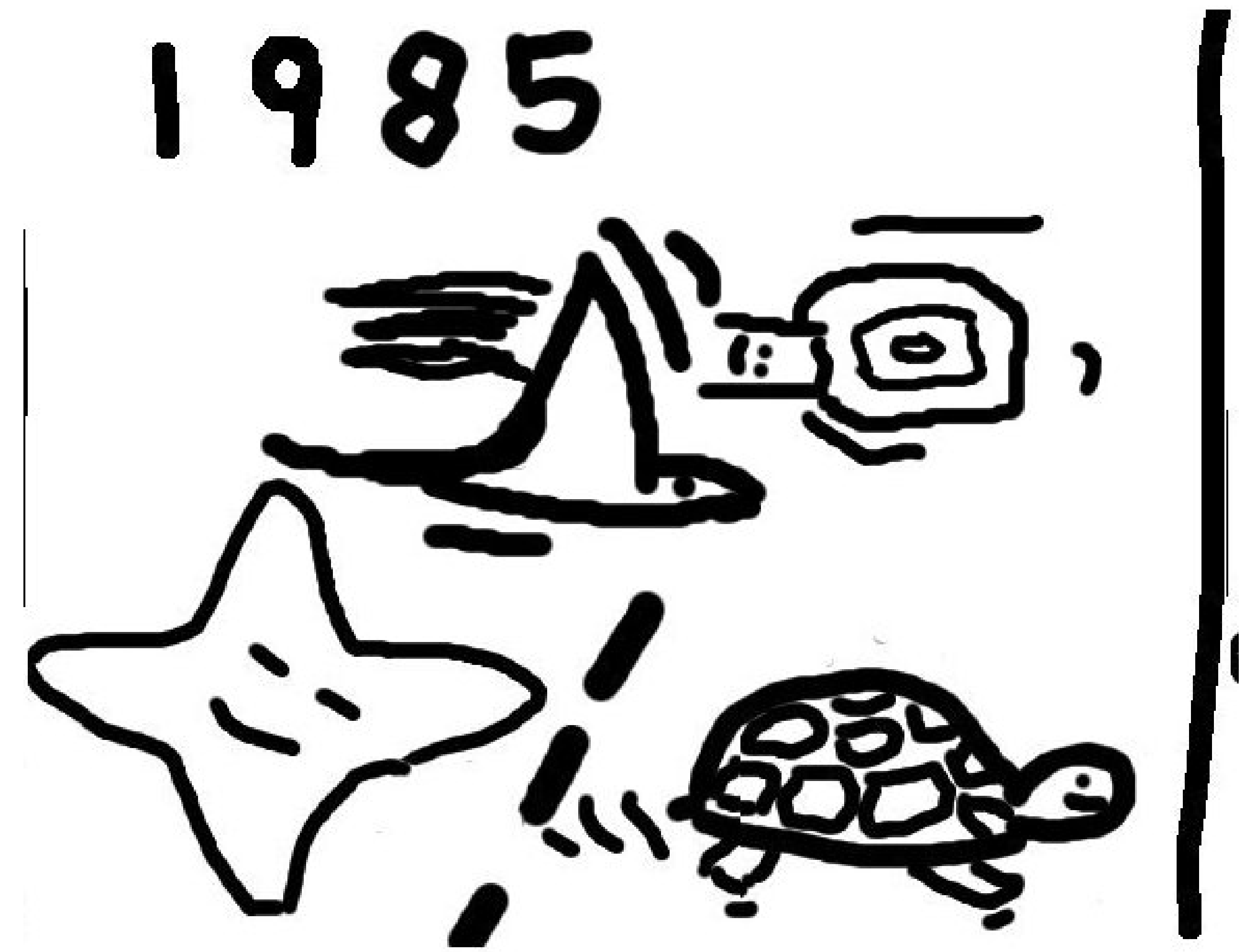
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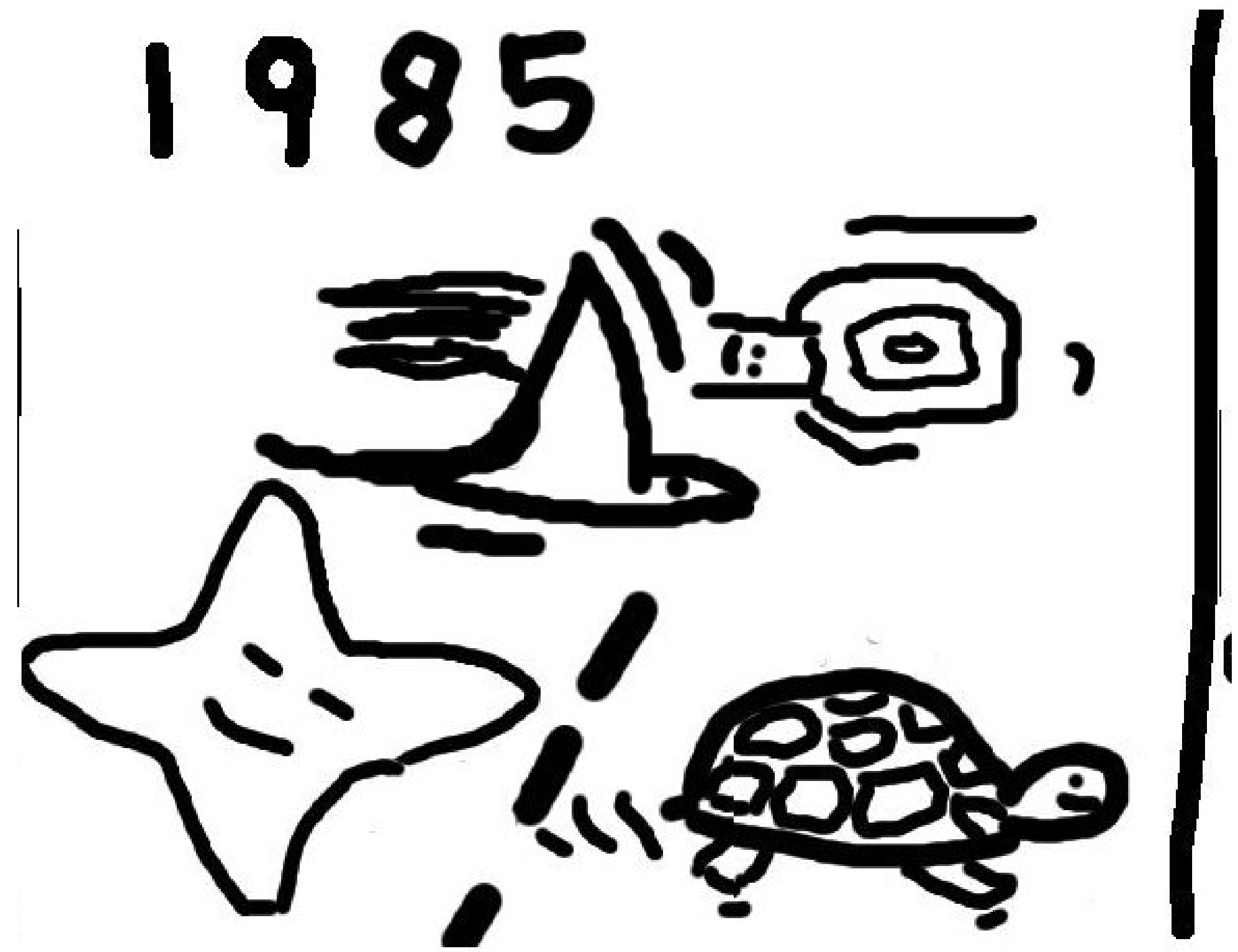
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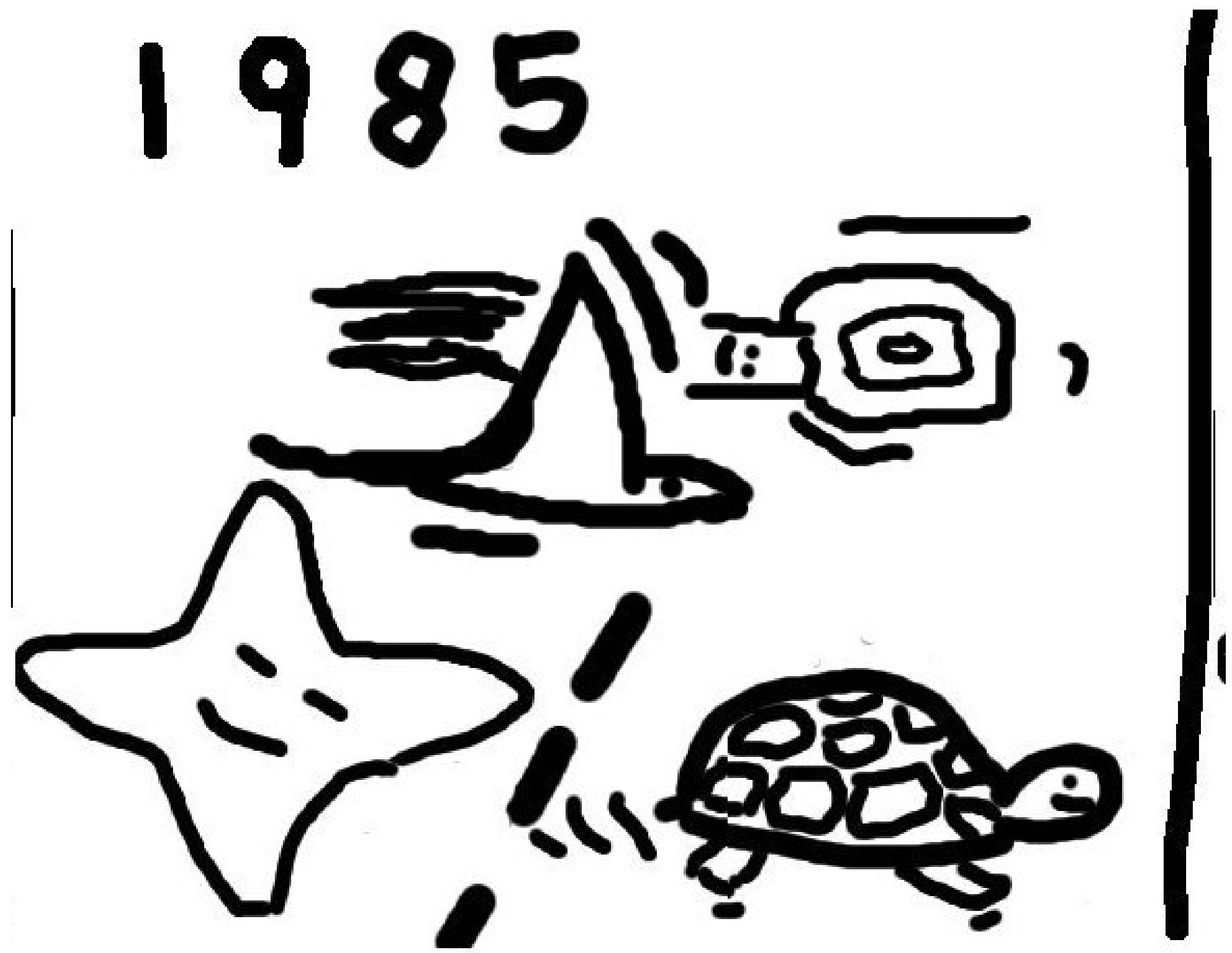
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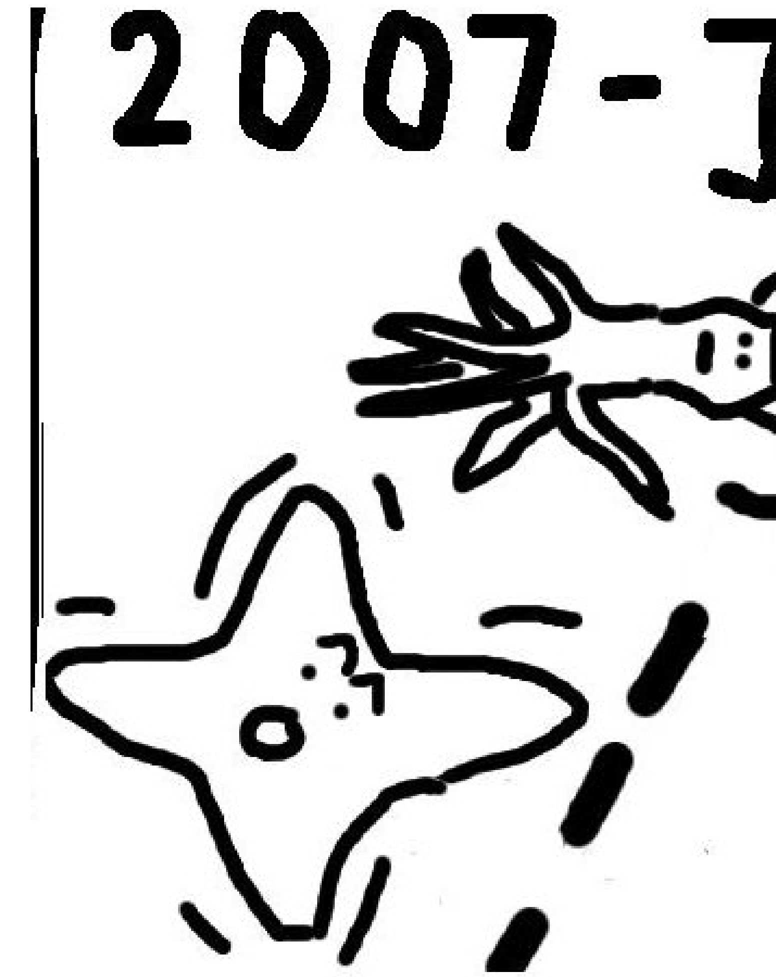
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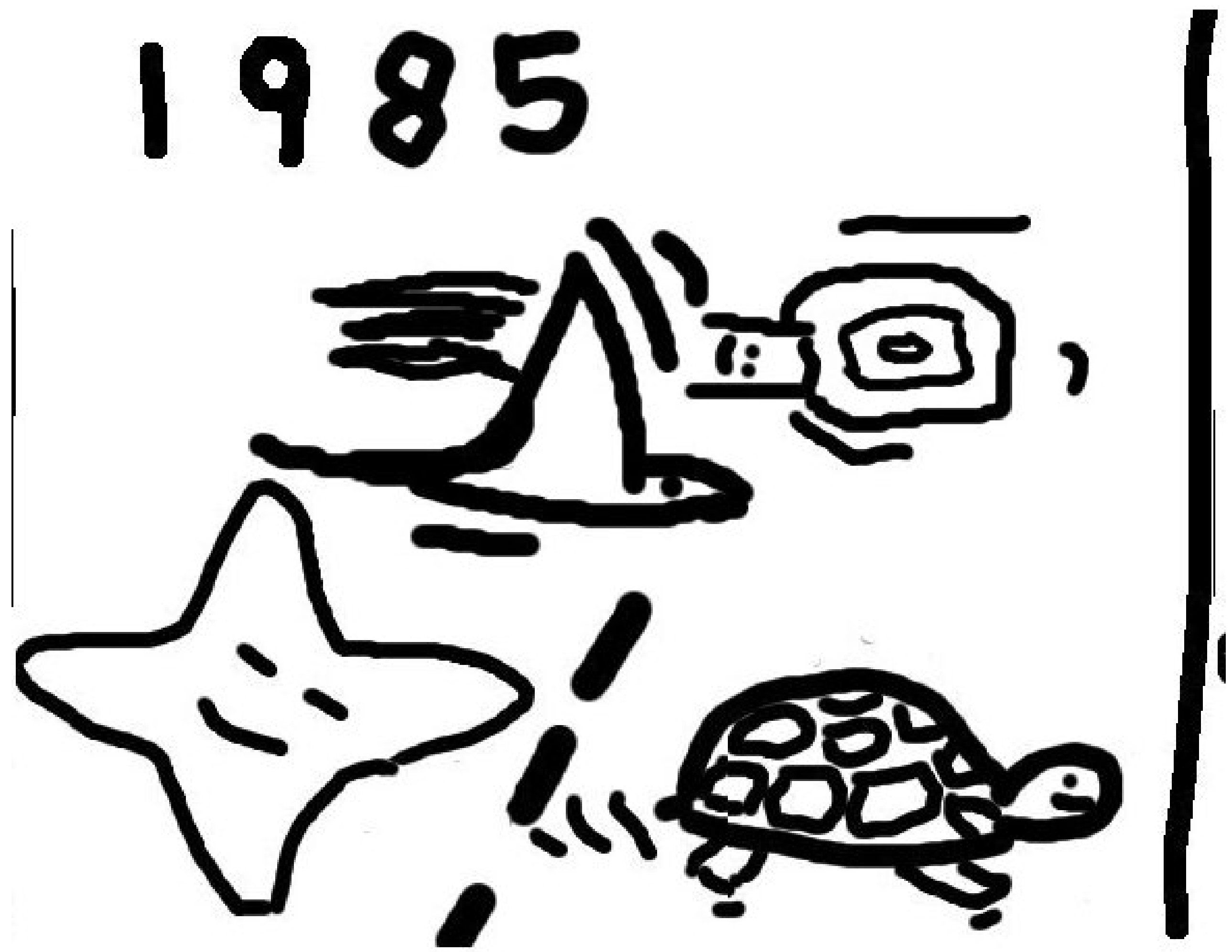
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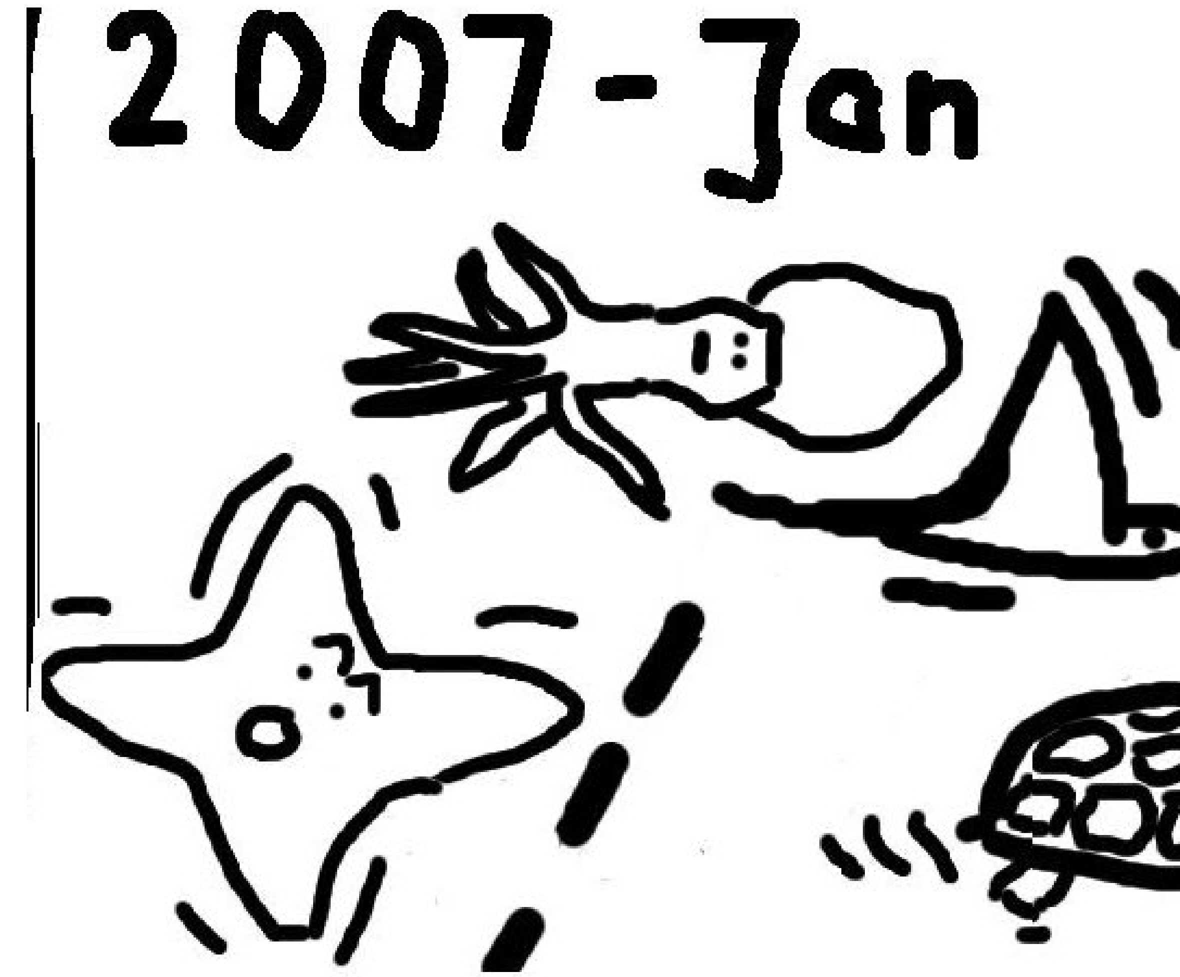
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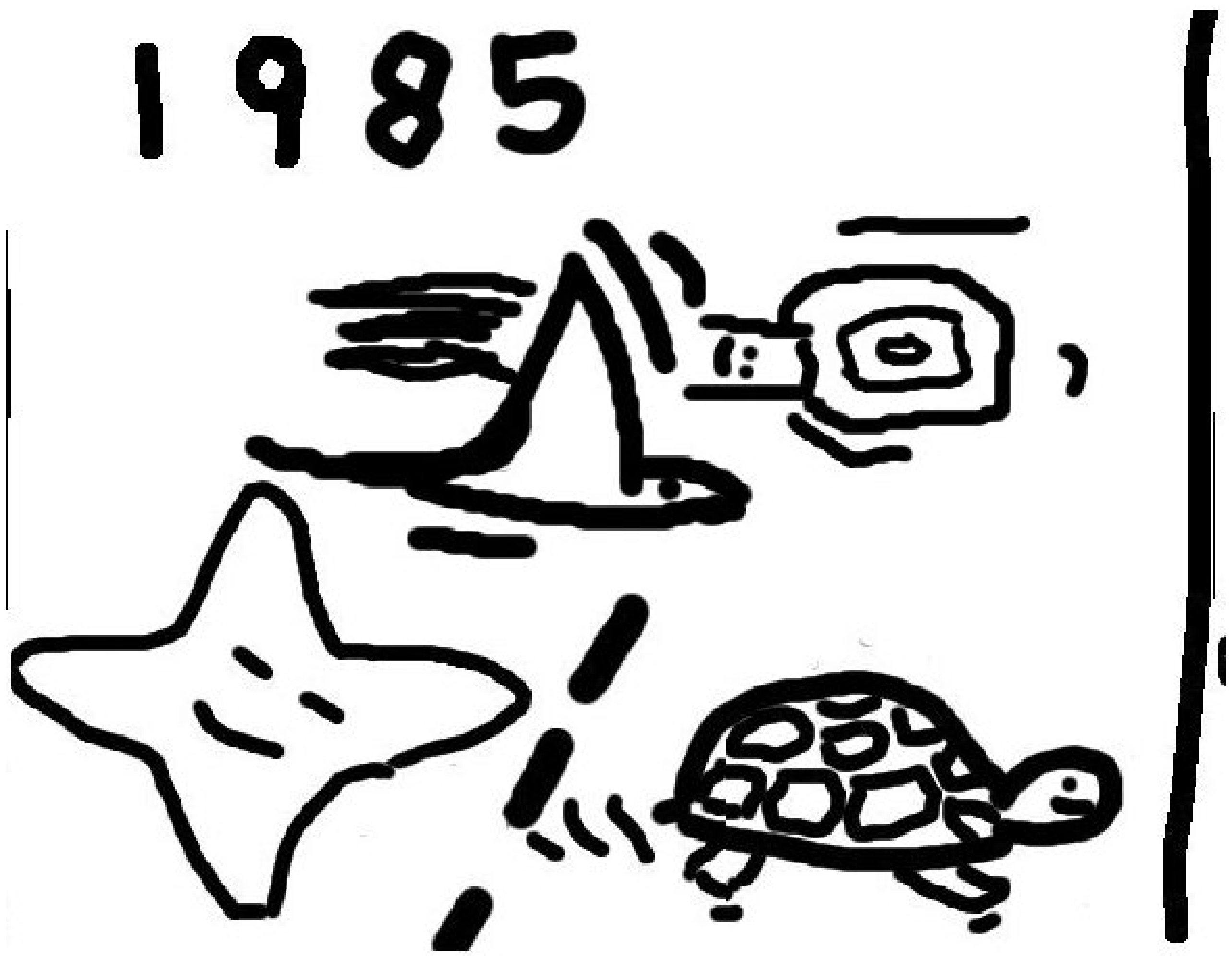
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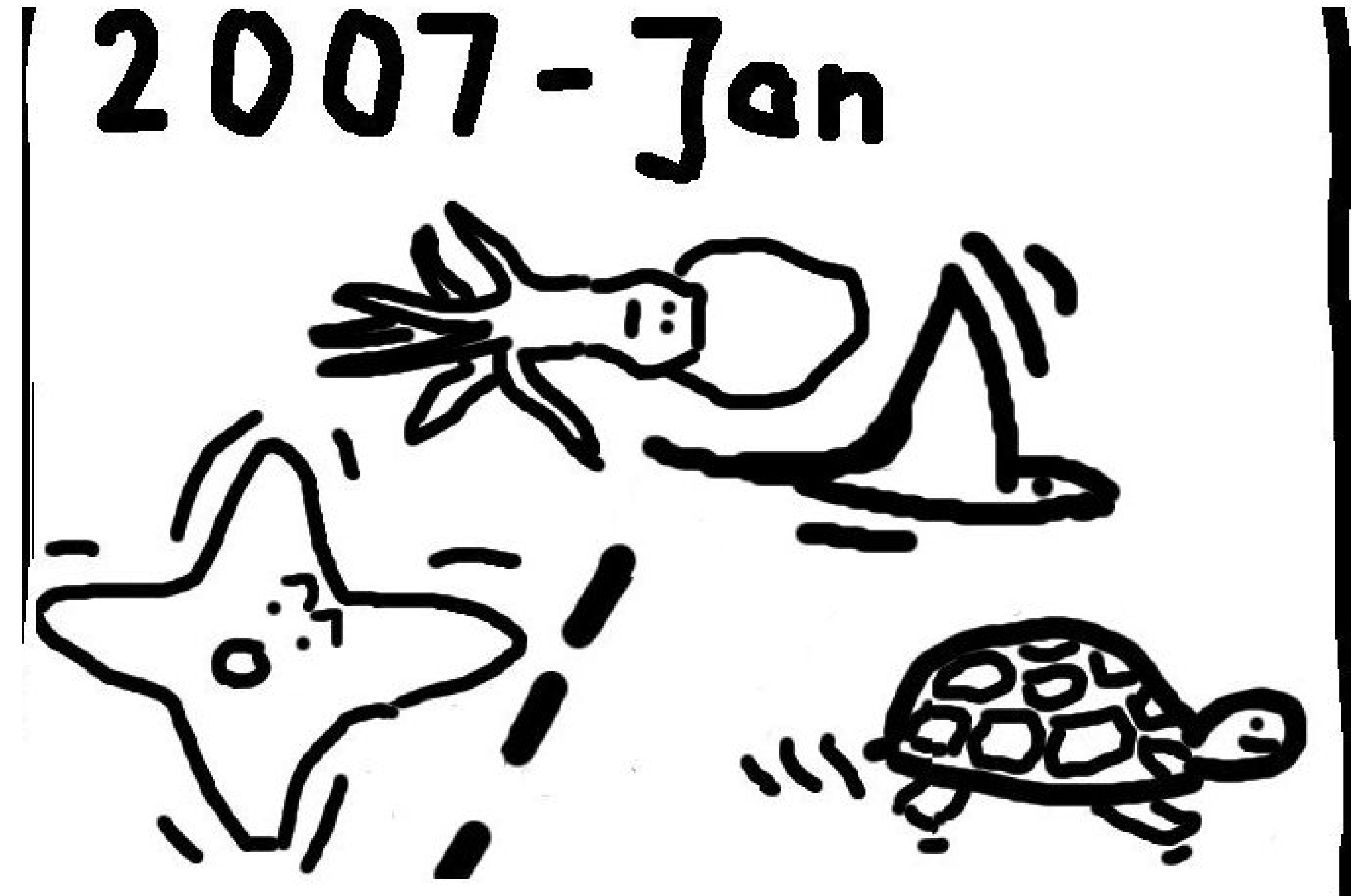
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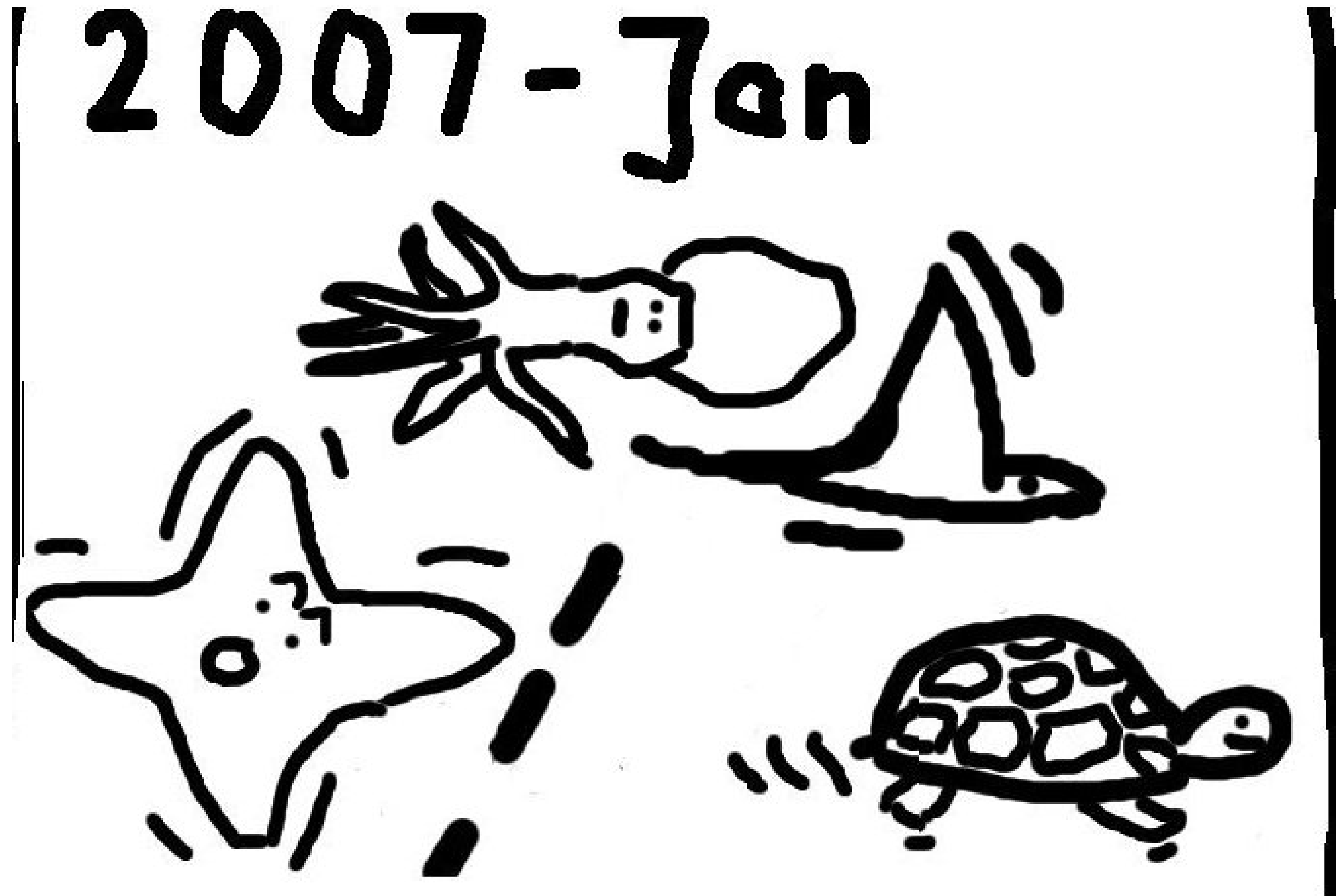
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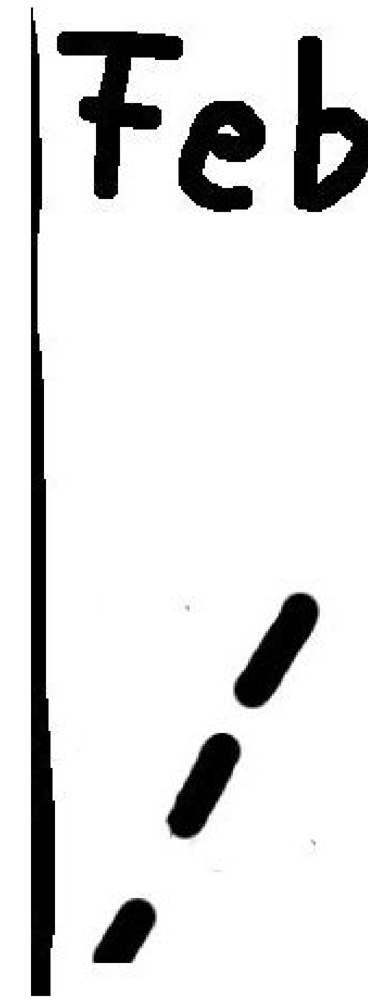
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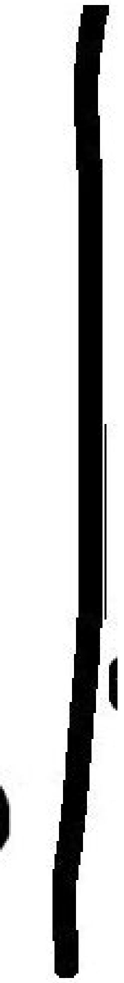

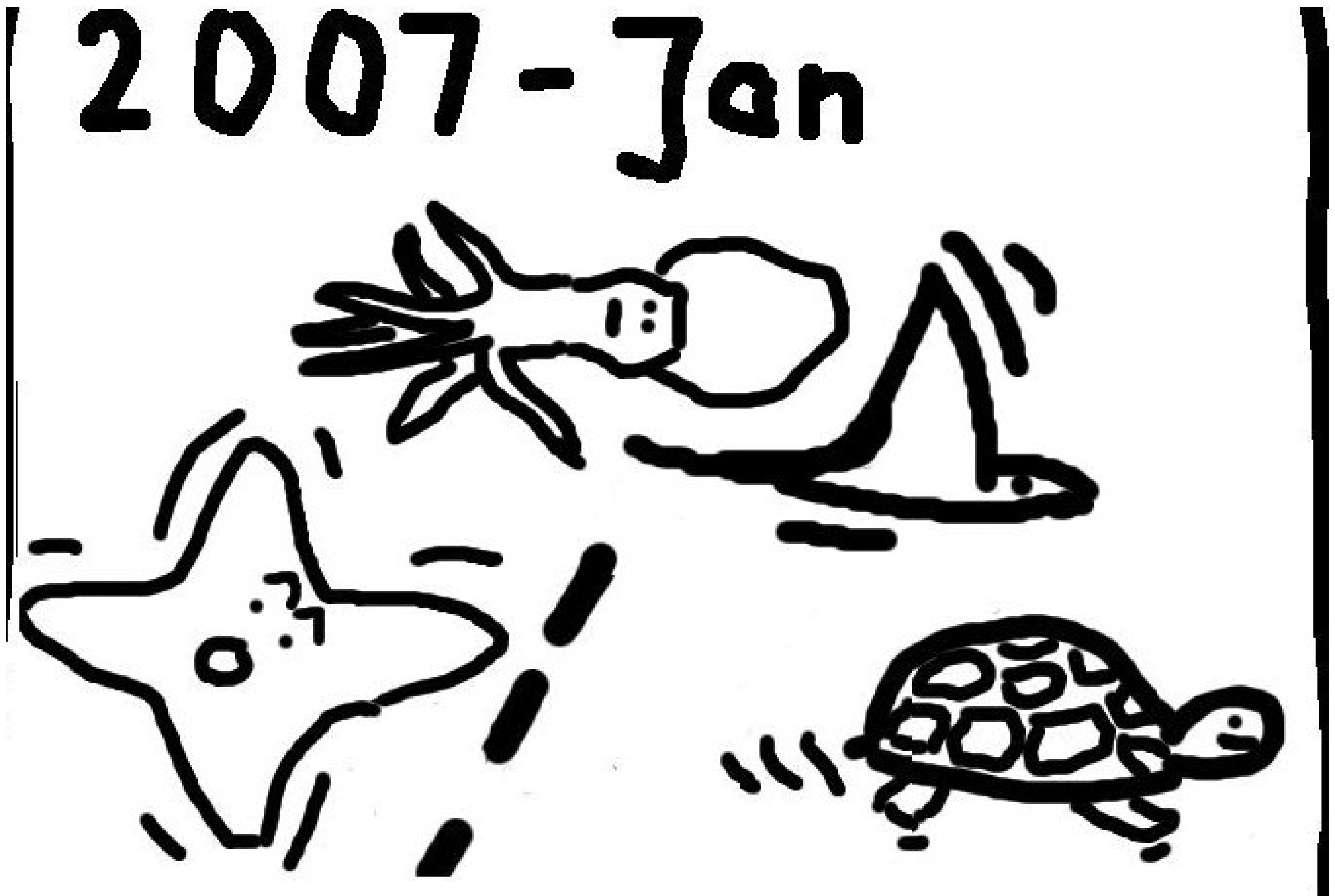
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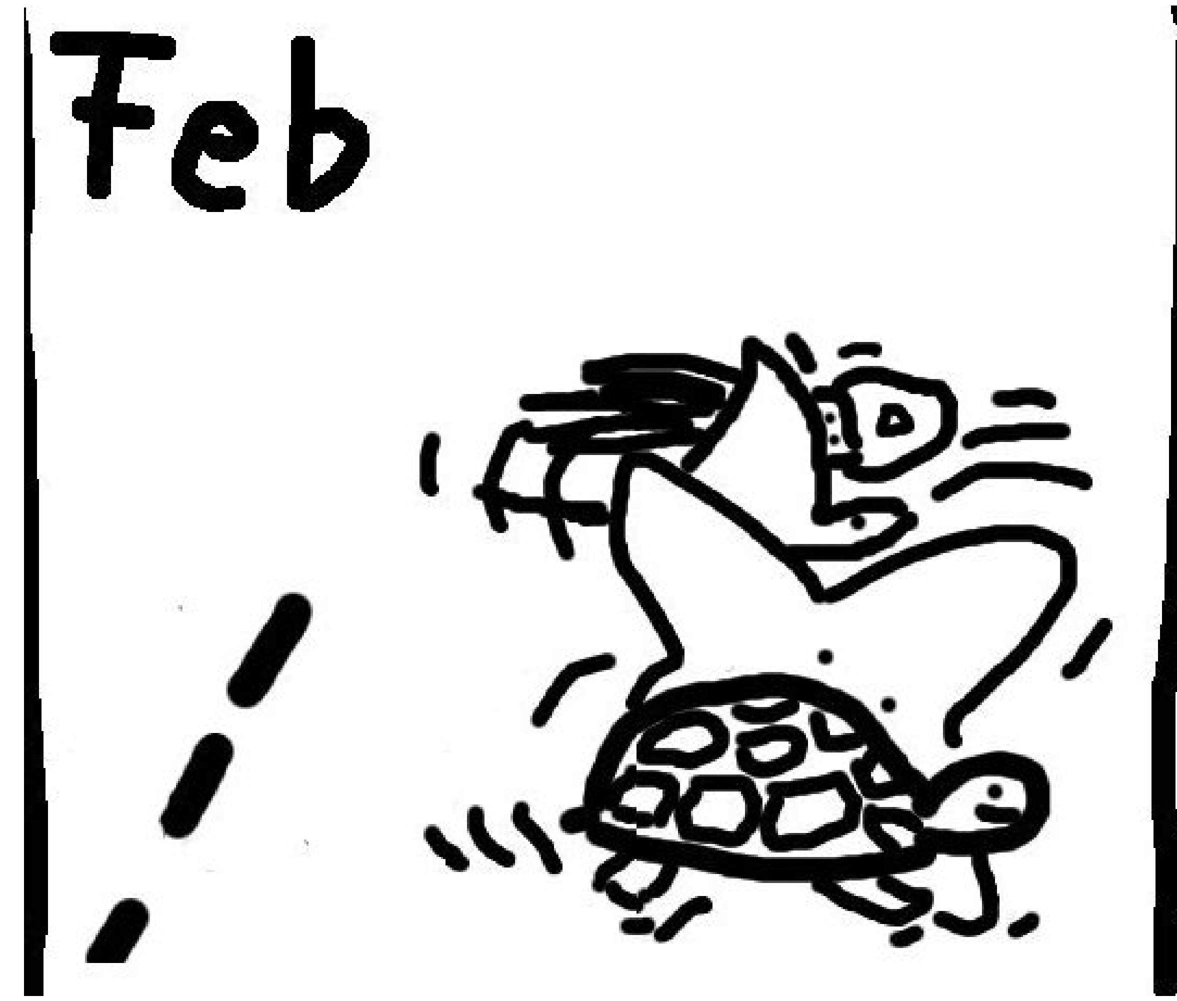
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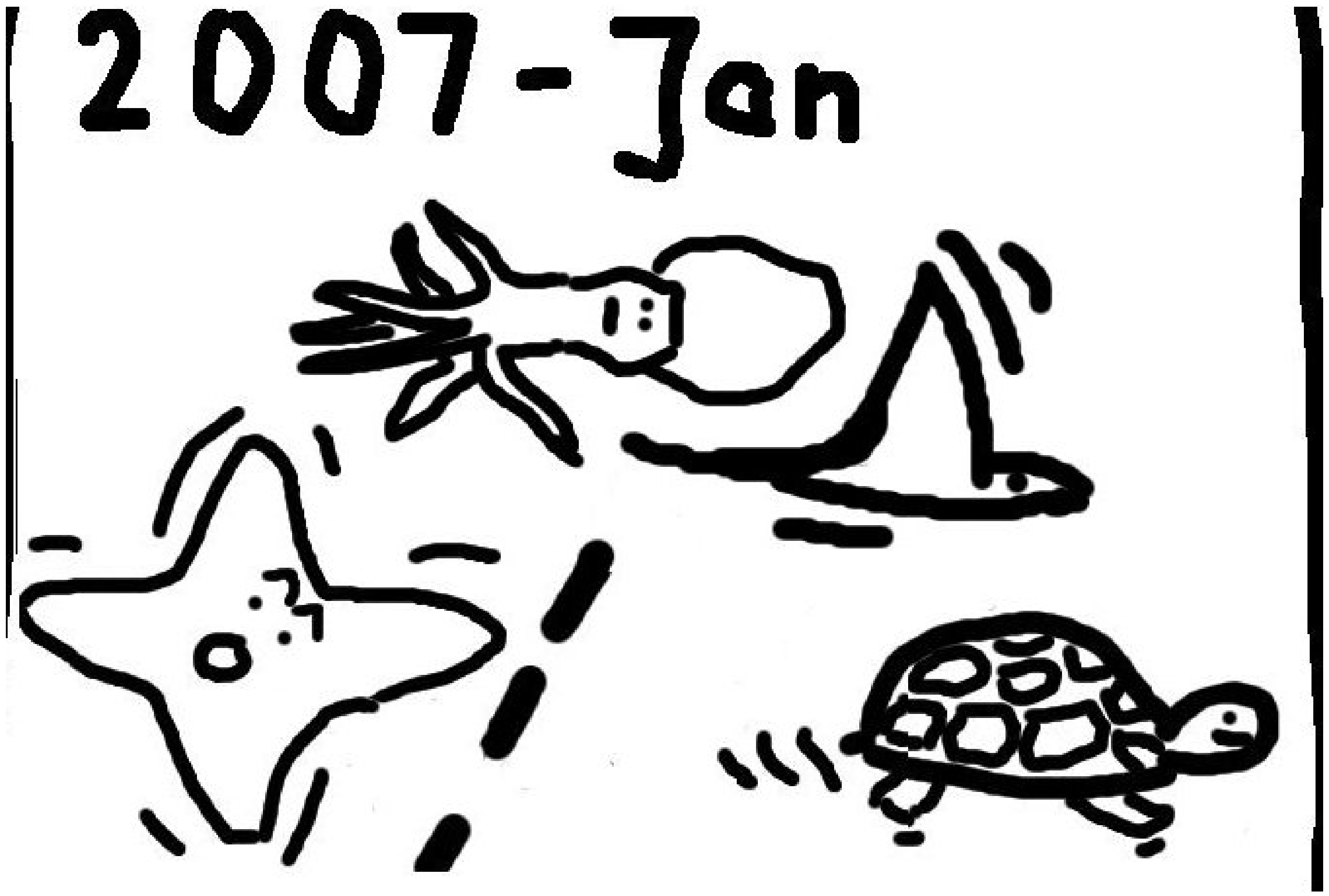
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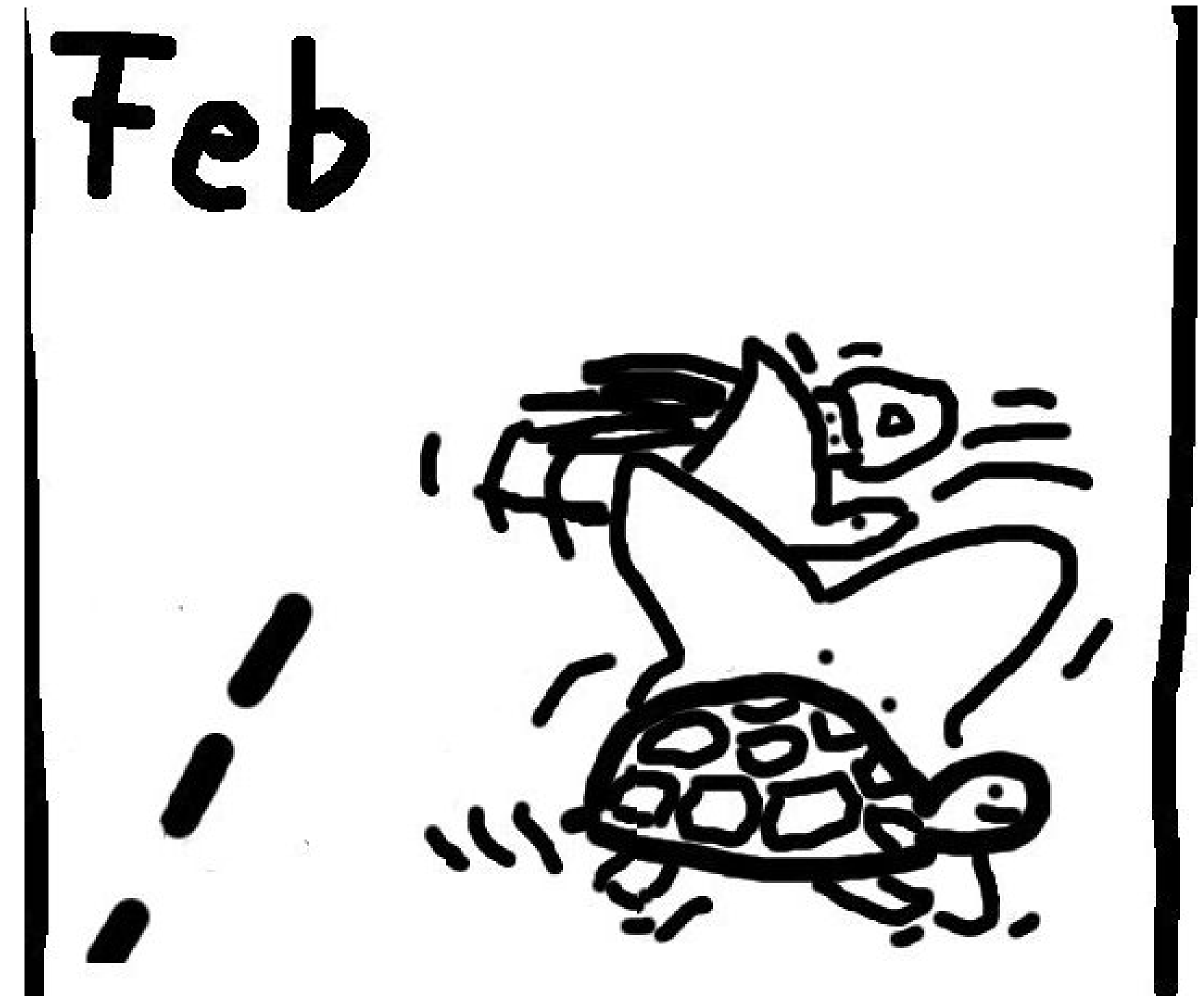
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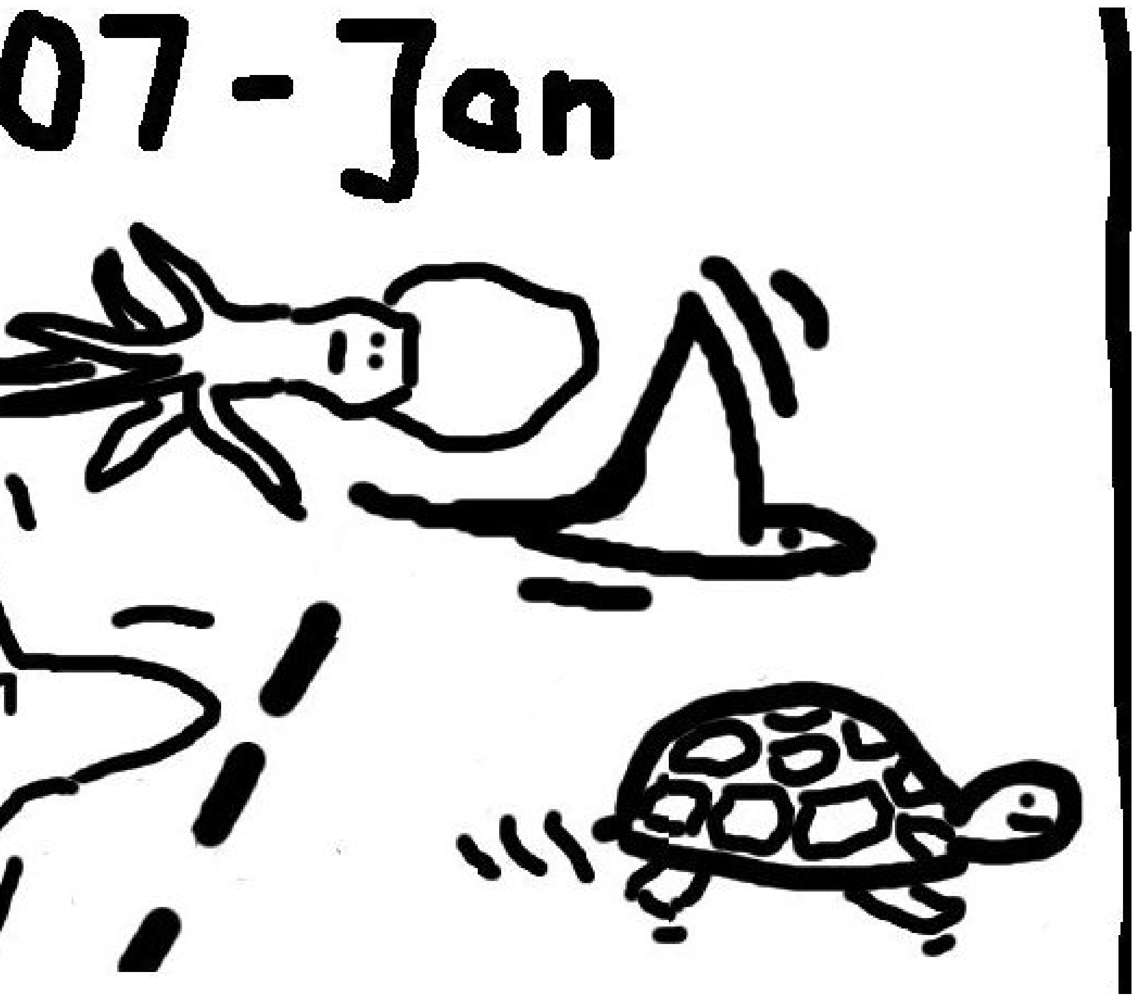
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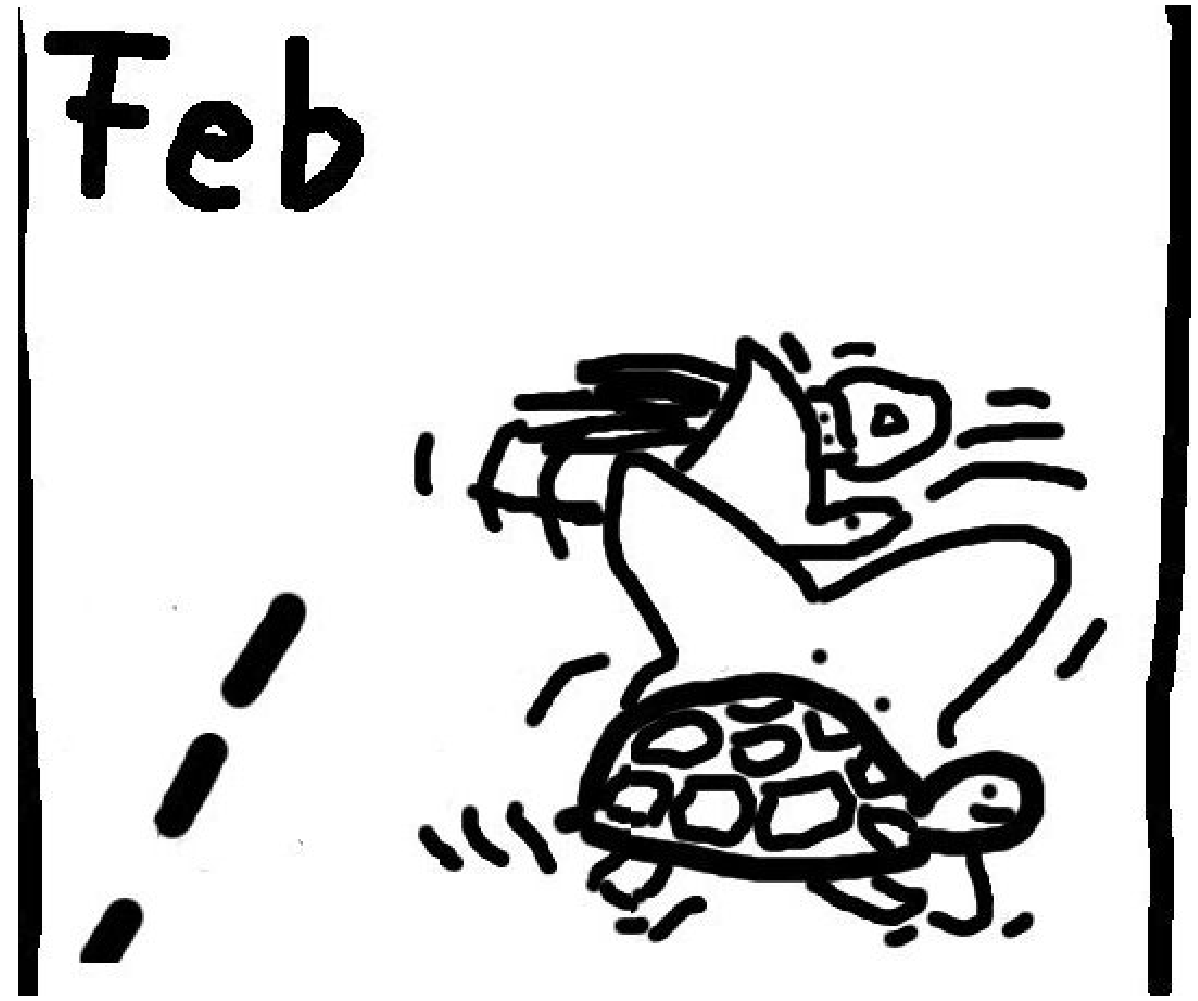
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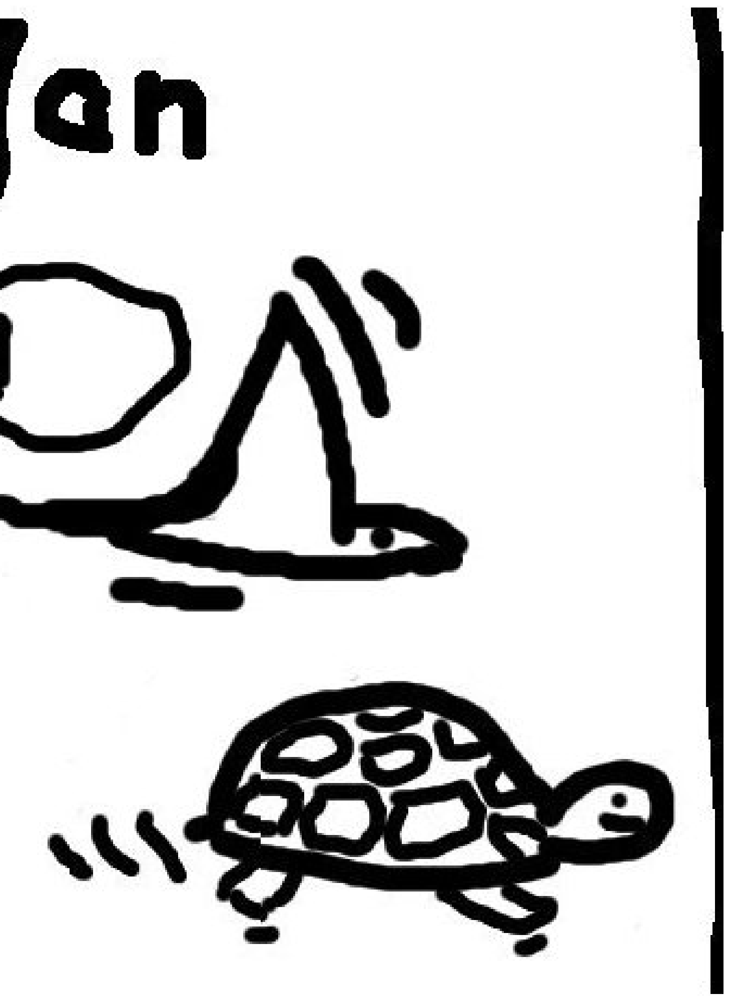
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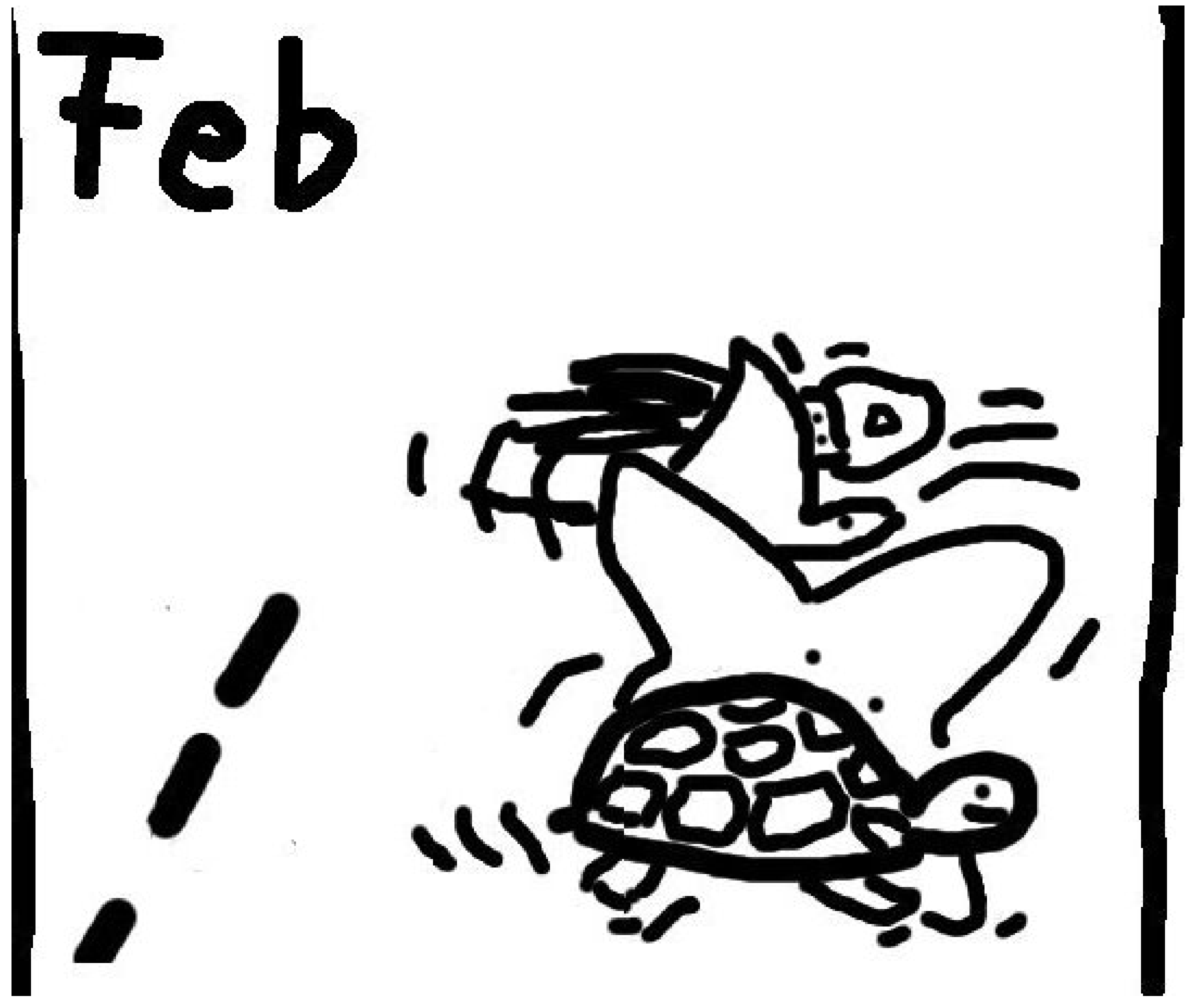
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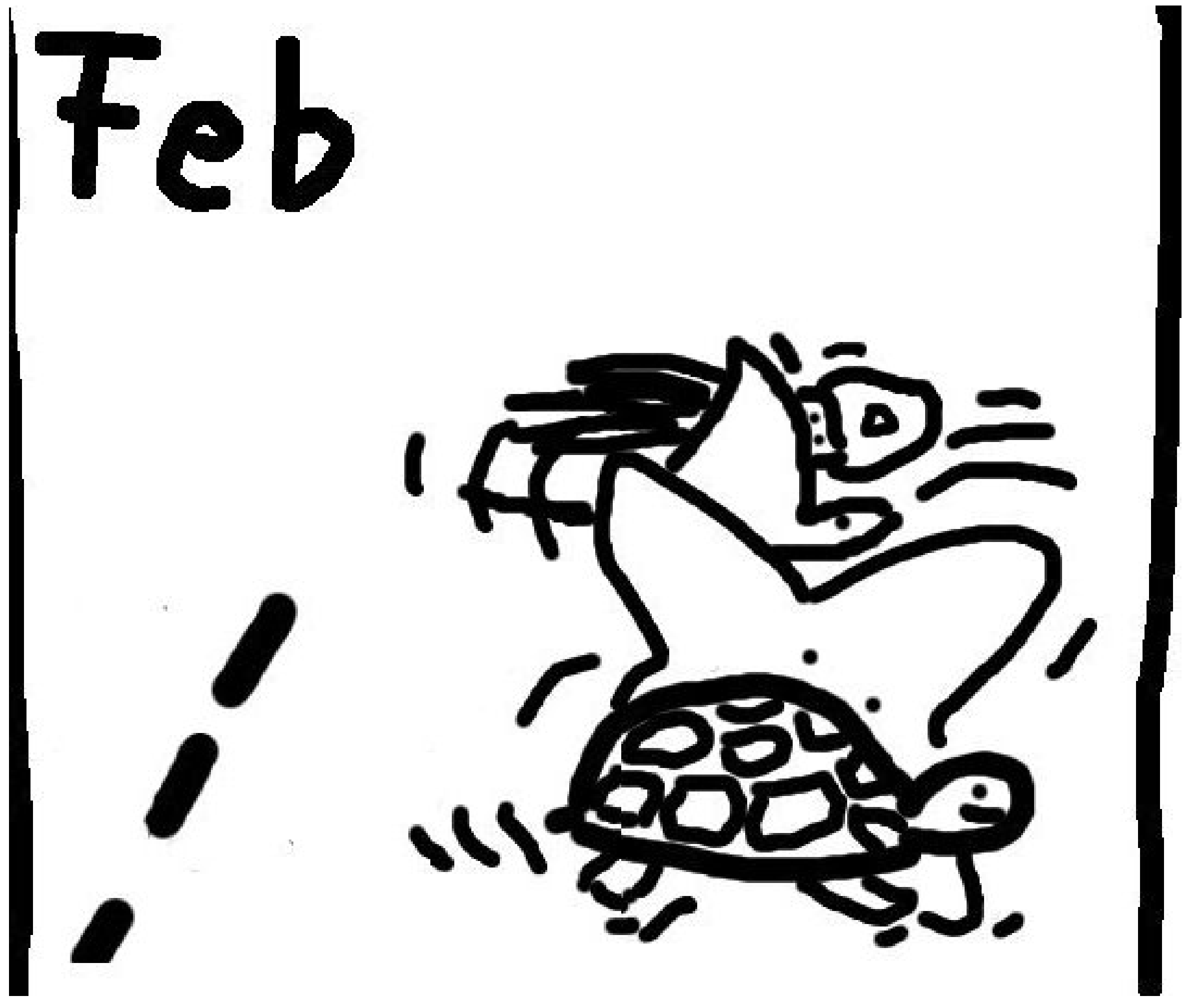
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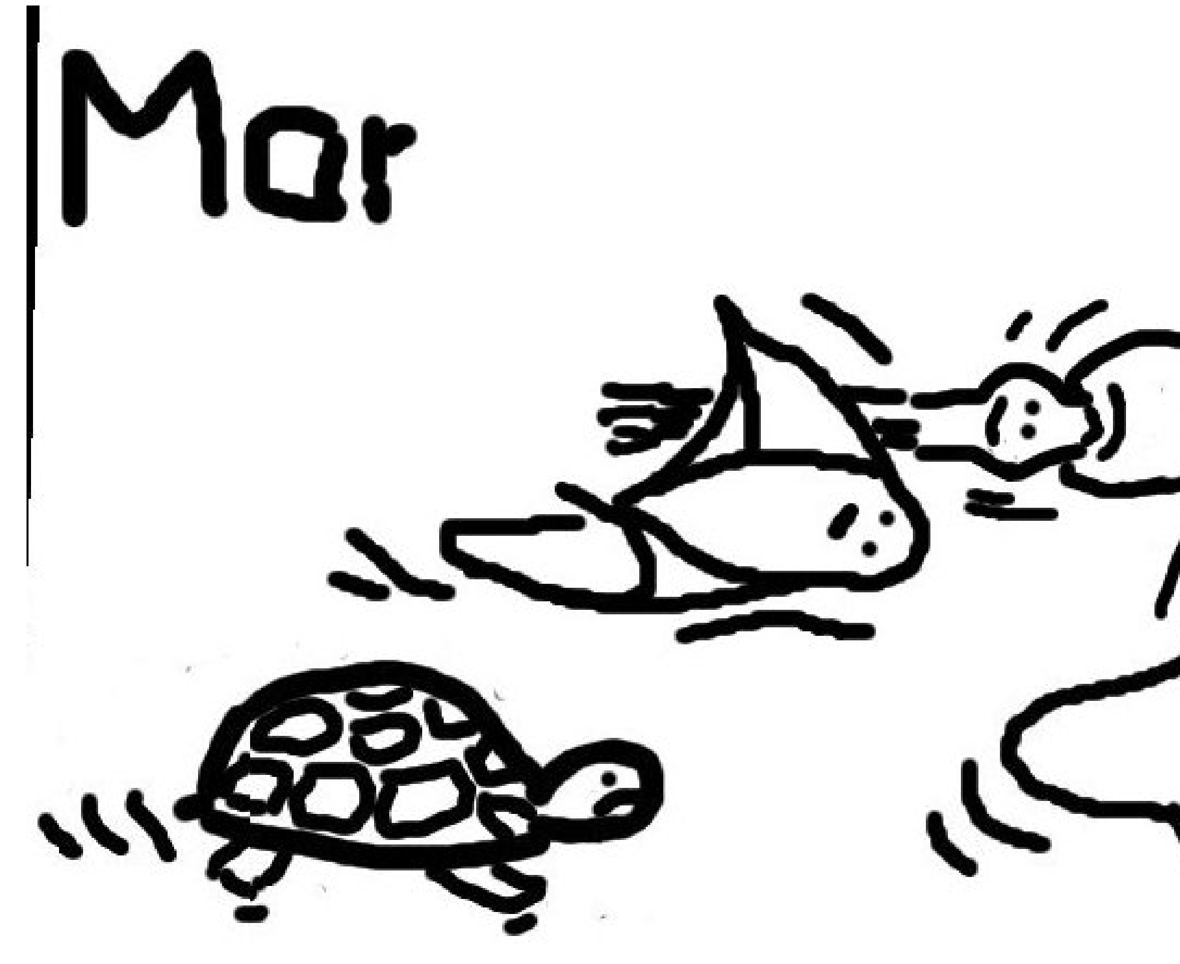
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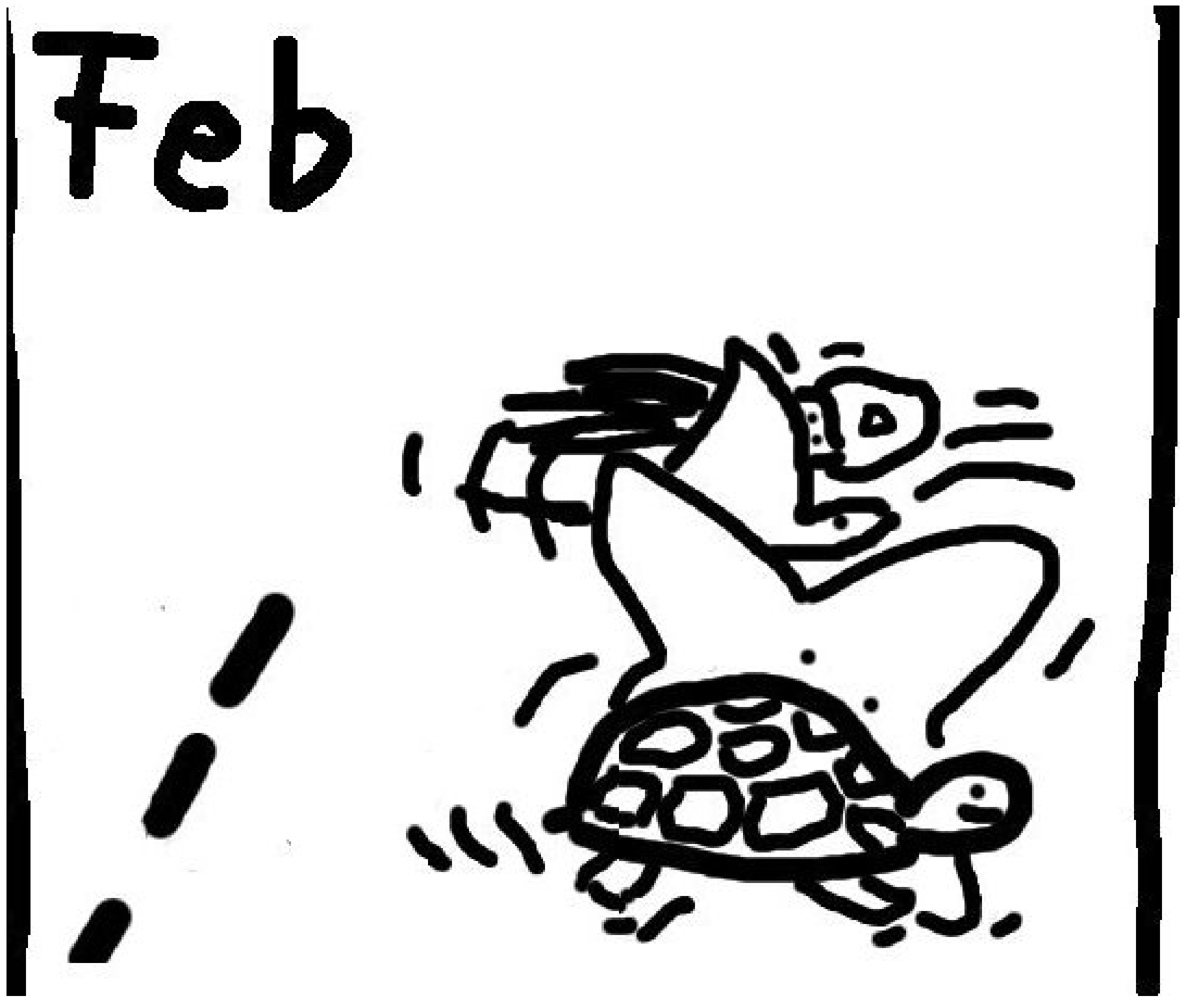
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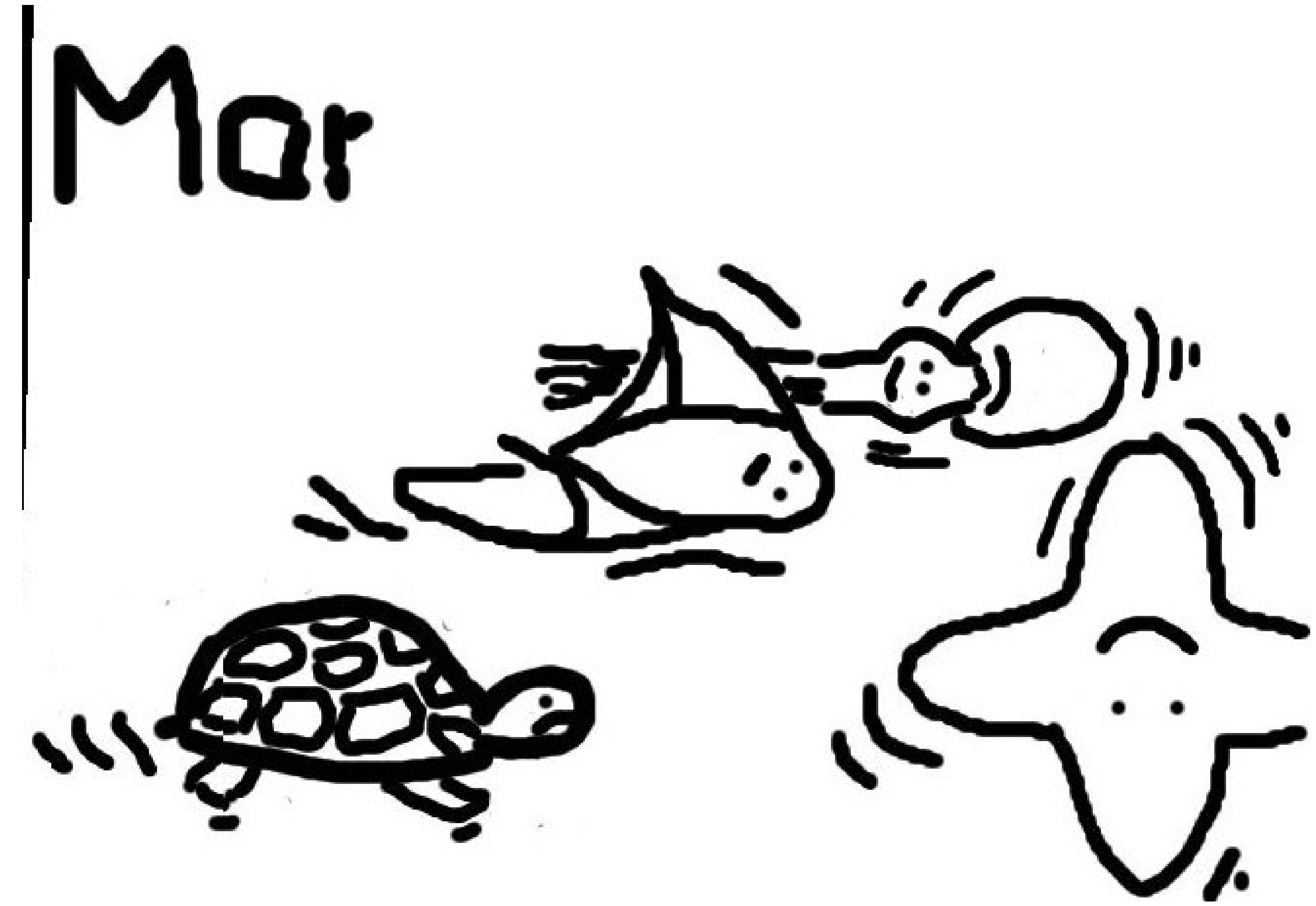
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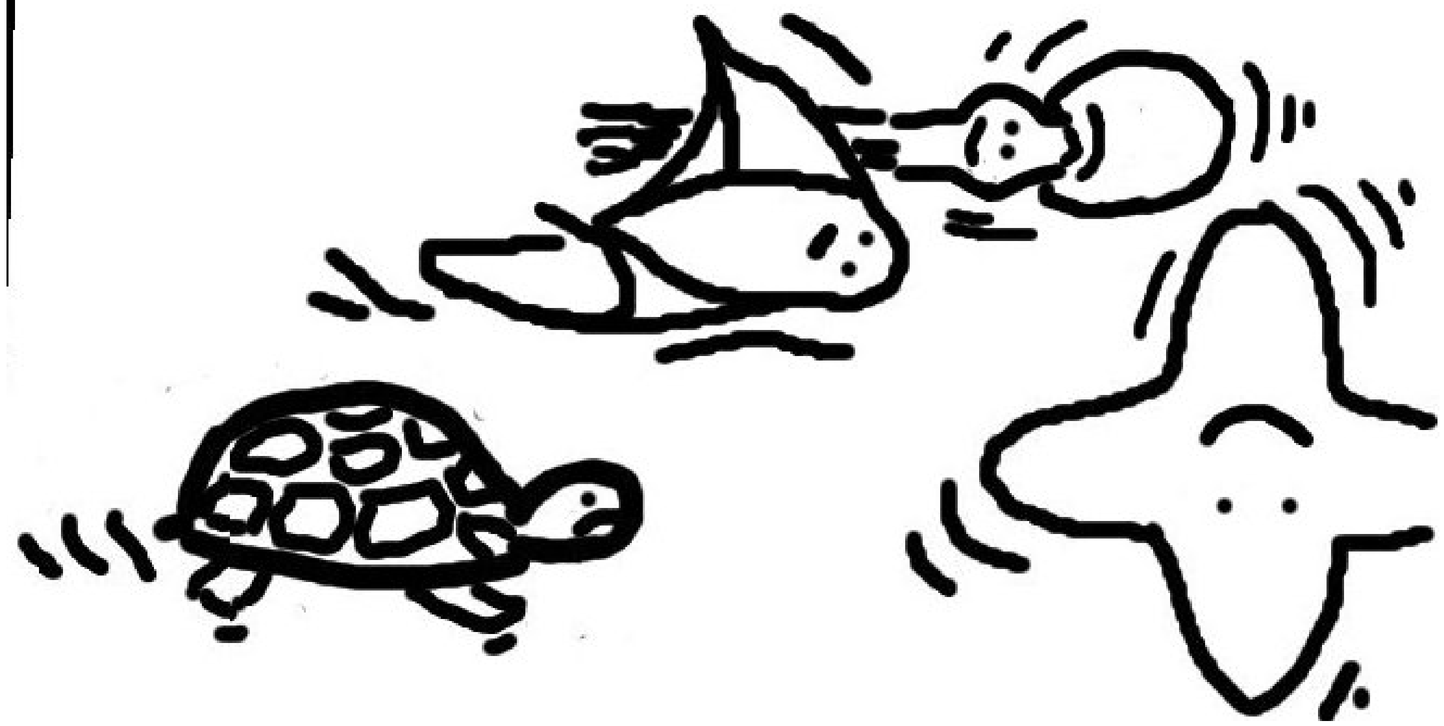

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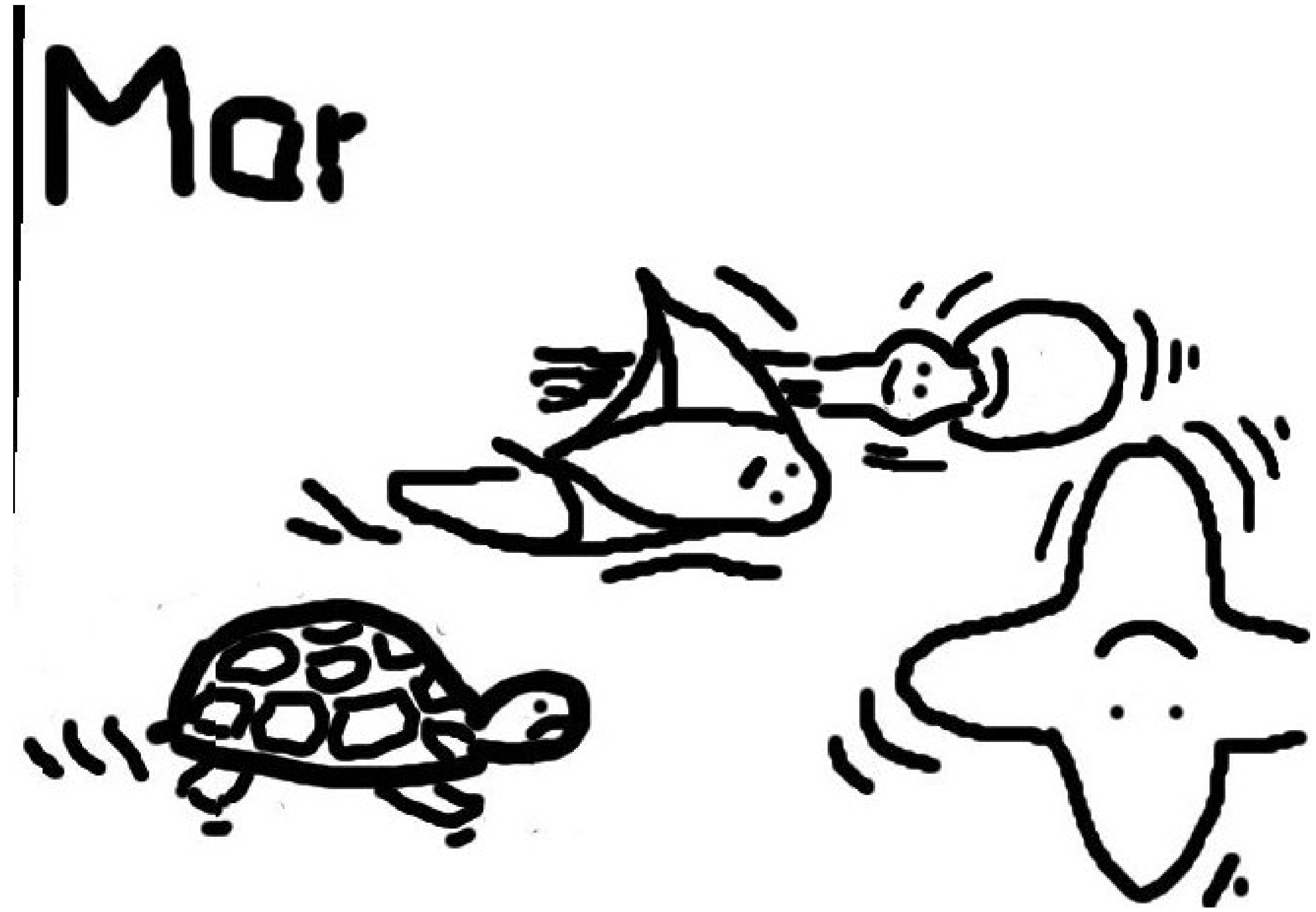

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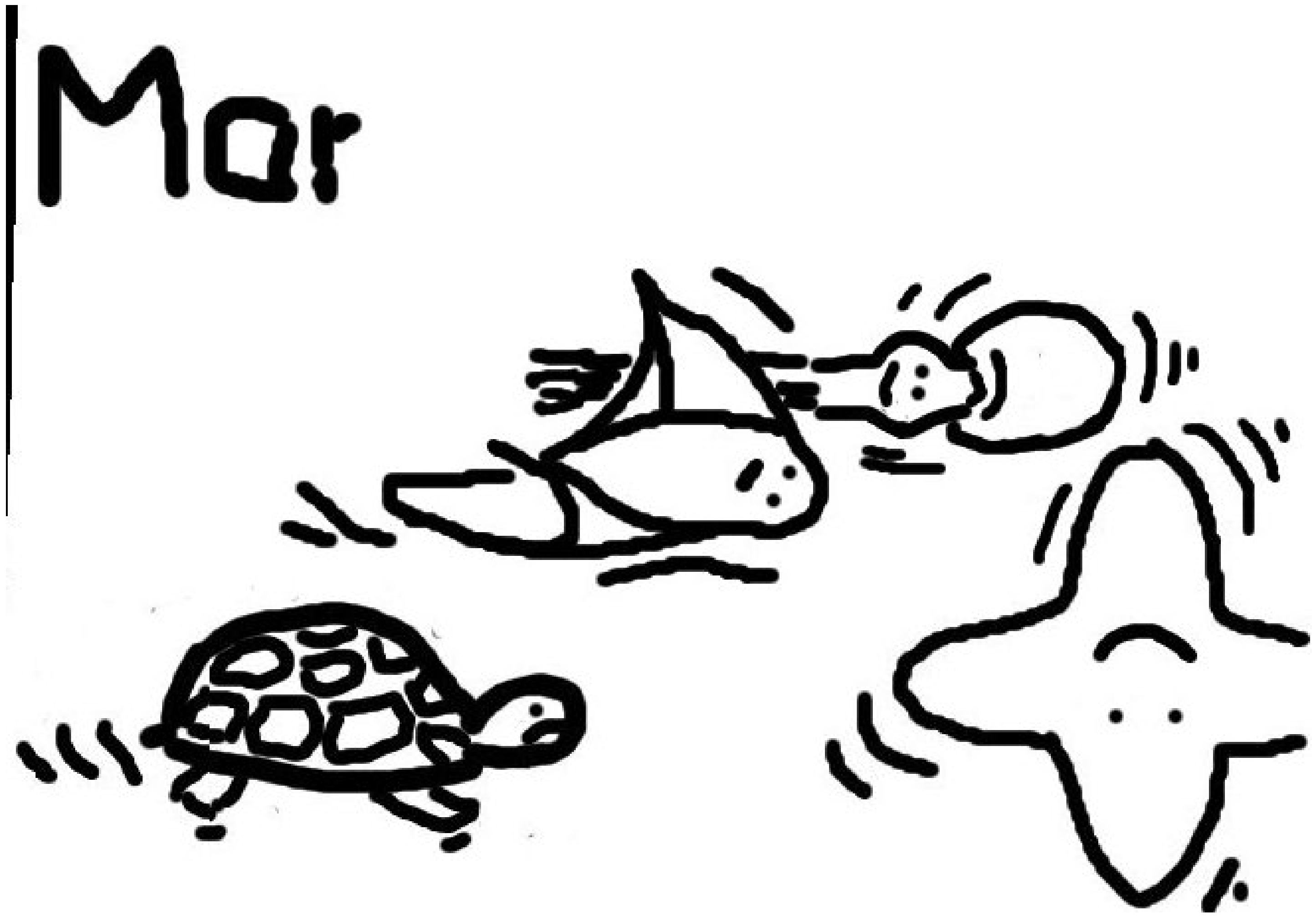

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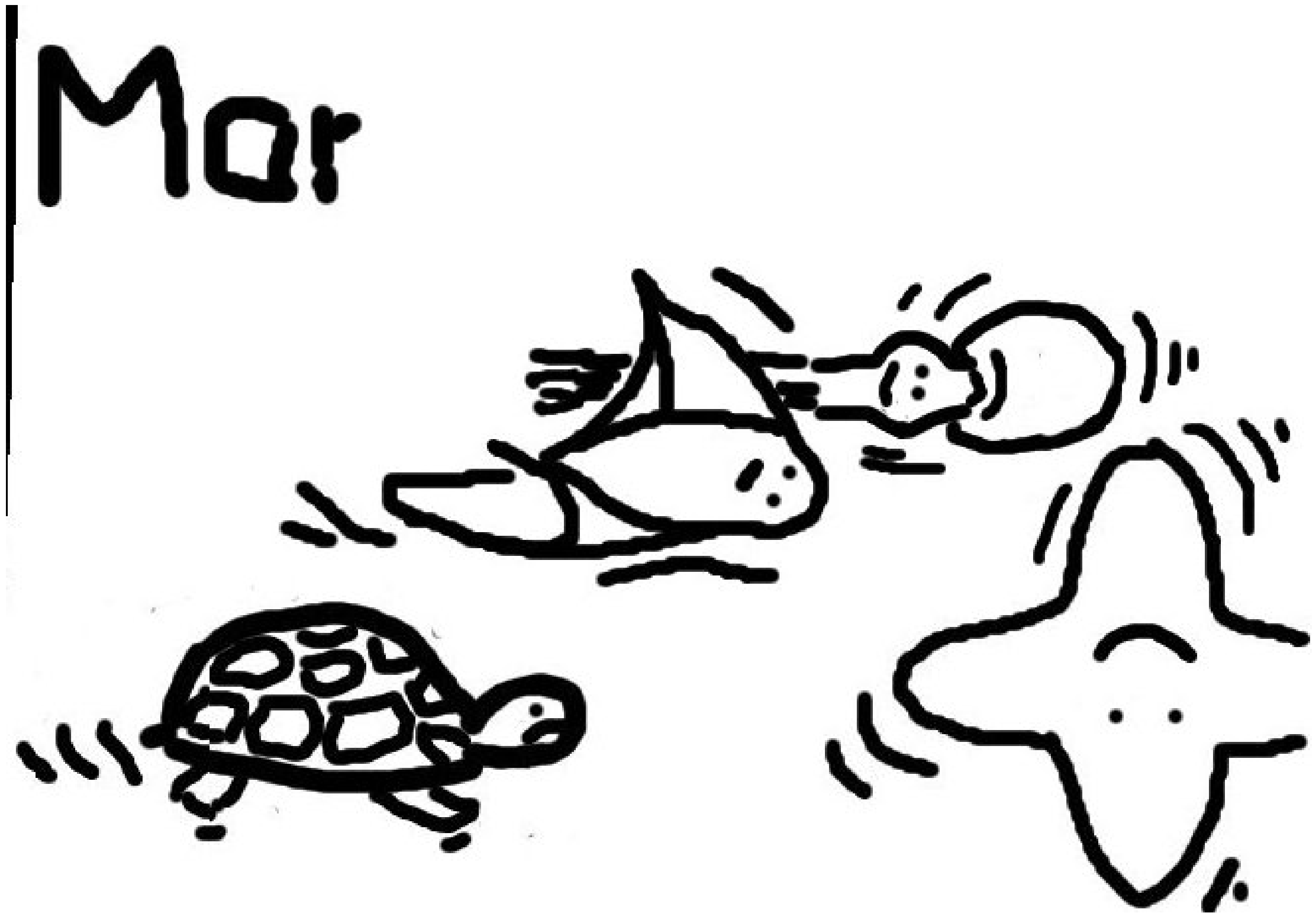

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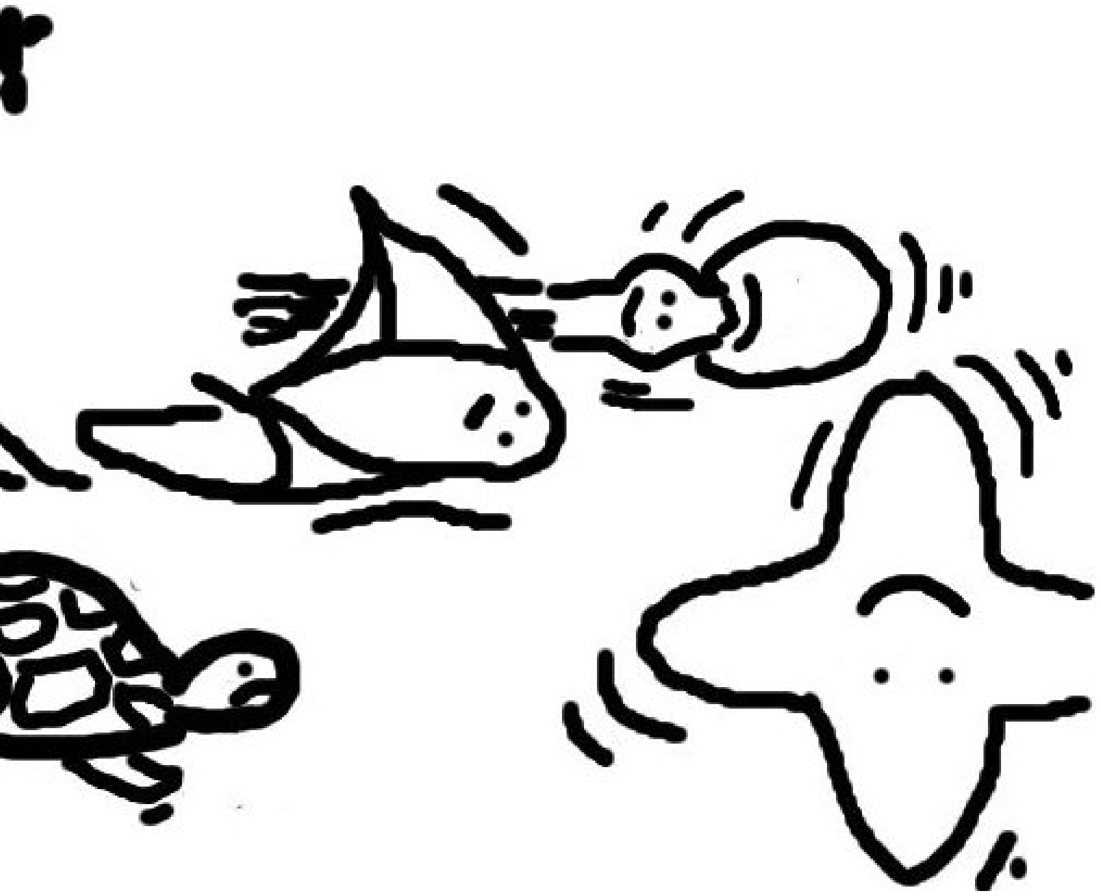

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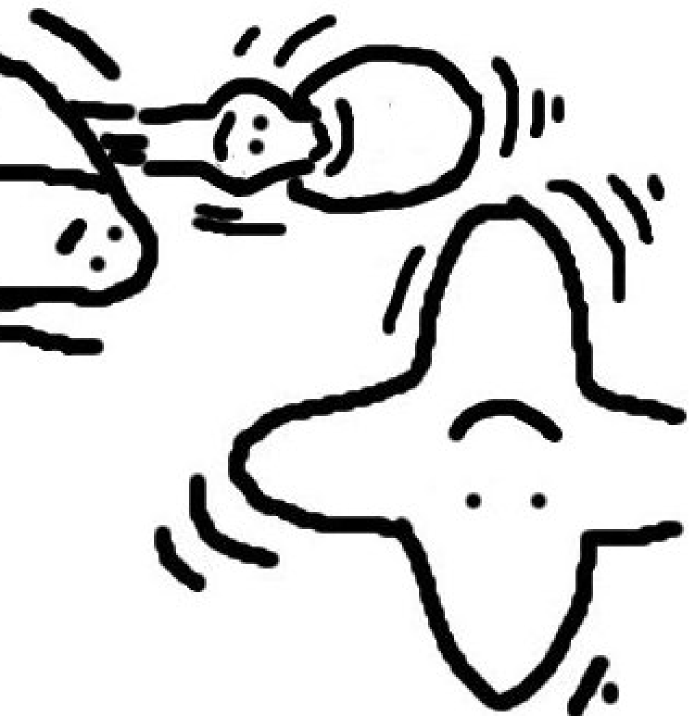

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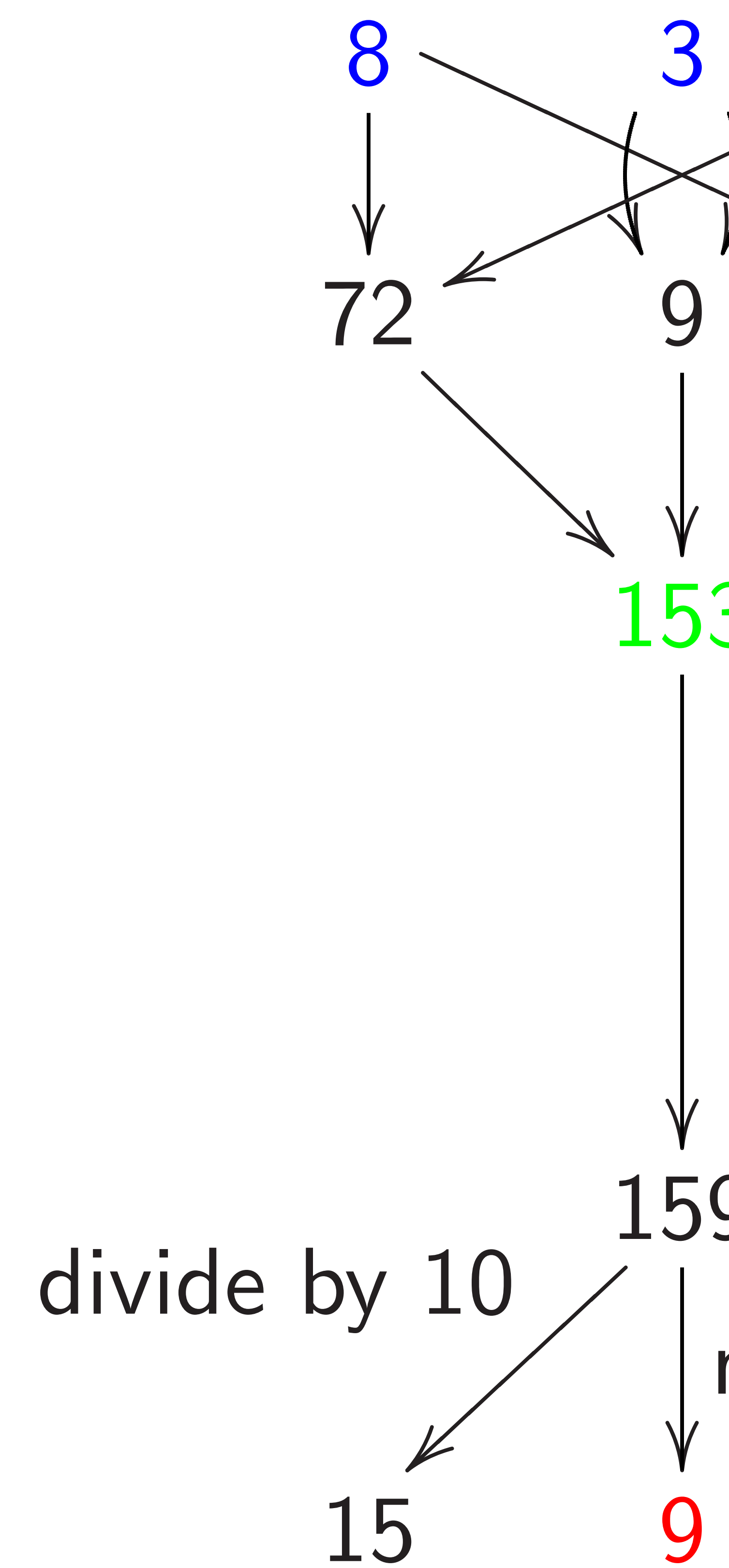

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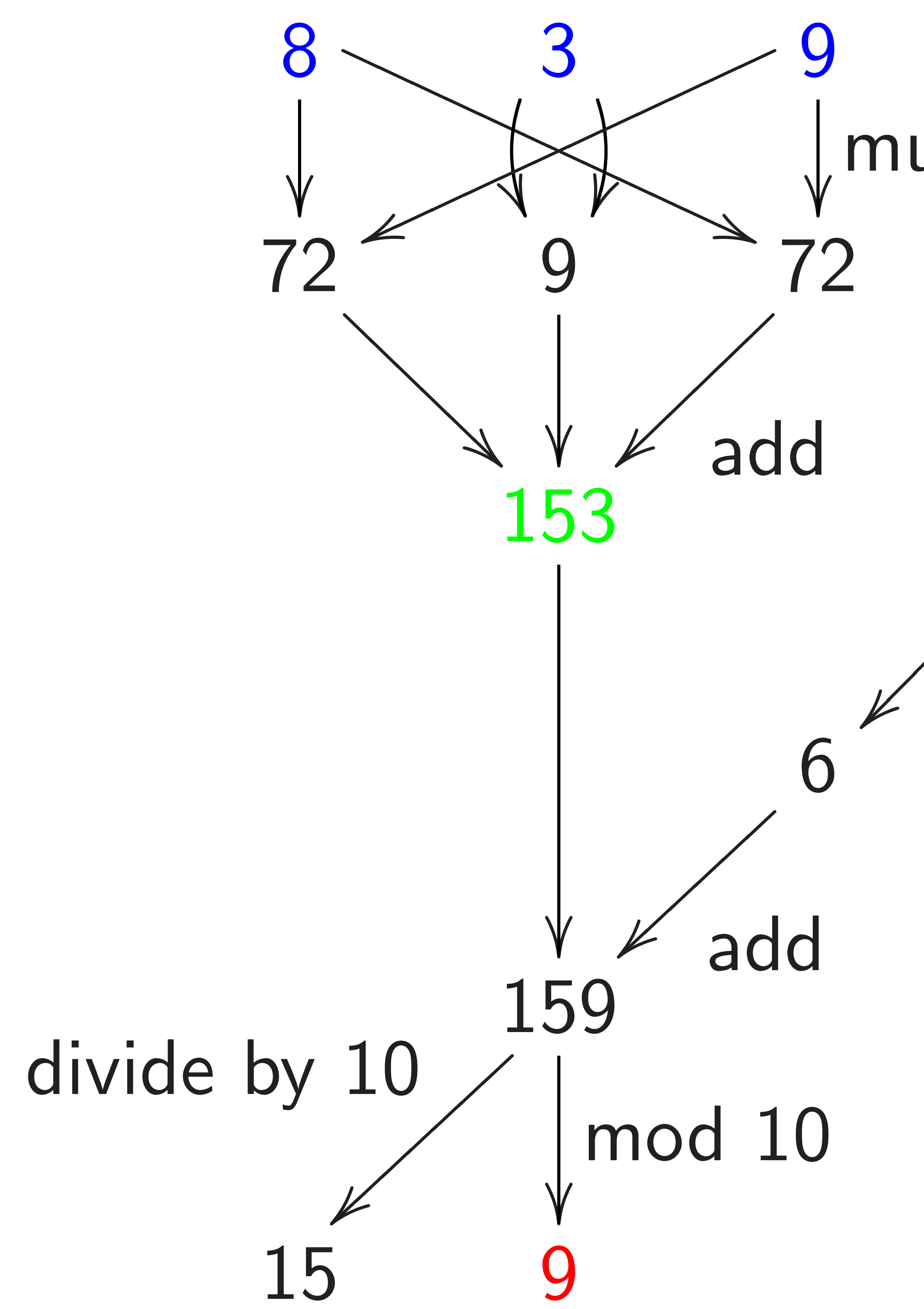

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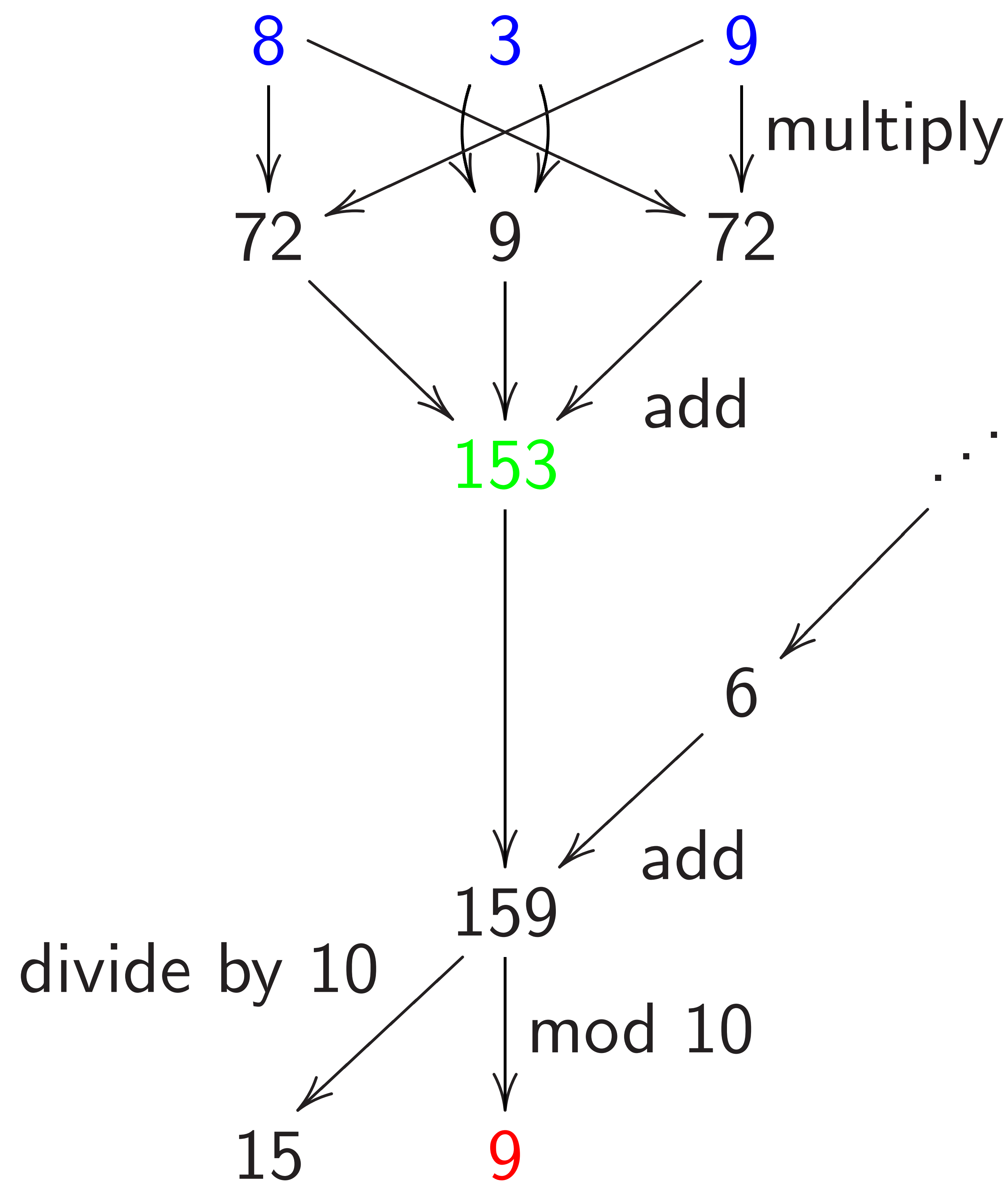

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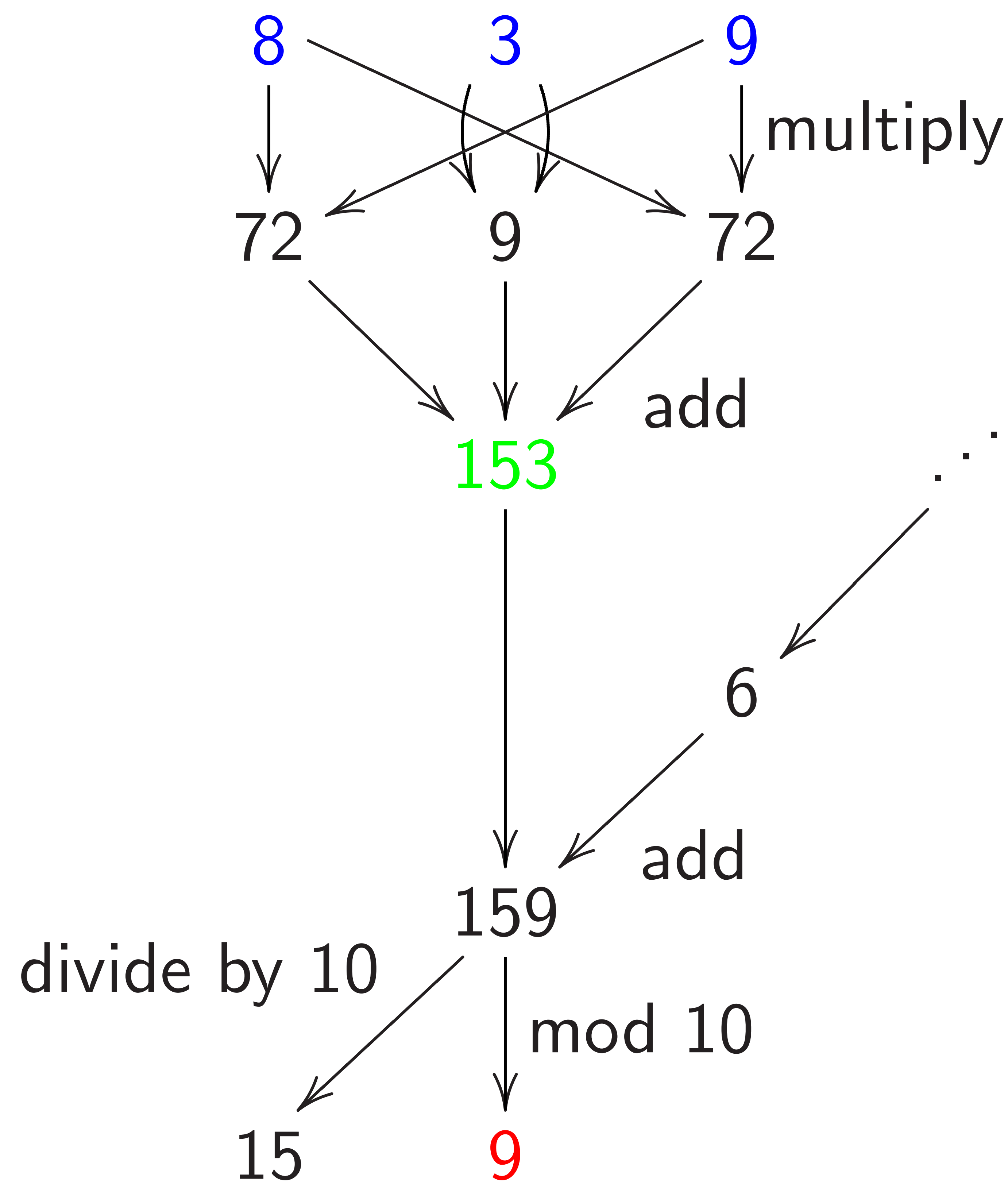

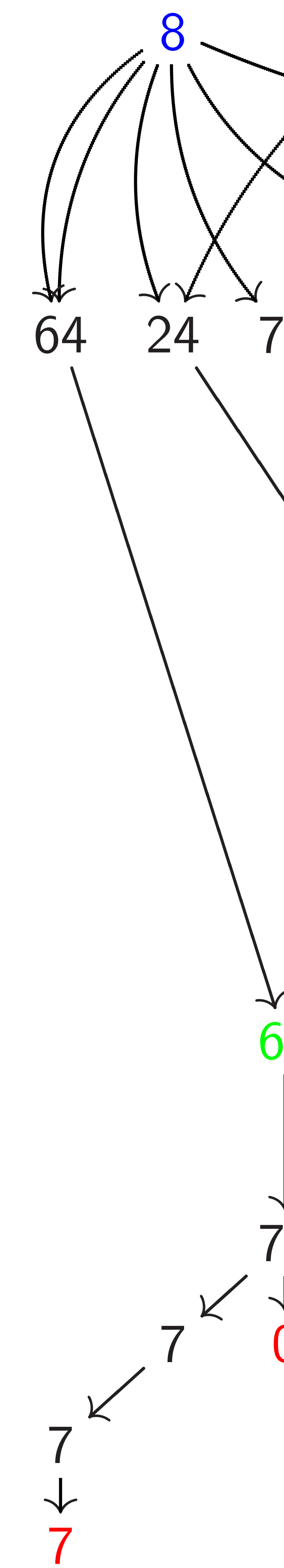

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839:

$t^2 + 54t^1 + 81t^0$;

$t^2 + 62t^1 + 1t^0$;

$t^2 + 2t^1 + 1t^0$;

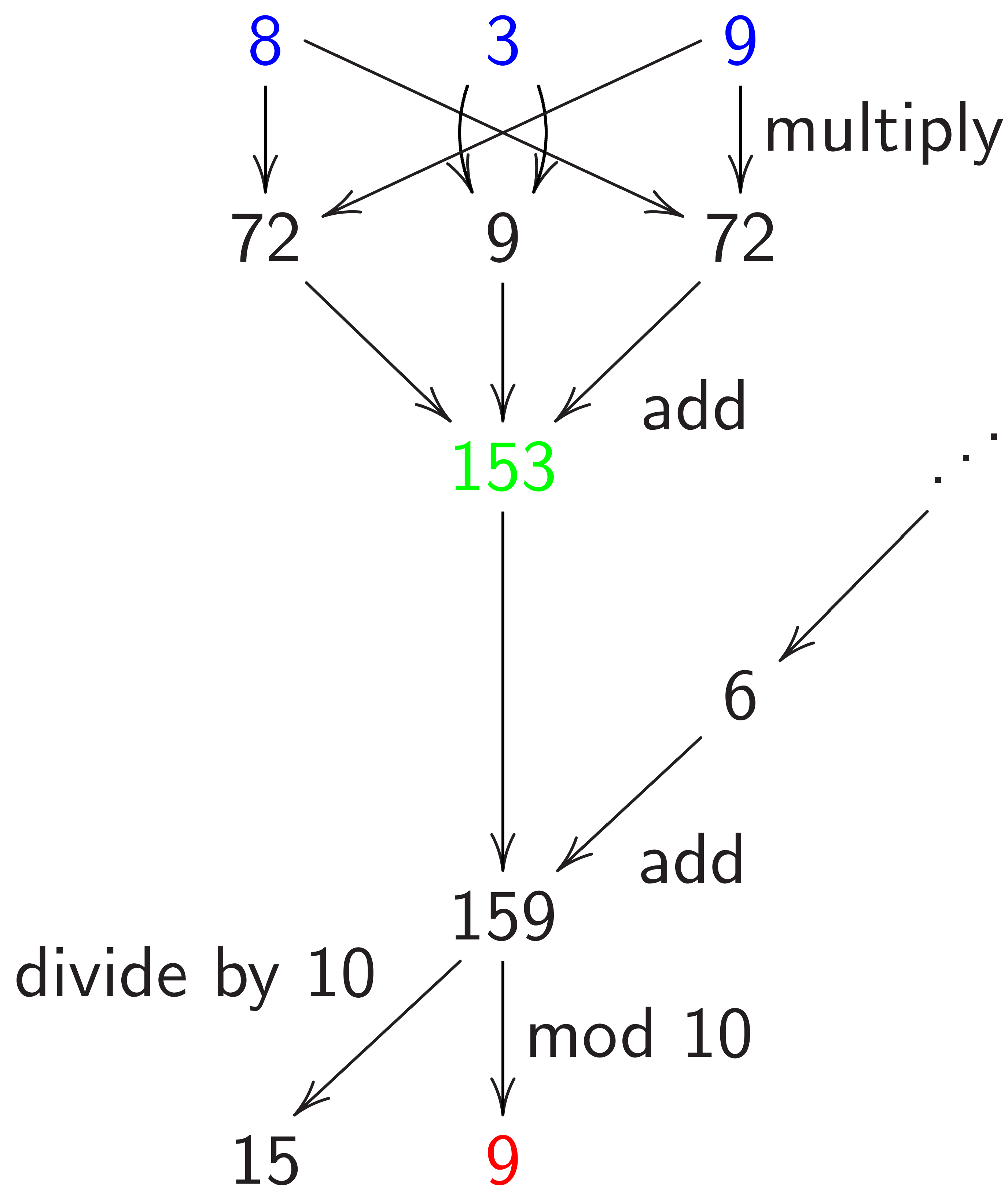
$+ 2t^1 + 1t^0$;

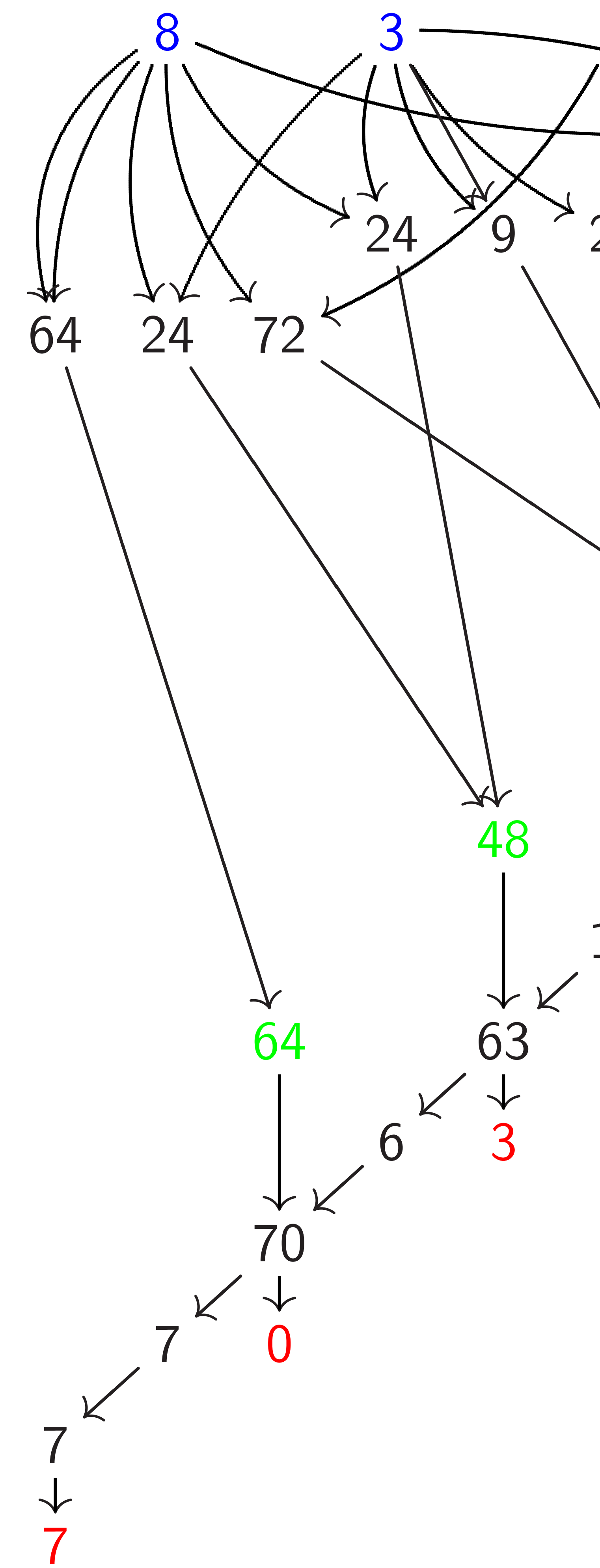
$- 2t^1 + 1t^0$;

$9t^2 + 2t^1 + 1t^0$.

$9^2 = 703921$.

What operations were used here?





$10)t^j$.

$+\ 81t^0$;

$+\ 1t^0$;

$-\ 1t^0$;

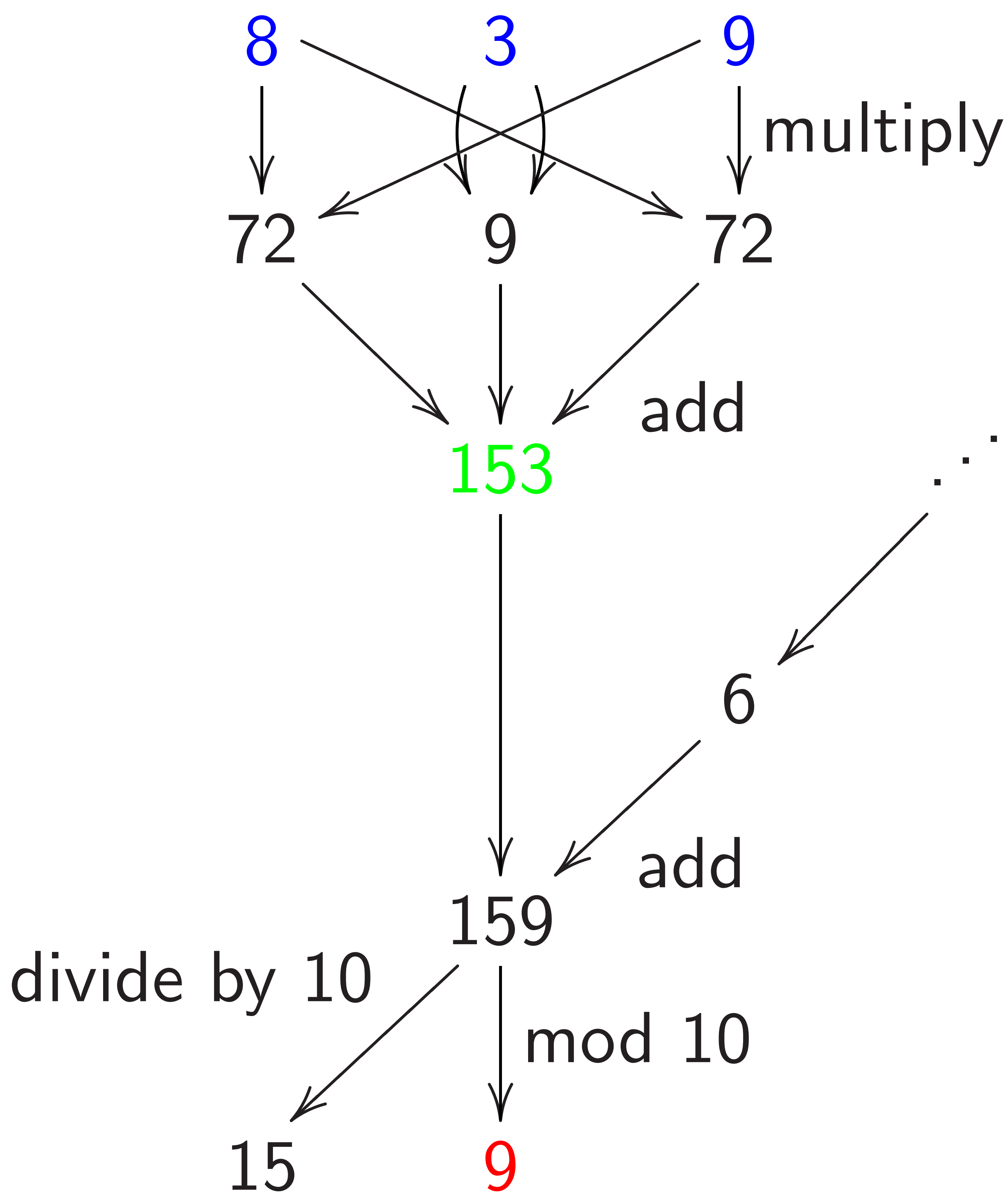
t^0;

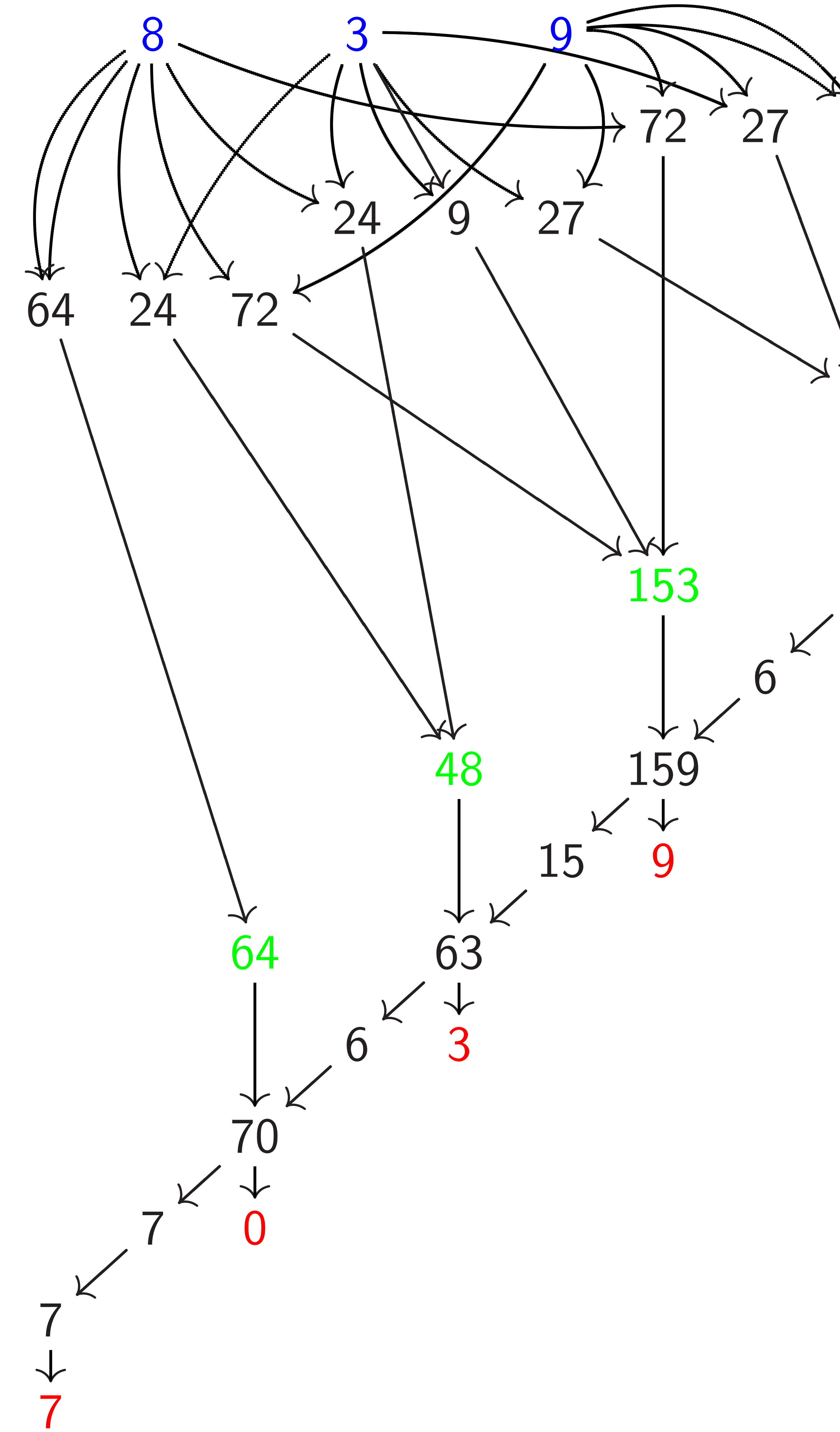
^{0};

$+\ 1t^0$.

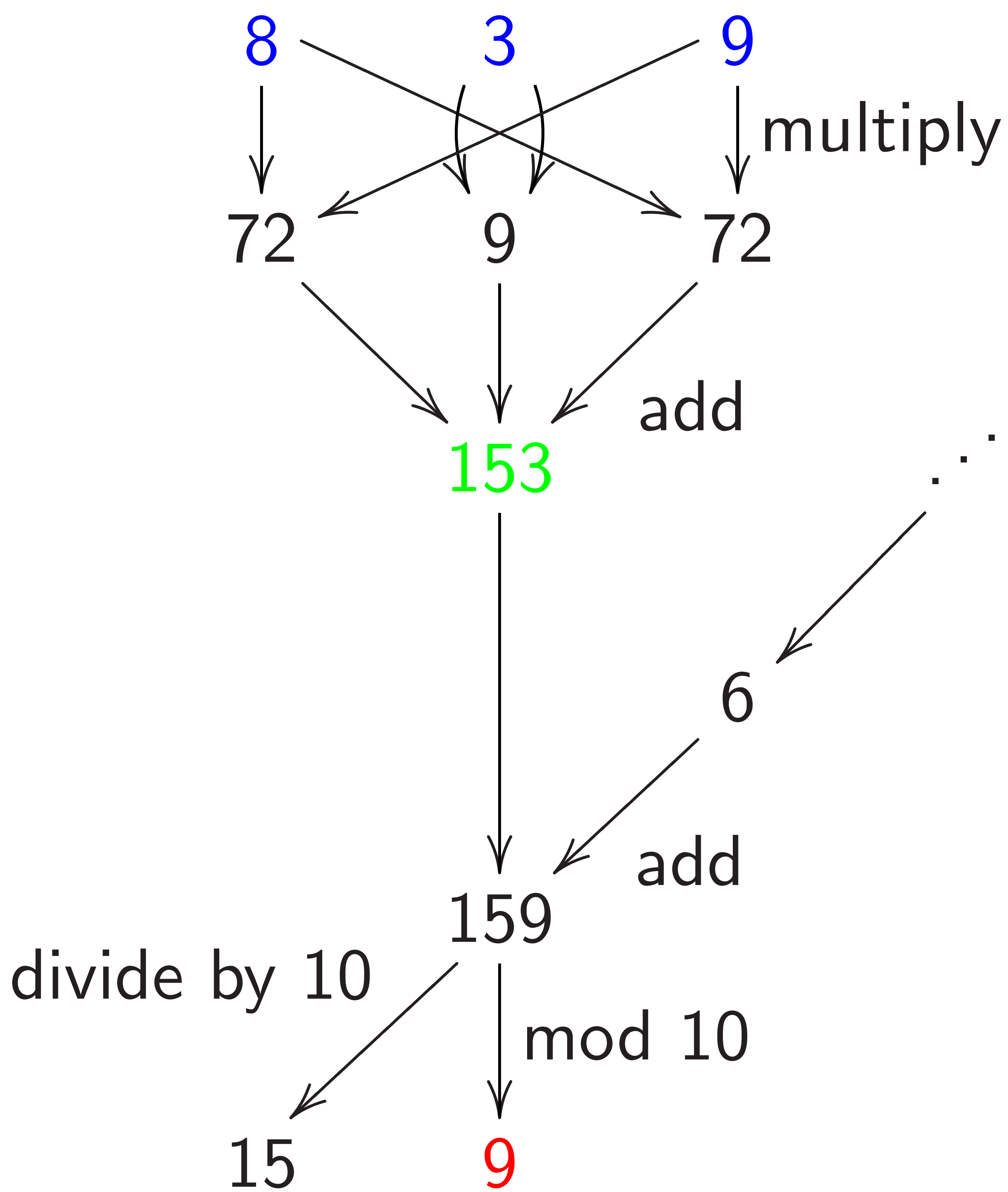
921.

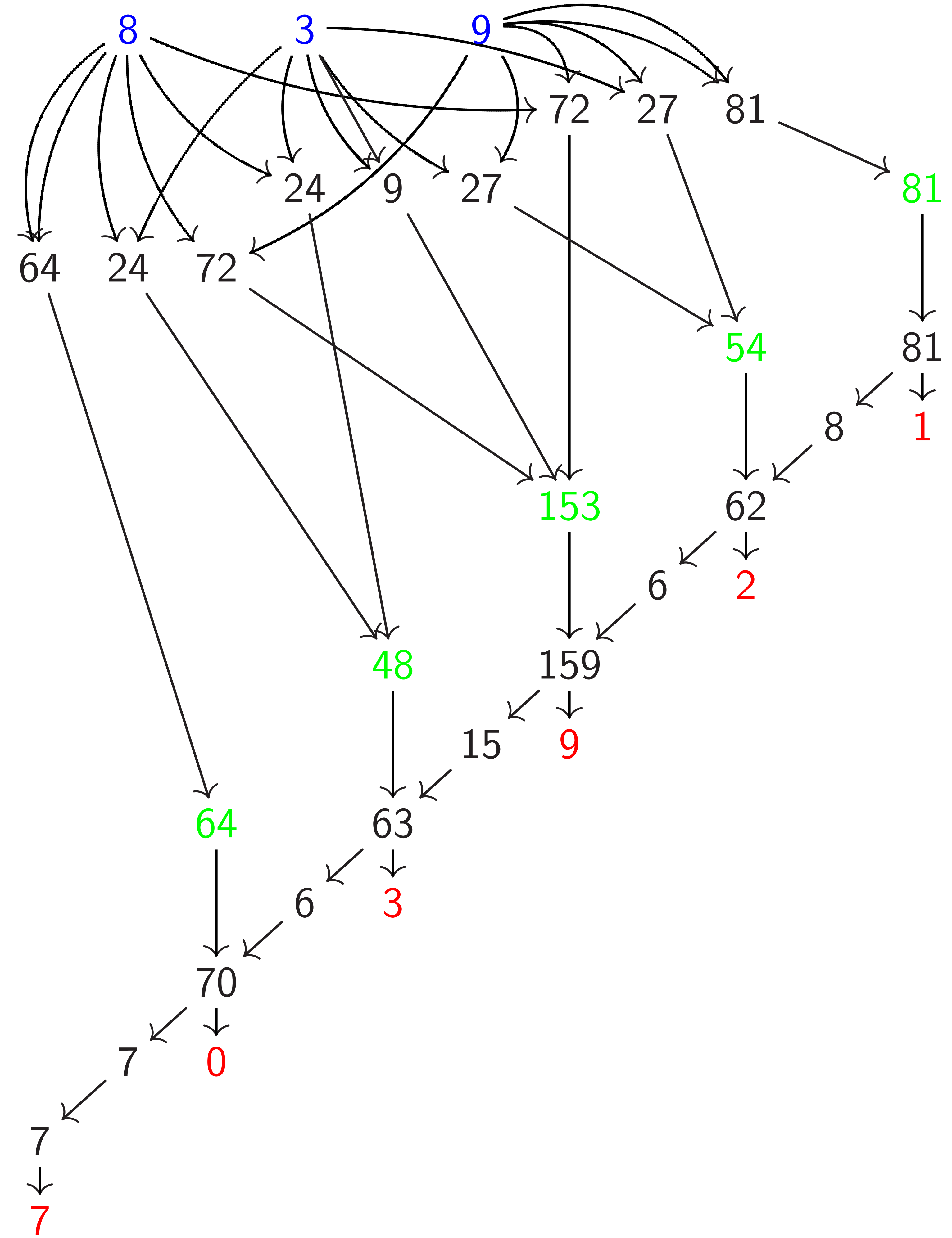
What operations were used here?





What operations were used here?





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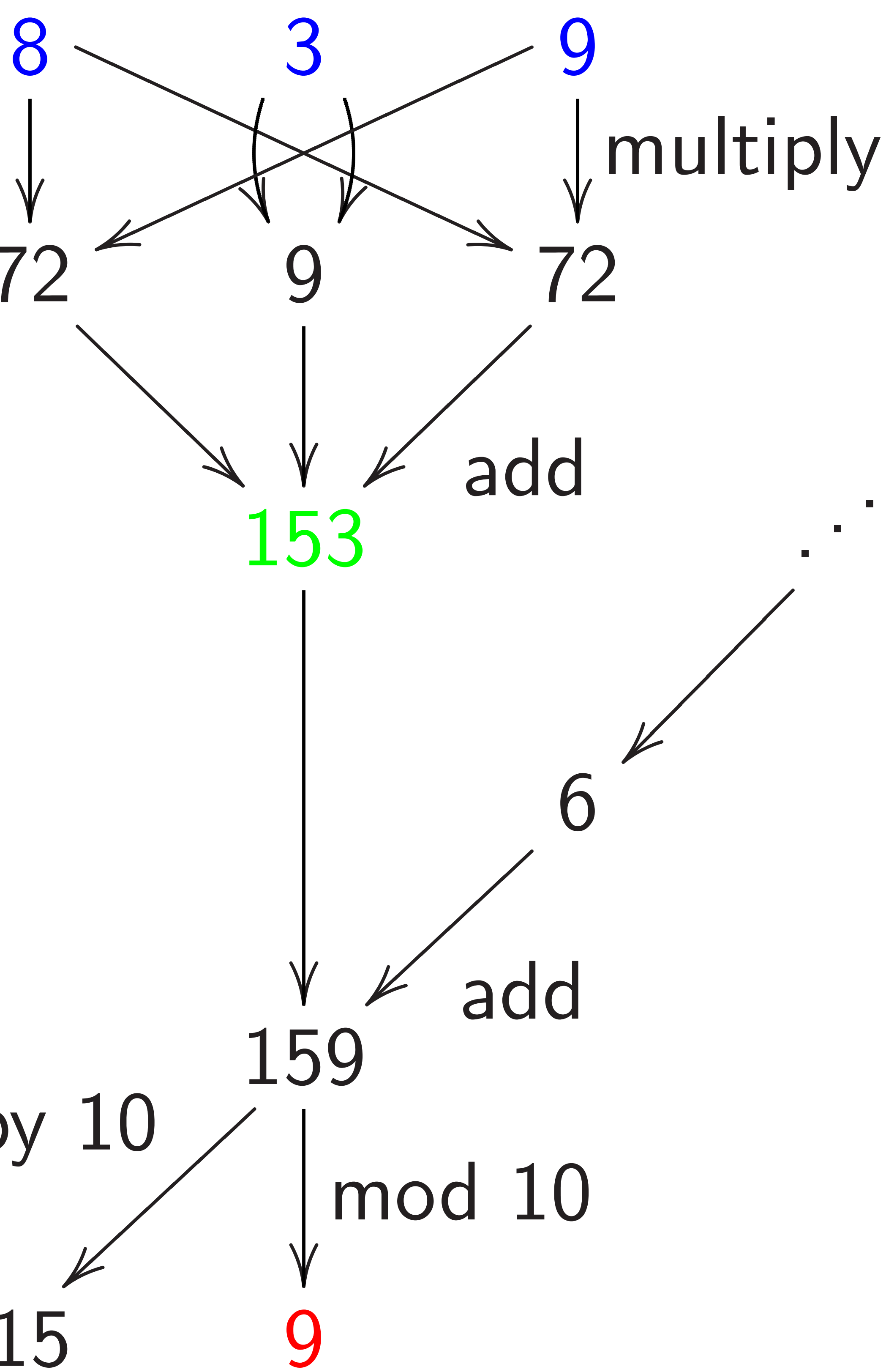
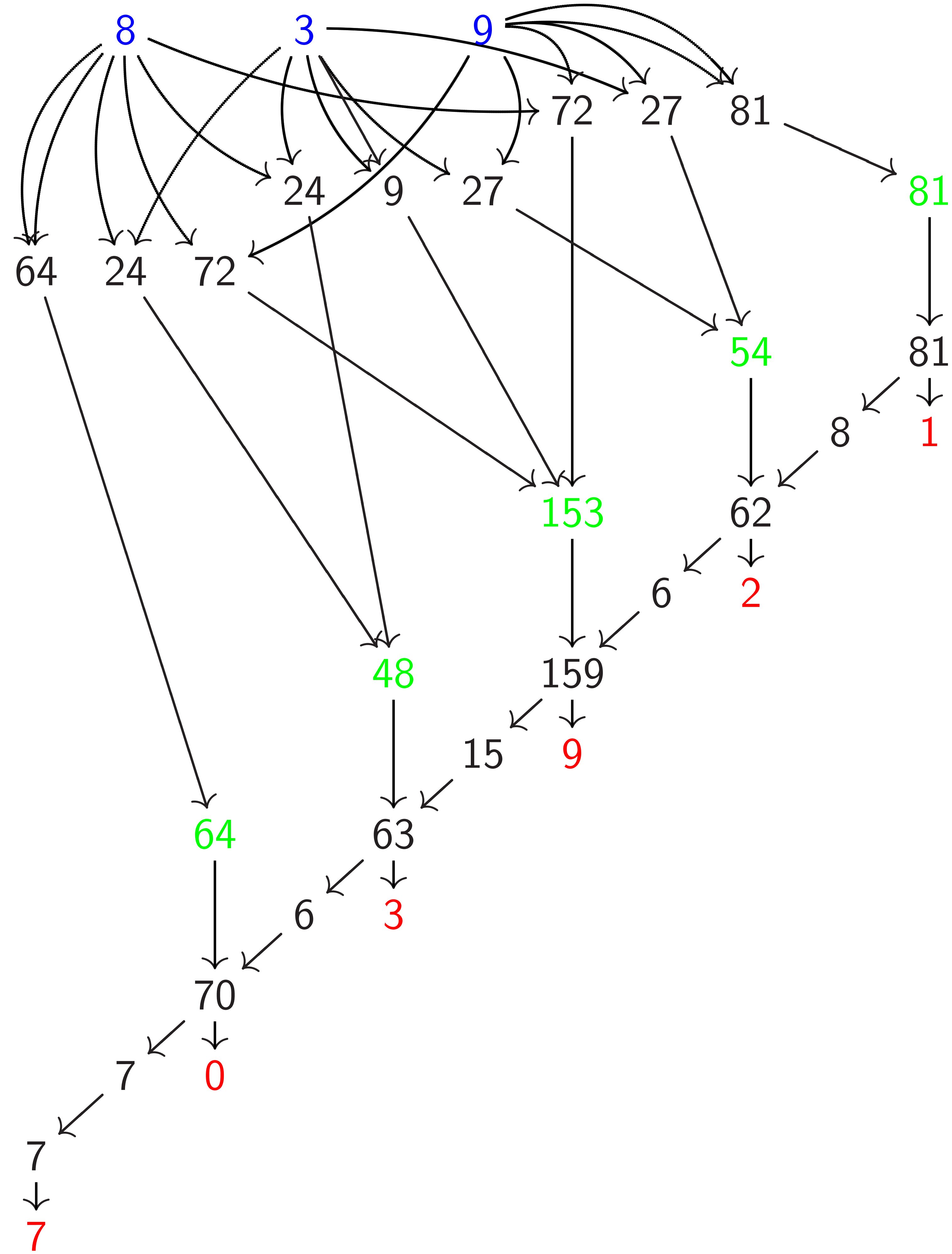




The scal

839 = 8

value (a

$800t^2 +$

Squaring

640000t

$540t^1 +$

Carrying

640000t

$540t^1 +$

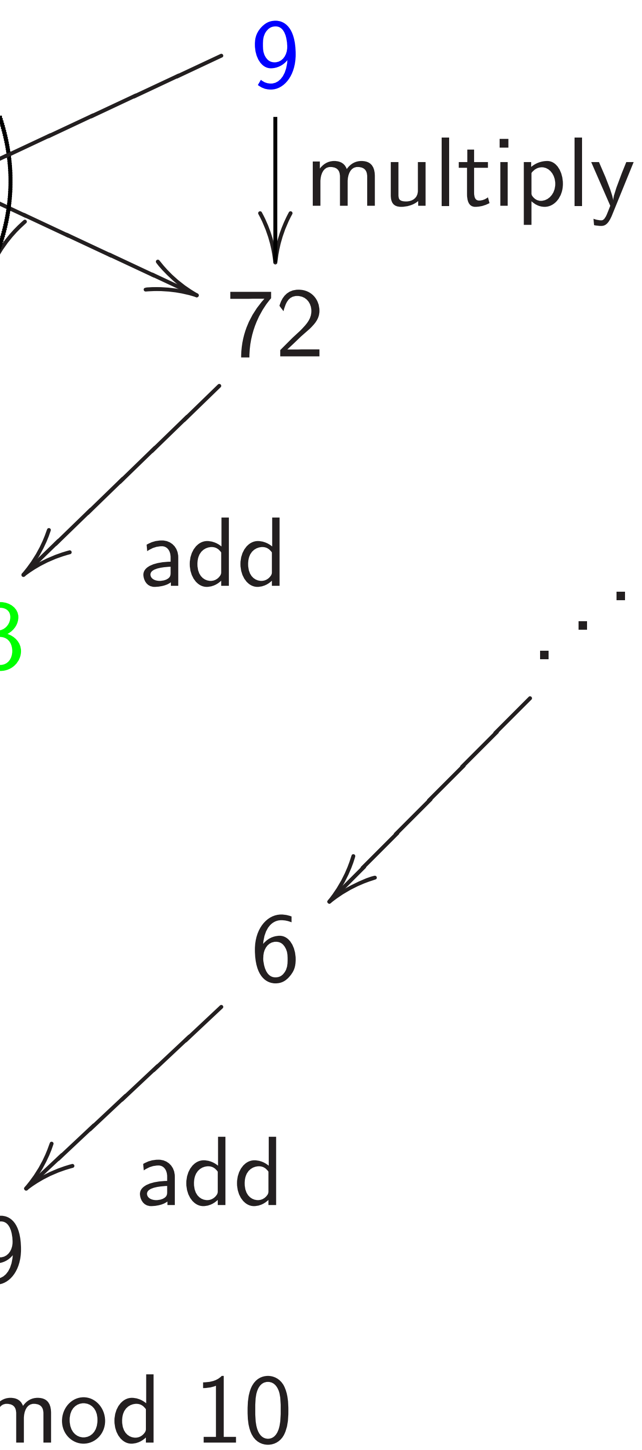
640000t

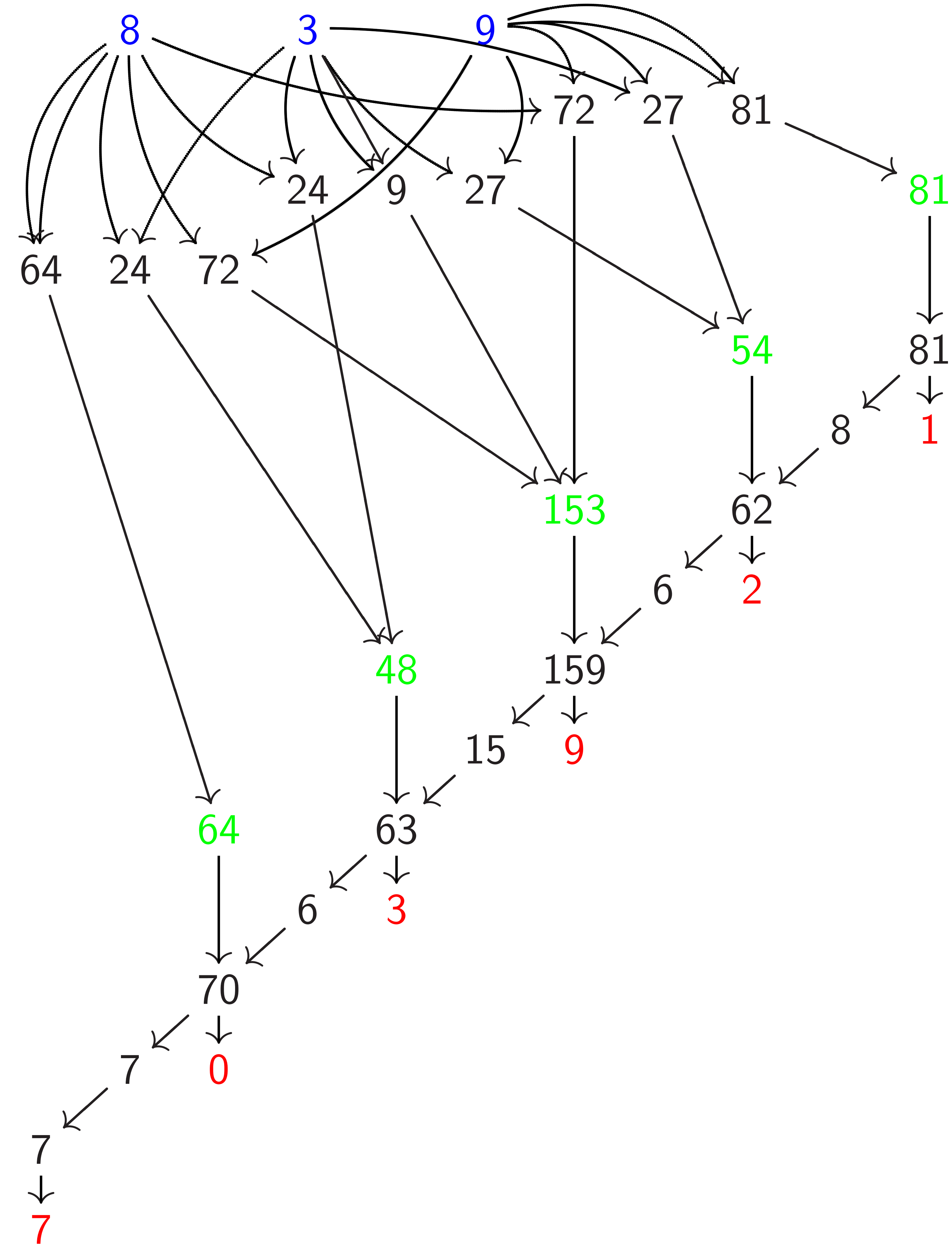
$620t^1 +$

700000t

$20t^1 + 1$

were used here?





<u>The scaled variatio</u>

$839 = 800 + 30 +$

value (at $t = 1$) of

$800t^2 + 30t^1 + 9t^0$

Squaring: $(800t^2 -$

$640000t^4 + 48000$

$540t^1 + 81t^0.$

Carrying:

$640000t^4 + 48000$

$540t^1 + 81t^0;$

$640000t^4 + 48000$

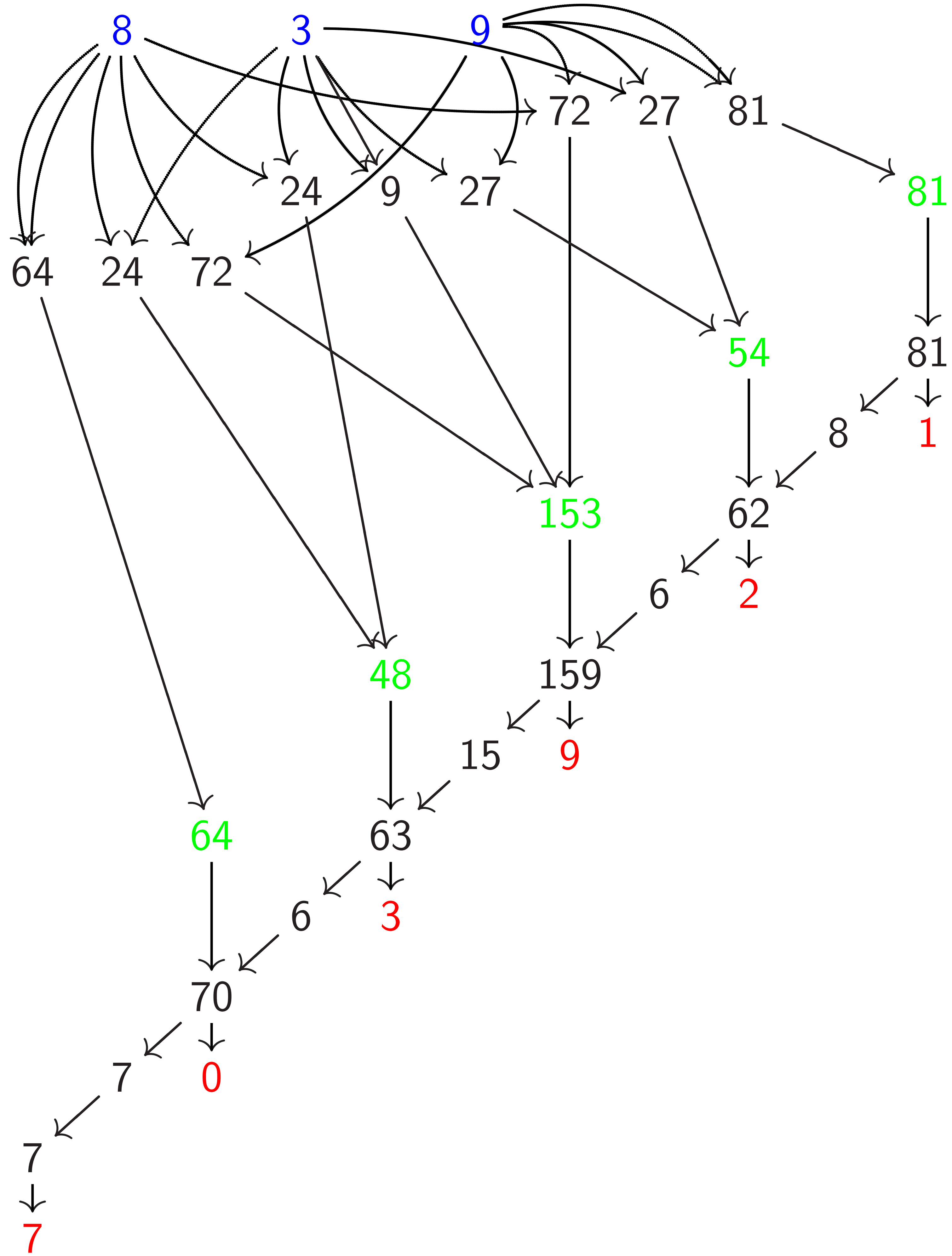
$620t^1 + 1t^0;$.

$700000t^5 + 0t^4 + 3$

$20t^1 + 1t^0.$

here?

ultiply



The scaled variation

$839 = 800 + 30 + 9 =$

value (at $t = 1$) of polynom

$800t^2 + 30t^1 + 9t^0$.

Squaring: $(800t^2 + 30t^1 + 9t$

$640000t^4 + 48000t^3 + 1530$

$540t^1 + 81t^0$.

Carrying:

$640000t^4 + 48000t^3 + 1530$

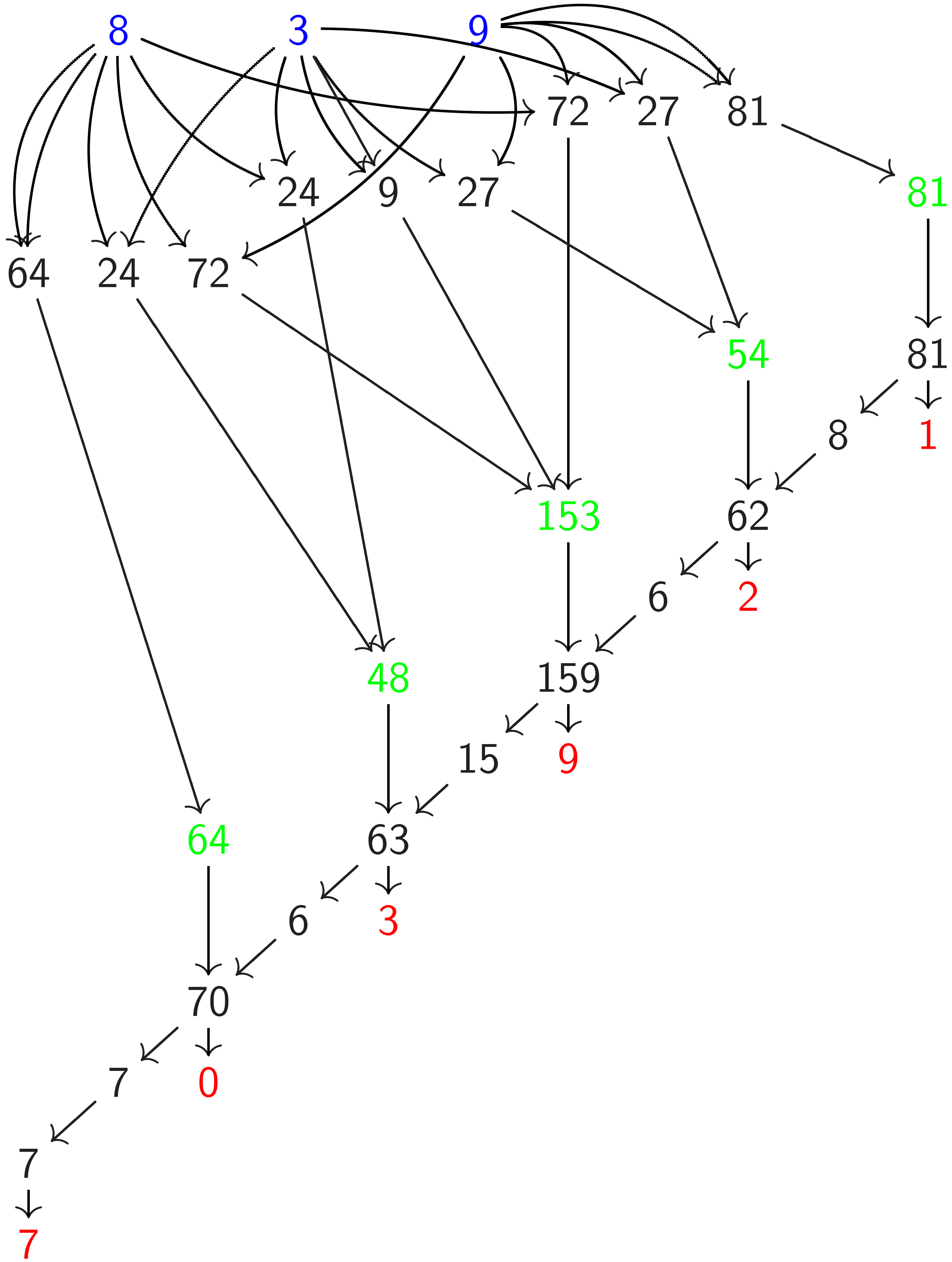
$540t^1 + 81t^0$;

$640000t^4 + 48000t^3 + 1530$

$620t^1 + 1t^0$; $\ldots$

$700000t^5 + 0t^4 + 3000t^3 + 9$

$20t^1 + 1t^0$.

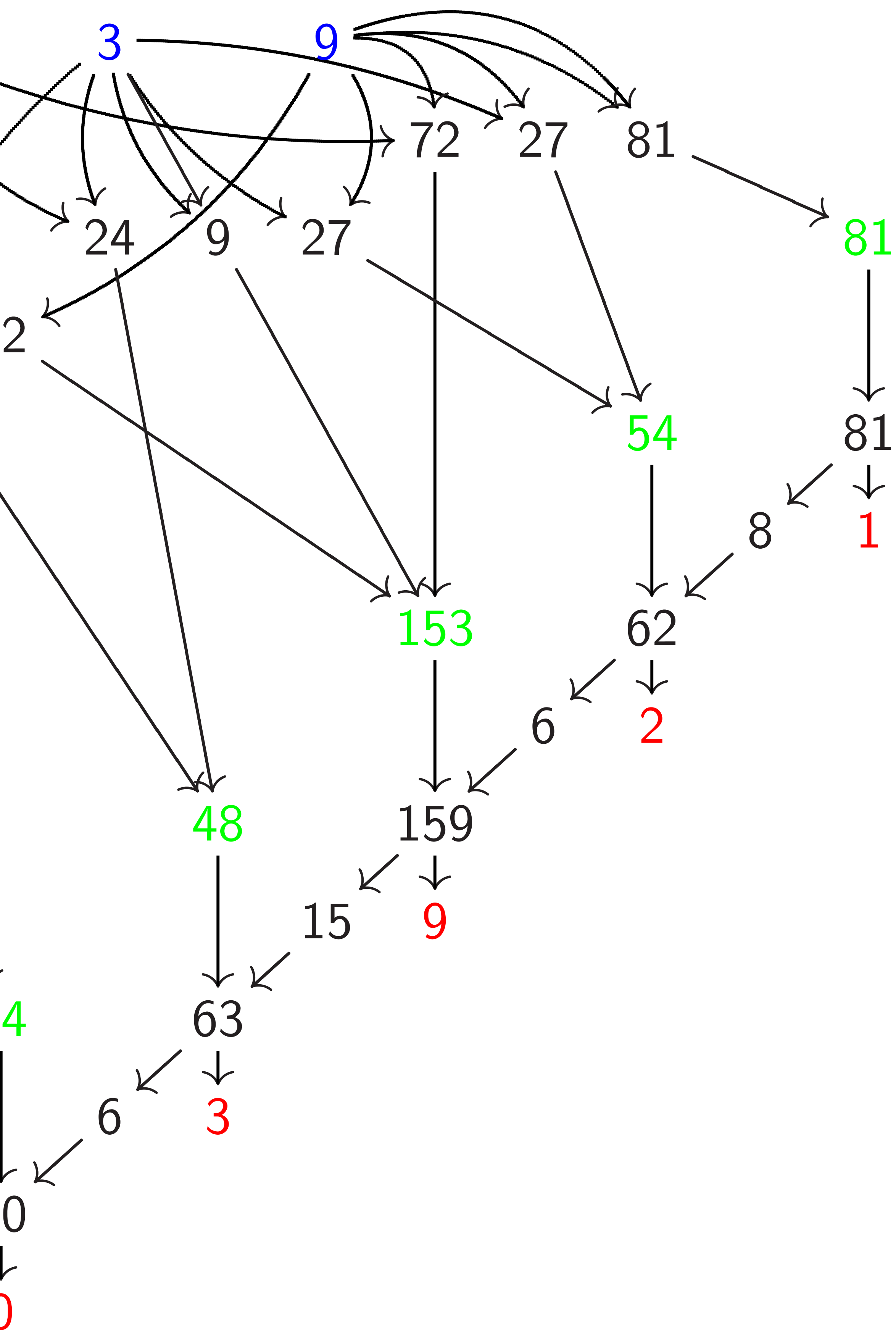


The scaled variation

$839 = 800 + 30 + 9 =$
value (at $t = 1$) of polynomial
$800t^2 + 30t^1 + 9t^0$.

Squaring: $(800t^2 + 30t^1 + 9t^0)^2 =$
$640000t^4 + 48000t^3 + 15300t^2 + 540t^1 + 81t^0$.

Carrying:
$640000t^4 + 48000t^3 + 15300t^2 + 540t^1 + 81t^0$;
$640000t^4 + 48000t^3 + 15300t^2 + 620t^1 + 1t^0$; $\quad\ldots$
$700000t^5 + 0t^4 + 3000t^3 + 900t^2 + 20t^1 + 1t^0$.



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Squaring: $(800t^2 + 30t^1 + 9t^0)^2 =$
$640000t^4 + 48000t^3 + 15300t^2 +$
$540t^1 + 81t^0$.

Carrying:
$640000t^4 + 48000t^3 + 15300t^2 +$
$540t^1 + 81t^0$;
$640000t^4 + 48000t^3 + 15300t^2 +$
$620t^1 + 1t^0$; ...
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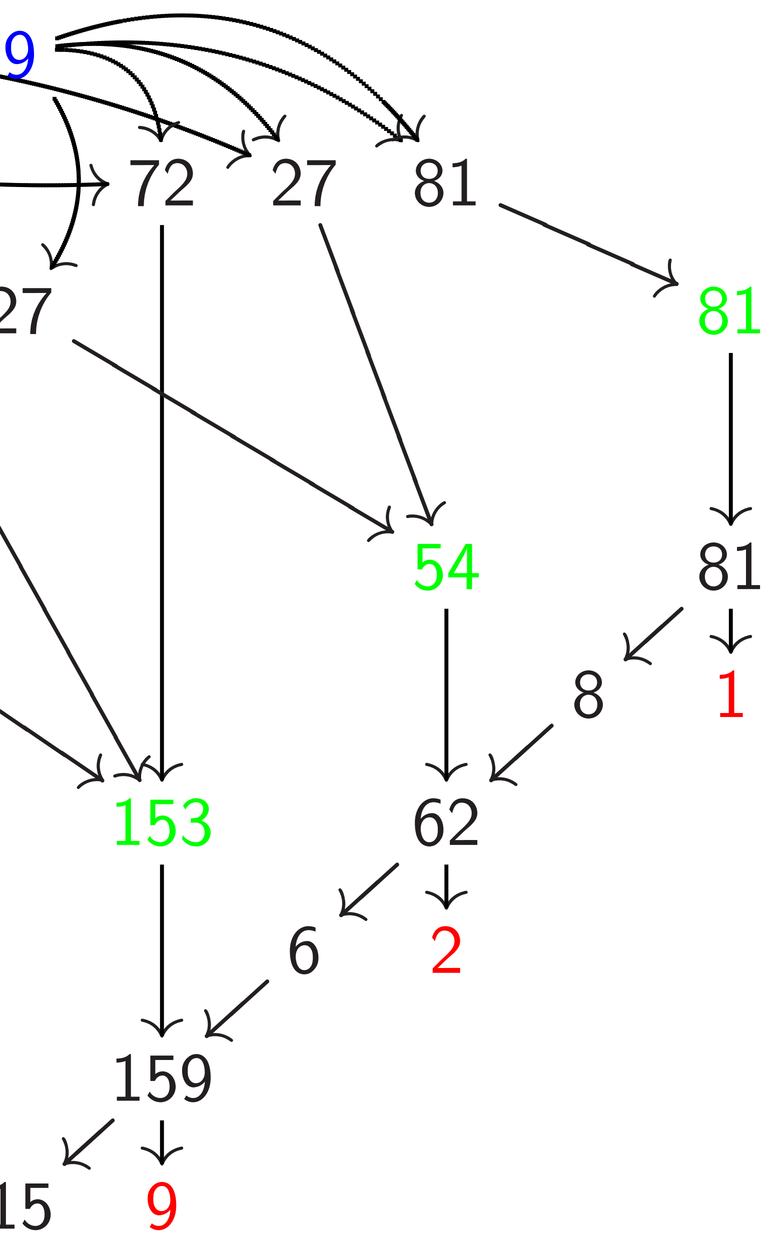
$20t^1 + 1t^0$.

What op



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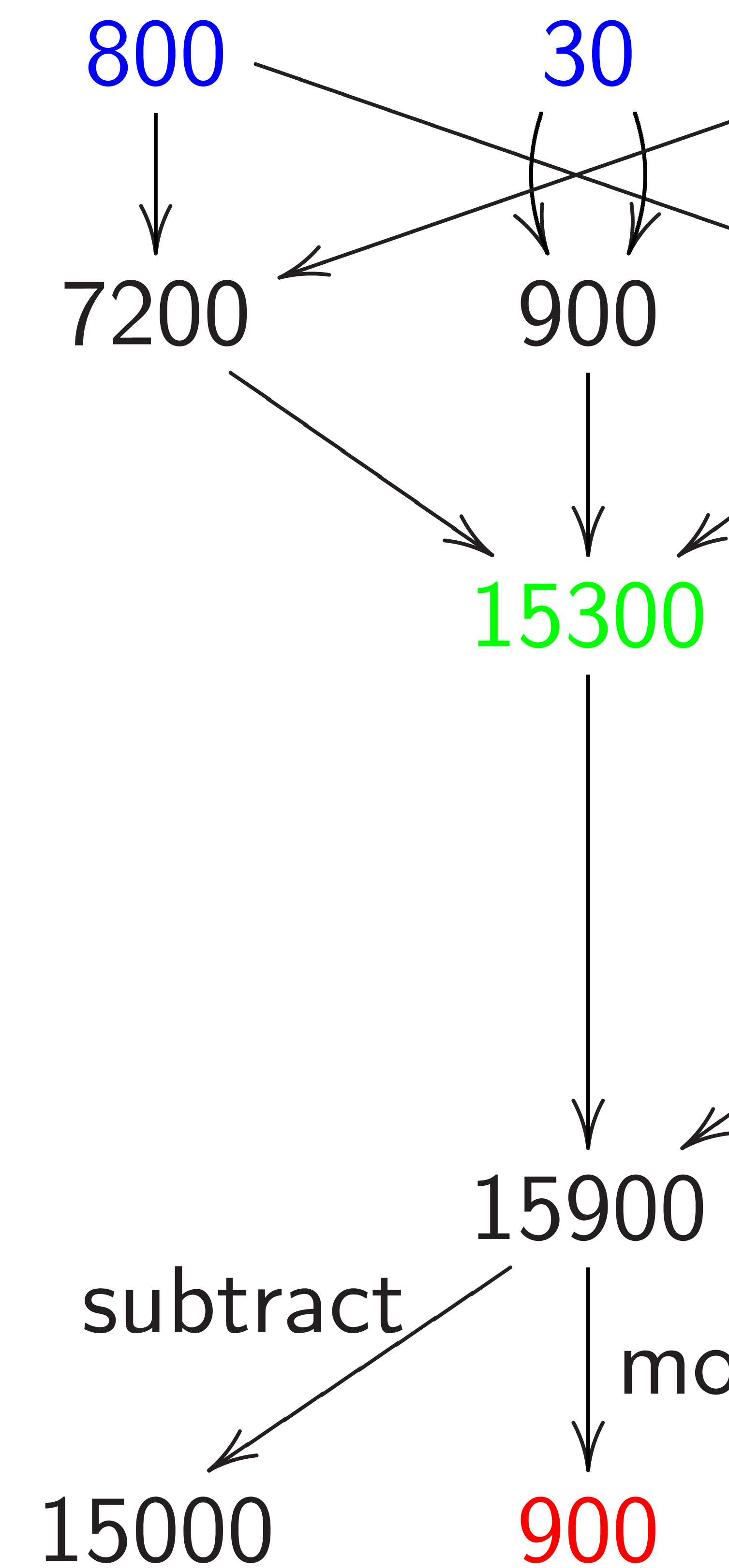
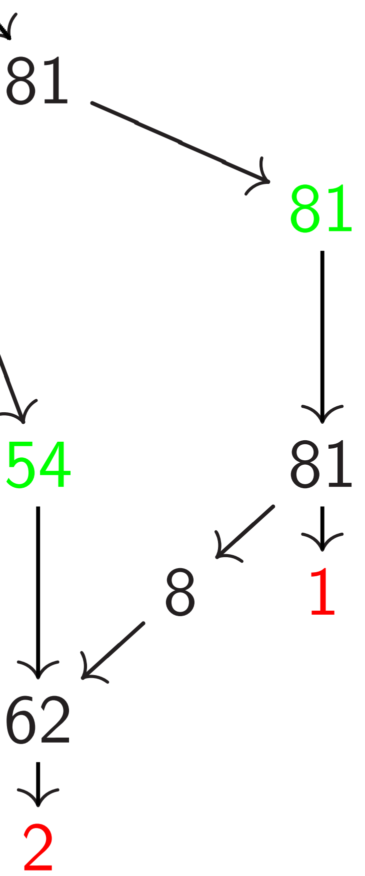
What operations w





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Carrying:
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$620t^1 + 1t^0$; $\quad \ldots$
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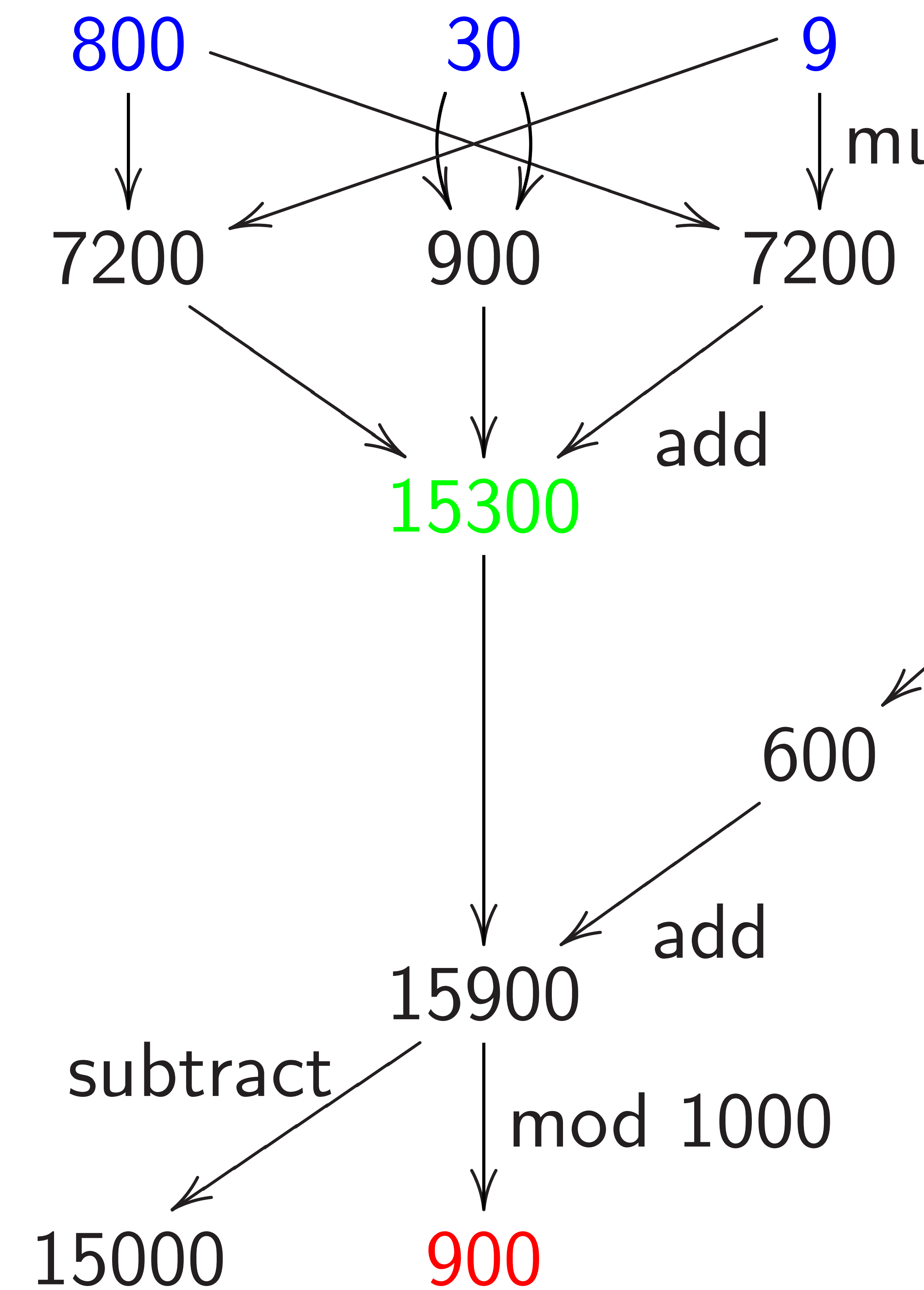
What operations were used



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Squaring: $(800t^2 + 30t^1 + 9t^0)^2 =$
$640000t^4 + 48000t^3 + 15300t^2 +$
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Carrying:
$640000t^4 + 48000t^3 + 15300t^2 +$
$540t^1 + 81t^0$;
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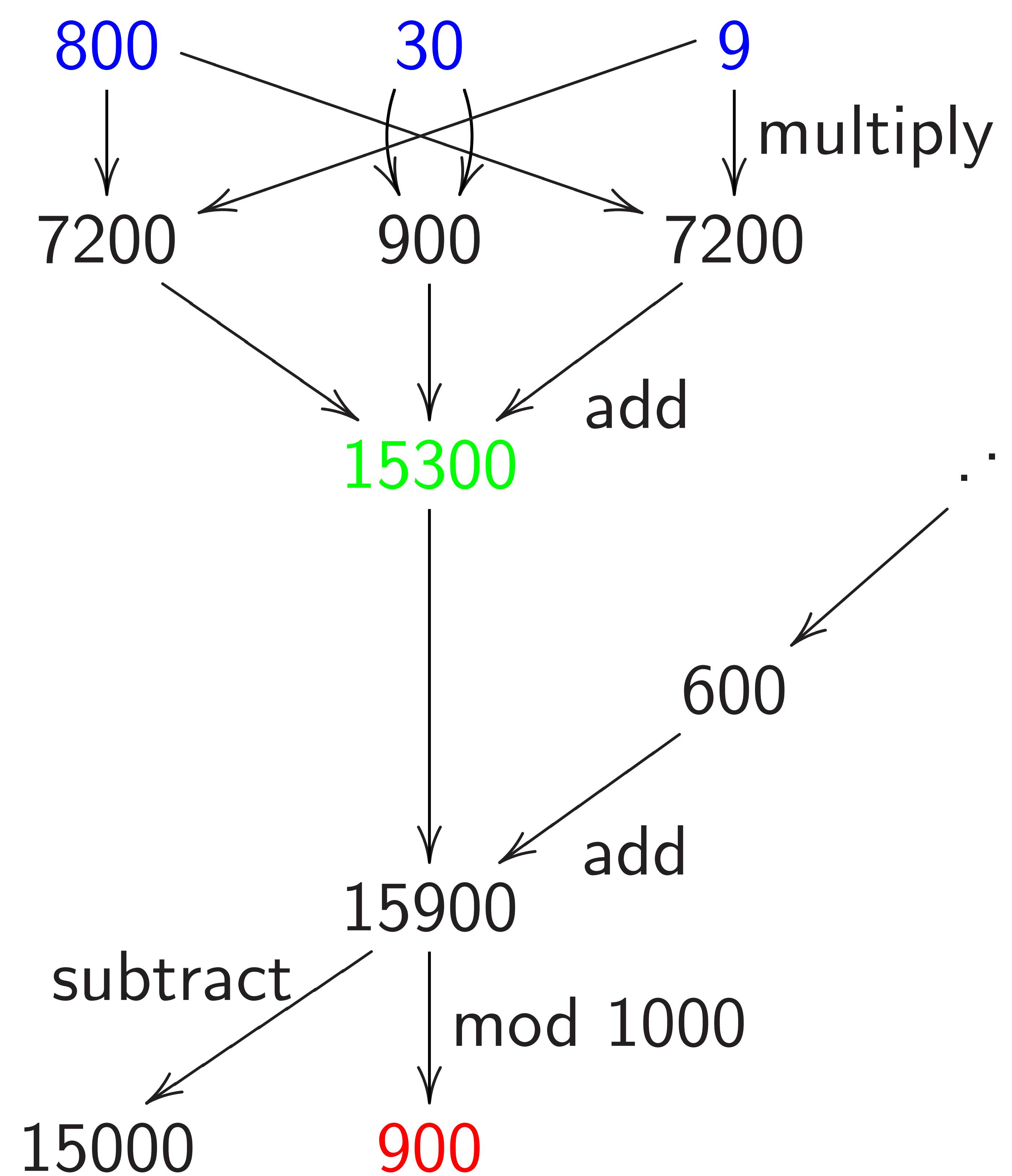


led variation

$00 + 30 + 9 =$

t $t = 1$) of polynomial

$30t^1 + 9t^0$.

g: $(800t^2 + 30t^1 + 9t^0)^2 =$

$^4 + 48000t^3 + 15300t^2 +$

$81t^0$.

g:

$^4 + 48000t^3 + 15300t^2 +$

$81t^0$;

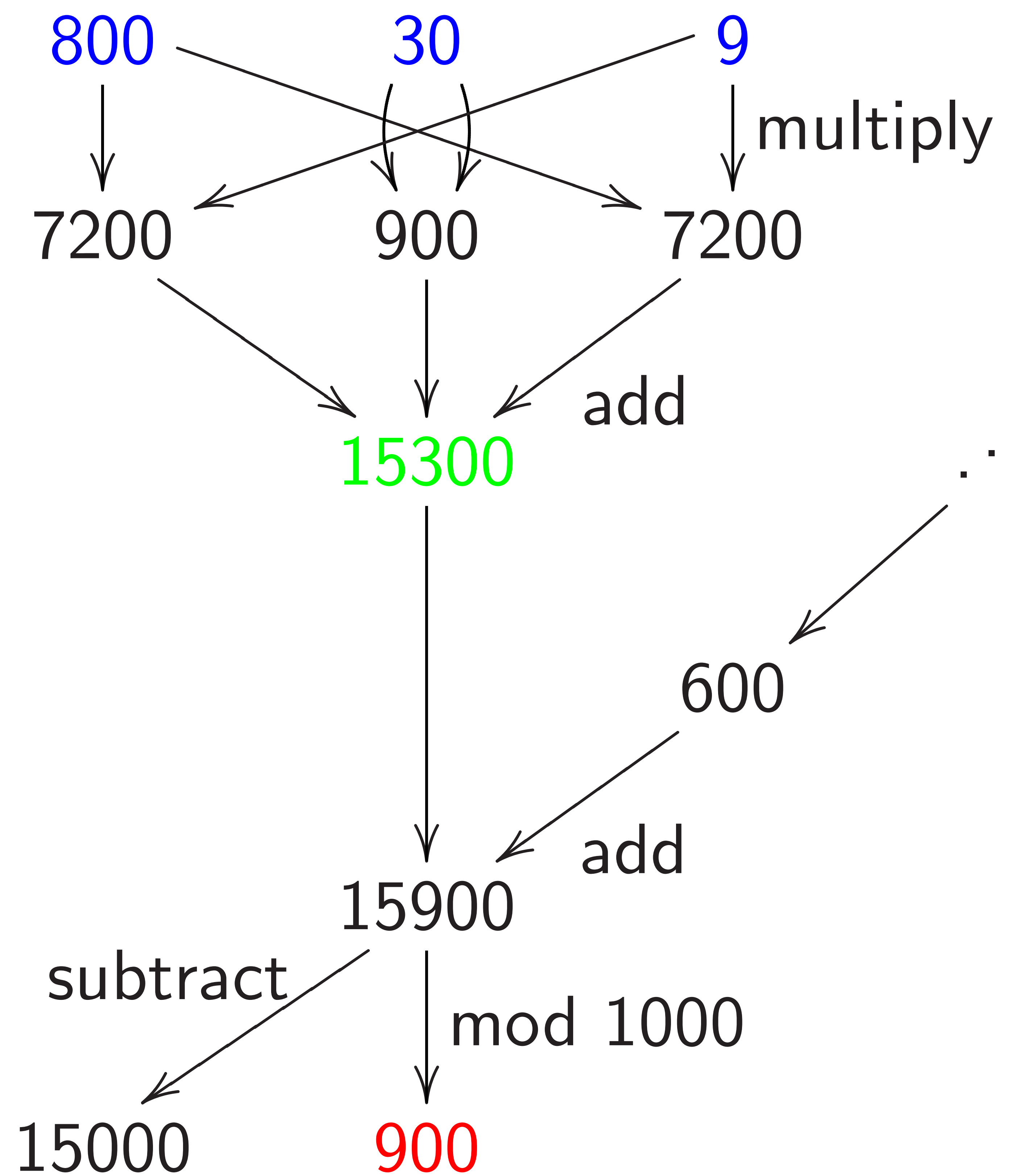
$^4 + 48000t^3 + 15300t^2 +$

$1t^0$; $\ldots$

$^5 + 0t^4 + 3000t^3 + 900t^2 +$

t^0.

What operations were used here?



Speedup

$(\cdots + f_2$

has coef

$f_4 f_0 + f$

5 mults,

on

$9 =$

f polynomial

^{0}.

$+30t^1+9t^0)^2 =$

$t^3 + 15300t^2 +$

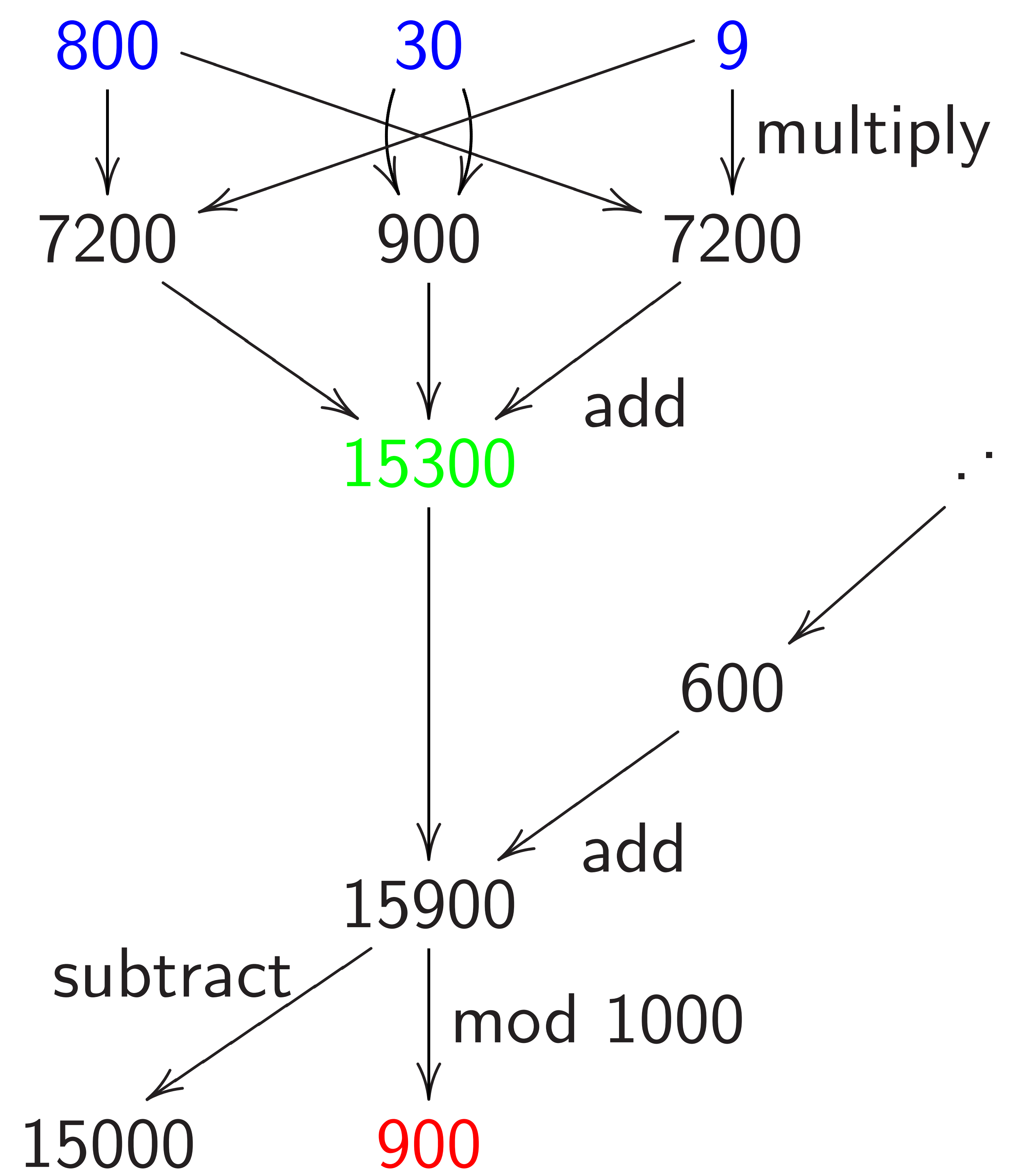
$t^3 + 15300t^2 +$

$t^3 + 15300t^2 +$

. .

$3000t^3 + 900t^2 +$

What operations were used here?



Speedup: double i

$(\cdots + f_2t^2 + f_1t^1$

has coefficients su

$f_4f_0 + f_3f_1 + f_2f$

5 mults, 4 adds.

ial

$^0)^2 =$

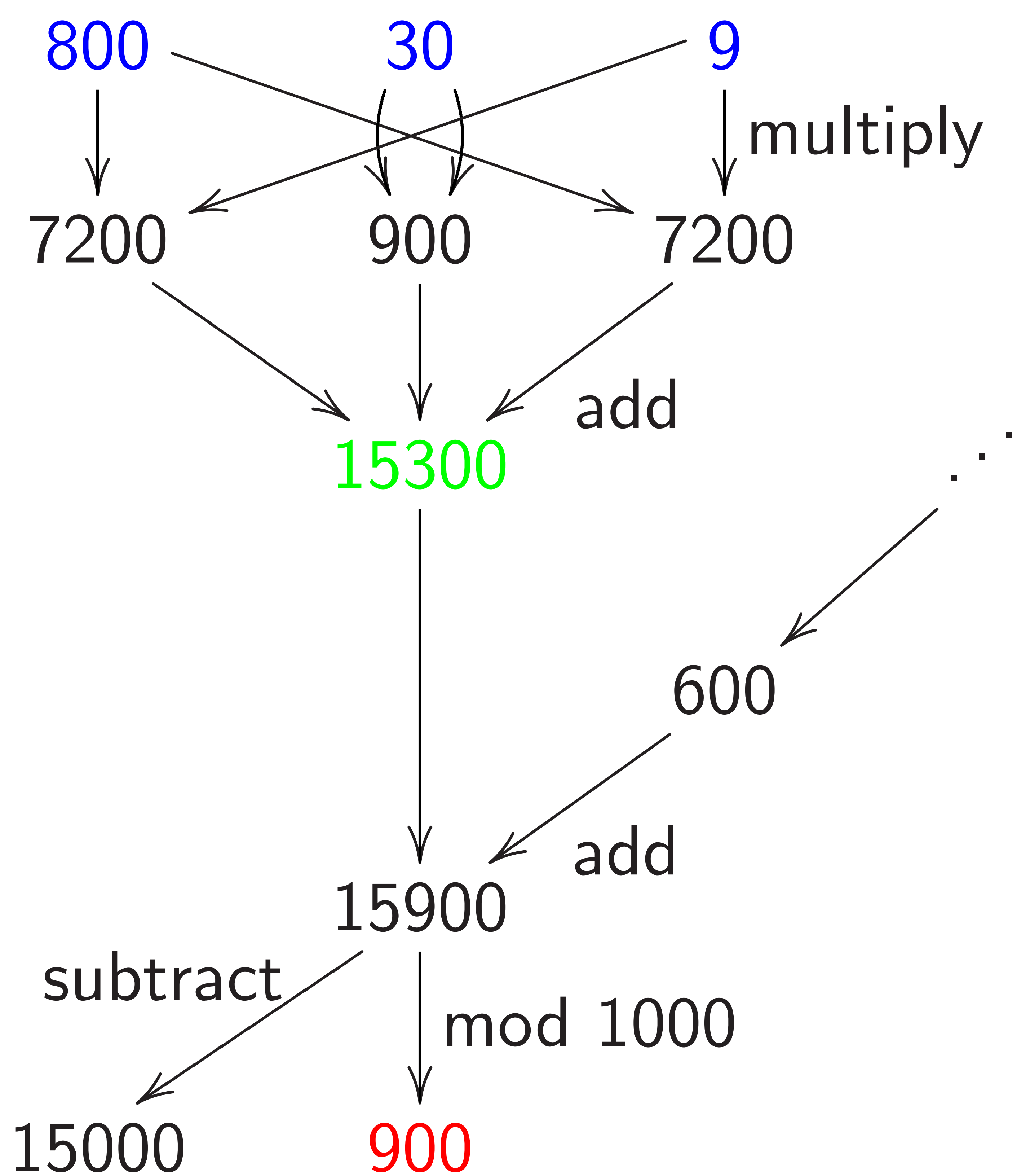
$0t^2 +$

$0t^2 +$

$0t^2 +$

$00t^2 +$

What operations were used here?



<u>Speedup: double inside squa</u>

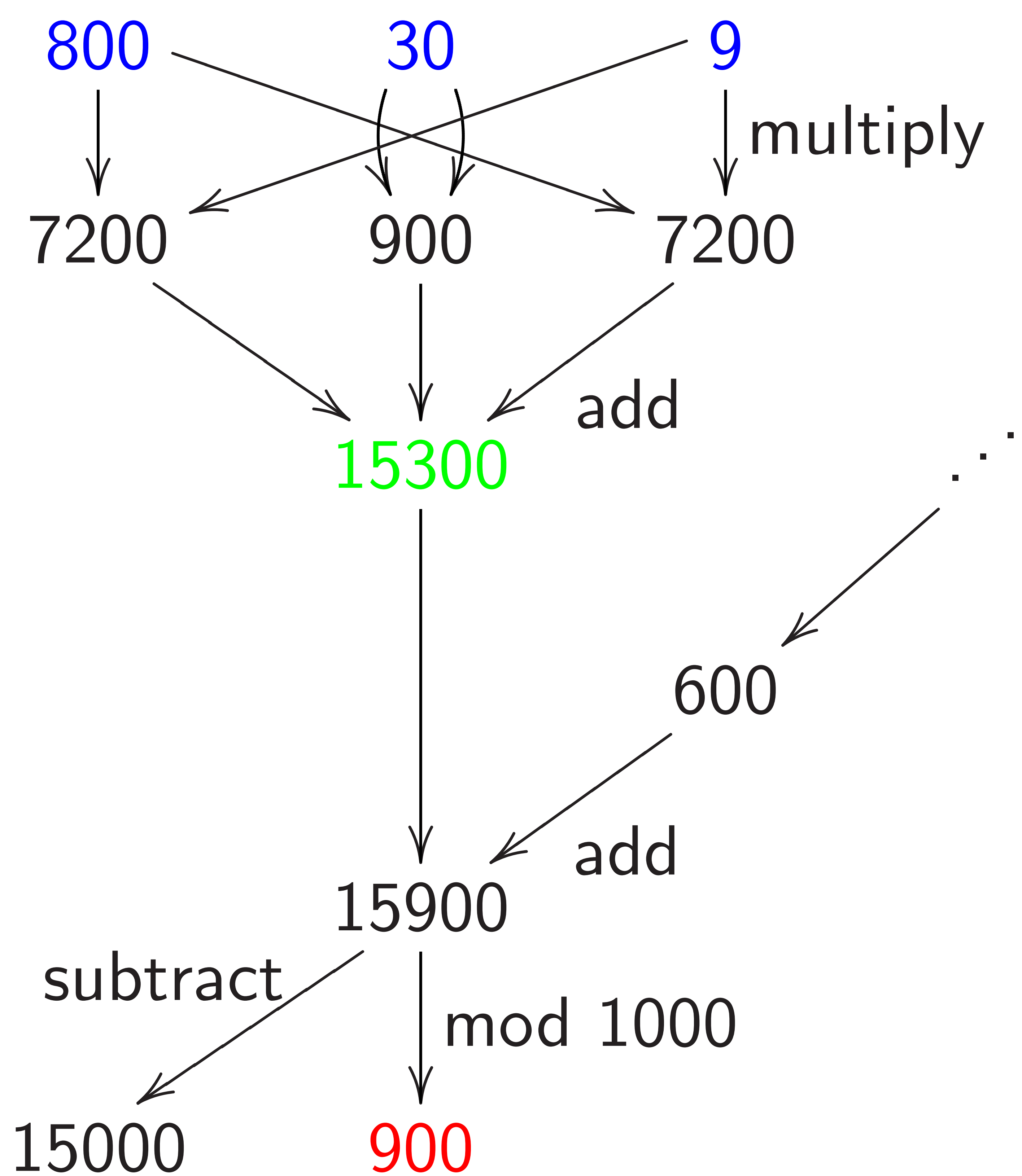
$(\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0)^2$

has coefficients such as

$f_4 f_0 + f_3 f_1 + f_2 f_2 + f_1 f_3 +$

5 mults, 4 adds.

What operations were used here?



<u>Speedup: double inside squaring</u>

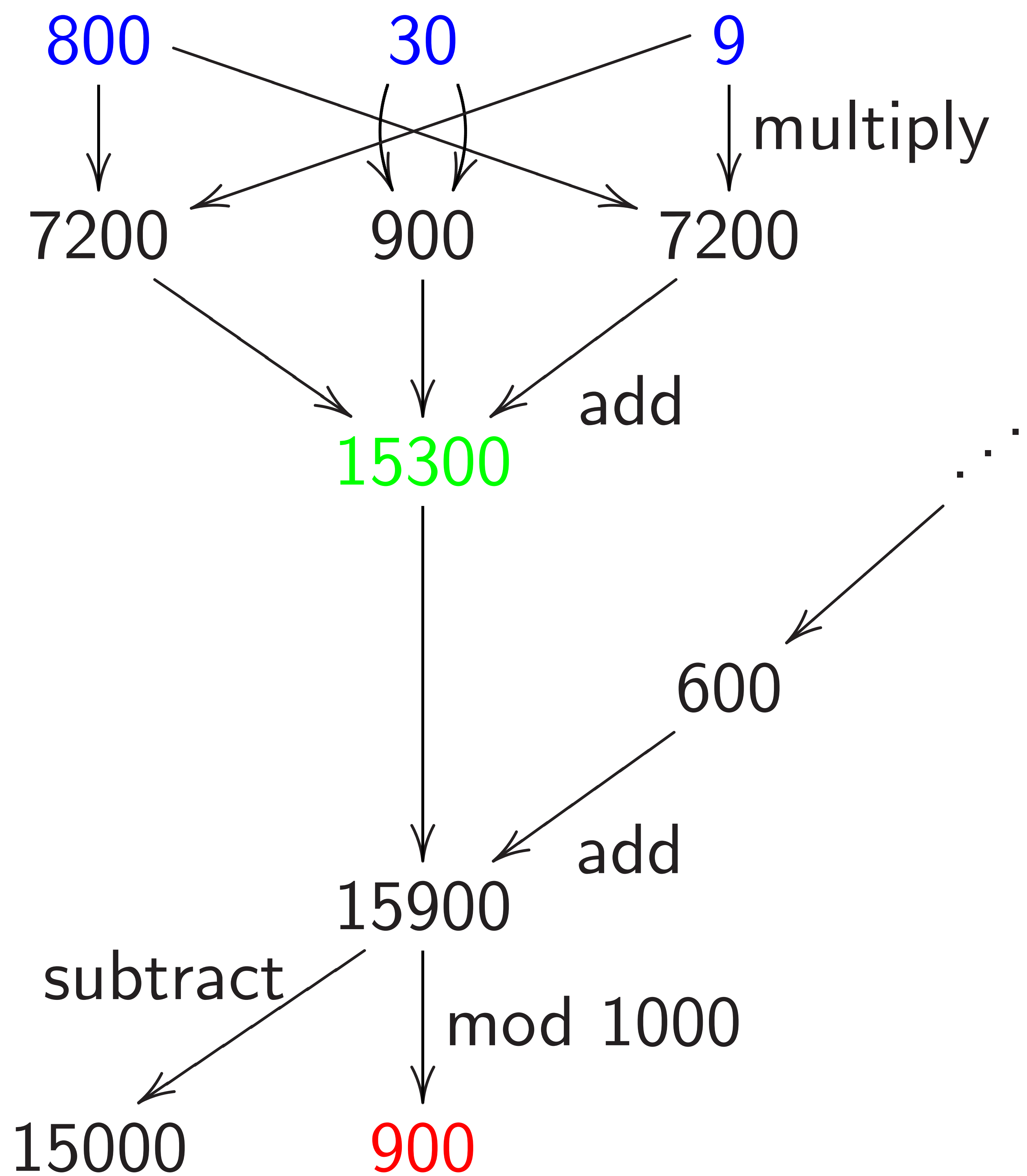
$(\cdots + f_2t^2 + f_1t^1 + f_0t^0)^2$

has coefficients such as

$f_4f_0 + f_3f_1 + f_2f_2 + f_1f_3 + f_0f_4$.

5 mults, 4 adds.

What operations were used here?



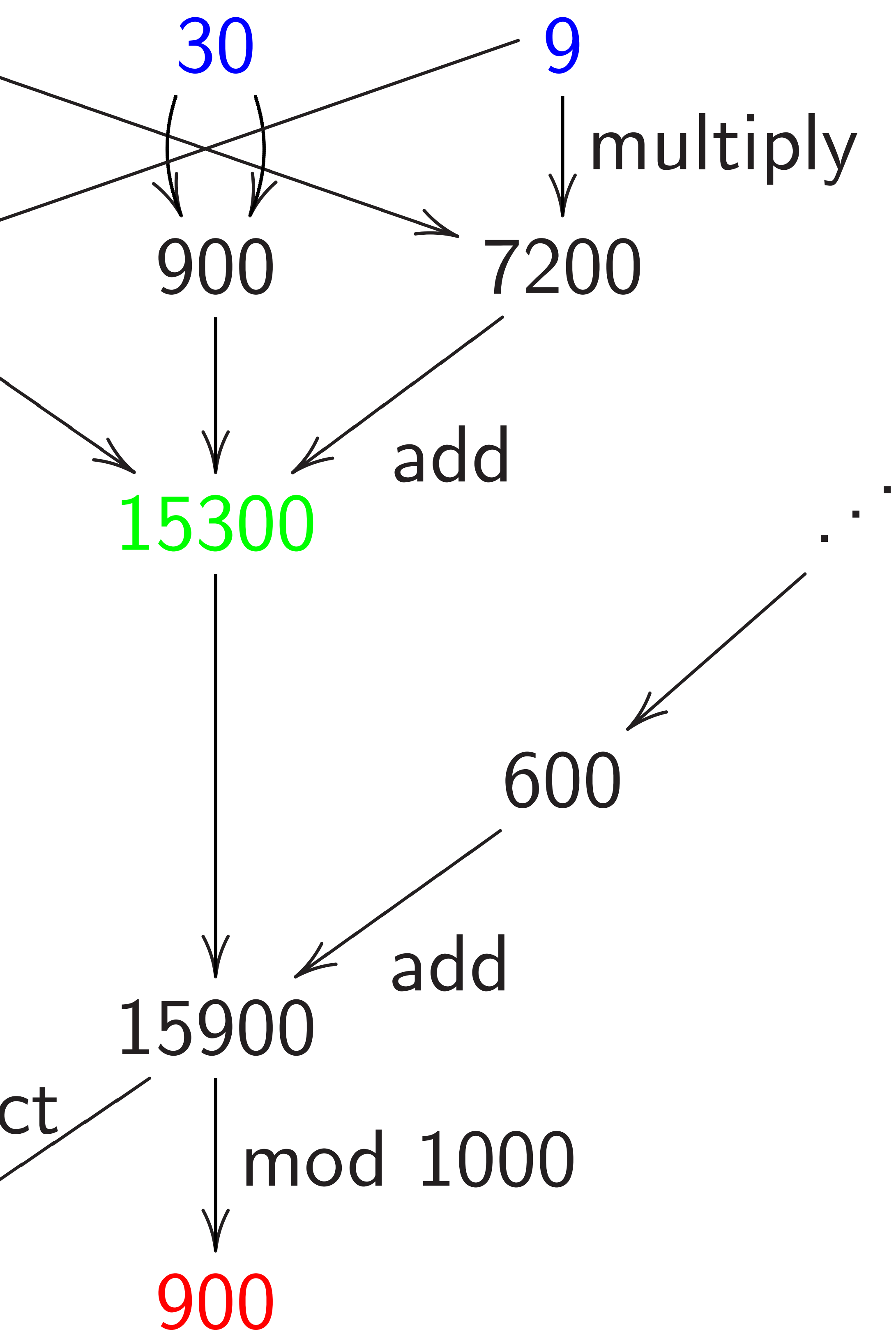
Speedup: double inside squaring

$(\cdots + f_2t^2 + f_1t^1 + f_0t^0)^2$
has coefficients such as
$f_4f_0 + f_3f_1 + f_2f_2 + f_1f_3 + f_0f_4$.
5 mults, 4 adds.

Compute more efficiently as
$2f_4f_0 + 2f_3f_1 + f_2f_2$.
3 mults, 2 adds, 2 doublings.

Save $\approx 1/2$ of the mults
if there are many coefficients.

perations were used here?



Speedup: double inside squaring

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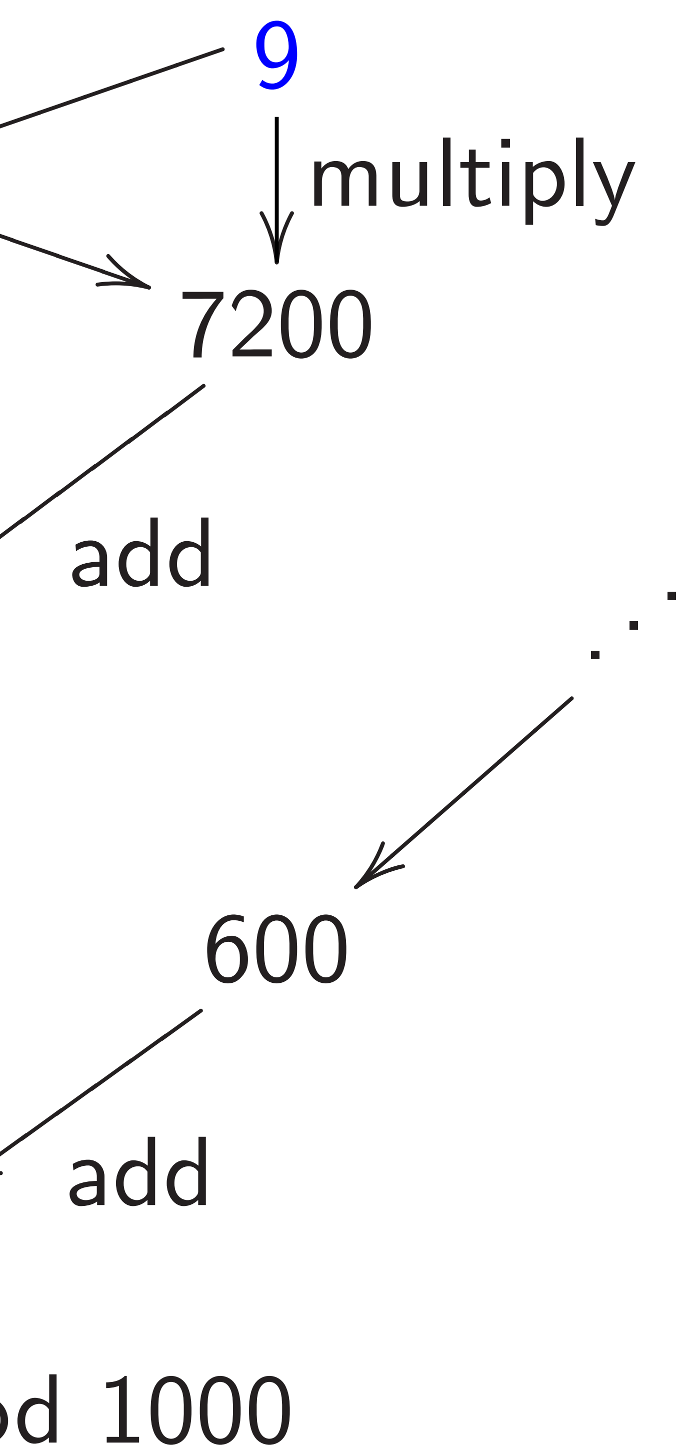
Faster a
$2(f_4 f_0 +$
3 mults,

Save $\approx$
if there a

were used here?



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$(\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0)^2$
has coefficients such as
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3 mults, 2 adds, 2 doublings.

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if there are many coefficients.

Faster alternative:
$2(f_4 f_0 + f_3 f_1) +$
3 mults, 2 adds, 1

Save $\approx 1/2$ of the
if there are many

here?

ultiply

Speedup: double inside squaring

$(\cdots + f_2t^2 + f_1t^1 + f_0t^0)^2$
has coefficients such as
$f_4f_0 + f_3f_1 + f_2f_2 + f_1f_3 + f_0f_4$.
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Save $\approx 1/2$ of the mults
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Faster alternative:
$2(f_4f_0 + f_3f_1) + f_2f_2$.
3 mults, 2 adds, 1 doubling.

Save $\approx 1/2$ of the adds
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Speedup: double inside squaring

$(\cdots + f_2 t^2 + f_1 t^1 + f_0 t^0)^2$
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3 mults, 2 adds, 1 doubling.

Save $\approx 1/2$ of the adds
if there are many coefficients.

Even faster alternative:
$(2f_0)f_4 + (2f_1)f_3 + f_2f_2$,
after precomputing $2f_0, 2f_1, \ldots$.

3 mults, 2 adds, 0 doublings.
Precomputation ≈ 0.5 doublings.

: double inside squaring

$_2t^2 + f_1t^1 + f_0t^0)^2$

fficients such as

$f_3f_1 + f_2f_2 + f_1f_3 + f_0f_4$.

4 adds.

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2 adds, 2 doublings.

1/2 of the mults

are many coefficients.

Faster alternative:

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Speedup

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coefficients.

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Precomputation ≈ 0.5 doublings.

Speedup: allow ne

Recall $159 \mapsto 15$, 9

Scaled: $15900 \mapsto$

Alternative: $159 \mapsto$

Scaled: $15900 \mapsto$

Use digits $\{-5, -$
instead of $\{0, 1, \ldots$
Small disadvantag
Several small adva
easily handle nega
easily handle subtr
reduce products a

aring

$- f_0 f_4$.

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Faster alternative:

$2(f_4 f_0 + f_3 f_1) + f_2 f_2$.

3 mults, 2 adds, 1 doubling.

Save $\approx 1/2$ of the adds if there are many coefficients.

Even faster alternative:

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after precomputing $2f_0, 2f_1, \ldots$.

3 mults, 2 adds, 0 doublings.

Precomputation ≈ 0.5 doublings.

Speedup: allow negative coe

Recall $159 \mapsto 15, 9$.

Scaled: $15900 \mapsto 15000, 900$

Alternative: $159 \mapsto 16, -1$.

Scaled: $15900 \mapsto 16000, -1$

Use digits $\{-5, -4, \ldots, 4, 5$

instead of $\{0, 1, \ldots, 9\}$.

Small disadvantage: need $-$

Several small advantages:

easily handle negative intege

easily handle subtraction;

reduce products a bit.

Faster alternative:

$2(f_4f_0 + f_3f_1) + f_2f_2$.

3 mults, 2 adds, 1 doubling.

Save $\approx 1/2$ of the adds if there are many coefficients.

Even faster alternative:

$(2f_0)f_4 + (2f_1)f_3 + f_2f_2$,

after precomputing $2f_0, 2f_1, \ldots$.

3 mults, 2 adds, 0 doublings.

Precomputation ≈ 0.5 doublings.

Speedup: allow negative coeffs

Recall $159 \mapsto 15, 9$.

Scaled: $15900 \mapsto 15000, 900$.

Alternative: $159 \mapsto 16, -1$.

Scaled: $15900 \mapsto 16000, -100$.

Use digits $\{-5, -4, \ldots, 4, 5\}$ instead of $\{0, 1, \ldots, 9\}$.

Small disadvantage: need $-$.

Several small advantages:

easily handle negative integers;

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reduce products a bit.

lternative:

$- f_3 f_1) + f_2 f_2$.

2 adds, 1 doubling.

1/2 of the adds

are many coefficients.

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$+ (2f_1)f_3 + f_2 f_2$,

ecomputing $2f_0, 2f_1, \ldots$.

2 adds, 0 doublings.

putation $\approx$ 0.5 doublings.

Speedup: allow negative coeffs

Recall 159 $\mapsto$ 15, 9.

Scaled: 15900 $\mapsto$ 15000, 900.

Alternative: 159 $\mapsto$ 16, -1.

Scaled: 15900 $\mapsto$ 16000, -100.

Use digits $\{-5, -4, \ldots, 4, 5\}$

instead of $\{0, 1, \ldots, 9\}$.

Small disadvantage: need $-$.

Several small advantages:

easily handle negative integers;

easily handle subtraction;

reduce products a bit.

Speedup

Computi

multiply

square c

e.g. $a =$

$(3t^2 + 1t$

$6t^4 + 23$

carry: 8

As befor

$64t^4 + 4$

$7t^5 + 0t$

$+$: $7t^5 +$

$7t^5 + 8t$

$f_2 f_2$.
doubling.

adds
coefficients.

ative:
$+ f_2 f_2$,
g $2f_0, 2f_1, \ldots$.

doublings.
$\approx$ 0.5 doublings.

Speedup: allow negative coeffs

Recall $159 \mapsto 15, 9$.
Scaled: $15900 \mapsto 15000, 900$.

Alternative: $159 \mapsto 16, -1$.
Scaled: $15900 \mapsto 16000, -100$.

Use digits $\{-5, -4, \ldots, 4, 5\}$
instead of $\{0, 1, \ldots, 9\}$.
Small disadvantage: need $-$.
Several small advantages:
easily handle negative integers;
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reduce products a bit.

Speedup: delay ca

Computing (e.g.)
multiply a, b polyn
square c poly, carr

e.g. $a = 314$, $b =$
$(3t^2 + 1t^1 + 4t^0)(2t$
$6t^4 + 23t^3 + 18t^2$
carry: $8t^4 + 5t^3 +$

As before $(8t^2 + 3$
$64t^4 + 48t^3 + 153t$
$7t^5 + 0t^4 + 3t^3 +$

$+$: $7t^5 + 8t^4 + 8t^3 +$
$7t^5 + 8t^4 + 9t^3 +$

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Speedup: allow negative coeffs

Recall $159 \mapsto 15, 9$.

Scaled: $15900 \mapsto 15000, 900$.

Alternative: $159 \mapsto 16, -1$.

Scaled: $15900 \mapsto 16000, -100$.

Use digits $\{-5, -4, \ldots, 4, 5\}$
instead of $\{0, 1, \ldots, 9\}$.
Small disadvantage: need $-$.
Several small advantages:
easily handle negative integers;
easily handle subtraction;
reduce products a bit.

Speedup: delay carries

Computing (e.g.) big $ab + c$
multiply a, b polynomials, ca
square c poly, carry, add, ca

e.g. $a = 314$, $b = 271$, $c =$
$(3t^2 + 1t^1 + 4t^0)(2t^2 + 7t^1 + 1$
$6t^4 + 23t^3 + 18t^2 + 29t^1 +$
carry: $8t^4 + 5t^3 + 0t^2 + 9t^1$

As before $(8t^2 + 3t^1 + 9t^0)^2$
$64t^4 + 48t^3 + 153t^2 + 54t^1 +$
$7t^5 + 0t^4 + 3t^3 + 9t^2 + 2t^1$

$+$: $7t^5 + 8t^4 + 8t^3 + 9t^2 + 11t$
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1$

Speedup: allow negative coeffs

Recall $159 \mapsto 15, 9$.

Scaled: $15900 \mapsto 15000, 900$.

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Speedup: delay carries

Computing (e.g.) big $ab + c^2$: multiply a, b polynomials, carry, square c poly, carry, add, carry.

e.g. $a = 314$, $b = 271$, $c = 839$:

$(3t^2+1t^1+4t^0)(2t^2+7t^1+1t^0) =$

$6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0$;

carry: $8t^4 + 5t^3 + 0t^2 + 9t^1 + 4t^0$.

As before $(8t^2 + 3t^1 + 9t^0)^2 =$

$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0$;

$7t^5 + 0t^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$.

$+$: $7t^5+8t^4+8t^3+9t^2+11t^1+5t^0$;

$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

p: allow negative coeffs

59 $\mapsto$ 15, 9.

15900 $\mapsto$ 15000, 900.

ive: 159 $\mapsto$ 16, -1.

15900 $\mapsto$ 16000, -100.

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As before $(8t^2 + 3t^1 + 9t^0)^2 =$
$64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0$;
$7t^5 + 0t^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$.

$+$: $7t^5 + 8t^4 + 8t^3 + 9t^2 + 11t^1 + 5t^0$;
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

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square c

$(6t^4 + 2$

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$= 70t^4 +$

$7t^5 + 8t$

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Speedup: delay carries

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$+$: $7t^5 + 8t^4 + 8t^3 + 9t^2 + 11t^1 + 5t^0$;
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

Faster: multiply a
square c polynomi

$(6t^4 + 23t^3 + 18t^2$
$(64t^4 + 48t^3 + 153$
$= 70t^4 + 71t^3 + 17$
$7t^5 + 8t^4 + 9t^3 +$

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Speedup: delay carries

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multiply a, b polynomials, carry,
square c poly, carry, add, carry.

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$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

Faster: multiply a, b polynor
square c polynomial, add, ca

$(6t^4 + 23t^3 + 18t^2 + 29t^1 +$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 +$
$= 70t^4 + 71t^3 + 171t^2 + 83t^1 +$
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1$

Eliminate intermediate carrie
Outweighs cost of handling
slightly larger coefficients.

Important to carry between
multiplications (and squarin
to reduce coefficient size;
but carries are usually a bad
before additions, subtractior

Speedup: delay carries

Computing (e.g.) big $ab + c^2$: multiply a, b polynomials, carry, square c poly, carry, add, carry.

e.g. $a = 314$, $b = 271$, $c = 839$:
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$+$: $7t^5+8t^4+8t^3+9t^2+11t^1+5t^0$;
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0)$
$= 70t^4+71t^3+171t^2+83t^1+85t^0$;
$7t^5 + 8t^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

Eliminate intermediate carries. Outweighs cost of handling slightly larger coefficients.

Important to carry between multiplications (and squarings) to reduce coefficient size; but carries are usually a bad idea before additions, subtractions, etc.

p: delay carries

ing (e.g.) big $ab + c^2$:
a, b polynomials, carry,
poly, carry, add, carry.

$= 314$, $b = 271$, $c = 839$:
$t^1 + 4t^0)(2t^2 + 7t^1 + 1t^0) =$
$t^3 + 18t^2 + 29t^1 + 4t^0$;
$t^4 + 5t^3 + 0t^2 + 9t^1 + 4t^0$.

re $(8t^2 + 3t^1 + 9t^0)^2 =$
$8t^3 + 153t^2 + 54t^1 + 81t^0$;
$^4 + 3t^3 + 9t^2 + 2t^1 + 1t^0$.

$8t^4 + 8t^3 + 9t^2 + 11t^1 + 5t^0$;
$^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

Faster: multiply a, b polynomials,
square c polynomial, add, carry.

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0)$
$= 70t^4 + 71t^3 + 171t^2 + 83t^1 + 85t^0$;
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Speedup

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$f = f_0$ -

$g = g_0$ +

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Faster:

$F_0 = f_0$

$F_1 = f_{10}$

Similarly

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big $ab + c^2$:

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271, $c = 839$:

$t^2 + 7t^1 + 1t^0) =$

$+ 29t^1 + 4t^0$;

$0t^2 + 9t^1 + 4t^0$.

$3t^1 + 9t^0)^2 =$

$t^2 + 54t^1 + 81t^0$;

$9t^2 + 2t^1 + 1t^0$.

$+9t^2 + 11t^1 + 5t^0$;

$0t^2 + 1t^1 + 5t^0$.

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0)$
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Speedup: polynom

How much work t

$f = f_0 + f_1 t + \cdot\cdot$

$g = g_0 + g_1 t + \cdots$

Using the obvious

400 coeff mults, 3

Faster: Write f as

$F_0 = f_0 + f_1 t + \cdot$

$F_1 = f_{10} + f_{11} t +$

Similarly write g a

Then $fg = (F_0 +$

$+ (F_0 G_0 - F_1 G_1 t^1$

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839:

$t^0) =$

$4t^0$;

$+ 4t^0$.

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$+ 81t^0$;

$+ 1t^0$.

$^1 + 5t^0$;

$+ 5t^0$.

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
$(64t^4 + 48t^3 + 153t^2 + 54t^1 + 81t^0)$
$= 70t^4 + 71t^3 + 171t^2 + 83t^1 + 85t^0$;
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Eliminate intermediate carries. Outweighs cost of handling slightly larger coefficients.

Important to carry between multiplications (and squarings) to reduce coefficient size; but carries are usually a bad idea before additions, subtractions, etc.

Speedup: polynomial Karats

How much work to multiply
$f = f_0 + f_1 t + \cdots + f_{19} t^{19}$,
$g = g_0 + g_1 t + \cdots + g_{19} t^{19}$?

Using the obvious method:
400 coeff mults, 361 coeff a

Faster: Write f as $F_0 + F_1 t$
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
$F_1 = f_{10} + f_{11} t + \cdots + f_{19} t$
Similarly write g as $G_0 + G_1$

Then $fg = (F_0 + F_1)(G_0 +$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10}$

Faster: multiply a, b polynomials, square c polynomial, add, carry.

$(6t^4 + 23t^3 + 18t^2 + 29t^1 + 4t^0) +$
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Eliminate intermediate carries. Outweighs cost of handling slightly larger coefficients.

Important to carry between multiplications (and squarings) to reduce coefficient size;
but carries are usually a bad idea before additions, subtractions, etc.

Speedup: polynomial Karatsuba

How much work to multiply polys
$f = f_0 + f_1 t + \cdots + f_{19} t^{19}$,
$g = g_0 + g_1 t + \cdots + g_{19} t^{19}$?

Using the obvious method:
400 coeff mults, 361 coeff adds.

Faster: Write f as $F_0 + F_1 t^{10}$;
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
$F_1 = f_{10} + f_{11} t + \cdots + f_{19} t^9$.
Similarly write g as $G_0 + G_1 t^{10}$.

Then $fg = (F_0 + F_1)(G_0 + G_1) t^{10}$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

multiply a, b polynomials,
polynomial, add, carry.

$3t^3 + 18t^2 + 29t^1 + 4t^0) +$
$48t^3 + 153t^2 + 54t^1 + 81t^0)$
$71t^3 + 171t^2 + 83t^1 + 85t^0$;
$^4 + 9t^3 + 0t^2 + 1t^1 + 5t^0$.

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400 coeff mults, 361 coeff adds.

Faster: Write f as $F_0 + F_1 t^{10}$;
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
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Similarly write g as $G_0 + G_1 t^{10}$.

Then $fg = (F_0 + F_1)(G_0 + G_1) t^{10}$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

20 adds
300 mul
$F_0 G_0$, F
243 add
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Speedup: polynomial Karatsuba

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$f = f_0 + f_1 t + \cdots + f_{19} t^{19}$,
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Using the obvious method:
400 coeff mults, 361 coeff adds.

Faster: Write f as $F_0 + F_1 t^{10}$;
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
$F_1 = f_{10} + f_{11} t + \cdots + f_{19} t^9$.
Similarly write g as $G_0 + G_1 t^{10}$.

Then $fg = (F_0 + F_1)(G_0 + G_1)t^{10}$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

20 adds for $F_0 +$
300 mults for thre
$F_0 G_0$, $F_1 G_1$, $(F_0 +$
243 adds for those
9 adds for $F_0 G_0 -$
with subs counted
and with delayed
19 adds for $\cdots(1$
19 adds to finish.

Total 300 mults, 3
Larger coefficients
still saves time.

Can apply idea rec
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$+81t^0)$

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$+ 5t^0.$

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Speedup: polynomial Karatsuba

How much work to multiply polys
$f = f_0 + f_1 t + \cdots + f_{19} t^{19}$,
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Faster: Write f as $F_0 + F_1 t^{10}$;
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
$F_1 = f_{10} + f_{11} t + \cdots + f_{19} t^9$.
Similarly write g as $G_0 + G_1 t^{10}$.

Then $fg = (F_0 + F_1)(G_0 + G_1) t^{10}$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

20 adds for $F_0 + F_1$, $G_0 + G$
300 mults for three products
$F_0 G_0$, $F_1 G_1$, $(F_0 + F_1)(G_0 +$
243 adds for those products
9 adds for $F_0 G_0 - F_1 G_1 t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots (1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
Larger coefficients, slight ex
still saves time.

Can apply idea recursively
as poly degree grows.

Speedup: polynomial Karatsuba

How much work to multiply polys
$f = f_0 + f_1 t + \cdots + f_{19} t^{19}$,
$g = g_0 + g_1 t + \cdots + g_{19} t^{19}$?

Using the obvious method:
400 coeff mults, 361 coeff adds.

Faster: Write f as $F_0 + F_1 t^{10}$;
$F_0 = f_0 + f_1 t + \cdots + f_9 t^9$;
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Similarly write g as $G_0 + G_1 t^{10}$.

Then $fg = (F_0 + F_1)(G_0 + G_1) t^{10}$
$+ (F_0 G_0 - F_1 G_1 t^{10})(1 - t^{10})$.

20 adds for $F_0 + F_1$, $G_0 + G_1$.
300 mults for three products
$F_0 G_0$, $F_1 G_1$, $(F_0 + F_1)(G_0 + G_1)$.
243 adds for those products.
9 adds for $F_0 G_0 - F_1 G_1 t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots(1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
Larger coefficients, slight expense;
still saves time.

Can apply idea recursively
as poly degree grows.

: polynomial Karatsuba

ch work to multiply polys

$+ f_1 t + \cdots + f_{19} t^{19}$,

$- g_1 t + \cdots + g_{19} t^{19}$?

ne obvious method:

ff mults, 361 coeff adds.

Write f as $F_0 + F_1 t^{10}$;

$+ f_1 t + \cdots + f_9 t^9$;

$_0 + f_{11} t + \cdots + f_{19} t^9$.

write g as $G_0 + G_1 t^{10}$.

$= (F_0 + F_1)(G_0 + G_1) t^{10}$

$_0 - F_1 G_1 t^{10})(1 - t^{10})$.

20 adds for $F_0 + F_1$, $G_0 + G_1$.
300 mults for three products
$F_0 G_0$, $F_1 G_1$, $(F_0 + F_1)(G_0 + G_1)$.
243 adds for those products.
9 adds for $F_0 G_0 - F_1 G_1 t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots(1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
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$\cdot + g_{19}t^{19}$?

method:
61 coeff adds.

s $F_0 + F_1 t^{10}$;
$\cdots + f_9 t^9$;
$\cdots + f_{19} t^9$.
s $G_0 + G_1 t^{10}$.

$F_1)(G_0 + G_1)t^{10}$
$^{0})(1 - t^{10})$.

20 adds for $F_0 + F_1$, $G_0 + G_1$.
300 mults for three products
$F_0 G_0$, $F_1 G_1$, $(F_0 + F_1)(G_0 + G_1)$.
243 adds for those products.
9 adds for $F_0 G_0 - F_1 G_1 t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots(1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
Larger coefficients, slight expense;
still saves time.

Can apply idea recursively
as poly degree grows.

Many other algebr
in polynomial mult
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polynomial degree
$O(n \lg n \lg \lg n)$ co
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20 adds for $F_0 + F_1$, $G_0 + G_1$.
300 mults for three products
F_0G_0, F_1G_1, $(F_0 + F_1)(G_0 + G_1)$.
243 adds for those products.
9 adds for $F_0G_0 - F_1G_1t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots(1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
Larger coefficients, slight expense;
still saves time.

Can apply idea recursively
as poly degree grows.

Many other algebraic speedu
in polynomial multiplication
"Toom," "FFT," etc.

Increasingly important as
polynomial degree grows.
$O(n \lg n \lg\lg n)$ coeff operat
to compute n-coeff product

Useful for sizes of n
that occur in cryptography?
In some cases, yes!
But Karatsuba is the limit
for prime-field ECC/ECDLP
on most current CPUs.

20 adds for $F_0 + F_1$, $G_0 + G_1$.
300 mults for three products
F_0G_0, F_1G_1, $(F_0 + F_1)(G_0 + G_1)$.
243 adds for those products.
9 adds for $F_0G_0 - F_1G_1t^{10}$
with subs counted as adds
and with delayed negations.
19 adds for $\cdots(1 - t^{10})$.
19 adds to finish.

Total 300 mults, 310 adds.
Larger coefficients, slight expense;
still saves time.

Can apply idea recursively
as poly degree grows.

Many other algebraic speedups
in polynomial multiplication:
"Toom," "FFT," etc.

Increasingly important as
polynomial degree grows.
$O(n \lg n \lg \lg n)$ coeff operations
to compute n-coeff product.

Useful for sizes of n
that occur in cryptography?
In some cases, yes!
But Karatsuba is the limit
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$_1G_1$, $(F_0 + F_1)(G_0 + G_1)$.
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or $F_0G_0 - F_1G_1t^{10}$
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<u>Modular reduction</u>

How to compute f

Can use definition
$f \bmod p = f - p$ |
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<u>Modular reduction</u>

How to compute $f \bmod p$?

Can use definition: $f \bmod p = f - p \lfloor f/p \rfloor$. Can multiply f by a precomputed $1/p$ approxima easily adjust to obtain $\lfloor f/p$

Slight speedup: "2-adic inve "Montgomery reduction."

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"Montgomery reduction."

e.g. 314

Precomp
$\lfloor 100000$
$= 36787$

Comput
314159
$= 11557$

Comput
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$= 57823$
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Can multiply f by a
precomputed $1/p$ approximation;
easily adjust to obtain $\lfloor f/p \rfloor$.

Slight speedup: "2-adic inverse";
"Montgomery reduction."

e.g. 31415926535

Precompute
$\lfloor 1000000000000/2$
$= 3678796$.

Compute
$314159 \cdot 3678796$
$= 1155726872564$

Compute
$31415926535 8 - 1$
$= 578230$.
Oops, too big:
$578230 - 271828$
$306402 - 271828$

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<u>Modular reduction</u>

How to compute $f \bmod p$?

Can use definition:
$f \bmod p = f - p\lfloor f/p \rfloor$.
Can multiply f by a precomputed $1/p$ approximation; easily adjust to obtain $\lfloor f/p \rfloor$.

Slight speedup: "2-adic inverse"; "Montgomery reduction."

e.g. 314159265358 mod 271

Precompute
$\lfloor 1000000000000/271828 \rfloor$
$= 3678796$.

Compute
$314159 \cdot 3678796$
$= 1155726872564$.

Compute
$314159265358 - 1155726 \cdot 2$
$= 578230$.
Oops, too big:
$578230 - 271828 = 306402$.
$306402 - 271828 = 34574$.

Modular reduction

How to compute f mod p?

Can use definition:
$f \bmod p = f - p\lfloor f/p \rfloor$.
Can multiply f by a precomputed $1/p$ approximation; easily adjust to obtain $\lfloor f/p \rfloor$.

Slight speedup: "2-adic inverse"; "Montgomery reduction."

e.g. 314159265358 mod 271828:

Precompute
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We can do better: normally
p is chosen with a special fo
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Special primes hurt security
for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$), etc.,
but not for elliptic curves!

gls1271: $p = 2^{127} - 1$,
with degree-2 extension.

Curve25519: $p = 2^{255} - 19.$

NIST P-224: $p = 2^{224} - 2^{96}$

secp112r1: $p = (2^{128} - 3)/7$
Divides special form.

e.g. 314159265358 mod 271828:

Precompute
$\lfloor 1000000000000/271828 \rfloor$
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Compute
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NIST P-224: $p = 2^{224} - 2^{96} + 1$.

secp112r1: $p = (2^{128} - 3)/76439$.
Divides special form.

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Small example: p

Then $1000000a +$

e.g. 31415926535

$314159 \cdot 1000000$

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Easily adjust $b - 3$

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e.g. $-677119 \equiv 3$

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Small example: $p = 100000$

Then $1000000a + b \equiv b - 3$

e.g. $314159265358 =$
$314159 \cdot 1000000 + 265358$
$314159(-3) + 265358 =$
$-942477 + 265358 =$
-677119.

Easily adjust $b - 3a$ to the range $\{0, 1, \ldots, p - 1$ by adding/subtracting a few e.g. $-677119 \equiv 322884$.

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Special primes hurt security for $\mathbf{F}_p^*$, Clock($\mathbf{F}_p$), etc., but not for elliptic curves!

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Small example: $p = 1000003$. Then $1000000a + b \equiv b - 3a$.

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$14t^7 + 4$

$82t^3 + 4$

Reduce:

$(-3c_i)t^i$

$64t^3 - 3$

Carry: 8

$1t^3 + 2t$

$= 1000003.$

$b \equiv b - 3a.$

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Speedup: Skip the adjustment for intermediate results. "Lazy reduction." Adjust only for output.

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e.g. To square 314

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$3t^5 + 1t^4 + 4t^3 +$

obtaining $9t^{10} + 6$

$14t^7 + 48t^6 + 72t^5$

$82t^3 + 43t^2 + 90t^1$

Reduce: replace (c

$(-3c_i)t^i$, obtainin

$64t^3 - 32t^2 + 48t^1$

Carry: $8t^6 - 4t^5 -$

$1t^3 + 2t^2 + 2t^1 -$

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$b - 3a$ is small enough to continue computations.

Can delay carries until after multiplication by 3.

e.g. To square 314159 in $\mathbf{Z}/1000003$: Square poly $3t^5 + 1t^4 + 4t^3 + 1t^2 + 5t^1$ obtaining $9t^{10} + 6t^9 + 25t^8$ $14t^7 + 48t^6 + 72t^5 + 59t^4 +$ $82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce: replace $(c_i)t^{6+i}$ by $(-3c_i)t^i$, obtaining $72t^5 + 3$ $64t^3 - 32t^2 + 48t^1 - 63t^0$.

Carry: $8t^6 - 4t^5 - 2t^4 +$ $1t^3 + 2t^2 + 2t^1 - 3t^0$.

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$25t^8 + 1$
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Finish re
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Reduce: replace $(c_i)t^{6+i}$ by $(-3c_i)t^i$, obtaining $72t^5 + 32t^4 + 64t^3 - 32t^2 + 48t^1 - 63t^0$.

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To minimize poly
mix reduction and
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e.g. Start from squ
$25t^8 + 14t^7 + 48t^6$
$82t^3 + 43t^2 + 90t$

Reduce $t^{10} \to t^4$
$t^5 \to t^6$: $6t^9 + 25t$
$5t^5 + 2t^4 + 82t^3 + 4$

Finish reduction:
$64t^3 - 32t^2 + 48t$
$t^0 \to t^1 \to t^2 \to$
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Can delay carries until after multiplication by 3.

e.g. To square 314159 in $\mathbf{Z}/1000003$: Square poly $3t^5 + 1t^4 + 4t^3 + 1t^2 + 5t^1 + 9t^0$, obtaining $9t^{10} + 6t^9 + 25t^8 + 14t^7 + 48t^6 + 72t^5 + 59t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce: replace $(c_i)t^{6+i}$ by $(-3c_i)t^i$, obtaining $72t^5 + 32t^4 + 64t^3 - 32t^2 + 48t^1 - 63t^0$.

Carry: $8t^6 - 4t^5 - 2t^4 + 1t^3 + 2t^2 + 2t^1 - 3t^0$.

To minimize poly degree, mix reduction and carrying, carrying the top sooner.

e.g. Start from square $9t^{10}$ + $25t^8 + 14t^7 + 48t^6 + 72t^5 +$ $82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce $t^{10} \to t^4$ and carry $t^5 \to t^6$: $6t^9 + 25t^8 + 14t^7 +$ $5t^5 + 2t^4 + 82t^3 + 43t^2 + 90t^1$

Finish reduction: $-5t^5 + 2t$ $64t^3 - 32t^2 + 48t^1 - 87t^0$.
$t^0 \to t^1 \to t^2 \to t^3 \to t^4$
$-4t^5 - 2t^4 + 1t^3 + 2t^2 - 1t^1$

Can delay carries until after multiplication by 3.

e.g. To square 314159 in $\mathbf{Z}/1000003$: Square poly $3t^5 + 1t^4 + 4t^3 + 1t^2 + 5t^1 + 9t^0$, obtaining $9t^{10} + 6t^9 + 25t^8 + 14t^7 + 48t^6 + 72t^5 + 59t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce: replace $(c_i)t^{6+i}$ by $(-3c_i)t^i$, obtaining $72t^5 + 32t^4 + 64t^3 - 32t^2 + 48t^1 - 63t^0$.

Carry: $8t^6 - 4t^5 - 2t^4 + 1t^3 + 2t^2 + 2t^1 - 3t^0$.

To minimize poly degree, mix reduction and carrying, carrying the top sooner.

e.g. Start from square $9t^{10} + 6t^9 + 25t^8 + 14t^7 + 48t^6 + 72t^5 + 59t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce $t^{10} \to t^4$ and carry $t^4 \to t^5 \to t^6$: $6t^9 + 25t^8 + 14t^7 + 56t^6 - 5t^5 + 2t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Finish reduction: $-5t^5 + 2t^4 + 64t^3 - 32t^2 + 48t^1 - 87t^0$. Carry $t^0 \to t^1 \to t^2 \to t^3 \to t^4 \to t^5$: $-4t^5 - 2t^4 + 1t^3 + 2t^2 - 1t^1 + 3t^0$.

ay carries until after
cation by 3.

square 314159
00003: Square poly
$^4 + 4t^3 + 1t^2 + 5t^1 + 9t^0$,
g $9t^{10} + 6t^9 + 25t^8 +$
$8t^6 + 72t^5 + 59t^4 +$
$3t^2 + 90t^1 + 81t^0$.

replace $(c_i)t^{6+i}$ by
, obtaining $72t^5 + 32t^4 +$
$2t^2 + 48t^1 - 63t^0$.

$t^6 - 4t^5 - 2t^4 +$
$^2 + 2t^1 - 3t^0$.

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Speedup

$p = 2^{61}$

Five coe
$f_4t^4 + f$
Most co

Square ·
Coeff of

Reduce:
$\cdots + (2$
Coeff co
Very litt
addition
on 32-bi

until after

3.

4159

uare poly

$1t^2 + 5t^1 + 9t^0$,

$t^9 + 25t^8 +$

$^5 + 59t^4 +$

$^1 + 81t^0$.

$c_i)t^{6+i}$ by

g $72t^5 + 32t^4 +$

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$- 2t^4 +$

$3t^0$.

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Speedup: non-inte

$p = 2^{61} - 1$.

Five coeffs in radi
$f_4t^4 + f_3t^3 + f_2t^2$
Most coeffs could

Square $\cdots + 2(f_4 f$
Coeff of t^5 could b

Reduce: $2^{65} = 2^4$
$\cdots + (2^5(f_4 f_1 +$
Coeff could be $>$
Very little room fo
additions, delayed
on 32-bit platform

$+ 9t^0$,

$+$

$-$

$32t^4 +$

To minimize poly degree, mix reduction and carrying, carrying the top sooner.

e.g. Start from square $9t^{10} + 6t^9 + 25t^8 + 14t^7 + 48t^6 + 72t^5 + 59t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce $t^{10} \to t^4$ and carry $t^4 \to t^5 \to t^6$: $6t^9 + 25t^8 + 14t^7 + 56t^6 - 5t^5 + 2t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Finish reduction: $-5t^5 + 2t^4 + 64t^3 - 32t^2 + 48t^1 - 87t^0$. Carry $t^0 \to t^1 \to t^2 \to t^3 \to t^4 \to t^5$: $-4t^5 - 2t^4 + 1t^3 + 2t^2 - 1t^1 + 3t^0$.

Speedup: non-integer radix

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}? $f_4t^4 + f_3t^3 + f_2t^2 + f_1t^1 +$

Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4f_1 + f_3f_2)t$

Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61}$

$\cdots + (2^5(f_4f_1 + f_3f_2) + f$

Coeff could be $> 2^{29}$.

Very little room for additions, delayed carries, et on 32-bit platforms.

To minimize poly degree, mix reduction and carrying, carrying the top sooner.

e.g. Start from square $9t^{10} + 6t^9 + 25t^8 + 14t^7 + 48t^6 + 72t^5 + 59t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Reduce $t^{10} \to t^4$ and carry $t^4 \to t^5 \to t^6$: $6t^9 + 25t^8 + 14t^7 + 56t^6 - 5t^5 + 2t^4 + 82t^3 + 43t^2 + 90t^1 + 81t^0$.

Finish reduction: $-5t^5 + 2t^4 + 64t^3 - 32t^2 + 48t^1 - 87t^0$. Carry $t^0 \to t^1 \to t^2 \to t^3 \to t^4 \to t^5$: $-4t^5 - 2t^4 + 1t^3 + 2t^2 - 1t^1 + 3t^0$.

<u>Speedup: non-integer radix</u>

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}? $f_4t^4 + f_3t^3 + f_2t^2 + f_1t^1 + f_0t^0$. Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4f_1 + f_3f_2)t^5 + \cdots$. Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61} - 1)$; $\cdots + (2^5(f_4f_1 + f_3f_2) + f_0^2)t^0$. Coeff could be $> 2^{29}$. Very little room for additions, delayed carries, etc. on 32-bit platforms.

mize poly degree,
uction and carrying,
the top sooner.

rt from square $9t^{10}+6t^9+$
$4t^7+48t^6+72t^5+59t^4+$
$3t^2+90t^1+81t^0$.

$t^{10}\to t^4$ and carry $t^4\to$
$6t^9+25t^8+14t^7+56t^6-$
$+82t^3+43t^2+90t^1+81t^0$.

eduction: $-5t^5+2t^4+$
$2t^2+48t^1-87t^0$. Carry
$\to t^2\to t^3\to t^4\to t^5$:
$2t^4+1t^3+2t^2-1t^1+3t^0$.

Speedup: non-integer radix

$p=2^{61}-1$.

Five coeffs in radix 2^{13}?
$f_4t^4+f_3t^3+f_2t^2+f_1t^1+f_0t^0$.
Most coeffs could be 2^{12}.

Square $\cdots+2(f_4f_1+f_3f_2)t^5+\cdots$.
Coeff of t^5 could be $>2^{25}$.

Reduce: $2^{65}=2^4$ in $\mathbf{Z}/(2^{61}-1)$;
$\cdots+(2^5(f_4f_1+f_3f_2)+f_0^2)t^0$.
Coeff could be $>2^{29}$.
Very little room for
additions, delayed carries, etc.
on 32-bit platforms.

Scaled:
f_4 is mu
f_3 is mu
f_2 is mu
f_1 is mu
f_0 is mu
$\cdots+(2$

Better:
f_4 is mu
f_3 is mu
f_2 is mu
f_1 is mu
f_0 is mu
Saves a

degree,
carrying,
ooner.

uare $9t^{10} + 6t^9 +$
$+ 72t^5 + 59t^4 +$
$^1 + 81t^0$.

and carry $t^4 \to$
$^8 + 14t^7 + 56t^6 -$
$3t^2 + 90t^1 + 81t^0$.

$-5t^5 + 2t^4 +$
$^1 - 87t^0$. Carry
$t^3 \to t^4 \to t^5$:
$2t^2 - 1t^1 + 3t^0$.

Speedup: non-integer radix

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}?
$f_4 t^4 + f_3 t^3 + f_2 t^2 + f_1 t^1 + f_0 t^0$.
Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4 f_1 + f_3 f_2) t^5 + \cdots$.
Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61} - 1)$;
$\cdots + (2^5(f_4 f_1 + f_3 f_2) + f_0^2) t^0$.
Coeff could be $> 2^{29}$.
Very little room for
additions, delayed carries, etc.
on 32-bit platforms.

Scaled: Evaluate a
f_4 is multiple of 2
f_3 is multiple of 2
f_2 is multiple of 2
f_1 is multiple of 2
f_0 is multiple of 2
$\cdots + (2^{-60}(f_4 f_1 -$

Better: Non-intege
f_4 is multiple of 2
f_3 is multiple of 2
f_2 is multiple of 2
f_1 is multiple of 2
f_0 is multiple of 2
Saves a few bits ir

$+6t^9+$

$59t^4+$

$t^4 \rightarrow$

$56t^6-$

$+81t^0.$

$^4+$

Carry

$\rightarrow t^5$:

$+3t^0.$

Speedup: non-integer radix

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}?

$f_4t^4 + f_3t^3 + f_2t^2 + f_1t^1 + f_0t^0$.

Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4f_1 + f_3f_2)t^5 + \cdots$.

Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61}-1)$;

$\cdots + (2^5(f_4f_1 + f_3f_2) + f_0^2)t^0$.

Coeff could be $> 2^{29}$.

Very little room for additions, delayed carries, etc. on 32-bit platforms.

Scaled: Evaluate at $t = 1$.

f_4 is multiple of 2^{52};

f_3 is multiple of 2^{39};

f_2 is multiple of 2^{26};

f_1 is multiple of 2^{13};

f_0 is multiple of 2^0. Reduce

$\cdots + (2^{-60}(f_4f_1 + f_3f_2) +$

Better: Non-integer radix 2^1

f_4 is multiple of 2^{49};

f_3 is multiple of 2^{37};

f_2 is multiple of 2^{25};

f_1 is multiple of 2^{13};

f_0 is multiple of 2^0.

Saves a few bits in coeffs.

Speedup: non-integer radix

$p = 2^{61} - 1$.

Five coeffs in radix 2^{13}?
$f_4t^4 + f_3t^3 + f_2t^2 + f_1t^1 + f_0t^0$.
Most coeffs could be 2^{12}.

Square $\cdots + 2(f_4f_1 + f_3f_2)t^5 + \cdots$.
Coeff of t^5 could be $> 2^{25}$.

Reduce: $2^{65} = 2^4$ in $\mathbf{Z}/(2^{61} - 1)$;
$\cdots + (2^5(f_4f_1 + f_3f_2) + f_0^2)t^0$.
Coeff could be $> 2^{29}$.
Very little room for
additions, delayed carries, etc.
on 32-bit platforms.

Scaled: Evaluate at $t = 1$.
f_4 is multiple of 2^{52};
f_3 is multiple of 2^{39};
f_2 is multiple of 2^{26};
f_1 is multiple of 2^{13};
f_0 is multiple of 2^0. Reduce:
$\cdots + (2^{-60}(f_4f_1 + f_3f_2) + f_0^2)t^0$.

Better: Non-integer radix $2^{12.2}$.
f_4 is multiple of 2^{49};
f_3 is multiple of 2^{37};
f_2 is multiple of 2^{25};
f_1 is multiple of 2^{13};
f_0 is multiple of 2^0.
Saves a few bits in coeffs.