Elliptic vs. hyperelliptic, part 1

D. J. Bernstein


Elliptic
vs.
Hyper-
elliptic
Part I

Goal: Protect all Internet packets against forgery, eavesdropping.

We aren't anywhere near the goal. Most Internet packets have little or no protection.

Why not deploy cryptography? Why `http://www.google.com`, not `https://www.google.com`?

Common answer: Cryptography takes too much CPU time.

Obvious response, maybe enough: Faster cryptography!

Streamlining protocols

Often quite easy to save time in cryptographic protocols by recognizing and eliminating wasteful cryptographic structures.

Example #1 of waste:
Sender feeds a message through "public-key encryption" and then "public-key signing."

Improvement: "Signcryption."
No need to partition into encryption and signing; combined algorithms are faster.

Example #2: Sender signcrypts two messages for same receiver.

Improvement: Signcrypt one key and use secret-key cryptography to protect both messages.

Example #3: Sender signcrypts randomly generated secret key.

Improvement: Diffie-Hellman, generating unique shared secret for each pair of public keys. Obtain randomness of secret from randomness of public keys. No need for extra randomness.

Streamlined structure to protect private communication:

Alice has secret key a,
long-term public key $G(a)$.
Alice, Bob have long-term shared secret $G(ab)$.
Alice, Bob use shared secret to encrypt and authenticate any number of packets.

(Public communication has a different streamlined structure. This talk will focus on private communication.)

How much does this cost?

Key generation: one evaluation of $a \mapsto G(a)$ for each user.

Shared secrets: one evaluation of $a, G(b) \mapsto G(ab)$ for each pair of communicating users.

Encryption and authentication: secret-key operations for each byte communicated.

This talk will focus on applications with many pairs of communicating users and with not much data communicated between each pair.

Bottleneck is $a, G(b) \mapsto G(ab)$.
How fast is this?

Answer depends on CPU,
on choice of G, and on
choice of method to compute G.

Many parameters.
Many interactions across levels.
Choices are not easy
to analyze and optimize.

Elliptic vs. hyperelliptic

PKC 2006: Analyzed wide range of elliptic-curve functions G and methods of computing G.

Obtained new speed records for $a, G(b) \mapsto G(ab)$ on today's most common CPUs. `http://cr.yp.to/ecdh.html`

The big questions for today: Can we obtain higher speeds at comparable security levels using genus-2 hyperelliptic curves? How fast is hyperelliptic-curve scalar multiplication?

Basic advantage of genus 2:
use much smaller field
for same conjectured security.

This talk will focus on a
comfortable security level:
$> 2^{128}$ bit ops for known attacks.

PKC 2006 genus-1 records
used field size $2^{255} - 19$.
$\approx 2^{255}$ points on curve.

Jacobian of genus-2 curve
over field of size $2^{127} - 1$
has $\approx 2^{254}$ points.
Much smaller field, so
much faster field mults.

Basic disadvantage of genus 2: many more field mults.

PKC 2006 genus-1 records used Montgomery-form curve $y^2 = x^3 + 486662x^2 + x$, $G(a) = X_0(aP)$, standard P. 10 mults per bit of a.

Culmination of extensive work on eliminating field mults for similar $G(a)$ defined by genus-2 hyperelliptic curve: 25 mults per bit. (2005 Gaudry)

Does the advantage
outweigh the disadvantage?

Superficial analysis: Yes!
Half as many bits in field
means, uhhh, $4\times$ faster? $3\times$?
Anyway, $(3 \times 10)/25 = 1.2$.
That's a 20% gap!
Genus-2 field mults have
finally been reduced enough
to beat genus 1!

This analysis has several flaws.
Let's do a serious analysis.

What are the formulas?

Genus-1 setup: Field k, big char. Specify elliptic curve $E \subset \mathbf{P}^2$ by equation $y^2 z = x^3 + a_2 x^2 z + x z^2$. (Full moduli space *if* $k = \overline{k}$.) Rational map $(x : y : z) \mapsto (x : z)$ induces $X : E/\{\pm 1\} \hookrightarrow \mathbf{P}^1$.

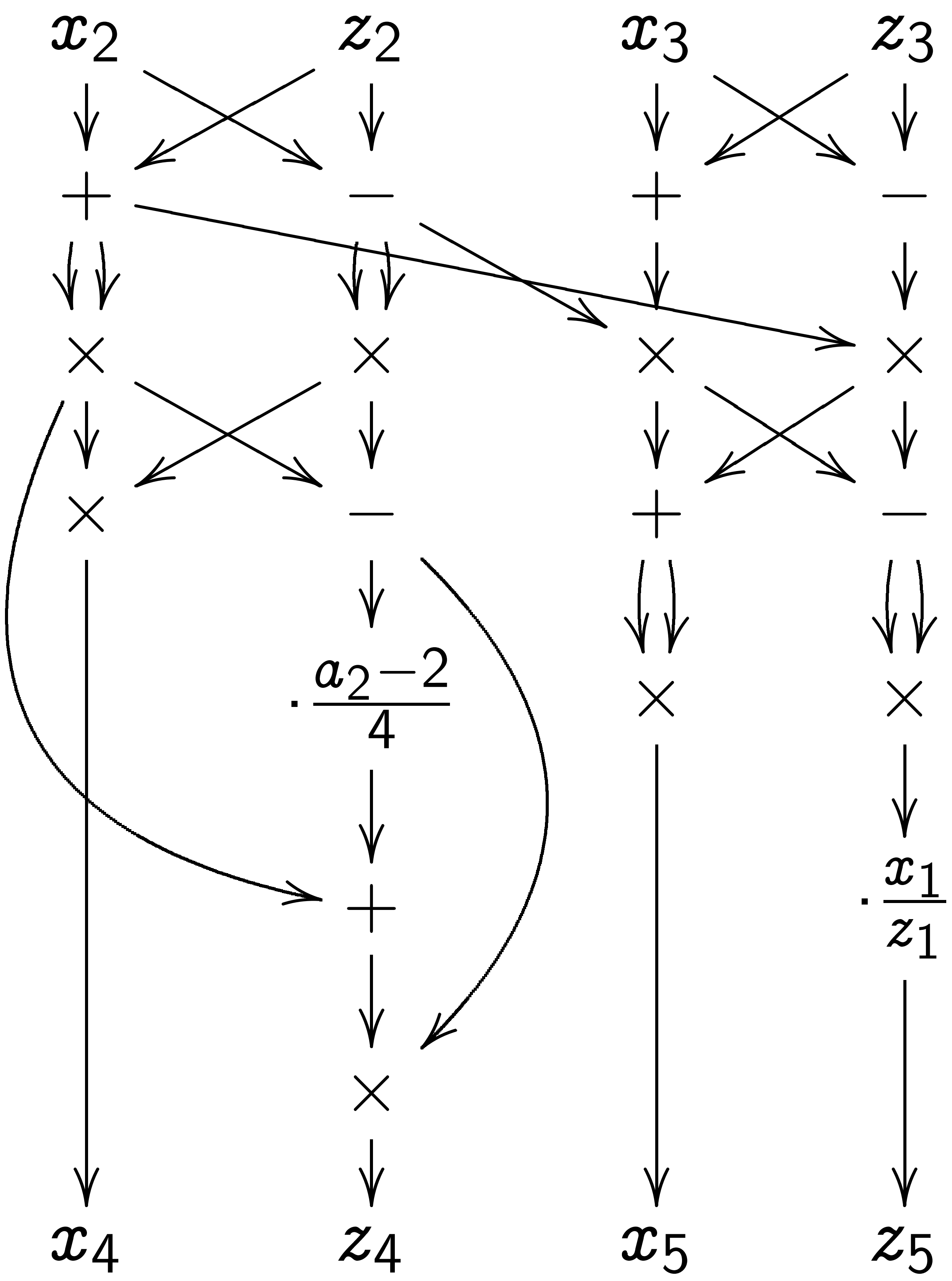
Analogous genus-2 setup: Specify genus-2 curve C by particular parametrization. Build "Kummer surface" $K \subset \mathbf{P}^3$ and particular rational map $X : (\operatorname{Jac} C)/\{\pm 1\} \hookrightarrow K$.

Recursively build rational functions $F_1, F_2, \ldots$ with $X(nQ) = F_n(X(Q))$ generically.

Recursion uses very fast rational functions $X(nQ) \mapsto X(2nQ)$ and $X(Q), X(nQ), X((n+1)Q) \mapsto X((2n+1)Q)$.

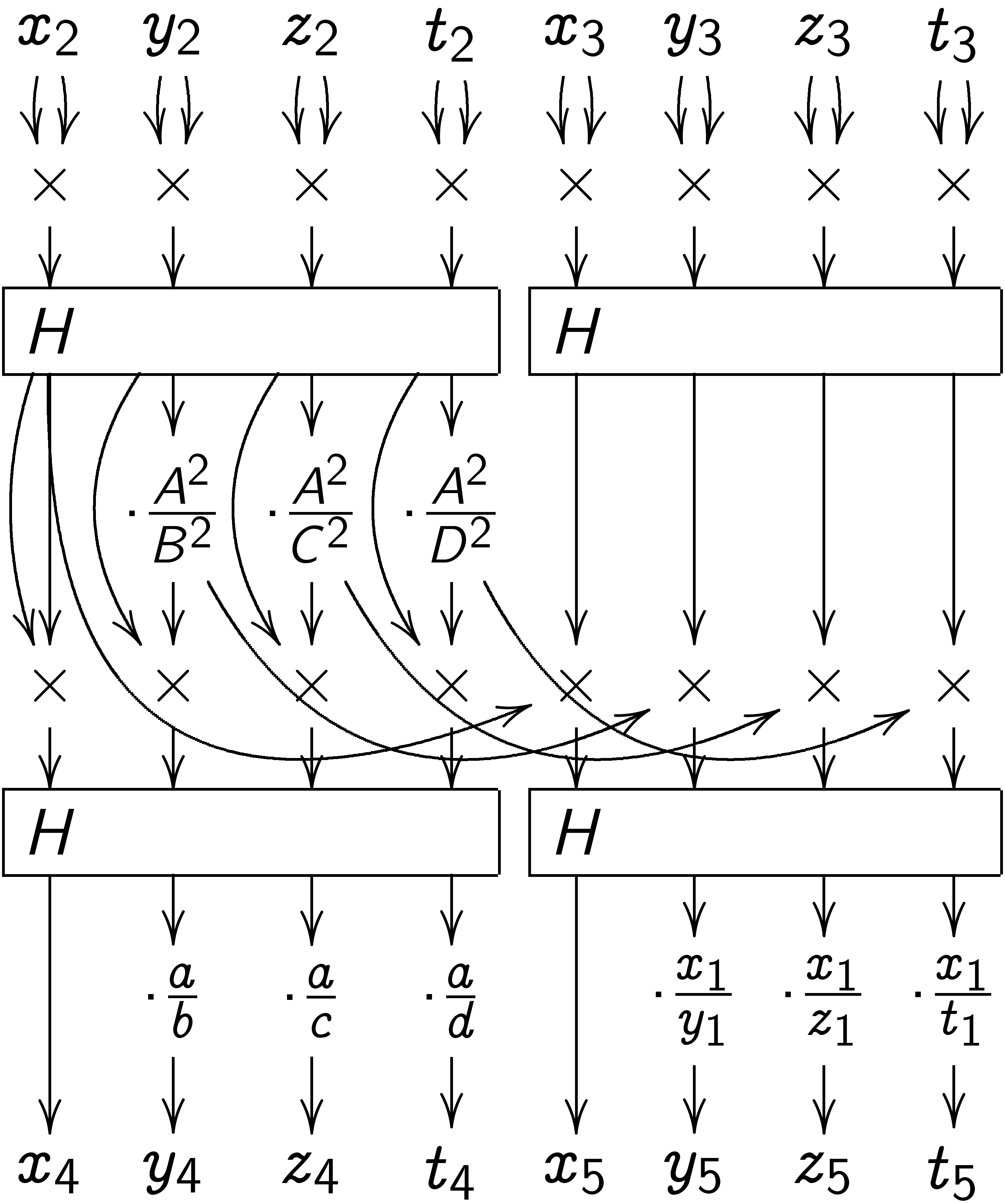
(genus 1: 1986 Chudnovsky, Chudnovsky; independently 1987 Montgomery; 10 mults: 1987 Montgomery; genus 2: 1986 Chudnovsky, Chudnovsky; 25 mults: 2005 Gaudry)

Montgomery's recursion for genus 1, $X(nQ) = (x_n : z_n)$:



Gaudry's recursion for genus 2,

$X(nQ) = (x_n : y_n : z_n : t_n)$:



$H(\alpha,\beta,\gamma,\delta) =$

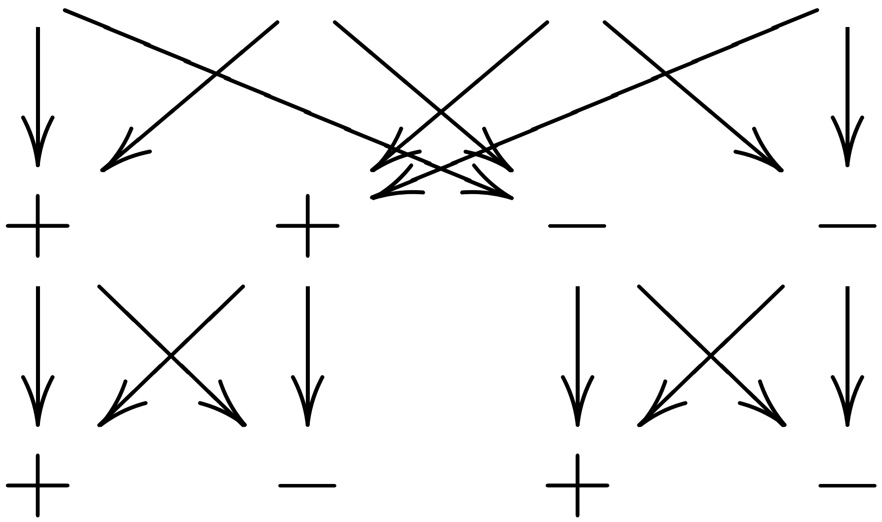
$(\alpha+\beta+\gamma+\delta,$

$\alpha+\beta-\gamma-\delta,$

$\alpha-\beta+\gamma-\delta,$

$\alpha-\beta-\gamma+\delta).$

Easy 8-addition chain ("fast Hadamard transform"):



Total Gaudry field operations: 25 mults, 32 adds.

$X(nQ) = F_n(X(Q))$ generically:
"Generically" allows failures.
Maybe trouble for cryptography!

Can detect failures by
testing for zero at each step.
Can we avoid these tests?

For genus 1: Yes,
after replacing X by X_0.
`cr.yp.to/papers.html`
`#curvezero`, Theorem 5.1.

Similar in genus 2?
Looks like painful calculations.
Let me know if you have
ideas for tackling this.

Curve specialization

Montgomery-form curves can be specialized to save time.
For $y^2 = x^3 + 486662x^2 + x$,
1 of the 10 mults is by 121665;
much faster than general mult.

Do Gaudry-form surfaces allow similar specialization?
Gaudry: Out of 25 mults,
6 "are multiplications by constants that depend only on the surface . . . Therefore by choosing an appropriate surface, a few multiplications can be saved."

What's "a few"?

Let's look at the formulas.

Gaudry has params $(a : b : c : d)$.
Also $(A : B : C : D)$ satisfying
$H(A^2, B^2, C^2, D^2) =$
(a^2, b^2, c^2, d^2).

Gaudry's 6 mults are by
$a/b, a/c, a/d$,
$(A/B)^2, (A/C)^2, (A/D)^2$.

Can choose small B, C, D,
small $A \in B\mathbf{Z} \cap C\mathbf{Z} \cap D\mathbf{Z}$.
Then solve for a, b, c, d.

Can scale formulas to have multiplications by, e.g., $(BCD)^2$, $(ACD)^2$, $(ABD)^2$, $(BCD)^2$.
Choose any small A, B, C, D.
Can also hope for some of a, b, c, d to be small.

More flexibility:
Can choose small A^2, B^2, C^2, D^2.
e.g. $A^2 = 21$, $B^2 = 16$, $C^2 = 8$, $D^2 = 4$, $a = 7$, $b = 5$, $c = 3$, $d = 1$.
Scale $1, a/b, a/c, a/d$ to bcd, acd, abd, abc.

Apparently “a few” is “all 6”!

Products with $a/b, a/c, a/d$ will be squared before use.

Convenient to change K by squaring coordinates. (as in 1986 Chudnovsky, Chudnovsky)

In data-flow diagram, roll top squarings to bottom and through a, b, c, d layer. No loss in speed. (2006 André Augustyniak)

Thus have even more flexibility: small a^2, b^2, c^2, d^2 suffice.

Unfortunately,
these specialized surfaces
have a big security problem:
genus-2 point counting
is too slow to reach 256 bits.

Our only secure genus-2 curves
are from CM. How to locate
a secure *specialized* surface
over, e.g., $\mathbf{Z}/(2^{127}-1)$?

Maybe can speed up
genus-2 point counting.
Inspiring news: speed records
for Schoof's original algorithm.
(2006 Nikki Pitcher)

Squarings and other operations

For Montgomery-form curves:
4 of the 9 big mults
are squarings;
faster than general mults.

For Gaudry-form surfaces:
9 squarings out of 25 mults.

$4S + 5M$ in big field
comparable to, uhhh,
$12S + 15M$ in small field?
$9S + 16M$ still slightly better,
but gap is only $\approx 5\%$,
depending on S/M ratio.

Gaudry understated benefit of specialized surfaces.

One of Gaudry's speedups: compute $(a/b)u^2, (a/b)uv$ by first computing $(a/b)u$. $3M$. Total: $9S + 16M$. Specialized: $2M$. Specialized total: $9S + 10M$.

Better when a/b is small: simply undo this speedup. $S + 3M$. Total: $12S + 16M$. Specialized: $S + M$. Specialized total: $12S + 7M$.

The $3\times, 4\times$ myths

Why do some people say that half as many bits in field means $4\times$ speedup?

Answer: "n-bit arithmetic takes time n^2."

Why do some people say that half as many bits in field means $3\times$ speedup?

Answer: "n-bit arithmetic takes time $n^{\lg 3}$."

Reality: Both n^2 and $n^{\lg 3}$ are horribly inaccurate models.

Field speed is CPU-dependent.
Today let's focus on
one common CPU: Pentium M.

Experience says:
Fastest Pentium M arithmetic
uses floating-point operations.
$\#\{\text{fp ops}\}/\#\{\text{cycles}\} \le 1$;
optimized code always close to 1,
very little variation.

PKC 2006 speed records
for $y^2 = x^3 + 486662x^2 + x$
over $\mathbf{Z}/(2^{255} - 19)$:
640838 cycles; 92% fp ops.

Accurately (but not perfectly) analyze cycles by counting fp ops.

e.g. $\mathbf{Z}/(2^{255} - 19)$ arithmetic in PKC 2006 records:
10 fp ops for $f, g \mapsto f + g$.
55 fp ops for $f \mapsto 121665f$.
162 fp ops for $f \mapsto f^2$.
243 fp ops for $f, g \mapsto fg$.

Where do these numbers come from? How do they scale?
Is $\mathbf{Z}/(2^{127} - 1)$ really $4\times$ faster?
Or at least $3\times$ faster?

Element of $\mathbf{Z}/(2^{255} - 19)$ is represented as 10-coeff poly.

Field add is poly add: 10 fp adds. In context, can skip carries.

Field mult is poly mult
and reduction mod $2^{255} - 19$
and carrying:
10^2 fp mults for poly,
$(10 - 1)^2$ fp adds for poly,
$10 - 1$ fp mults for reduce,
$10 - 1$ fp adds for reduce,
$4 \cdot 10 + 4$ fp adds for carry.

Squaring: save $(10 - 1)^2$ ops.

Element of $\mathbf{Z}/(2^{127}-1)$ is represented as 5-coeff poly.

Field add is 5 fp ops; $2\times$ faster.

Poly mult is $5^2+(5-1)^2$
but reduce is $(5-1)+(5-1)$
and carry is $4\cdot 5+4$.
73 fp ops; $3.329\times$ faster.

Squaring saves $(5-1)^2$ ops.
57 fp ops; $2.842\times$ faster.

Surprisingly small ratios,
even *without* Karatsuba.
Heavy optimization of mults
makes linear effects more visible.

Montgomery uses 8 adds,
1 mult by 121665,
4 squarings, 5 mults:
$8 \cdot 10 + 1 \cdot 55 + 4 \cdot 162 + 5 \cdot 243$
$= 1998$.

Gaudry uses 32 adds,
9 squarings, 16 mults:
$32 \cdot 5 + 9 \cdot 57 + 16 \cdot 73 = 1841$.

Gaudry loses in adds,
wins in squarings,
wins in other mults.

Specialized Gaudry: $[1355, 1659]$
depending on exact coeff size.
Far fewer than 1998 ops!

<u>Reciprocals</u>

What about divisions?

At end of computation,
$(x : y : z : t) \mapsto (x/t, y/t, z/t)$
for transmission.
Three multiplications and
one reciprocal in $\mathbf{Z}/(2^{127} - 1)$.

Montgomery needs division
in $\mathbf{Z}/(2^{255} - 19)$;
more than twice as slow.
Not big part of computation
but still a disadvantage.

Space disadvantage for Gaudry:
$\approx$ 384 bits in $(x/t, y/t, z/t)$.

Standard 512-bit alternative:
blinding. Choose random r,
send $(xr : yr : zr : tr)$.
Negligible computation cost.
Also negligible for Montgomery.

Standard 256-bit alternative:
point compression.
Transmit, e.g., $(x/t, y/t)$.
Then have to solve quartic.
Disadvantage for Gaudry.

Open: Compression method
allowing faster decompression?

Extra Gaudry division problem: recall multiplications by $x_1/y_1, x_1/z_1, x_1/t_1$.

Even if we're given $t_1 = 1$, have to divide by y_1, z_1.
How to avoid extra division?
Can't merge with final division.
Scaling $(1 : x_1/y_1 : x_1/z_1 : x_1)$ is bad: extra mult for each bit.

Easy solution: Don't send $(x/t, y/t, z/t)$. Instead send $(t/x, t/y, t/z)$ or $(x/y, x/z, x/t)$.
Sender can merge divisions.

Software speed measurements

Using `qhasm` tools, wrote Pentium M implementation of scalar multiplication (with no input-dependent branches, indices, etc.) on a Gaudry-form surface.

$n, P \mapsto nP$. Coords $(x/y, x/z, x/t)$ for P, nP. Arbitrary params a, b, c, d.

Recall the competition, speed record from PKC 2006: 640838 cycles for genus 1.

Genus 2: 582363 cycles.
New Diffie-Hellman speed record!

Try the software yourself:
`cr.yp.to/hecdh.html`

Standardize genus-2 curve for cryptography? Use CM to generate secure a, b, c, d?

I think that's premature. Very small choices of a, b, c, d will provide a big speedup. Let's wait for point counting, *then* standardize.

Halftime advertising, part 1

Part 1 was brought to you by . . .

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