Using fast power-series arithmetic in the Kedlaya-Denef-Vercauteren algorithm

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The problem. Let K be a field of characteristic 0. Fix a positive integer g. We're given $f, h, Q \in K[x]$ where f is monic, $\deg f = 2g + 1$, and $\deg h \le g$. How do we compute $P, R \in K[x]$ with $\deg R < 2g$ and $Q - R = (2f' + hh')P + (1/3)(4f + h^2)P'$? One can take $(P, R) = (0, Q)$ if $\deg Q < 2g$, so assume that $\deg Q \ge 2g$.

Tiny example: Define $K = \mathbf{C}$, $g = 2$, $f = x^5 + x^4 + 1$, $h = x$, and $Q = x^7 + 11x^5 + x + 1$. How do we compute $P, R \in \mathbf{C}[x]$ with

$$x^7 + 11x^5 + x + 1 - R = (10x^4 + 8x^3 + x)P + (1/3)(4x^5 + 4x^4 + x^2 + 4)P'$$

and $\deg R < 4$?

Application. Kedlaya introduced an algorithm for computing the zeta function of a genus-g hyperelliptic curve over a finite field of size p^n when p is odd. Kedlaya's algorithm uses roughly $g^4 n^3$ bit operations for fixed p.

Denef and Vercauteren adapted Kedlaya's algorithm to the case $p = 2$. The Kedlaya-Denef-Vercauteren algorithm uses roughly $g^4 n^3$ bit operations for a "typical" curve but roughly $g^5 n^3$ bit operations for some other curves.

At a meeting in Oberwolfach I asked Kedlaya about the discrepancy between $g^4 n^3$ and $g^5 n^3$. He explained the problem of computing P, R from f, h, Q and told me that this was one of the bottlenecks in the $p = 2$ case.

A slow solution. Apparently Denef and Vercauteren use the equation $Q - R = (2f' + hh')P + (1/3)(4f + h^2)P'$ to determine the coefficients of P one at a time. The algebraic complexity of this computation over K—the number of additions, subtractions, multiplications, and divisions of coefficients in K—grows quadratically with g in the typical case $\deg Q = 4g$.

Tiny example: Consider again the problem of finding $P, R \in \mathbf{C}[x]$ with $x^7 + 11x^5 + x + 1 - R = (10x^4 + 8x^3 + x)P + (1/3)(4x^5 + 4x^4 + x^2 + 4)P'$ and $\deg R < 4$. Assume that P will have degree at most 3; write P as $P_3x^3 + P_2x^2 + P_1x + P_0$; write R as $R_3x^3 + R_2x^2 + R_1x + R_0$. The problem is now to find $P_3, P_2, P_1, P_0, R_3, R_2, R_1, R_0$ such that

$$\begin{aligned} &x^7 + 11x^5 + x + 1 - (R_3x^3 + R_2x^2 + R_1x + R_0) \\ &\quad = (10x^4 + 8x^3 + x)(P_3x^3 + P_2x^2 + P_1x + P_0) \\ &\qquad + (1/3)(4x^5 + 4x^4 + x^2 + 4)(3P_3x^2 + 2P_2x + P_1). \end{aligned}$$

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Extract the coefficients of $x^7, x^6, x^5, x^4, x^3, x^2, x^1, x^0$ from this equation to form a lower-triangular system of linear equations:

$$\begin{pmatrix} 1 \\ 0 \\ 11 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 38/3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 32/3 & 34/3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 28/3 & 10 & 0 & 0 & 0 & 0 \\ 2 & 2/3 & 0 & 8 & 1 & 0 & 0 & 0 \\ 4 & 2 & 1/3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 8/3 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4/3 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_3 \\ P_2 \\ P_1 \\ P_0 \\ R_3 \\ R_2 \\ R_1 \\ R_0 \end{pmatrix}.$$

Use substitution to solve this system one variable at a time: use the first equation $1 = 14P_3$ to determine $P_3 = 1/14$, then use the second equation $0 = 12P_3 + (38/3)P_2$ to determine $P_2 = -9/133$, etc.

A faster solution. The following solution produces the same output but is much more efficient than the one-at-a-time solution when g and $\deg Q - 2g$ are large. This solution relies on standard FFT-based subroutines for fast power-series multiplication, division, and square root. The higher-level aspects of the solution are also standard, so I'd be embarrassed to receive any credit for the solution; my interests here are purely expository, advertising yet another reason that novices should learn how to use fast multiplication. Anyway, here's the solution:

- Compute $(4f + h^2)^{1/2} = (2)x^{g+1/2} + (\cdots)x^{g-1/2} + (\cdots)x^{g-3/2} + \cdots$ to high precision in the field $K((1/\sqrt{x}))$.
- Multiply by $3Q$, producing $3Q(4f + h^2)^{1/2}$ to high precision in $K((1/\sqrt{x}))$.
- Integrate with respect to x, producing $\int 3Q(4f + h^2)^{1/2}\,dx$ to high precision in $K((1/\sqrt{x}))$.
- Divide by $(4f + h^2)^{3/2} = (8)x^{3g+3/2} + (\cdots)x^{3g+1/2} + (\cdots)x^{3g-1/2} + \cdots$, producing $(4f + h^2)^{-3/2}\int 3Q(4f + h^2)^{1/2}\,dx$ to high precision in $K((1/\sqrt{x}))$.
- Round to a polynomial $P \in K[x]$.
- Compute $R = Q - (2f' + hh')P - (1/3)(4f + h^2)P'$ in $K[x]$.

Why does this work? Answer: Write $\epsilon = P - (4f + h^2)^{-3/2}\int 3Q(4f + h^2)^{1/2}\,dx$. Multiply by $(4f + h^2)^{3/2}$, differentiate, and divide by $3(4f + h^2)^{1/2}$ to see that $R = (2f' + hh')\epsilon + (1/3)(4f + h^2)\epsilon'$. By construction $\epsilon = (\cdots)x^{-1} + (\cdots)x^{-2} + \cdots$ so $R = (2(2g+1)x^{2g} + \cdots)((\cdots)x^{-1} + \cdots) + (1/3)(4x^{2g+1} + \cdots)((\cdots)x^{-2} + \cdots) = (\cdots)x^{2g-1} + \cdots$; i.e., $\deg R < 2g$ as desired.

I omitted one important detail above: What does "high precision" mean? Answer: We compute the first $\deg Q - 2g + 1$ coefficients of each series; this is enough information to determine $P \in K[x]$. This means that we compute

- the coefficients of $x^{g+1/2}, x^{g-1/2}, \ldots, x^{3g-\deg Q+1/2}$ in $(4f + h^2)^{1/2}$;
- the coefficients of $x^{\deg Q+g+1/2}, x^{\deg Q+g-1/2}, \ldots, x^{3g+1/2}$ in $3Q(4f + h^2)^{1/2}$;
- the coefficients of $x^{\deg Q+g+3/2}, \ldots, x^{3g+3/2}$ in $\int 3Q(4f + h^2)^{1/2}\,dx$; and
- the coefficients of $x^{\deg Q-2g}, \ldots, x^0$ in $(4f + h^2)^{-3/2}\int 3Q(4f + h^2)^{1/2}\,dx$.

Rounding to $P \in K[x]$ means simply copying the coefficients of $x^{\deg Q - 2g}, \ldots, x^0$.

This computation has algebraic complexity essentially *linear* in g, rather than quadratic in g, in the typical case $\deg Q = 4g$. More precisely, this computation has algebraic complexity $O(g \lg g \lg \lg g)$, with the $\lg \lg g$ disappearing for some choices of K. The complexity here is within a constant factor of the complexity of multiplication, division, and square root; I haven't analyzed or optimized the constant factor. Similar comments apply to other ranges of $\deg Q$.

Tiny example: Consider once again the problem of finding $P, R \in \mathbf{C}[x]$ with $x^7 + 11x^5 + x + 1 - R = (10x^4 + 8x^3 + x)P + (1/3)(4x^5 + 4x^4 + x^2 + 4)P'$ and $\deg R < 4$. Compute the first 4 coefficients of each of the following series:

$$\begin{aligned}
&(4x^5 + 4x^4 + x^2 + 4)^{1/2} \\
&\qquad = 2x^{5/2} + 1x^{3/2} - (1/4)x^{1/2} + (3/8)x^{-1/2} + \cdots; \\
&3(x^7 + 11x^5 + x + 1)(4x^5 + 4x^4 + x^2 + 4)^{1/2} \\
&\qquad = 6x^{19/2} + 3x^{17/2} + (261/4)x^{15/2} + (273/8)x^{13/2} + \cdots; \\
&\int 3(x^7 + 11x^5 + x + 1)(4x^5 + 4x^4 + x^2 + 4)^{1/2}\,dx \\
&\qquad = (12/21)x^{21/2} + (6/19)x^{19/2} + (261/34)x^{17/2} + (273/60)x^{15/2} + \cdots; \\
&(4x^5 + 4x^4 + x^2 + 4)^{-3/2} \int 3(x^7 + 11x^5 + x + 1)(4x^5 + 4x^4 + x^2 + 4)^{1/2}\,dx \\
&\qquad = (1/14)x^3 - (9/133)x^2 + (4677/4522)x^1 - (22149/22610)x^0 + \cdots.
\end{aligned}$$

Now round to $P = (1/14)x^3 - (9/133)x^2 + (4677/4522)x^1 - (22149/22610)x^0$ and compute $R = x^7 + 11x^5 + x + 1 - (10x^4 + 8x^3 + x)P - (1/3)(4x^5 + 4x^4 + x^2 + 4)P' = (89871/11305)x^3 - (3764/2261)x^2 + (6977/3230)x - (857/2261)$.

Impact on the application. Consider the cost of computing the zeta function of a genus-g hyperelliptic curve $y^2 + h(x)y = f(x)$ over a field of size 2^n. "Cost" here refers to bit operations.

The Denef-Vercauteren "Theorem 1" reports cost "$O((g^\lambda + g^\nu)g^{4+\epsilon}n^{3+\epsilon})$." As a mathematician I feel compelled to point out that the order of quantifiers here is horribly unclear. Do the authors mean "for each $\epsilon > 0$ there exists n_0 such that for each $n \geq n_0$ there exist g_0, c such that for each $g \geq g_0$ the cost is at most $c(g^\lambda + g^\nu)g^{4+\epsilon}n^{3+\epsilon}$"? Do they mean "for each $\epsilon > 0$ there exist c, d_0 such that for each n, g with $ng \geq d_0$ the cost is at most $c(g^\lambda + g^\nu)g^{4+\epsilon}n^{3+\epsilon}$"? There are many other possibilities. How is a reader supposed to apply this "theorem" without redoing the analysis?

Anyway, the Denef-Vercauteren parameters λ and ν refer to the size and ramification of the polynomial h in the curve $y^2 + h(x)y = f(x)$. Specifically, g^λ is (modulo further O confusion) shorthand for $\deg f - 2 \deg h$, and g^ν is shorthand for the maximum exponent in the factorization of h.

For a uniform random curve, usually $\deg h = g$, and usually h has very few repeated factors, so $g^\lambda + g^\nu$ is close to 1. On the other hand, I can imagine users selecting curves where g^λ is much larger. Consider, for example, the Lange-Stevens hyperelliptic-curve addition formulas; one reason that these formulas are so fast is that they force h to have small degree. Perhaps users are also interested in curves where g^ν is large.

Evidently there are two different ways that the Denef-Vercauteren cost can grow more quickly than $g^{4+o(1)}n^{3+o(1)}$:

- $g^\lambda = \deg f - 2 \deg h$ can grow more quickly than $g^{o(1)}$; e.g., $\deg h$ could be around $g - \sqrt{g}$, or around $g/2$. My impression is that the problem here is exactly the problem I've addressed, and that the one-at-a-time solution is the Denef-Vercauteren bottleneck; I speculate that the fast-arithmetic solution eliminates this bottleneck.
- g^ν, the maximum exponent in the factorization of h, can grow more quickly than $g^{o(1)}$; for example, $h(x)$ could be $x^{g/2}(x-1)(x-2)\cdots(x-g/2)$. My impression is that this is a completely different problem, caused by Denef and Vercauteren working modulo, e.g., $(x(x-1)\cdots(x-g/2))^{g/2}$. Without looking more closely at the computation—which I'm certainly not planning to do any time soon—I can't guess whether such a large modulus is really necessary.

Bottom line: I speculate that fast power-series arithmetic expands the set of "g^4n^3 curves" to allow small h degrees. I have no idea whether the set can be further expanded to allow large powers in h.