S-unit attacks

Daniel J. Bernstein

University of Illinois at Chicago; Ruhr University Bochum

Includes new joint work with
Kirsten Eisenträger, Tanja Lange, Karl Rubin,
Alice Silverberg, and Christine van Vredendaal.

Builds upon vast previous literature;
see upcoming paper for credits.

Algebraic geometry: the line over $\mathbf{C}$

$f = x^4 + 6x^3 + 5x^2 = (x+1)^1(x+5)^1x^2 \in \mathbf{C}[x]$:

$$\begin{aligned} f(10) &= f \bmod x - 10 = 16500 & \operatorname{ord}_{10} f &= 0 \\ f(-1) &= f \bmod x + 1 = 0 & \operatorname{ord}_{-1} f &= 1 \\ f(-5) &= f \bmod x + 5 = 0 & \operatorname{ord}_{-5} f &= 1 \\ f(0) &= f \bmod x - 0 = 0 & \operatorname{ord}_{0} f &= 2 \end{aligned}$$

... and consider $\mathbf{C}[1/x] \subset \mathbf{C}(x)$: $\operatorname{ord}_\infty f = -4$

"$\operatorname{ord}_r f$" $= x - r$ exponent in f. "$\operatorname{ord}_\infty$" $= -\deg$.
This f is an "S-unit" if $\{\infty, 0, -1, -5\} \subseteq S$.

Fundamental thm of algebra: $\sum_{\rho \in \mathbf{C} \cup \{\infty\}} \operatorname{ord}_\rho f = 0$.
f is almost determined by the vector $\rho \mapsto \operatorname{ord}_\rho f$.

Intermediate: the line over $\mathbf{F}_7$

$f = x^4+3x^3+x^2+5x+2 = (x-2)^2(x^2-3)^1 \in \mathbf{F}_7[x]$:

$$
\begin{array}{rcr}
f \bmod x-0 = 2 & \operatorname{ord}_x f = 0 & |f|_x = 1 \\
f \bmod x-2 = 0 & \operatorname{ord}_{x-2} f = 2 & |f|_{x-2} = 1/7^2 \\
f \bmod x^2+1 \ne 0 & \operatorname{ord}_{x^2+1} f = 0 & |f|_{x^2+1} = 1 \\
f \bmod x^2-3 = 0 & \operatorname{ord}_{x^2-3} f = 1 & |f|_{x^2-3} = 1/7^2 \\
 & \operatorname{ord}_\infty f = -4 & |f|_\infty = 7^4
\end{array}
$$

$|f|_P = 1/\#(\mathbf{F}_7[x]/P)^{\operatorname{ord}_P f}$ for "finite place" P.

"Product formula": $\prod_\rho |f|_\rho = 1$; $\sum_\rho \log|f|_\rho = 0$; here ρ ranges over $\{$monic irreds in $\mathbf{F}_7[x]\} \cup \{\infty\}$. f is almost determined by the vector $\rho \mapsto \operatorname{ord}_\rho f$.

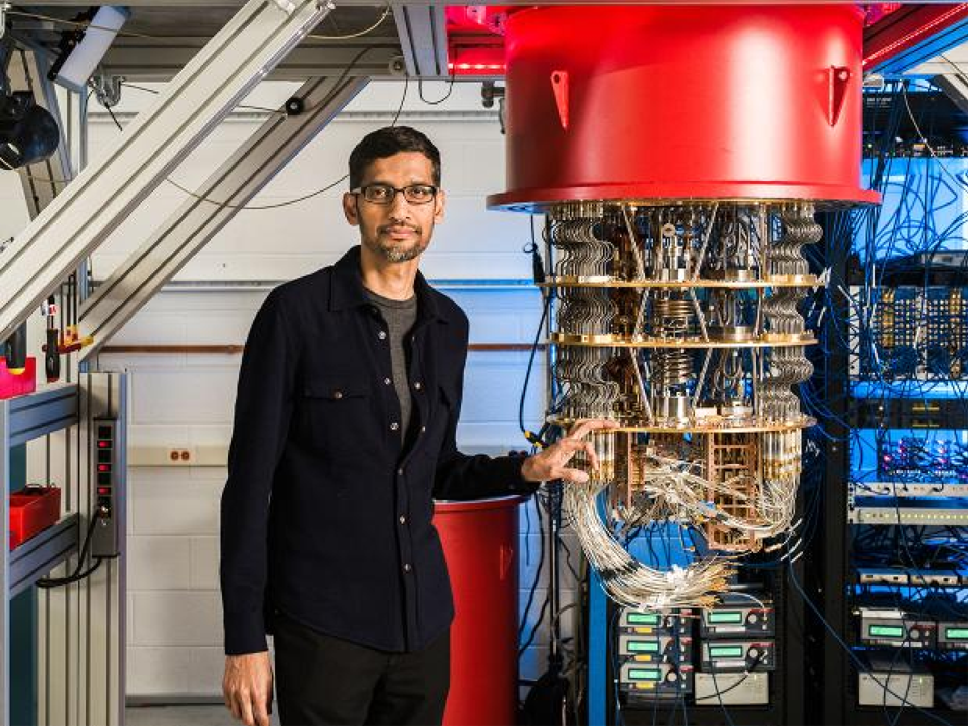
Number theory: $\mathbf{Z}$

$f = -50421 = -3^1 7^5 \in \mathbf{Z}$:

$$
\begin{array}{lll}
f \bmod 2 = 1 & \operatorname{ord}_2 f = 0 & |f|_2 = 1 \\
f \bmod 3 = 0 & \operatorname{ord}_3 f = 1 & |f|_3 = 1/3^1 \\
f \bmod 5 = 4 & \operatorname{ord}_5 f = 0 & |f|_5 = 1 \\
f \bmod 7 = 0 & \operatorname{ord}_7 f = 5 & |f|_7 = 1/7^5 \\
 & & |f|_\infty = 50421
\end{array}
$$

$|f|_P = 1/\#(\mathbf{Z}/P)^{\operatorname{ord}_P f}$ for "finite place" P.

"Product formula": $\prod_\rho |f|_\rho = 1$; $\sum_\rho \log |f|_\rho = 0$;
here ρ ranges over $\{\text{prime numbers}\} \cup \{\infty\}$.
f is almost determined by the vector $\rho \mapsto \operatorname{ord}_\rho f$.



Lattice-based cryptography

2010 LPR proved "very strong hardness guarantees":

Assume "worst-case problems on ideal lattices are hard for polynomial-time quantum algorithms"

$\Downarrow$

"the ring-LWE distribution is pseudorandom"

$\Downarrow$

security for a "truly practical lattice-based public-key cryptosystem"

Concrete parameters in cryptosystems are chosen assuming much more than polynomial hardness.

What's the supposedly hard problem?

Parameters: Define $R = \mathbf{Z}[x]/(x^n + 1)$ for some $n \in \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots\}$.
[Can generalize, but this talk focuses on these rings R.]

Problem: Given a nonzero ideal $I \subseteq R$, find a "short" nonzero element $g \in I$.

"Given" I: given $v_1, v_2, \ldots, v_n \in R$ such that $I = \mathbf{Z}v_1 + \mathbf{Z}v_2 + \cdots + \mathbf{Z}v_n$.

e.g.
$$\begin{aligned} v_1 &= x^3 + 817 \quad \longrightarrow \quad & g &= 2v_1 + 3v_2 - 5v_3 - 2v_4 \\ v_2 &= x^2 + 540 & &= 2x^3 + 3x^2 - 5x + 1 \\ v_3 &= x + 247 & & \\ v_4 &= 1009 & & \end{aligned}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

817	0	0	1
540	0	1	0
247	1	0	0
1009	0	0	0

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 817 & 0 & 0 & 1 \\ 540 & 0 & 1 & 0 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{matrix} 277 & 0 & -1 & 1 \\ 540 & 0 & 1 & 0 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{matrix}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 277 & 0 & -1 & 1 \\ 263 & 0 & 2 & -1 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{matrix} 14 & 0 & -3 & 2 \\ 263 & 0 & 2 & -1 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{matrix}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 247 & 1 & 0 & 0 \\ 192 & 0 & 0 & -1 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 55 & 1 & 0 & 1 \\ 192 & 0 & 0 & -1 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 55 & 1 & 0 & 1 \\ 137 & -1 & 0 & -2 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{matrix} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 55 & 1 & 0 & 1 \\ 82 & -2 & 0 & -3 \end{matrix}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 55 & 1 & 0 & 1 \\ 27 & -3 & 0 & -4 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr}
14 & 0 & -3 & 2 \\
16 & -1 & 2 & -1 \\
28 & 4 & 0 & 5 \\
27 & -3 & 0 & -4
\end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 1 & 7 & 0 & 9 \\ 27 & -3 & 0 & -4 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{matrix} 14 & 0 & -3 & 2 \\ 16 & -1 & 2 & -1 \\ 1 & 7 & 0 & 9 \\ 11 & -2 & -2 & -3 \end{matrix}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 14 & 0 & -3 & 2 \\ 2 & -1 & 5 & -3 \\ 1 & 7 & 0 & 9 \\ 11 & -2 & -2 & -3 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ 1 & 7 & 0 & 9 \\ 11 & -2 & -2 & -3 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{matrix} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ 1 & 7 & 0 & 9 \\ 9 & -1 & -7 & 0 \end{matrix}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ -2 & 5 & 1 & 4 \\ 9 & -1 & -7 & 0 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ -2 & 5 & 1 & 4 \\ 6 & -3 & -6 & -5 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{matrix} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ -2 & 5 & 1 & 4 \\ 4 & 2 & -5 & -1 \end{matrix}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ -5 & 3 & 2 & -1 \\ 4 & 2 & -5 & -1 \end{array}$$

Doesn't look so hard . . .

Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ -5 & 3 & 2 & -1 \\ -1 & 5 & -3 & -2 \end{array}$$

Doesn't look so hard . . .

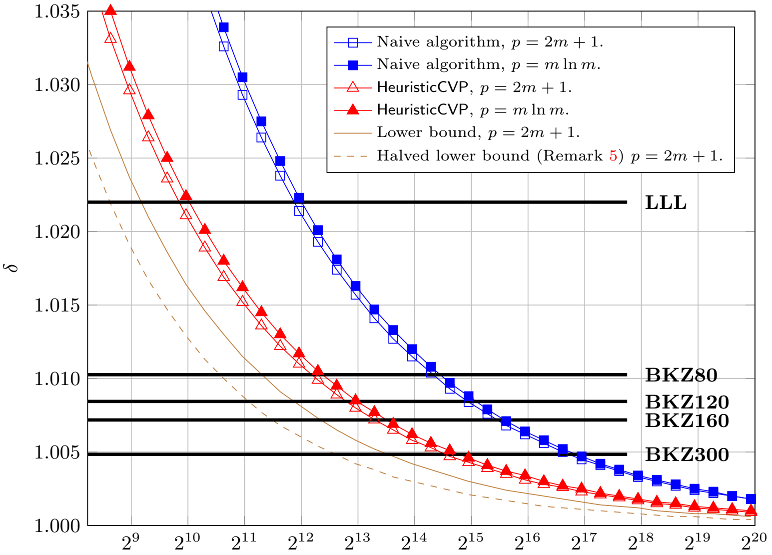
Naive lattice-basis reduction: Reduce largest row by subtracting closest multiple of another row.

$$\begin{array}{rrrr} 3 & 2 & -1 & 5 \\ 2 & -1 & 5 & -3 \\ -5 & 3 & 2 & -1 \\ -1 & 5 & -3 & -2 \end{array}$$

But this doesn't reach "short" when n is large.

[This difficulty is only for number theory, not geometry. Analogous short-vector problem for sublattice of $\mathbf{F}_7[y]^n$: naive algorithm gives shortest basis in poly time.]

Big picture: screenshot from 2019 DPW



How well the algorithms do

Given nonzero ideal $I \subseteq R = \mathbf{Z}[x]/(x^n + 1)$, algorithm finds nonzero $g = g_0 + \cdots + g_{n-1}x^{n-1} \in I$ with $(g_0^2 + \cdots + g_{n-1}^2)^{1/2} = \eta \cdot (\#(R/I))^{1/n}$.

Algorithms using only additive structure of I:

- LLL (fast): $\eta^{1/n} \approx 1.022$.
- BKZ-80 (not hard): $\eta^{1/n} \approx 1.010$.
- BKZ-160 (public attack): $\eta^{1/n} \approx 1.007$.
- BKZ-300 (large-scale attack): $\eta^{1/n} \approx 1.005$.

Algorithms also using multiplicative structure of R: blue/red curves; $\eta \in 2^{n^{1/2+o(1)}}$ but worse η than LLL below "rank 1000". Thin curves: "lower bound".

Major research directions

Many papers analyzing+optimizing BKZ-β: e.g.,

- Last century: $\exp(\Theta(\beta \log \beta))$ ops.
- 2001: $\exp((0.415\ldots + o(1))\beta)$ ops.
- 2015: $\exp((0.292\ldots + o(1))\beta)$ ops.
- 2015: $\exp((0.265\ldots + o(1))\beta)$ quantum ops.
- 2021: $\exp((0.257\ldots + o(1))\beta)$ quantum ops.
- Many more speedups hidden inside the $o(1)$.

This talk focuses on multiplicative attacks:

- Part 2 of talk: How multiplicative attacks work.
- Part 3 of talk: Better multiplicative attacks.

Part 2
How multiplicative attacks work

Infinite places of $K = \mathbf{Q}[x]/(x^n + 1)$

Define $\zeta_m = \exp(2\pi i/m) \in \mathbf{C}$ for nonzero $m \in \mathbf{Z}$.

For any $c \in 1 + 2\mathbf{Z}$ have $(\zeta_{2n}^c)^n + 1 = 0$ so there is a unique ring morphism $\iota_c : K \to \mathbf{C}$ taking x to ζ_{2n}^c.

All $x^n + 1$ roots in $\mathbf{C}$: $\zeta_{2n}^1, \ldots, \zeta_{2n}^{n-1}, \zeta_{2n}^{-(n-1)}, \ldots, \zeta_{2n}^{-1}$.
All $\iota : K \to \mathbf{C}$: $\iota_1, \ldots, \iota_{n-1}, \iota_{-(n-1)}, \ldots, \iota_{-1}$.

Define $|g|_c = |\iota_c(g)|^2 = \iota_c(g)\iota_{-c}(g)$.
The maps $g \mapsto |g|_c$ are the **infinite places** of K.
All places: $g \mapsto |g|_1, g \mapsto |g|_3, \ldots, g \mapsto |g|_{n-1}$.
Same as: $g \mapsto |g|_{-1}, g \mapsto |g|_{-3}, \ldots, g \mapsto |g|_{-n-1}$.

$$\sum_{c \in \{1,3,\ldots,n-1\}} |g_0 + \cdots + g_{n-1}x^{n-1}|_c = \frac{n}{2}(g_0^2 + \cdots + g_{n-1}^2).$$

Finite places of $K = \mathbf{Q}[x]/(x^n + 1)$

Nonzero ideals of R factor into prime ideals.

For each nonzero prime ideal P of R, define $|g|_P = \#(R/P)^{-\mathrm{ord}_P g}$. "Norm of P" is $\#(R/P)$. The maps $g \mapsto |g|_P$ are the **finite places** of K.

For each prime number p: Factor $x^n + 1$ in $\mathbf{F}_p[x]$ to see the prime ideals of R containing p.

e.g. $p = 2$: Prime ideal $2R + (x + 1)R = (x + 1)R$.

e.g. "unramified degree-1 primes": $p \in 1 + 2n\mathbf{Z} \Rightarrow$ exactly n nth roots $r_1, \ldots, r_n$ of -1 in $\mathbf{F}_p$. $x^n + 1 = (x - r_1)(x - r_2) \ldots (x - r_n)$ in $\mathbf{F}_p[x]$. Prime ideals $pR + (x - r_1)R, \ldots, pR + (x - r_n)R$.

Example: $n = 4$; $R = \mathbf{Z}[x]/(x^4 + 1)$

$$g = g_0 + g_1 x + g_2 x^2 + g_3 x^3, \qquad \zeta_8 = \exp(2\pi i/8):$$

$$\iota_{-1}(g) = g_0 + g_1\zeta_8^{-1} + g_2\zeta_8^{-2} + g_3\zeta_8^{-3};$$

$$\iota_1(g) = g_0 + g_1\zeta_8 + g_2\zeta_8^2 + g_3\zeta_8^3; \qquad |g|_1 = |\iota_1(g)|^2.$$

$$\iota_{-3}(g) = g_0 + g_1\zeta_8^{-3} + g_2\zeta_8^{-6} + g_3\zeta_8^{-9};$$

$$\iota_3(g) = g_0 + g_1\zeta_8^3 + g_2\zeta_8^6 + g_3\zeta_8^9; \qquad |g|_3 = |\iota_3(g)|^2.$$

$P_{17,2} = 17R + (x-2)R$: $\quad |g|_{17,2} = 17^{-\operatorname{ord}_{P_{17,2}} g}$.
$P_{17,8} = 17R + (x-8)R$: $\quad |g|_{17,8} = 17^{-\operatorname{ord}_{P_{17,8}} g}$.
$P_{17,-8} = 17R + (x+8)R$: $\quad |g|_{17,-8} = 17^{-\operatorname{ord}_{P_{17,-8}} g}$.
$P_{17,-2} = 17R + (x+2)R$: $\quad |g|_{17,-2} = 17^{-\operatorname{ord}_{P_{17,-2}} g}$.
$P_{41,3} = 41R + (x-3)R$: $\quad |g|_{41,3} = 41^{-\operatorname{ord}_{P_{41,3}} g}$.
etc.

S-units of $K = \mathbf{Q}[x]/(x^n + 1)$

Assume $\infty \subseteq S \subseteq \{\text{places of } K\}$.
Useful special case: S has all primes $\le$something.
[Warning: Often people rename $S - \infty$ as S.]

$g \in K^*$ is an ***S*-unit**
$\Leftrightarrow gR = \prod_{P \in S} P^{e_P}$ for some e_P
$\Leftrightarrow |g|_\rho = 1$ for all $\rho \in \{\text{places of } K\} - S$
$\Leftrightarrow$ the vector $\rho \mapsto \log |g|_\rho$ is 0 outside S.

***S*-unit lattice:** set of such vectors $\rho \mapsto \log |g|_\rho$.

e.g. Temporarily allowing $n = 1$, $K = \mathbf{Q}$:
$\{\{\infty, 2, 3\}\text{-units in } \mathbf{Q}\} = \pm 2^{\mathbf{Z}} 3^{\mathbf{Z}}$. (“3-smooth”.)
Lattice: $(\log 2, -\log 2, 0)\mathbf{Z} + (\log 3, 0, -\log 3)\mathbf{Z}$.

S-unit attacks

0. Choose a finite set S of places.
1. Input a nonzero ideal I of R.
2. Find an S-generator of I:
   some g with $gR = I \prod_{P \in S} P^{e_P}$.
   This has a poly-time quantum algorithm,
   and surprisingly fast non-quantum algorithms.
3. Find an S-unit u "close to g/I".
   This is an S-unit-lattice close-vector problem.
4. Output g/u.

Critical for Step 3 speed: constructing short vectors in the S-unit lattice. We'll see several constructions!

Special case: unit attacks

0. Define $S = \infty$.
   $\{\infty\text{-units of } K\} = \{\text{units of } R\} = R^*$.
1. Input a nonzero ideal I of R.
2. Find a generator of I: some g with $gR = I$.
3. Find a unit u "close to g".
4. Output g/u.

Questions coming up later in this talk:

- How small is g/u compared to I?
- What happens if I isn't principal?
- Is this special case as good as the general case?

“Cyclotomic units” in $R = \mathbf{Z}[x]/(x^n + 1)$

$\pm 1, \pm x, \pm x^2, \ldots, \pm x^{n-1} = \mp 1/x$ are units.

$(1 - x^3)/(1 - x) = 1 + x + x^2 \in R$. Unit since
$(1 - x)/(1 - x^3) = (1 - x^{2n^2+1})/(1 - x^3) \in R$.

For $c \in 1 + 2\mathbf{Z}$: R has automorphism $\sigma_c : x \mapsto x^c$.
$\sigma_c(1 + x + x^2) = 1 + x^c + x^{2c}$ is a unit.
Useful to symmetrize: define $u_c = 1 + x^c + x^{-c}$.

$x^{\mathbf{Z}} \prod_c u_c^{\mathbf{Z}}$ has finite index in R^*. Index is called h^+.
Assume $h^+ = 1$. Proven, assuming GRH, for
$n \in \{2, 4, 8, \ldots, 256\}$; heuristics say always true.

[Note to number theorists: This talk is only for powers of 2.]

Unit lattice for $n = 8$

$|u_1|_1 = |1 + \zeta_{16} + \zeta_{16}^{-1}|^2 \approx \exp 2.093.$
$|u_1|_3 = |1 + \zeta_{16}^{3} + \zeta_{16}^{-3}|^2 \approx \exp 1.137.$
$|u_1|_5 = |1 + \zeta_{16}^{5} + \zeta_{16}^{-5}|^2 \approx \exp -2.899.$
$|u_1|_7 = |1 + \zeta_{16}^{7} + \zeta_{16}^{-7}|^2 \approx \exp -0.330.$

Define $\operatorname{Log}_\infty f = (\log|f|_1, \log|f|_3, \log|f|_5, \log|f|_7)$.
$\operatorname{Log}_\infty u_1 \approx (2.093, 1.137, -2.899, -0.330)$.
$\operatorname{Log}_\infty u_3 \approx (1.137, -0.330, 2.093, -2.899)$.
$\operatorname{Log}_\infty u_5 \approx (-2.899, 2.093, -0.330, 1.137)$.
$\operatorname{Log}_\infty u_7 \approx (-0.330, -2.899, 1.137, 2.093)$.

$\operatorname{Log}_\infty R^*$ is lattice of dim $n/2 - 1 = 3$ in hyperplane $\{(\ell_1, \ell_3, \ell_5, \ell_7) \in \mathbf{R}^4 : \ell_1 + \ell_3 + \ell_5 + \ell_7 = 0\}$.
Short lattice basis: $\operatorname{Log}_\infty u_1$, $\operatorname{Log}_\infty u_3$, $\operatorname{Log}_\infty u_5$.

Reducing mod units

Start with $g = g_0 + g_1x + \cdots + g_{n-1}x^{n-1}$.
Compute $\mathrm{Log}_\infty g = (\log|g|_1, \log|g|_3, \ldots, \log|g|_{n-1})$.

Try to reduce $\mathrm{Log}_\infty g$ modulo unit lattice:
adjust $\mathrm{Log}_\infty g$ by subtracting closest vector from some precomputed combinations of basis vectors;
repeat several times; keep smallest $g_0^2 + \cdots + g_{n-1}^2$.

Replacing g with gu replaces $|g|_c$ with $|g|_c|u|_c$.
Easy to track $\sum_c |g|_c = (n/2)(g_0^2 + \cdots + g_{n-1}^2)$.

Note that unit hyperplane is orthogonal to norm:
$\#(R/I) = \#(R/g) = \prod_c |g|_c = \exp \sum_c \log|g|_c$.

Experiments for small n

Geometric average of $\eta^{1/n}$ over 100000 experiments:

n	Model	Attack	Tweak	Shortest
4	1.01516	1.01518	1.01518	1.01518
8	1.01968	1.01972	1.01696	1.01696
16	1.01861	1.01860	1.01628	1.01627

"Shortest": Take I, find a shortest nonzero vector g, output $\eta = (g_0^2 + \cdots + g_{n-1}^2)^{1/2}/\#(R/I)^{1/n}$.
[*Assuming* BKZ-n software produces shortest nonzero vector.]

"Attack": Same I, find a generator, reduce mod unit lattice $\to g$, output $(g_0^2 + \cdots + g_{n-1}^2)^{1/2}/\#(R/I)^{1/n}$.

"Model": Take a hyperplane point, reduce mod unit lattice $\to \operatorname{Log}_\infty g$, output $(g_0^2 + \cdots + g_{n-1}^2)^{1/2}$.

Wasn't this attack supposed to be useless?

Geometric average of 100000 runs of model for $32, 64, 128, 256, 512, 1024$: 1.01570, 1.01332, 1.01118, 1.00950, 1.00804, (10000:) 1.00667.

Why did 2019 DPW say $>$1.022 for *n* below 1000?

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Geometric average of 100000 runs of model for $32, 64, 128, 256, 512, 1024$: 1.01570, 1.01332, 1.01118, 1.00950, 1.00804, (10000:) 1.00667.

Why did 2019 DPW say $>$1.022 for n below 1000?

Aha: 2019 DPW applies unit attack to principal IJ.

Multiplying J into I
$\Rightarrow$ multiplying $\#(R/J)$ into $\#(R/I)$
$\Rightarrow$ multiplying $\#(R/J)^{1/n}$ into $\#(R/I)^{1/n}$
$\Rightarrow$ expanding η by $\#(R/J)^{1/n}$
$\Rightarrow$ expanding $\eta^{1/n}$ by $\#(R/J)^{1/n^2}$.

Finding a close principal multiple IJ

Prime $p \in 1 + 2n\mathbf{Z}$ is contained in n prime ideals P_c.
"Augmented Stickelberger": known rank-n lattice $\Lambda \subseteq \mathbf{Z}^n$ with $e \in \Lambda \Rightarrow \prod_c P_c^{e_c}$ principal; e.g., $P_c P_{-c}$.

Poly-time quantum algorithm + minor assumption $\Rightarrow$ some vector v such that $I \prod_c P_c^{v_c}$ is principal.

Search some $e \in \Lambda$, trying to minimize $\sum_c |v_c - e_c|$.
Use principal $P_c P_{-c}$ to force $e_c \le v_c$.
Define $J = \prod_c P_c^{v_c - e_c}$. Then IJ is principal.
Replace I with IJ, and apply unit attack.

Contribution to $\eta^{1/n}$: $\#(R/J)^{1/n^2} = (p^{1/n^2})^{\sum_c |v_c - e_c|}$.

Constructing the 2019 DPW graph

Reverse-engineered procedure to build the graph:

- Experiments for $\sum_c |v_c - e_c|$ (for red curve; blue: limit search; thin: "lower bound").
- Experiments for reducing mod unit lattice.
- Insert $n^{1/2}$ factor because of notation choices.
- Combine appropriately to obtain $n^{1/2}\eta$.
- Multiply by $n^{-1/2}$ to obtain η. Graph $\eta^{1/n}$.

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- Typo: Omit the "$-$" in the previous line.

Big impact of typo: e.g., $n^{1/n} \approx 1.012$ for $n = 512$.
Attack is much more effective than graph shows.

Part 3
Better multiplicative attacks

Prime factors of some random integers

$2 \cdot 3 \cdot 59 \cdot 73 \cdot 14051 \cdot 57977 \cdot 1492315939$

$136652609 \cdot 229896280545203$

$2^2 \cdot 43973 \cdot 2825227 \cdot 63219409867$

$3 \cdot 7 \cdot 13 \cdot 115076653977648103973$

$2 \cdot 5 \cdot 41 \cdot 4259 \cdot 1799112727475127 7$

$11 \cdot 17 \cdot 167407 \cdot 3365381 \cdot 298195039$

$2^3 \cdot 3^4 \cdot 29 \cdot 92401 \cdot 150959 \cdot 119850869$

$43 \cdot 730602942695300753131$

$2 \cdot 79 \cdot 379 \cdot 577 \cdot 5009 \cdot 382979 \cdot 473971$

$3 \cdot 5 \cdot 209439510239319549230 9$

$2^2 \cdot 7 \cdot 337 \cdot 332936906908625820 1$

$23 \cdot 4363 \cdot 14153 \cdot 2212016270092 1$

Traditional method to find S-units

Take random small element $u \in R$:
e.g. $u = x^{31} - x^{41} + x^{59} + x^{26} - x^{53}$.
1. Does $\#(R/u)$ factor into primes $\leq y$?
2. Is u an S-unit for $S = \infty \cup \{P : \#(R/P) \leq y\}$?

Small primes $\Rightarrow$ fast non-quantum factorization.
[Helpful speedups: $\#(R/P) \in 1 + 2n\mathbf{Z}$. Batch factorization.]

Standard heuristics $\Rightarrow$ $y^{2+o(1)}$ choices of u include $y^{1+o(1)}$ S-units, spanning all S-units, for

- appropriate $n^{1/2+o(1)}$ choice for $\log y$,
- appropriate $n^{1/2+o(1)}$ choice for $\sum_i u_i^2$.

Total time $\exp(n^{1/2+o(1)})$. [Extension NFS: $1/3 + o(1)$?]

Automorphisms and subrings

Apply each σ_c to quickly amplify each u found into, typically, n independent S-units.

What if u is invariant under (say) two σ_c? Great! Start with u from proper subrings. Makes $\#(R/u)$ much more likely to factor into small primes.

Examples of useful subrings of $R = \mathbf{Z}[x]/(x^n + 1)$:

- $\mathbf{Z}[x^2]/(x^n + 1) = \{u \in R : \sigma_{n+1}(u) = u\}$.
- $R^+ = \{u \in R : \sigma_{-1}(u) = u\}$.

Also use subrings to speed up $\#(R/u)$ computation for any $u \in R$: $v = u\sigma_{n+1}(u)$, $w = v\sigma_{n/2+1}(v)$, ... $n^{1+o(1)}$ times faster than “fast” resultant methods.

More cyclotomic fun: Gauss sums

For each prime number $p \in 1 + 2n\mathbf{Z}$,
and each group morphism $\chi : \mathbf{F}_p^* \to \zeta_{2n}^{\mathbf{Z}}$, define

$$\mathrm{Gauss}\Sigma_p(\chi) = \sum_{a \in \mathbf{F}_p^*} \chi(a)\zeta_p^a.$$

Exercise: $|\mathrm{Gauss}\Sigma_p(\chi)|^2 = p$ if $\chi \neq 1$.
So $\mathrm{Gauss}\Sigma_p(\chi)$ is an S-unit for $S = \infty \cup p$.

e.g. $n = 16$, $\zeta_{2n} = \zeta_{32}$, $p = 97 \in 1 + 2n\mathbf{Z}$:
There is a morphism $\chi : \mathbf{F}_{97}^* \to \zeta_{32}^{\mathbf{Z}}$ with $\chi(5) = \zeta_{32}$.
$\mathrm{Gauss}\Sigma_p(\chi) = \zeta_{32}^0\zeta_{97}^1 + \zeta_{32}^1\zeta_{97}^5 + \zeta_{32}^2\zeta_{97}^{25} + \cdots$.
$\mathrm{Gauss}\Sigma_p(\chi^2) = \zeta_{32}^0\zeta_{97}^1 + \zeta_{32}^2\zeta_{97}^5 + \zeta_{32}^4\zeta_{97}^{25} + \cdots$.

Many S-units for $S = \infty \cup p$

Magic fact: $\text{Gauss}\Sigma_p(\chi)^3/\text{Gauss}\Sigma_p(\chi^3) \in \mathbf{Z}[\zeta_{2n}]$.

Pull back via ι_1 to an element of $R = \mathbf{Z}[x]/(x^n + 1)$.

Factor element into prime ideals for, e.g., $n = 16$: $P_{11}P_{13}P_{15}P_{-15}P_{-13}P_{-11}P_{-9}^2P_{-7}^2P_{-5}^2P_{-3}^2P_{-1}^2$ where $P_{\pm 1}, P_{\pm 3}, \ldots, P_{\pm 15}$ are the prime ideals containing p.

Similarly $\text{Gauss}\Sigma_p(\chi)^5/\text{Gauss}\Sigma_p(\chi^5)$ etc. $\Rightarrow$ More principal products of powers of $P_{\pm 1}, P_{\pm 3}, \ldots, P_{\pm 15}$.

Λ is generated by exponent vectors for (1) these S-units and (2) $P_c P_{-c}$ (principal since $h^+ = 1$).

[Note to number theorists: labeling here is $P_c = \sigma_c^{-1}(P_1)$.]

Explaining the magic: Jacobi sums

Define $\text{Jacobi}\Sigma_p(\chi_1, \chi_2) = \sum_{a \in \mathbf{F}_p^* - \{1\}} \chi_1(a)\chi_2(1-a)$.

Exercise: If $\chi_1\chi_2 \neq 1$ then $\text{Jacobi}\Sigma_p(\chi_1, \chi_2) = \text{Gauss}\Sigma_p(\chi_1)\,\text{Gauss}\Sigma_p(\chi_2)/\text{Gauss}\Sigma_p(\chi_1\chi_2)$.

So $|\text{Jacobi}\Sigma_p(\chi_1, \chi_2)|^2 = p$ if $1 \notin \{\chi_1, \chi_2, \chi_1\chi_2\}$.

e.g. $n = 16$, $\zeta_{2n} = \zeta_{32}$, $p = 97$, $\chi(5) = \zeta_{32}$:
$\text{Jacobi}\Sigma_p(\chi, \chi) = \zeta_{32}^{1+20} + \zeta_{32}^{2+28} + \zeta_{32}^{3+66} + \cdots$,
$\text{Jacobi}\Sigma_p(\chi^2, \chi) = \zeta_{32}^{2+20} + \zeta_{32}^{4+28} + \zeta_{32}^{6+66} + \cdots$
since $1 - 5^1 = 5^{20}$, $1 - 5^2 = 5^{28}$, etc. in $\mathbf{F}_{97}$.

Λ', improving Λ by a factor 2

$\mathrm{Jacobi}\Sigma_p(\chi^i, \chi)$ for $i = 1$, $i = 2$, etc.:
$\mathrm{Gauss}\Sigma_p(\chi)^2/\mathrm{Gauss}\Sigma_p(\chi^2)$,
$\mathrm{Gauss}\Sigma_p(\chi^2)\mathrm{Gauss}\Sigma_p(\chi)/\mathrm{Gauss}\Sigma_p(\chi^3)$,
$\mathrm{Gauss}\Sigma_p(\chi^3)\mathrm{Gauss}\Sigma_p(\chi)/\mathrm{Gauss}\Sigma_p(\chi^4)$,
$\mathrm{Gauss}\Sigma_p(\chi^4)\mathrm{Gauss}\Sigma_p(\chi)/\mathrm{Gauss}\Sigma_p(\chi^5)$, etc.

Multiply:
$\mathrm{Gauss}\Sigma_p(\chi)^2/\mathrm{Gauss}\Sigma_p(\chi^2)$ (wasn't used in Λ),
$\mathrm{Gauss}\Sigma_p(\chi)^3/\mathrm{Gauss}\Sigma_p(\chi^3)$ (was used in Λ),
$\mathrm{Gauss}\Sigma_p(\chi)^4/\mathrm{Gauss}\Sigma_p(\chi^4)$ (wasn't used in Λ),
$\mathrm{Gauss}\Sigma_p(\chi)^5/\mathrm{Gauss}\Sigma_p(\chi^5)$ (was used in Λ), etc.

Define Λ' using *all* Jacobi sums: all base-field combinations of Gauss sums. $\#(\mathbf{Z}^n/\Lambda) = 2\#(\mathbf{Z}^n/\Lambda')$.

Λ'', improving Λ by a factor $2^{n/2}$

Fact: More products $\prod_c P_c^{e_c}$ are principal if $n \geq 4$.
Typical case: P_c generates the "class group"; then Λ' has index $2^{n/2-1}$ inside lattice of "class relations". Class group $= \{\text{ideals} \neq 0\}/\{\text{principal ideals} \neq 0\}$.

Start from all known S-units: group generated by cyclotomic units, Jacobi sums, generators of $P_c P_{-c}$. Successively extend set by adjoining square roots.

How to find square products of powers of current generators? Map the group in many ways to $\mathbf{F}_2$: use known exponents of P_c; use random quadratic characters (squareness mod random prime ideals Q). Then fast linear algebra over $\mathbf{F}_2$ finds squares.

Example: $n = 8$

Take $p = 17$, $\chi(3) = \zeta_{16}$, $u_c = 1 + x^c + x^{-c}$.
Find generator $g_7 = x^6 - x^5 + x^3 - x^2 - 1$ of $P_7 P_{-7}$.
Compute $\Sigma_i = \text{Jacobi}\Sigma_p(\chi^i, \chi)$ pulled back to R.

S-unit	ideal factorization
$\Sigma_1 = 2x^7 + 2x^6 - x^4 + 2x^2 - 2x$	$P_{-7}P_{-5}P_{-3}P_{-1}$
$\Sigma_2 = x^7 - 2x^6 - 3x^5 + x^4 - x^3 - x$	$P_7 P_{-5}P_{-3}P_{-1}$
Σ_2/Σ_1	P_7/P_{-7}
g_7	$P_7 P_{-7}$
$g_7\Sigma_2/\Sigma_1$	P_7^2
$(u_5 g_7 \Sigma_2/\Sigma_1)^{1/2} = x^7 - x^4 + x^3$	P_7

Scaling up to $n = 256$: All sqrts in 10 minutes.

End of the story for $n = 4$, $n = 8$, $n = 16$

For $n = 16$: $\#(\mathbf{Z}^{16}/\Lambda) = 256$. "Lower bound" $2 \Rightarrow$ expand $\#(R/I)^{1/n^2}$ by $p^{2/n^2} = 97^{2/n^2} \approx 1.03639$, on top of ≈ 1.01861 for unit-lattice model.

End of the story for $n = 4$, $n = 8$, $n = 16$

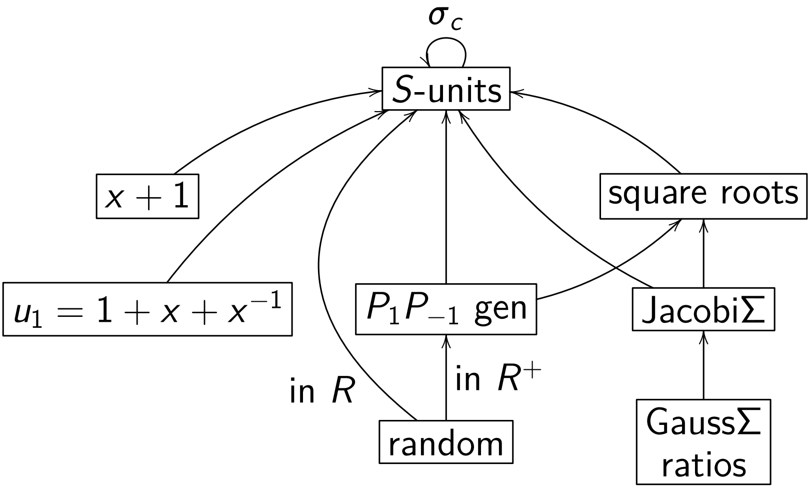
For $n = 16$: $\#(\mathbf{Z}^{16}/\Lambda) = 256$. "Lower bound" $2 \Rightarrow$ expand $\#(R/I)^{1/n^2}$ by $p^{2/n^2} = 97^{2/n^2} \approx 1.03639$, on top of ≈ 1.01861 for unit-lattice model.

Instead construct more S-units: $\#(\mathbf{Z}^{16}/\Lambda'') = 1$. The input ideal was principal in the first place! Find generator of I. Reduce mod units.

"Tweak": Multiply by $x + 1$, reduce, repeat for $I, (x+1)I, (x+1)^2 I, (x+1)^3 I, (x+1)^4 I, \ldots$. Often $(x+1)^e g$ is closer to unit lattice than g. Take smallest generator found across all $(x+1)^e I$. When to stop? Compare $2^e \#(R/I)$ to best g.

[Faster: reduce in log space mod units and $x + 1$.]

Recap: Constructing small S-units



Impact for larger values of n

For $n = 32$: $\#(\mathbf{Z}^{32}/\Lambda) = 1114112$.
"Lower bound" 5 $\Rightarrow$ expand by ≈ 1.02603,
on top of ≈ 1.01570 for unit-lattice model.

Impact for larger values of n

For $n = 32$: $\#(\mathbf{Z}^{32}/\Lambda) = 1114112$.
"Lower bound" 5 $\Rightarrow$ expand by ≈ 1.02603,
on top of ≈ 1.01570 for unit-lattice model.

Instead construct more S-units: $\#(\mathbf{Z}^{32}/\Lambda'') = 17$.
"Class number" $= \#(\text{class group}) = 17$.

Chance $1/17$: I principal. Expansion factor 1.

Chance $16/17$: I non-principal. IP principal
for some prime ideal P with $\#(R/P) = 193$.
Expansion factor $193^{1/n^2} \approx 1.00515$.

[Note to number theorists: upcoming labels use $P_{p,c} = \sigma_c(P_{p,1})$,
with $P_{p,1} = pR + (x + a)R$ for smallest a in $\{0, 1, \ldots, p-1\}$.]

Broader $n = 32$ search example, part 1

32 prime ideals $P_{193,c}$ have $\#(R/P_{193,c}) = 193$.
32 prime ideals $P_{257,c}$ have $\#(R/P_{257,c}) = 257$.
32 prime ideals $P_{449,c}$ have $\#(R/P_{449,c}) = 449$.
Note $449^{1/n^2} \approx 1.00598$ vs. $193^{1/n^2} \approx 1.00515$.

Precompute S-units, including generators $\gamma_{193}, \gamma_{257}, \gamma_{449}, \gamma_{577}, \gamma_{641}, \gamma_{769}, \ldots$ of $P_{193,31}P_{193,1}^{-1}$, $P_{257,-19}P_{193,1}^{-1}$, $P_{449,-19}P_{193,1}^{-1}$, $P_{577,15}P_{193,1}^{-1}$, $P_{641,19}P_{193,1}^{-1}$, $P_{769,5}P_{193,1}^{-1}$, $\ldots$.

Broader $n = 32$ search example, part 1

32 prime ideals $P_{193,c}$ have $\#(R/P_{193,c}) = 193$.
32 prime ideals $P_{257,c}$ have $\#(R/P_{257,c}) = 257$.
32 prime ideals $P_{449,c}$ have $\#(R/P_{449,c}) = 449$.
Note $449^{1/n^2} \approx 1.00598$ vs. $193^{1/n^2} \approx 1.00515$.

Precompute S-units, including generators $\gamma_{193}, \gamma_{257}, \gamma_{449}, \gamma_{577}, \gamma_{641}, \gamma_{769}, \ldots$ of $P_{193,31}P_{193,1}^{-1}$, $P_{257,-19}P_{193,1}^{-1}$, $P_{449,-19}P_{193,1}^{-1}$, $P_{577,15}P_{193,1}^{-1}$, $P_{641,19}P_{193,1}^{-1}$, $P_{769,5}P_{193,1}^{-1}$, $\ldots$.

Random example of a target: $I = 3141592653589793238462643383280129R + (x + 13443234652173688219737012017423)R$.
Initial S-generator computation: $gR = IP_{193,13}$.

Broader $n = 32$ search example, part 2

Multiply by precomputed S-units for more S-gens of I. (Don't repeat the quantum computations!)

$gR = IP_{193,13}$.	Attack: 1.02549; tweak: 1.01901.
$g\sigma_{13}(\gamma_{193})R = IP_{193,19}$.	1.01709; 1.01709.
$g\sigma_{13}(\gamma_{257})R = IP_{257,9}$.	1.02179; 1.02103.
$g\sigma_{13}(\gamma_{193})\sigma_{19}(\gamma_{257})R = IP_{257,23}$.	1.02517; 1.01588.
$g\sigma_{13}(\gamma_{449})R = IP_{449,9}$.	1.02100; 1.02100.
$g\sigma_{13}(\gamma_{193})\sigma_{19}(\gamma_{449})R = IP_{449,23}$.	1.02584; 1.01830.
$g\sigma_{13}(\gamma_{577})R = IP_{577,3}$.	1.02634; 1.02456.
$g\sigma_{13}(\gamma_{193})\sigma_{19}(\gamma_{577})R = IP_{577,29}$.	1.02682; 1.02224.
$g\sigma_{13}(\gamma_{641})R = IP_{641,-9}$.	1.01810; 1.01810.
$g\sigma_{13}(\gamma_{193})\sigma_{19}(\gamma_{641})R = IP_{641,-23}$.	1.00990; 1.00990.

End of the story for $n = 32$

Geometric average of $\eta^{1/n}$ over 10000 experiments:

n	Attack10	Attack12	Attack14	Shortest
32	1.01660	1.01622	1.01599	1.01576

"Attack10": Tweaked unit attack starting from 12 gens of ideals $IP_{p,c}$ with $p < 2^{10}$.

"Attack12": Tweaked unit attack starting from same I pool, 32 gens of ideals $IP_{p,c}$ with $p < 2^{12}$.

"Attack14": Tweaked unit attack starting from same I pool, 124 gens of ideals $IP_{p,c}$ with $p < 2^{14}$.

(If I is principal, take gen of I. Could also try IJ.)

Generalizing to any n

Find S-unit lattice: generators of $\prod_{P \in S} P^{e_P}$.
Typically see small $P_{\ell,1} \in S$ generating class group;
for each $Q \in S$, find generator of some $Q \prod_c P_{\ell,c}^{e_c}$.

Find S-generator of I: $gR = I \prod_{P \in S} P^{v_P}$.
No more quantum steps required after this.

Try $J = R$, $J = Q$, $J = QQ'$, etc. For each J,
immediately see generator of some $IJ \prod_c P_{\ell,c}^{e_c}$.
Fast reduction mod $\Lambda'' \Rightarrow$ gen of small multiple of I.
(For $n = 32$, jumped to J with IJ principal.)
Fast reduction mod unit lattice and $x + 1 \Rightarrow$ short.

Much shorter vectors than pure unit attack.

Using more primes for $n = 64$

$\#(\mathbf{Z}^{64}/\Lambda'') = 17 \cdot 21121 = 359057$.
Again precompute S-units.

Given I, compute S-generator: $gR = I \prod_c P_{257,c}^{v_c}$.
Basic attack: Reduce exponent vector mod Λ'',
finding generator of small $I \prod_c P_{257,c}^{v_c - e_c}$.

"Small": 1000 experiments in $\sum_c |v_c - e_c|$ model $\Rightarrow$
25.2% 5, 64.8% 4, 9.6% 3, 0.3% 2, 0.1% 1.

$257^{4/n^2} \approx 1.00543$; $257^{1/n^2} \approx 1.00136$.

Further options: $I \prod_c P_{641,c}^{v_c}$. Many more options:
$IP_{641,b} \prod_c P_{257,c}^{v_c}$; $IP_{769,a} P_{641,b} \prod_c P_{257,c}^{v_c}$; etc.
Paying 2 primes gains many tries at closeness.

A meet-in-the-middle search for $n = 64$

Efficiently index each ideal class by $e \in \mathbf{Z}/359057$:
I has class $e \Leftrightarrow IP_{257,1}^{-e}$ principal.
σ_{-1}, σ_3 act as mults by $-1, 29301$ on $\mathbf{Z}/359057$.
Precompute classes of $P_{257,1}, P_{641,1}, P_{769,1}, P_{1153,1}$
(via small S-units): 1, 25489, 99282, 201437.

Start with S-generator of $I \Rightarrow$ class of I.
Tabulate 64^2 classes of $IP_{1153,a}P_{769,b}$.
Tabulate 64^2 classes of $P_{641,c}^{-1}P_{257,d}^{-1}$.
Rough estimate: $64^4/359057 \approx 47$ collisions.

Collision $\Rightarrow IP_{1153,a}P_{769,b}P_{641,c}P_{257,d}$ principal.
Reconstruct $IP_{1153,a}P_{769,b}P_{641,c}P_{257,d}$ generator.
Reduce each generator mod units, and apply tweak.

A numerical example for $n = 64$

Took ideal $I \subset R$ containing the random prime 314159265358979323846264338327950288419716939937510593.

Examples of short $g \in I$ found by meet-in-the-middle search of principal IJ_1J_2 with odd $\#(R/J_j) < 2^{22}$:

Ideal generated by g	$\eta^{1/n}$
$(1+x)^8 IP_{641,\ldots}P_{769,\ldots}P_{78977,\ldots}$	1.01399
$(1+x)^5 IP_{398977,\ldots}$	1.01389
$IP_{641,\ldots}P_{1340033,\ldots}$	1.01385
$(1+x)^4 IP_{257,\ldots}P_{1153,\ldots}P_{11777,\ldots}P_{39041,\ldots}$	1.01350
$(1+x)^3 IP_{35969,\ldots}P_{2350081,\ldots}$	1.01288

For comparison, shortest nonzero vector in I:

$(1+x)IP_{6525293171851009,\ldots}$ 1.01243

Conjectured scalability: $\exp(n^{1/2+o(1)})$

Simple algorithm variant, skipping many speedups:

Take traditional $\log y \in n^{1/2+o(1)}$.
Take $S = \infty \cup \{P : \#(R/P) \le y\}$.
Precompute $\left\{S\text{-unit } u \in R\text{: } \sum_i u_i^2 \le n^{1/2+o(1)}\right\}$.

Compute S-generator g of I.
Replace g with gu/v having log vector closest to I;
repeat until stable $\Rightarrow$ small S-generator of I.
Multiply by $P_c P_{-c}$ gens $\Rightarrow$ short element of I.
Repeat $y^{O(1)}$ times, avoiding cycles; take shortest.

Heuristics $\Rightarrow$ $\eta \le n^{1/2+o(1)}$, time $\exp(n^{1/2+o(1)})$.
"Vector within ϵ of shortest in subexponential time."