Lattice-based cryptography, day 1: simplicity

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2000 Cohen cryptosystem

Public key: vector of integers
$K = (K_1, \ldots, K_N) \in \{-X, \ldots, X\}^N$.

Encryption:

1. Input message $m \in \{0, 1\}$.

2. Generate $r_1, \ldots, r_N \in \{0, 1\}$.
i.e. $r = (r_1, \ldots, r_N) \in \{0, 1\}^N$.

(Cohen says pick "half of the integers in the public key at random": I guess this means $N \in 2\mathbf{Z}$ and $\sum r_i = N/2$.)

3. Compute and send ciphertext
$C = (-1)^m (r_1 K_1 + \cdots + r_N K_N)$.

How can receiver decrypt?

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Key generation:

Generate $s \in \{1, \ldots, Y\}$;

$$u_1, \ldots, u_N \in \left\{0, \ldots, \left\lfloor \frac{s-1}{2N} \right\rfloor \right\};$$

$K_i \in (u_i + s\mathbf{Z}) \cap \{-X, \ldots, X\}$.

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Decryption:

$m = 0$ if $C \bmod s \le (s-1)/2$;

otherwise $m = 1$.

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otherwise $m = 1$.

Why this works:

$K_i \bmod s = u_i \leq (s-1)/2N$ so

$r_1 K_1 + \cdots + r_N K_N \bmod s \leq \frac{s-1}{2}$.

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$$r_1 K_1 + \cdots + r_N K_N \bmod s \le \frac{s-1}{2}.$$

(Be careful! What if all $r_i = 0$?)

Let's try this on the computer.

Debian: `apt install sagemath`

Fedora: `dnf install sagemath`

Source: `www.sagemath.org`

Web (use `print(X)` to see `X`):

`sagecell.sagemath.org`

Sage is Python 3

+ many math libraries

+ a few syntax differences:

```
sage: 10^6 # power, not xor
1000000
sage: factor(314159265358979323)
317213509 * 990371647
sage:
```

For integers `C`, `s` with `s` > 0,
Sage's "`C%s`" always produces
outputs between 0 and `s` $- 1$.

Matches standard math definition:
$C \bmod s = C - \lfloor C/s \rfloor s$.

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$\texttt{C} < 0$ produces $\texttt{C\%s} < 0$
in lower-level languages, so nonzero output leaks input sign.

Warning: For polynomials `C`,
Sage can make the same mistake.

sage:

```
sage: N=10
sage:
```

```
sage: N=10
sage: X=2^50
sage:
```

```
sage: N=10
sage: X=2^50
sage: Y=2^20
sage:
```

```
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sage: X=2^50
sage: Y=2^20
sage: Y
1048576
sage:
```

```
sage: N=10
sage: X=2^50
sage: Y=2^20
sage: Y
1048576
sage: s=randrange(1,Y+1)
sage:
```

```
sage: N=10
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sage: Y
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sage: s
359512
sage:
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sage: X=2^50
sage: Y=2^20
sage: Y
1048576
sage: s=randrange(1,Y+1)
sage: s
359512
sage: u=[randrange(
....:        (s-1)//(2*N)+1)
....:     for i in range(N)]
sage:
```

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sage: X=2^50
sage: Y=2^20
sage: Y
1048576
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sage: s
359512
sage: u=[randrange(
....:       (s-1)//(2*N)+1)
....:     for i in range(N)]
sage: u
[14485, 7039, 6945, 15890,
 10493, 17333, 1397, 8656,
 8213, 6370]
```

```
sage: K=[ui+s*randrange(
....:         ceil(-(X+ui)/s),
....:         floor((X-ui)/s)+1)
....:       for ui in u]
sage:
```

```
sage: K=[ui+s*randrange(
....:          ceil(-(X+ui)/s),
....:          floor((X-ui)/s)+1)
....:      for ui in u]
sage: K
[870056918917829,
 822006576592695,
 -294765544345815,
 -669275100080982,
 528958455221029,
 426006001074157,
 -641940176080531,
 501543495923784,
 -583064075392587,
 46109390243834]
```

```
sage: [Ki%s for Ki in K]
[14485, 7039, 6945, 15890,
 10493, 17333, 1397, 8656,
 8213, 6370]
sage: u
[14485, 7039, 6945, 15890,
 10493, 17333, 1397, 8656,
 8213, 6370]
sage:
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sage: u
[14485, 7039, 6945, 15890,
 10493, 17333, 1397, 8656,
 8213, 6370]
sage: sum(K)%s
96821
sage: sum(u)
96821
sage:
```

```
sage: [Ki%s for Ki in K]
[14485, 7039, 6945, 15890,
 10493, 17333, 1397, 8656,
 8213, 6370]
sage: u
[14485, 7039, 6945, 15890,
 10493, 17333, 1397, 8656,
 8213, 6370]
sage: sum(K)%s
96821
sage: sum(u)
96821
sage: s//2
179756
sage:
```

```
sage: m=randrange(2)
sage:
```

```
sage: m=randrange(2)
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....:     for i in range(N)]
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```

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....:    for i in range(N)]
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sage: C
-202215856043576
sage:
```

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-202215856043576
sage: C%s
47024
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....:   for i in range(N))
sage: C
-202215856043576
sage: C%s
47024
sage: m
0
sage:
```

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sage: C=(-1)^m*sum(r[i]*K[i]
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sage: C
-202215856043576
sage: C%s
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sage: m
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Some problems with cryptosystem

1. Functionality problem:
System can't encrypt messages
that have more than 1 bit.

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System can't encrypt messages that have more than 1 bit.

2. Security problem:
We want cryptosystems to resist "chosen-ciphertext attacks" where attacker can see decryptions of other ciphertexts.

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1. Functionality problem:
System can't encrypt messages that have more than 1 bit.

2. Security problem:
We want cryptosystems to resist "chosen-ciphertext attacks" where attacker can see decryptions of other ciphertexts.

Chosen-ciphertext attack against this system:
Decrypt $-C$. Flip result.

(Works whenever $C \neq 0$.)

2000 Cohen: cryptosystem fixing both of these problems.

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B-bit input message $m = (m_1, \ldots, m_B) \in \{0, 1\}^B$.
For each $i \in \{1, \ldots, B\}$:
Generate $r_{i,1}, \ldots, r_{i,N} \in \{0, 1\}$.

Ciphertext C:
$(-1)^{m_1}(r_{1,1}K_1 + \cdots + r_{1,N}K_N)$,
$\ldots$,
$(-1)^{m_B}(r_{B,1}K_1 + \cdots + r_{B,N}K_N)$.

2. Derandomize encryption, and reencrypt during decryption.

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Derandomization: Generate r as cryptographic hash $H(m)$, using standard hash function H. (Watch out: Is m guessable?)

Decryption with reencryption:
1. Input C'. (Maybe $C' \neq C$.)
2. Decrypt to obtain m'.
3. Recompute $r' = H(m')$.
4. Recompute C'' from m', r'.
5. Abort if $C'' \neq C'$.

Subset-sum attacks

Attacker searches all possibilities for $(r_1, \ldots, r_N)$, checks $r_1 K_1 + \cdots + r_N K_N$ against $\pm C_1$.

This takes 2^N easy operations: e.g. 1024 operations for $N = 10$.

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"This finds only one bit m_1."

— This is a problem in some applications. Should design encryption to leak *no* information.

— Also, can easily modify attack to find all bits of message.

Modified attack:

For each $(r_1, \ldots, r_N)$, look up $r_1 K_1 + \cdots + r_N K_N$ in hash table containing $\pm C_1, \pm C_2, \ldots, \pm C_B$.

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For each $(r_1, \ldots, r_N)$, look up $r_1K_1 + \cdots + r_NK_N$ in hash table containing $\pm C_1, \pm C_2, \ldots, \pm C_B$.

Multi-target attack:
Apply this not just to B bits in one message, but all bits in all messages sent to this key.

Finding all bits in all messages: total 2^N operations.

Finding 1% of all bits in all messages, huge information leak: total $0.01 \cdot 2^N$ operations.

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— Standard subset-sum attacks take only $2^{N/2}$ operations to find $(r_1, \ldots, r_N) \in \{0, 1\}^N$ with $r_1 K_1 + \cdots + r_N K_N = C$.

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— Standard subset-sum attacks take only $2^{N/2}$ operations to find $(r_1, \ldots, r_N) \in \{0,1\}^N$ with $r_1 K_1 + \cdots + r_N K_N = C$.

Make hash table containing $C - r_{N/2+1} K_{N/2+1} - \cdots - r_N K_N$ for all $(r_{N/2+1}, \ldots, r_N)$.

Look up $r_1 K_1 + \cdots + r_{N/2} K_{N/2}$ in hash table for each $(r_1, \ldots, r_{N/2})$.

These attacks exploit linear structure of problem to convert one target C into many targets.

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(Actually have $2B$ targets $\pm C_1, \ldots, \pm C_B$ for one message. Convert into $B^{1/2}2^{N/2}$ targets: total $B^{1/2}2^{N/2}$ operations to find all B bits. Also, maybe have more messages to attack.)

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There are even more ways to exploit the linear structure.

1981 Schroeppel–Shamir: $2^{N/2}$ operations, space $2^{N/4}$.

2010 Howgrave-Graham–Joux: claimed $2^{0.311N}$ operations. 2011 May–Meurer correction: $2^{0.337N}$.

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Quantum attacks: various papers.

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Quantum attacks: various papers.

Multi-target speedups: probably!

Variants of cryptosystem

2003 Regev: Cohen cryptosystem (without credit), but replace $(-1)^m(r_1K_1 + \cdots + r_NK_N)$ with $m(K_1/2) + r_1K_1 + \cdots + r_NK_N$.

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modify keygen to force $K_1 \in 2\mathbf{Z}$ and $(K_1 - u_1)/s \in 1 + 2\mathbf{Z}$.
Also be careful with u_i bounds.

2009 van Dijk–Gentry–Halevi–Vaikuntanathan: $K_i \in 2u_i + s\mathbf{Z}$;
$C = m + r_1K_1 + \cdots + r_NK_N$;
$m = (C \bmod s) \bmod 2$.
Be careful to take $s \in 1 + 2\mathbf{Z}$.

Homomorphic encryption

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Take two ciphertexts:
$C = m + 2\epsilon + sq$,
$C' = m' + 2\epsilon' + sq'$
with small $\epsilon, \epsilon' \in \mathbf{Z}$.

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Take two ciphertexts:
$C = m + 2\epsilon + sq$,
$C' = m' + 2\epsilon' + sq'$
with small $\epsilon, \epsilon' \in \mathbf{Z}$.

$C + C' = m + m' + 2(\epsilon + \epsilon') + s(q + q')$. This decrypts to $m + m'$ mod 2 if $\epsilon + \epsilon'$ is small.

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Take two ciphertexts:
$C = m + 2\epsilon + sq$,
$C' = m' + 2\epsilon' + sq'$
with small $\epsilon, \epsilon' \in \mathbf{Z}$.

$C + C' = m + m' + 2(\epsilon + \epsilon') + s(q + q')$. This decrypts to $m + m' \bmod 2$ if $\epsilon + \epsilon'$ is small.

$CC' = mm' + 2(\epsilon m' + \epsilon' m + 2\epsilon\epsilon') + s(\cdots)$. This decrypts to mm' if $\epsilon m' + \epsilon' m + 2\epsilon\epsilon'$ is small.

```
sage: N=10
sage:
```

```
sage: N=10
sage: E=2^10
sage:
```

```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage:
```

```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage:
```

```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*randrange(Y/4,Y/2)
sage: s
984887308997925
sage:
```

```
sage: N=10
sage: E=2^10
sage: Y=2^50
sage: X=2^80
sage: s=1+2*randrange(Y/4,Y/2)
sage: s
984887308997925
sage: u=[randrange(E)
....:      for i in range(N)]
sage: u
[247, 418, 365, 738, 123, 735,
 772, 209, 673, 47]
sage:
```

```
sage:
```

```
sage: K=[2*ui+s*randrange(
....:         ceil(-(X+2*ui)/s),
....:         floor((X-2*ui)/s)+1)
....:       for ui in u]
sage:
```

```
sage: K=[2*ui+s*randrange(
....:         ceil(-(X+2*ui)/s),
....:         floor((X-2*ui)/s)+1)
....:     for ui in u]
sage: K
[587473338058640662659869,
 -1111539179100720083770339,
 794301459533783434896055,
 688178021083749589017 51,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381]
```

```
sage: m=randrange(2)
sage: r=[randrange(2)
....:     for i in range(N)]
sage:
```

```
sage: m=randrange(2)
sage: r=[randrange(2)
....:      for i in range(N)]
sage: C=m+sum(r[i]*K[i]
....:      for i in range(N))
sage: C
2094088748748247210016703
sage:
```

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sage: r=[randrange(2)
....:     for i in range(N)]
sage: C=m+sum(r[i]*K[i]
....:     for i in range(N))
sage: C
20940887487482472l0016703
sage: C%s
2703
sage:
```

```
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....:      for i in range(N)]
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....:      for i in range(N))
sage: C
2094088748748247210016703
sage: C%s
2703
sage: (C%s)%2
1
sage:
```

```
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sage: r=[randrange(2)
....:      for i in range(N)]
sage: C=m+sum(r[i]*K[i]
....:      for i in range(N))
sage: C
2094088748748247210016703
sage: C%s
2703
sage: (C%s)%2
1
sage: m
1
sage:
```

```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:       for i in range(N)]
sage:
```

```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:      for i in range(N)]
sage: C2=m2+sum(r2[i]*K[i]
....:      for i in range(N))
sage: C2
-5172235373798273727O129
sage:
```

```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:      for i in range(N)]
sage: C2=m2+sum(r2[i]*K[i]
....:      for i in range(N))
sage: C2
-5172235373798273727O129
sage: C2%s
4971
sage:
```

```
sage: m2=randrange(2)
sage: r2=[randrange(2)
....:       for i in range(N)]
sage: C2=m2+sum(r2[i]*K[i]
....:       for i in range(N))
sage: C2
-51722353737982737270129
sage: C2%s
4971
sage: (C2%s)%2
1
sage:
```

```
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sage: C2
-5172235373798273727O129
sage: C2%s
4971
sage: (C2%s)%2
1
sage: m2
1
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```

```
sage: (C+C2)%s
7674
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13436613
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Because $C \bmod s$ and $C' \bmod s$ are small enough compared to s, have $C + C' \bmod s = (C \bmod s) + (C' \bmod s)$ and $CC' \bmod s = (C \bmod s)(C' \bmod s)$.

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Refinements: add more noise to ciphertexts, bootstrap (2009 Gentry) to control noise, etc.

Lattices

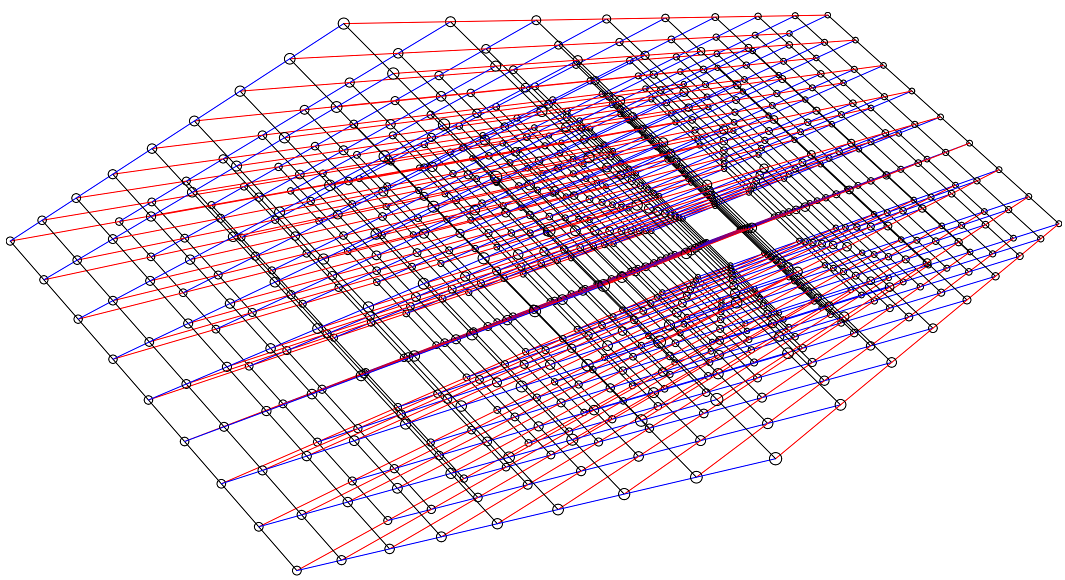
Lattices

This is a lettuce:

Lattices

This is a lettuce:

This is a lattice:



Lattices, mathematically

Assume that $V_1, \ldots, V_D \in \mathbf{R}^N$ are $\mathbf{R}$-linearly independent, i.e., $\mathbf{R}V_1 + \cdots + \mathbf{R}V_D = \{r_1V_1 + \cdots + r_DV_D : r_1, \ldots, r_D \in \mathbf{R}\}$ is a D-dimensional vector space.

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$\mathbf{Z}V_1 + \cdots + \mathbf{Z}V_D =$
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is a rank-D length-N **lattice**.

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is a rank-D length-N **lattice**.

$V_1, \ldots, V_D$
is a **basis** of this lattice.

Short vectors in lattices

Given $V_1, V_2, \ldots, V_D \in \mathbf{Z}^N$,
what is shortest vector
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"SVP: shortest-vector problem": What is shortest nonzero vector?

1982 Lenstra–Lenstra–Lovász (LLL) algorithm runs in poly time, computes a nonzero vector in L with length at most $2^{D/2}$ times length of shortest nonzero vector. Typically $\approx 1.02^D$ instead of $2^{D/2}$.

Subset-sum lattices

One way to find $(r_1, \ldots, r_N)$ where $C = r_1 K_1 + \cdots + r_N K_N$:

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Choose λ. Define
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$V_1 = (K_1, \lambda, 0, \ldots, 0)$,
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$\ldots$,
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Define $L = \mathbf{Z}V_0 + \cdots + \mathbf{Z}V_N$.
L contains the short vector
$V_0 + r_1 V_1 + \cdots + r_N V_N =$
$(0, r_1\lambda, \ldots, r_N\lambda)$.

LLL is fast but almost never finds this short vector in L.

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Is this true? Open: What's the exponent of this algorithm?

Lattice attacks on DGHV keys

Recall $K_i = 2u_i + sq_i \approx sq_i$.

Each u_i is small: $u_i < E$.

Note $q_j K_i - q_i K_j = 2q_j u_i - 2q_i u_j$.

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Define
$V_1 = (E, K_2, K_3, \ldots, K_N)$;
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$V_3 = (0, 0, -K_1, \ldots, 0)$;
$\ldots$;
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Define $L = \mathbf{Z}V_1 + \cdots + \mathbf{Z}V_N$.
L contains $q_1 V_1 + \cdots + q_N V_N =$
$(q_1 E, q_1 K_2 - q_2 K_1, \ldots) =$
$(q_1 E, 2q_1 u_2 - 2q_2 u_1, \ldots)$.

```
sage: V=matrix.identity(N)
sage:
```

```
sage: V=matrix.identity(N)
sage: V=-K[0]*V
sage:
```

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sage: V=matrix.identity(N)
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sage: Vtop=copy(K)
sage:
```

```
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```

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sage: V[0]=Vtop
sage: q0=V.LLL()[0][0]/E
sage: q0
596487875
sage:
```

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sage: q0
596487875
sage: round(K[0]/q0)
984887308997925
sage:
```

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sage: q0
596487875
sage: round(K[0]/q0)
984887308997925
sage: s
984887308997925
sage:
```

```
sage: V[0]
(1024,
 -1111539179100720083770339,
 794301459533783434896055,
 68817802108374958901751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -2356289377850037705233381)
sage:
```

```
sage: V[0]
(1024,
 -1111539179100720083770339,
 794301459533783434896055,
 68817802108374958901751,
 742362470968200823035396,
 1023345827831539515054795,
 -357168679398558876730006,
 1121421619119964601051443,
 -1109674862276222495587129,
 -235628937785003770523381)
sage: V[1]
(0, -587473338058640662659869,
 0, 0, 0, 0, 0, 0, 0, 0)
sage:
```

```
sage: V.LLL()[0]
(610803584000, 1056189937254,
 37030242384, 845898454698,
 -225618319442, 363547143644,
 1100126026284, -313150978512,
 1359463649048, 174256676348)
sage:
```

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sage: V.LLL()[0]
(610803584000, 1056189937254,
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sage: q=[Ki//s for Ki in K]
sage:
```

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(610803584000, 1056189937254,
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sage: q=[Ki//s for Ki in K]
sage: q[0]*E
610803584000
sage:
```

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(610803584000, 1056189937254,
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sage: q=[Ki//s for Ki in K]
sage: q[0]*E
610803584000
sage: q[0]*K[1]-q[1]*K[0]
1056189937254
sage:
```

```
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(610803584000, 1056189937254,
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 1100126026284, -313150978512,
 1359463649048, 174256676348)
sage: q=[Ki//s for Ki in K]
sage: q[0]*E
610803584000
sage: q[0]*K[1]-q[1]*K[0]
1056189937254
sage: q[0]*K[9]-q[9]*K[0]
174256676348
sage:
```

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can choose key sizes where
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e.g. all attacks take $\geq 2^{72}$ cycles with public keys only 802MB.

2012 Chen–Nguyen: faster attack. Need bigger DGHV/CMNT keys.

Big attack surfaces are dangerous

1991 Chaum–van Heijst–Pfitzmann: choose p sensibly; define $C(x, y) = 4^x 9^y \bmod p$ for suitable ranges of x and y.

Simple, beautiful, structured. Very easy security reduction: finding C collision implies computing a discrete logarithm.

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Simple, beautiful, structured. Very easy security reduction: finding C collision implies computing a discrete logarithm.

Typical exaggerations: C is "provably secure"; C is "cryptographically collision-free"; "security follows from rigorous mathematical proofs".

Security losses in C include
1922 Kraitchik (index calculus);
1986 Coppersmith–Odlyzko–
Schroeppel (NFS predecessor);
1993 Gordon (general DL NFS);
1993 Schirokauer (faster NFS);
1994 Shor (quantum poly time);
many subsequent attack speedups
from people who care about
pre-quantum security.

C is very bad cryptography.
No matter what user's cost limit
is, obtain better security with
"unstructured" compression-
function designs such as BLAKE.

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Some mathematical structure
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but pursuing simple structures
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Pre-quantum example: DH is simpler than ECDH, but DH has suffered many more security losses than ECDH. State-of-the-art DH attacks are very complicated.

2013 Barbulescu–Gaudry–Joux–Thomé: pre-quantum quasi-poly break of small-characteristic DH.

The state-of-the-art attacks against Cohen's cryptosystem are much more complicated than the cryptosystem is. Scary!

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Lattice-based cryptosystems are advertised as "algorithmically simple", consisting mainly of "linear operations on vectors". Attacks exploit this structure!

For efficiency, lattice-based cryptosystems usually have features that expand the attack surface even more: e.g., rings and decryption failures.

NISTPQC

NIST received 82 submissions.
69 submissions in round 1,
from hundreds of people;
22 signature submissions,
47 encryption submissions.

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Round 3 starting soon.
My guesses: NIST will announce
short list of planned standards
+ short backup list; and will
overemphasize speed.

Lattice-based signature submissions:

- Dilithium: round 2.
- DRS: **broken**; eliminated.
- FALCON☢: round 2.
- pqNTRUSign☢: eliminated.
- qTESLA: mistaken security "theorems"; round 2; **some parameters broken**.

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☢: submitter claims patent on this submission. Warning: even without ☢, submission could be covered by other patents!

Lattice-based encryption submissions in round 2: Frodo, Kyber, LAC, NewHope, NTRU, NTRU Prime, Round5☢, SABER, ThreeBears (≈lattice).

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Other round-1 lattice-based encryption submissions: Compact LWE☢ (**broken**), Ding☢, EMBLEM, KINDI, LIMA, Lizard☢, LOTUS, Mersenne (≈lattice, big keys), Odd Manhattan (big keys), OKCN/AKCN/CNKE/KCL☢, Ramstake (≈lattice, big keys), Titanium.

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All lattice submissions have suffered security losses.

Examples of attack improvements after beginning of round 1:

2018 Laarhoven–Mariano: saves "between a factor 20 to 40" in sieving, asymptotically fastest SVP attack known.

2018 Bai–Stehlé–Wen: new BKZ variant, "bases of better quality" for the "same cost" of SVP.

2018 Aono–Nguyen–Shen: quantum enumeration. For cryptographic sizes, costs less than sieving in some cost metrics.

2018 Anvers–Vercauteren–Verbauwhede: “an attacker can significantly reduce the security of (Ring/Module)-LWE/LWR based schemes that have a relatively high failure rate”.

Frodo, Kyber, LAC, NewHope, Round5, SABER, ThreeBears have nonzero failure rates.

For LAC-128, “the failure rate is 2^{48} times bigger than estimated”. Failure rate is also what broke first version of Round5.

2019 Albrecht–Ducas–Herold–Kirshanova–Postlethwaite–Stevens: "Our solution for the SVP-151 challenge was found 400 times faster than the time reported for the SVP-150 challenge, the previous record."

2019 Pellet-Mary–Hanrot–Stehlé broke claimed half-exponential approximation-factor barrier for number-theoretic attacks against Ideal-SVP. (These attacks broke cyclotomic STOC 2009 Gentry FHE in quantum poly time.)

2019 Guo–Johansson–Yang: faster attacks against some systems that use error correction to reduce decryption failures. (Violates security claims for LAC.)

2020 Dachman-Soled–Ducas–Gong–Rossi: slightly faster attacks against constant-sum secrets (LAC, NTRU, Round5).

2020 Albrecht–Bai–Fouque–Kirchner–Stehlé–Wen: better exponent for enumeration and quantum enumeration.

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"Strong security guarantees from *worst-case* hardness" of problems that "have been deeply studied by some of the great mathematicians and computer scientists going back at least to Gauss".

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